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# Quotient of Deligne–Lusztig varieties

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ABSTRACT

We study the quotient of parabolic Deligne–Lusztig varieties by a finite unipotent group $\mathbf{U}^F$ where $\mathbf{U}$ is the unipotent radical of a rational parabolic subgroup $\mathbf{P} = \mathbf{LU}$. We show that in some particular cases the cohomology of this quotient can be expressed in terms of "smaller" parabolic Deligne–Lusztig varieties associated to the Levi subgroup $\mathbf{L}$.



## Introduction

The very first approach to the representation theory of finite reductive groups is the construction of representations via Harish-Chandra (or parabolic) induction. If $\mathbf{G}$ is a connected reductive group over $\mathbb{F} = \overline{\mathbb{F}}_p$ with an $\mathbb{F}_q$-structure associated to a Frobenius endomorphism $F : \mathbf{G} \longrightarrow \mathbf{G}$, and $\mathbf{P}$ is an $F$-stable parabolic subgroup with an $F$-stable Levi complement $\mathbf{L}$, one can define, over any ring $\Lambda$, the following functors

$$\mathrm{R}_{\mathbf{L}}^{\mathbf{G}} : \Lambda\mathbf{L}^F\text{-mod} \longrightarrow \Lambda\mathbf{G}^F\text{-mod}$$

and

$$^*\mathrm{R}_{\mathbf{L}}^{\mathbf{G}} : \Lambda\mathbf{G}^F\text{-mod} \longrightarrow \Lambda\mathbf{L}^F\text{-mod}$$

called Harish-Chandra induction and restriction functors. One of the main features of these functors is that they satisfy the so-called Mackey formula: if $\mathbf{Q}$ is another $F$-stable parabolic subgroup with

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$F$-stable Levi complement $\mathbf{M}$ then

$$^*\mathrm{R}_{\mathbf{M}}^{\mathbf{G}} \circ \mathrm{R}_{\mathbf{L}}^{\mathbf{G}} \simeq \sum \mathrm{R}_{\mathbf{L}\cap {}^x\mathbf{M}}^{\mathbf{L}} \circ {}^*\mathrm{R}_{\mathbf{L}\cap {}^x\mathbf{M}}^{{}^x\mathbf{M}} \circ \operatorname{ad} x$$

where $x$ runs over an explicit finite set associated to $\mathbf{L}$ and $\mathbf{M}$. In addition to being a powerful tool for studying an induced representation, this formula is also essential for proving that the Harish-Chandra functors depend on $\mathbf{L}$ only and not on the choice of $\mathbf{P}$.

It turns out that not all the representations of $\mathbf{G}^F$ can be obtained by Harish-Chandra induction (already for $\mathbf{G} = \mathrm{SL}_2(\mathbb{F})$, many representations are *cuspidal*). To resolve this problem Deligne and Lusztig defined in [6] a generalised induction in the case where $\mathbf{P}$ is no longer $F$-stable but $\mathbf{L}$ still is. They constructed morphisms between the Grothendieck groups

$$\mathrm{R}_{\mathbf{L}}^{\mathbf{G}} : K_0\big(\Lambda\mathbf{L}^F\text{-mod}\big) \longrightarrow K_0\big(\Lambda\mathbf{G}^F\text{-mod}\big)$$

and

$$^*\mathrm{R}_{\mathbf{L}}^{\mathbf{G}} : K_0\big(\Lambda\mathbf{G}^F\text{-mod}\big) \longrightarrow K_0\big(\Lambda\mathbf{L}^F\text{-mod}\big)$$

still satisfying the Mackey formula. These morphisms come from a virtual character given by the $\ell$-adic cohomology of a quasi-projective variety $\widetilde{\mathrm{X}}_{\mathbf{L},\mathbf{P}}$, the *parabolic Deligne–Lusztig variety* associated to $(\mathbf{L}, \mathbf{P})$. Here, $\Lambda$ is a finite extension of $\mathbb{Q}_\ell$, $\mathbb{Z}_\ell$ or $\mathbb{F}_\ell$.

When $\Lambda$ is field of characteristic zero, the category $\Lambda\mathbf{G}^F$-mod is semisimple, and its Grothendieck group encodes most of the information. However, in the modular framework, that is when $\Lambda = \mathbb{Z}_\ell$ or $\mathbb{F}_\ell$, the Deligne–Lusztig induction and restriction morphisms give only partial information on the category of modules. To obtain homological properties, one needs to consider the complex $\mathrm{R}\Gamma_c(\mathrm{X}, \Lambda)$ representing the cohomology of the variety in the derived category $D^b(\Lambda\mathbf{G}^F\text{-mod})$. Using this point of view, Bonnafé and Rouquier defined in [1] triangulated functors

$$\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : D^b\big(\Lambda\mathbf{L}^F\text{-mod}\big) \longrightarrow D^b\big(\Lambda\mathbf{G}^F\text{-mod}\big)$$

and

$$^*\mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}} : D^b\big(\Lambda\mathbf{G}^F\text{-mod}\big) \longrightarrow D^b\big(\Lambda\mathbf{L}^F\text{-mod}\big).$$

Unlike the previous functors, these are not expected to satisfy a naive Mackey formula as they highly depend on the choice of $\mathbf{P}$. However, there is a good evidence that the composition $^*\mathcal{R}_{\mathbf{M}\subset\mathbf{Q}}^{\mathbf{G}} \circ \mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ should be somehow related to functors associated to smaller Levi subgroups. The purpose of this paper is to investigate the case where $\mathbf{Q}$ is $F$-stable. If $\mathbf{U}$ denotes its unipotent radical, then the composition $^*\mathcal{R}_{\mathbf{M}\subset\mathbf{Q}}^{\mathbf{G}} \circ \mathcal{R}_{\mathbf{L}\subset\mathbf{P}}^{\mathbf{G}}$ is induced by the cohomology of the quotient variety $\mathbf{U}^F\backslash\widetilde{\mathrm{X}}_{\mathbf{L},\mathbf{P}}$.

In the original paper of Deligne and Lusztig [6], the Levi subgroup $\mathbf{L}$ is a torus and $\widetilde{\mathrm{X}}_{\mathbf{L},\mathbf{P}}$ corresponds to some element $w$ of the Weyl group $W$ of $\mathbf{G}$. The motivating example is when $(\mathbf{L}, \mathbf{P})$ represents a Coxeter torus, that is when $w$ is a Coxeter element of $W$. In that case, the variety $\mathrm{X}_{\mathbf{L},\mathbf{P}} = \widetilde{\mathrm{X}}_{\mathbf{L},\mathbf{P}}/\mathbf{L}^F$ is contained in the maximal Schubert cell and its quotient by $\mathbf{U}^F$ has been computed by Lusztig in [15]. In the case where $\Lambda = \overline{\mathbb{Q}}_\ell$ it is given by the following quasi-isomorphism of $\mathbf{M}^F$-modules:

$$\mathrm{R}\Gamma_c\big(\mathbf{U}^F\backslash\mathrm{X}_{\mathbf{L},\mathbf{P}}, \overline{\mathbb{Q}}_\ell\big) \simeq \mathrm{R}\Gamma_c(\mathrm{X}_{\mathbf{L}\cap\mathbf{M},\mathbf{P}\cap\mathbf{M}}, \overline{\mathbb{Q}}_\ell) \otimes \mathrm{R}\Gamma_c\big(\big(\mathbb{F}^\times\big)^d, \overline{\mathbb{Q}}_\ell\big)$$

where $d$ is the semisimple $\mathbb{F}_q$-index of $\mathbf{M}$ in $\mathbf{G}$. Surprisingly, this isomorphism does not come from an $\mathbf{M}^F$-equivariant isomorphism of varieties, and we will see that it is more natural to study the quotient of $\widetilde{\mathrm{X}}_{\mathbf{L},\mathbf{P}}$ instead of $\mathrm{X}_{\mathbf{L},\mathbf{P}}$.

In general, the variety $\mathrm{X}_{\mathbf{L},\mathbf{P}}$ is not contained in only one Schubert cell. The strategy towards the determination of the cohomology of $\mathbf{U}^F\backslash\widetilde{\mathrm{X}}_{\mathbf{L},\mathbf{P}}$ will consist in the following steps:

- decompose the variety $\widetilde{\mathrm{X}}_{\mathbf{L},\mathbf{P}}$ into *pieces* $\widetilde{\mathrm{X}}_x$ coming from the decomposition of $\mathbf{G}/\mathbf{P}$ into $\mathbf{Q}$-orbits (see Section 2);
- in some well-identified cases, express the cohomology of $\mathbf{U}^F\backslash\widetilde{\mathrm{X}}_x$ in terms of parabolic Deligne–Lusztig varieties associated to Levi subgroups of $\mathbf{M}$ (see Section 3).

The second step is undoubtedly the most difficult. We are able to provide a satisfactory solution to this problem in presumably very specific situations, namely when the pair $(\mathbf{L}\cap{}^x\mathbf{M},\mathbf{P}\cap{}^x\mathbf{M})$ is close to $(\mathbf{L},\mathbf{P})$ (see Theorem 3.11 for more details). However, it turns out that our main result is general enough to cover most of the Deligne–Lusztig varieties associated with unipotent $\Phi_d$-blocks with cyclic defect group. This should give many new results on the geometric version of Broué's abelian defect group conjecture. To illustrate this phenomenon, we compute in Section 3.3 the principal part of the cohomology of the parabolic variety associated to the principal $\Phi_{2n-2}$-block for a group of type $B_n$ as well as its Alvis–Curtis dual. In subsequent papers this baby example will be supplemented by the following results:

- for exceptional groups, the determination of the cohomology of varieties associated to principal $\Phi_d$-blocks when $d$ is the largest regular number besides the Coxeter number. This should be enriched with predictions for the corresponding Brauer trees [14];
- for groups of type $A_n$, the determination of the cohomology of varieties associated to any unipotent block from the knowledge of the cohomology of the variety $\mathrm{X}(\mathbf{w}_0^2)$ [13].

## 1. Parabolic Deligne–Lusztig varieties

Let $\mathbf{G}$ be a connected reductive algebraic group, together with an isogeny $F$, some power of which is a Frobenius endomorphism. In other words, there exists a positive integer $\delta$ such that $F^\delta$ defines a split $\mathbb{F}_{q^\delta}$-structure on $\mathbf{G}$ for a certain power $q^\delta$ of the characteristic $p$ (note that $q$ might not be an integer). For all $F$-stable algebraic subgroup $\mathbf{H}$ of $\mathbf{G}$, we will denote by $H=\mathbf{H}^F$ the finite group of fixed points.

We fix a Borel subgroup $\mathbf{B}$ containing a maximal torus $\mathbf{T}$ of $\mathbf{G}$ such that both $\mathbf{B}$ and $\mathbf{T}$ are $F$-stable. They define a root system $\Phi$ with basis $\Delta$, and a set of positive (resp. negative) roots $\Phi^+$ (resp. $\Phi^-$). Note that the corresponding Weyl group $W$ is endowed with an action of $F$, compatible with the isomorphism $W\simeq N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. The set of simple reflections will be denoted by $S$. We shall also consider representatives $\{\dot{w}\mid w\in W\}$ of $W$ in $N_{\mathbf{G}}(\mathbf{T})$ compatible with the action of $F$ (this is possible by [9, Proposition 8.21]).

To any subset $I\subset S$ one can associate a standard parabolic subgroup $\mathbf{P}_I$ containing $\mathbf{B}$ and a standard Levi subgroup $\mathbf{L}_I$ containing $\mathbf{T}$. If $\mathbf{U}_I$ denotes the unipotent radical of $\mathbf{P}_I$, the parabolic subgroup can be written as $\mathbf{P}_I=\mathbf{L}_I\mathbf{U}_I$. The corresponding root system will be denoted by $\Phi_I$. Let $\mathbf{U}$ (resp. $\mathbf{U}^-$) be the unipotent radical of $\mathbf{B}$ (resp. the opposite Borel subgroup $\mathbf{B}^-$). Each root $\alpha$ defines a one-parameter subgroup $\mathbf{U}_\alpha$, and we will denote by $u_\alpha:\mathbb{F}\longrightarrow\mathbf{U}_\alpha$ an isomorphism of algebraic group. In order to simplify the calculations, we shall choose these isomorphisms so that $u_\alpha(\lambda)\dot{s}_\alpha=u_{-\alpha}(\lambda^{-1})\alpha^\vee(\lambda)u_\alpha(-\lambda^{-1})$. Note that the groups $\mathbf{U}_\alpha$ might not be $F$-stable in general even though the groups $\mathbf{U}$ and $\mathbf{U}^-$ are.

Finally, we denote by $B_W^+$ (resp $B_W$) the Artin–Tits monoid (resp. Artin–Tits group) of $W$, and by $\mathbf{S}=\{\mathbf{s}_\alpha\mid\alpha\in\Delta\}$ its generating set. The reduced elements of $B_W^+$ form a set $\mathbf{W}$ which is in bijection with $W$ via the canonical projection $B_W\twoheadrightarrow W$. We shall also consider the semi-direct product $B_W\rtimes\langle F\rangle$ where $F\cdot\mathbf{b}={}^F\mathbf{b}\cdot F$.

Let $\mathbf{I}$ be a subset of $\mathbf{S}$ and denote by $B_{\mathbf{I}}^+$ the submonoid of $B_W^+$ generated by $\mathbf{I}$. Following [9], we will denote by $\mathbf{I}\xrightarrow{\mathbf{b}}{}^F\mathbf{I}$ any pair $(\mathbf{I},\mathbf{b})$ with $\mathbf{b}\in B_W^+$ satisfying the following properties:

- any left divisor of $\mathbf{b}$ in $B_{\mathbf{I}}^+$ is trivial;
- ${}^{\mathbf{b}F}\mathbf{I}=\mathbf{I}$, that is every $\mathbf{s}\in\mathbf{I}$ satisfies $\mathbf{b}^{-1}\mathbf{s}\mathbf{b}\in{}^F\mathbf{I}$.

Digne and Michel have constructed in [9] a *parabolic Deligne–Lusztig variety* $\mathrm{X}(\mathbf{I}, \mathbf{b}F)$ associated to any such pair. Note that when $\mathbf{b} = \mathbf{w} \in \mathbf{W}$ and if $w$ denotes its image by the canonical projection $B_W \twoheadrightarrow W$, the previous conditions are equivalent to $w$ being $I$-reduced with ${}^{wF}I = I$. In that case, the variety $\mathrm{X}(\mathbf{I}, \mathbf{w}F)$ can be written

$$\mathrm{X}(I, wF) = \{g \in \mathbf{G} \mid g^{-1}{}^F g \in \mathbf{P}_I w^F\mathbf{P}_I\}/\mathbf{P}_I.$$

As in the case of tori, we can construct a Galois covering of $\mathrm{X}(I, wF)$. It is well-defined up to a choice of a representative $n$ of $w$ in $N_{\mathbf{G}}(\mathbf{T})$:

$$\widetilde{\mathrm{X}}(I, nF) = \{g \in \mathbf{G} \mid g^{-1}{}^F g \in \mathbf{U}_I n^F\mathbf{U}_I\}/\mathbf{U}_I.$$

The natural projection $\mathbf{G}/\mathbf{U}_I \longrightarrow \mathbf{G}/\mathbf{P}_I$ makes $\widetilde{\mathrm{X}}(I, nF)$ an $\mathbf{L}_I^{nF}$-torsor over $\mathrm{X}(I, wF)$. By using an $F$-stable Tits homomorphism $t : B_W \longrightarrow N_{\mathbf{G}}(\mathbf{T})$ extending $w \in W \longmapsto \dot{w}$, Digne and Michel have generalised in [9] this construction to any element $\mathbf{I} \xrightarrow{\mathbf{b}} {}^F\mathbf{I}$. The corresponding variety will be denoted by $\widetilde{\mathrm{X}}(\mathbf{I}, \mathbf{b}F)$. It is an $\mathbf{L}_I^{t(\mathbf{b})F}$-torsor over $\mathrm{X}(\mathbf{I}, \mathbf{b}F)$. When $\mathbf{b} = \mathbf{w} \in \mathbf{W}$ we shall simply denote $t(\mathbf{w})$ by $\dot{w}$.

**Remark 1.1.** When $\mathbf{I}$ is empty, we obtain the usual Deligne–Lusztig varieties $\mathrm{X}(\mathbf{b}F)$ and $\widetilde{\mathrm{X}}(\mathbf{b}F)$ associated to any element $\mathbf{b}$ of the Braid monoid (as defined in [3] or [1]).

## 2. Decomposing the quotient of $\mathrm{X}(I, wF)$

Let $(I, w)$ be a pair consisting of an element $w$ of $W$ and a subset $I$ of $S$ such that $w$ is $I$-reduced and ${}^{wF}I = I$. Let $J$ be another subset of $S$. If $J$ is $F$-stable, then so is the corresponding standard parabolic subgroup $\mathbf{P}_J$ and its unipotent radical $\mathbf{U}_J$. In this section we are interested in describing the quotient of the parabolic Deligne–Lusztig variety

$$\mathrm{X}(I, wF) = \{g \in \mathbf{G} \mid g^{-1}{}^F g \in \mathbf{P}_I w^F\mathbf{P}_I\}/\mathbf{P}_I$$

by the finite unipotent group $U_J$. Our main goal is to express this quotient (or at least its cohomology) in terms of "smaller parabolic varieties" associated to the Levi subgroup $\mathbf{L}_J$.

Throughout this paper, $\Lambda$ will be any extension of the ring $\mathbb{Z}_\ell$ of $\ell$-adic integers. We shall always assume that $\ell$ is different from $p$, so that by cohomology over $\Lambda$ we mean the extension of the étale cohomology of quasi-projective varieties with coefficients in $\mathbb{Z}_\ell$. The properties of $\mathrm{R}\Gamma_c(-, \Lambda)$ that we will use are either classical or can be found in [16].

### 2.1. A general method

Recall that the partial flag variety $\mathbf{G}/\mathbf{P}_I$ admits a decomposition into $\mathbf{P}_J$-orbits $\mathbf{G}/\mathbf{P}_I = \coprod \mathbf{P}_J x\mathbf{P}_I$ where $x$ runs over any set of representatives of $W_J\backslash W/W_I$. The restriction of this decomposition to $\mathrm{X}(I, wF)$ can be written as

$$\mathrm{X}(I, wF) = \coprod_{x \in [W_J\backslash W/W_I]} \{px\mathbf{P}_I \in \mathbf{P}_J x\mathbf{P}_I/\mathbf{P}_I \mid p^{-1}{}^F p \in x(\mathbf{P}_I w^F\mathbf{P}_I)^F x^{-1}\}. \tag{2.1}$$

We will denote by $\mathrm{X}_x = \mathrm{X}(I, wF) \cap \mathbf{P}_J x\mathbf{P}_I/\mathbf{P}_I$ a piece of this decomposition. It is a locally closed $P_J$-subvariety of $\mathrm{X}(I, wF)$. Now, each of these pieces can be lifted up to $\mathbf{P}_J$. More precisely, if we define the variety

$$\mathrm{Z}_x = \{p \in \mathbf{P}_J \mid p^{-1}{}^F p \in x(\mathbf{P}_I w^F\mathbf{P}_I)^F x^{-1}\}$$

then the canonical projection $\mathbf{G} \longrightarrow \mathbf{G}/\mathbf{P}_I$ induces a fibration $\mathrm{Z}_x \longrightarrow \mathrm{X}_x$ with fiber isomorphic to $\mathbf{P}_J \cap {}^x\mathbf{P}_I$. Now if we assume that $x$ is $J$-reduced-$I$, the intersection $\mathbf{P}_J \cap {}^x\mathbf{P}_I$ can be decomposed as $\mathbf{P}_J \cap {}^x\mathbf{P}_I = (\mathbf{L}_J \cap {}^x\mathbf{P}_I) \cdot (\mathbf{U}_J \cap {}^x\mathbf{U})$. Furthermore, $\mathbf{L}_J \cap {}^x\mathbf{P}_I$ is a standard parabolic subgroup of $\mathbf{L}_J$ (it contains $\mathbf{L}_J \cap \mathbf{B}$) and hence it can be written $\mathbf{L}_J \cap \mathbf{P}_{K_x}$ with $K_x = J \cap {}^x\Phi_I$. The cohomology of $\mathrm{X}_x$ is thus given by

$$\mathrm{R}\Gamma_c(\mathrm{X}_x, \Lambda) \simeq \mathrm{R}\Gamma_c(\mathrm{Z}_x/\mathbf{L}_J \cap \mathbf{P}_{K_x}, \Lambda)\big[2\dim \mathbf{U}_J \cap {}^x\mathbf{U}\big]. \tag{2.2}$$

The advantage of this description is that the quotient of the variety $\mathrm{Z}_x$ by $U_J$ is easier to compute. If we decompose $p \in \mathbf{P}_J$ as $p = ul \in \mathbf{U}_J\mathbf{L}_J$ then the quotient variety can be written (see for example [11, Proposition 1.3])

$$U_J\backslash \mathrm{Z}_x = \big\{(\bar{p}, l) \in \big[\big(x\mathbf{P}_I w^F\mathbf{P}_I{}^F x^{-1}\big) \cap \mathbf{P}_J\big] \times \mathbf{L}_J \;\big|\; \pi_J(\bar{p}) = l^{-1}{}^F l\big\}$$

where $\pi_J : \mathbf{P}_J \longrightarrow \mathbf{L}_J$ is the canonical projection.

Our aim is to relate this variety to “smaller” parabolic Deligne–Lusztig varieties. For that purpose, we need to identify the double cosets in which $l^{-1}{}^F l$ lies, which amounts to decomposing the intersection $(x\mathbf{P}_I w^F\mathbf{P}_I{}^F x^{-1}) \cap \mathbf{P}_J$ as well as its image under $\pi_J$. Let $v \in W_J$ be a $K_x$-reduced-${}^F K_x$ element. We can decompose the double coset $\mathbf{P}_{K_x} v^F\mathbf{P}_{K_x}$ as follows:

$$\mathbf{P}_{K_x} v^F\mathbf{P}_{K_x} = (\mathbf{L}_J \cap \mathbf{P}_{K_x})\mathbf{U}_J v\big(\mathbf{L}_J \cap {}^F\mathbf{P}_{K_x}\big).$$

Since $\mathbf{L}_J \cap \mathbf{P}_{K_x} = \mathbf{L}_J \cap {}^x\mathbf{P}_I$ is a subgroup of ${}^x\mathbf{P}_I$, the intersection $(x\mathbf{P}_I w^F\mathbf{P}_I{}^F x^{-1}) \cap (\mathbf{P}_{K_x} v^F\mathbf{P}_{K_x})$ is non-empty if and only if $(x\mathbf{P}_I w^F\mathbf{P}_I{}^F x^{-1} v^{-1}) \cap \mathbf{U}_J$ is. In this case, the projection $\pi_J : \mathbf{P}_J \longrightarrow \mathbf{L}_J$ induces a fibration $(x\mathbf{P}_I w^F\mathbf{P}_I{}^F x^{-1}) \cap (\mathbf{P}_{K_x} v^F\mathbf{P}_{K_x}) \longrightarrow (\mathbf{L}_J \cap \mathbf{P}_{K_x})v(\mathbf{L}_J \cap {}^F\mathbf{P}_{K_x})$ with fiber isomorphic to $(x\mathbf{P}_I w^F\mathbf{P}_I{}^F x^{-1} v^{-1}) \cap \mathbf{U}_J$. If we define $\mathrm{Z}_x^v$ to be the variety

$$\mathrm{Z}_x^v = \big\{(\bar{p}, l) \in \big[\big(x\mathbf{P}_I w^F\mathbf{P}_I{}^F x^{-1}\big) \cap \big(\mathbf{P}_{K_x} v^F\mathbf{P}_{K_x}\big)\big] \times \mathbf{L}_J \;\big|\; \pi_J(\bar{p}) = l^{-1}{}^F l\big\}$$

then we obtain a decomposition of $U_J\backslash \mathrm{Z}_x$ into locally closed subvarieties together with $L_J$-equivariant maps

$$\mathrm{Z}_x^v \longrightarrow \big\{l \in \mathbf{L}_J \;\big|\; l^{-1}{}^F l \in \mathbf{L}_J \cap \mathbf{P}_{K_x} v^F(\mathbf{L}_J \cap \mathbf{P}_{K_x})\big\} \tag{2.3}$$

with fibers isomorphic to $(x\mathbf{P}_I w^F\mathbf{P}_I{}^F x^{-1} v^{-1}) \cap \mathbf{U}_J$.

**Remark 2.4.** In the case where ${}^{vF}K_x = K_x$, the quotient by $\mathbf{L}_J \cap \mathbf{P}_{K_x}$ of the variety on the right-hand side of (2.3) can be identified with the parabolic Deligne–Lusztig variety associated to $K_x \xrightarrow{v} {}^F K_x$. We shall, by convenient abuse of notation, denote it by $\mathrm{X}_{\mathbf{L}_J}(K_x, vF)$ even when $vF$ does not normalise $K_x$.

Finally, we set $\mathfrak{Z}_x^v = \mathrm{Z}_x^v/\mathbf{L}_J \cap \mathbf{P}_{K_x}$. The right action of $\mathbf{U}_J \cap {}^x\mathbf{U}$ on $\mathrm{Z}_x$ induces an action by $F$-conjugation on $\mathfrak{Z}_x^v$ and let $\mathfrak{X}_x^v = \mathfrak{Z}_x^v/\mathbf{U}_J \cap {}^x\mathbf{U}_I$ be the quotient (equivalently, it is the image of $\mathrm{Z}_x^v$ by the morphism $U_J\backslash \mathrm{Z}_x \twoheadrightarrow U_J\backslash \mathrm{X}_x$). At this point we have obtained:

- A decomposition of $U_J\backslash \mathrm{X}(I, wF)$ into some locally closed $L_J$-varieties $\mathfrak{X}_x^v$.
- A quasi-isomorphism $\mathrm{R}\Gamma_c(\mathfrak{X}_x^v, \Lambda) \simeq \mathrm{R}\Gamma_c(\mathfrak{Z}_x^v, \Lambda)[2\dim \mathbf{U}_J \cap {}^x\mathbf{U}]$ (obtained as in (2.2)).
- An $L_I$-equivariant morphism $\mathfrak{Z}_x^v \longrightarrow \mathrm{X}_{L_J}(K_x, vF)$ with fiber isomorphic to $(x\mathbf{P}_I w^F\mathbf{P}_I{}^F x^{-1} v^{-1}) \cap \mathbf{U}_J$.

Therefore, if we want to express the cohomology of $U_J\backslash \mathrm{X}(I, wF)$ in terms of the different varieties $\mathrm{X}_{L_J}(K_x, vF)$ that can appear we need to refine the description of the latter morphism. This will be done in Section 2.3 after discussing the case of parabolic varieties associated to elements of the Braid monoid.

**Remark 2.5.** When ${}^{vF}K_x = K_x$, we can actually be more precise: $l^{-1}{}^F l$ can be written uniquely as $l_1 \dot{v}^F l_2$ with $l_1 \in (\mathbf{L}_J \cap \mathbf{U}_{K_x}) \cap {}^{vF}(\mathbf{L}_J \cap \mathbf{U}_{K_x}^-)$ and $l_2 \in \mathbf{L}_J \cap \mathbf{P}_{K_x}$. Then for $z \in (x\mathbf{P}_I w^F \mathbf{P}_I{}^F x^{-1} v^{-1}) \cap \mathbf{U}_J$ we have $(l_1 z \dot{v}^F l_2, l) \in \mathrm{Z}_x^v$ and all the elements are obtained that way. In other words, we have the following isomorphism of varieties

$$\mathrm{Z}_x^v \simeq \big[\big(x\mathbf{P}_I w^F \mathbf{P}_I{}^F x^{-1} v^{-1}\big) \cap \mathbf{U}_J\big] \times \big\{ l \in \mathbf{L}_J \bigm| l^{-1}{}^F l \in \mathbf{L}_J \cap \mathbf{P}_{K_x} v^F(\mathbf{L}_J \cap \mathbf{P}_{K_x})\big\}.$$

Through this isomorphism the group $L_J$ (resp. $\mathbf{L}_J \cap \mathbf{P}_{K_x}$) acts on $l \in \mathbf{L}_J$ by left (resp. right) multiplication. However, it is more difficult to describe the action of $\mathbf{L}_J \cap \mathbf{P}_{K_x}$ on $(x\mathbf{P}_I w^F \mathbf{P}_I{}^F x^{-1} v^{-1}) \cap \mathbf{U}_J$. In particular, $\mathfrak{Z}_x^v$ is in general not isomorphic to $[(x\mathbf{P}_I w^F \mathbf{P}_I{}^F x^{-1} v^{-1}) \cap \mathbf{U}_J] \times \mathrm{X}_{\mathbf{L}_J}(K_x, vF)$. We shall nevertheless give many examples where the cohomology of these two varieties coincide.

### 2.2. Elements of the Braid monoid

By [9, Section 6] any element $\mathbf{I} \xrightarrow{\mathbf{b}} {}^F\mathbf{I}$ can be decomposed as $\mathbf{I} = \mathbf{I}_1 \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \xrightarrow{\mathbf{w}_2} \cdots \xrightarrow{\mathbf{w}_r} \mathbf{I}_{r+1} = {}^F\mathbf{I}$ where $\mathbf{w}_i \in \mathbf{W}$. Using this property one can easily generalise the previous constructions to $\mathrm{X}(\mathbf{I}, \mathbf{b}F)$: to each tuple $\mathbf{x} = (x_1, \ldots, x_r)$ with $x_i$ a $J$-reduced-$I_i$ element of $W$ one can associate varieties $\mathrm{X}_{\mathbf{x}}$ and $\mathrm{Z}_{\mathbf{x}}$ such that

$$\mathrm{Z}_{\mathbf{x}} = \left\{ (p_1, \ldots, p_r) \in (\mathbf{P}_J)^r \;\middle|\; \begin{array}{l} p_i^{-1} p_{i+1} \in x_i \mathbf{P}_{I_i} w_i \mathbf{P}_{I_{i+1}} x_{i+1}^{-1} \\ p_r^{-1}{}^F p_1 \in x_r \mathbf{P}_{I_r} w_r{}^F \mathbf{P}_{I_1}{}^F x_1^{-1} \end{array} \right\}$$

and

$$\mathrm{R}\Gamma_c(\mathrm{X}_{\mathbf{x}}, \Lambda) \simeq \mathrm{R}\Gamma_c\Big(\mathrm{Z}_{\mathbf{x}} \Big/ \prod \mathbf{L}_J \cap \mathbf{P}_{K_{x_i}}, \Lambda\Big)\Big[2 \sum \dim \mathbf{U}_J \cap {}^{x_i}\mathbf{U}\Big]$$

with $K_{x_i} = J \cap {}^{x_i}\Phi_{I_i}$.

By looking at the intersections of $x_i \mathbf{P}_{I_i} w_i \mathbf{P}_{I_{i+1}} x_{i+1}^{-1}$ with double cosets of the form $\mathbf{P}_{K_{x_i}} v_i \mathbf{P}_{K_{x_{i+1}}}$ one can decompose $U_J \backslash \mathrm{Z}_{\mathbf{x}}$ into locally closed subvarieties $\mathrm{Z}_{\mathbf{x}}^{\mathbf{v}}$ together with $L_J$-equivariant maps

$$\mathrm{Z}_{\mathbf{x}}^{\mathbf{v}} \longrightarrow \left\{ (l_1, \ldots, l_r) \in (\mathbf{L}_J)^r \;\middle|\; \begin{array}{l} l_i^{-1} l_{i+1} \in (\mathbf{L}_J \cap \mathbf{P}_{K_{x_i}}) v_i (\mathbf{L}_J \cap \mathbf{P}_{K_{x_{i+1}}}) \\ l_r^{-1}{}^F l_1 \in (\mathbf{L}_J \cap \mathbf{P}_{K_{x_r}}) v_r{}^F(\mathbf{L}_J \cap \mathbf{P}_{K_{x_1}}) \end{array} \right\} \tag{2.6}$$

with fibers isomorphic to

$$\mathbf{U}_J \cap \big(x_r \mathbf{P}_{I_r} w_r{}^F \mathbf{P}_{I_1}{}^F x_1^{-1} v_r^{-1}\big) \times \prod_{i=1}^{r-1} \mathbf{U}_J \cap \big(x_i \mathbf{P}_{I_i} w_i \mathbf{P}_{I_{i+1}} x_{i+1}^{-1} v_i^{-1}\big).$$

In the case where ${}^{v_i}K_{x_{i+1}} = K_{x_i}$ and ${}^{v_r F}K_{x_1} = K_{x_r}$, the quotient by $\prod \mathbf{L}_J \cap \mathbf{P}_{K_{x_i}}$ of the variety on the right-hand side of (2.6) can be identified with the parabolic Deligne–Lusztig variety $\mathrm{X}_{\mathbf{L}_J}(\mathbf{K}_{x_1}, \mathbf{v}_1 \cdots \mathbf{v}_r F)$.

### 2.3. A further decomposition

We now study the intersection $(x\mathbf{P}_I w^F \mathbf{P}_I{}^F x^{-1} v^{-1}) \cap \mathbf{U}_J$ in order to obtain information on the fibration $\mathfrak{Z}_x^v \longrightarrow \mathrm{X}_{L_J}(K_x, vF)$ defined at the end of Section 2.1. This will be achieved using the Curtis–Deodhar decomposition.

Let $x, w, w'$ be elements of $W$, and fix a reduced expression $w = s_1 \cdots s_r$ of $w$. Recall that a *subexpression* of $w$ (with respect to the decomposition $w = s_1 \cdots s_r$) is an element of $\Gamma = \{1, s_1\} \times$

$\cdots \times \{1, s_r\}$. Such a subexpression $\gamma = (\gamma_1, \ldots, \gamma_r)$ is said to be *x-distinguished* if $\gamma_i = s_i$ whenever $x\gamma_1 \cdots \gamma_{i-1} s_i > x\gamma_1 \cdots \gamma_{i-1}$. The main result in [7] and [5] gives a decomposition of the double Schubert cell $\mathbf{B}w\mathbf{B} \cap (\mathbf{B})^x w'\mathbf{B} \subset \mathbf{G}/\mathbf{B}$ in terms of certain $x$-distinguished subexpressions of $w$, as well as an explicit parametrisation of each piece (see [12, Section 2.2] for more details).

**Theorem 2.7** *(Deodhar, Curtis). Let $w, w', x$ be elements of the Weyl group and $w = s_1 \cdots s_r$ be a reduced expression of $w$. There exists a decomposition of $\mathbf{B}w\mathbf{B} \cap (\mathbf{B})^x w'\mathbf{B}$ into locally closed subvarieties*

$$\mathbf{B}w\mathbf{B} \cap (\mathbf{B})^x w'\mathbf{B} = \coprod_{\gamma \in \Gamma_{w'}} \Omega_\gamma w'\mathbf{B}$$

*where $\gamma$ runs over the set $\Gamma_{w'}$ of subexpression of $w$ whose product is $w'$. Furthermore, the decomposition has the following properties*:

(i) *Each cell $\Omega_\gamma w'\mathbf{B}$ is stable by multiplication by $\mathbf{U} \cap \mathbf{U}^x$*;
(ii) *$\Omega_\gamma \subset \mathbf{U}^x$ and the restriction of the map $\mathbf{B}^x \longrightarrow (\mathbf{B})^x w'\mathbf{B}/\mathbf{B}$ to $\Omega_\gamma$ is injective*;
(iii) *$\Omega_\gamma$ is non-empty if and only if $\gamma$ is x-distinguished*;
(iv) *If $\Omega_\gamma$ is non-empty, then it is isomorphic to $\mathbb{A}_{n_\gamma} \times (\mathbb{G}_m)^{m_\gamma}$ where*

$$n_\gamma = \#\{i = 1, \ldots, r \mid x\gamma_1 \cdots \gamma_{i-1} s_i > x\gamma_1 \cdots \gamma_{i-1}\}$$

*and*

$$m_\gamma = \#\{i = 1, \ldots, r \mid \gamma_i = 1\}.$$

**Remark 2.8.** For convenience, we will always denote by $\mathbb{G}_m$ the spectrum of the ring $\mathbb{F}[t, t^{-1}]$ although we will not necessarily use its group structure.

In order to use this result, we first write the fiber of (2.3) as

$$\big(x\mathbf{P}_I w^F \mathbf{P}_I{}^F x^{-1} v^{-1}\big) \cap \mathbf{U}_J = \big(x\mathbf{B}W_I w\mathbf{B}^F x^{-1} v^{-1}\big) \cap \mathbf{U}_J.$$

Let $y \in W_I$, and let $\gamma$ be an $x$-distinguished subexpression of $yw$ whose product is $w' = x^{-1} v^F x$. Then the map $(z, z') \in \Omega_\gamma \times \mathbf{U}^x \cap {}^{w'}\mathbf{U} \longmapsto zz'w' \in \mathbf{B}w\mathbf{B} \cap (\mathbf{U})^x w'$ is well-defined and it is injective by Theorem 2.7(ii). By taking the union over such subexpressions, we obtain the following decomposition

$$\mathbf{U} \cap x\mathbf{B}yw\mathbf{B}\big(v^F x\big)^{-1} = \coprod_{\gamma \in \Gamma_{x^{-1}v^F x}} \big({}^x\Omega_\gamma\big) \cdot \big(\mathbf{U} \cap {}^{v^F x}\mathbf{U}\big).$$

Note that we do not need to fix a reduced expression of $y$: indeed, since $x$ is reduced-$I$, the subexpression $\gamma$ will start with any reduced expression of $y$.

Furthermore, by Theorem 2.7(i), each coset $\Omega_\gamma x^{-1} v^F x\mathbf{B}$ is stable by left multiplication by $\mathbf{U} \cap \mathbf{U}^x$, and therefore all the varieties occurring in the previous decomposition are stable by the left action of ${}^x\mathbf{U} \cap \mathbf{U}$. Since $x$ is $J$-reduced, they are in particular stable by the action of $\mathbf{L}_J \cap \mathbf{U}$. Taking the image by the projection $\varpi_J : \mathbf{U} \longrightarrow \mathbf{U}_J$ associated to the decomposition $\mathbf{U} = (\mathbf{U} \cap \mathbf{L}_J)\mathbf{U}_J$ we obtain

$$\mathbf{U}_J \cap x\mathbf{B}yw\mathbf{B}\big(v^F x\big)^{-1} = \coprod_{\gamma \in \Gamma_{x^{-1}v^F x}} \varpi_J\big({}^x\Omega_\gamma \cdot \big(\mathbf{U} \cap {}^{v^F x}\mathbf{U}\big)\big) = \coprod_{\gamma \in \Gamma_{x^{-1}v^F x}} \Upsilon_\gamma. \tag{2.9}$$

In many interesting examples, the intersection $(x\mathbf{P}_I w^F \mathbf{P}_I{}^F x^{-1} v^{-1}) \cap \mathbf{U}_J$ will always consist of at most one cell $\Upsilon_\gamma$, which will be isomorphic to $(\mathbb{G}_m)^r \times \mathbb{A}_s$ for some integers $r, s$. Note that in this case, the cell is automatically stable by the action of $(\mathbf{L}_J \cap \mathbf{P}_{K_x}) \cap {}^{vF}(\mathbf{L}_J \cap \mathbf{P}_{K_x})$ by conjugation. If in addition one can find an equivariant embedding $\Upsilon_\gamma \subset \mathbb{A}_{r+s}$, then the cohomology of $U_J \backslash \mathrm{X}_x$ can be obtained by shifts of the cohomology of $\mathrm{X}_{L_J}(K_x, vF)$. We shall not make this claim more precise as we will encounter only the cases where $r = 0$ or 1.

**Remark 2.10.** The decomposition (2.9) gives a combinatorial test for the emptiness of a piece $\mathrm{X}_x$: it is non-empty if and only if there exist $y \in W_I$ and an $x$-distinguished subexpression $\gamma$ of $yw$ such that the product of the elements of $\gamma$ lies in $x^{-1} W_J{}^F x$.

*2.4. Examples*

In this section we give examples for which the previous method is effective. Some of them will nevertheless suggest that one should rather work with the variety $\widetilde{\mathrm{X}}(I, \dot{w}F)$ instead of $\mathrm{X}(I, wF)$.

*2.4.1. Fibers are affine spaces*

Let $J$ be an $F$-stable subset of $S$. Assume that there exists a $J$-reduced-$I$ element $x$ such that $xw^F x^{-1} \in W_J$ and let $v$ be the corresponding $W_{K_x}$-reduced element. Then we have $U_J \backslash \mathrm{Z}_x = \mathrm{Z}_x^v$ and the map

$$\mathfrak{Z}_x^v \twoheadrightarrow \mathrm{X}_{\mathbf{L}_J}(K_x, vF)$$

has affine fibers. In particular, the cohomology of the varieties $U_J \backslash \mathrm{X}_x$ and $\mathrm{X}_{\mathbf{L}_J}(K_x, vF)$ differs only by a shift.

Let $v' \in W_J$. We start by showing that the intersection $x\mathbf{P}_I w^F \mathbf{P}_I{}^F x^{-1} \cap \mathbf{U}_J v'$ is empty if $v'$ and $v$ are not in the same $W_{K_x}$-coset of $W_J$. Since $wF$ normalises $I$, the element $xw^F x^{-1} F$ normalises $W_{K_x}$ and so does $vF$. Thus we can write

$$x\mathbf{P}_I w^F \mathbf{P}_I{}^F x^{-1} \cap \mathbf{U}_J v' = x\mathbf{B} w^F W_I \mathbf{B}^F x^{-1} \cap \mathbf{U}_J v' = \left({}^x\mathbf{B}\right) v^F x^F W_I \mathbf{B}^F x^{-1} \cap \mathbf{U}_J v'.$$

By multiplying by ${}^F x\mathbf{B}$, we observe that if this set is non-empty, then one of the following double Bruhat cells

$$\left({}^x\mathbf{B}\right) v^F x^F W_I \mathbf{B} \cap \mathbf{B} v'^F x \mathbf{B}$$

is also non-empty. By Theorem 2.7, this means that there exists an $x^{-1}$-distinguished subexpression $\gamma$ of $v'^F x$ such that the product of the elements of $\gamma$ lies in the coset $v^F x^F W_I$. Since $x^{-1}$ is reduced-$J$, this subexpression has to start with a reduced decomposition of $v'$. The product of its elements is therefore of the form $v'^F x'$ with $x' \leqslant x$ for the Bruhat order. But then $v'^F x' \in v^F x^F W_I$ so that $x'$ is in the double coset $W_J x W_I$. This forces $x = x'$ since $x$ is the minimal element of this coset. Now, since $W_{K_x} = W_J \cap (W_I)^x$, the condition $v'^F x \in v^F x^F W_I$ implies $v' \in v^F W_{K_x}$ which, with $vF$-normalising $W_{K_x}$ is equivalent to $v' \in W_{K_x} v$.

Now, if we assume that $v'$ is $K_x$-reduced, we must have $v' = v$. In this case, the intersection $x\mathbf{P}_I w^F \mathbf{P}_I{}^F x^{-1} \cap \mathbf{U}_J v$ is just $x\mathbf{B} x^{-1} v^F x \mathbf{B}^F x^{-1} \cap \mathbf{U}_J v$. The Curtis–Deodhar cell $\Omega_\gamma$ associated to the unique $x$-distinguished subexpression of $x^{-1} v^F x$ giving $x^{-1} v^F x$ is contained in $\mathbf{U} \cap \mathbf{U}^x$. Since the product ${}^x\Omega_\gamma \cdot (\mathbf{U} \cap {}^{v^F x}\mathbf{U})$ is stable by left multiplication by $\mathbf{U} \cap {}^x\mathbf{U}$, we deduce that

$$x\mathbf{B} x^{-1} v^F x \mathbf{B}^F x^{-1} v^{-1} \cap \mathbf{U} = \left(\mathbf{U} \cap {}^x\mathbf{U}\right) \cdot \left(\mathbf{U} \cap {}^{v^F x}\mathbf{U}\right).$$

Finally, we can write $\mathbf{U}\cap {}^{v^F x}\mathbf{U} = (\mathbf{U}\cap \mathbf{L}_J \cap {}^{v^F x}\mathbf{U})\cdot(\mathbf{U}_J \cap {}^{v^F x}\mathbf{U})$ and use the fact that $\mathbf{U}\cap\mathbf{L}_J \subset \mathbf{U}\cap {}^x\mathbf{U}$ to obtain

$$x\mathbf{P}_I w^F \mathbf{P}_I{}^F x^{-1} v^{-1} \cap \mathbf{U}_J = \big(\mathbf{U}_J \cap {}^x\mathbf{U}\big)\cdot\big(\mathbf{U}_J \cap {}^{v^F x}\mathbf{U}\big).$$

This proves that the fibers of $U_J\backslash \mathrm{Z}_x/(\mathbf{L}_J \cap \mathbf{P}_{K_x}) \twoheadrightarrow \mathrm{X}_{\mathbf{L}_J}(K_x, vF)$ are affine spaces of same dimension.

**Remark 2.11.** Note that the previous statement remains true if we replace ${}^F x$ by $x'$ with $\ell(x') = \ell(x)$. More precisely, if $I^w = I'$ and $xwx'^{-1} = v \in W_J$ is such that $(K_x)^v = K_{x'}$ then $x\mathbf{P}_I w \mathbf{P}_{I'} x'^{-1} v'^{-1} \cap \mathbf{U}_J$ is empty unless $v'$ and $v$ are in the same $W_{K_x}$-coset and in that case

$$x\mathbf{P}_I w \mathbf{P}_{I'} x'^{-1} v^{-1} \cap \mathbf{U}_J = \big(\mathbf{U}_J \cap {}^x\mathbf{U}\big)\cdot\big(\mathbf{U}_J \cap {}^{vx'}\mathbf{U}\big).$$

The condition $\ell(x) = \ell(x')$ is essential, as several $W_{K_x}$-cosets of $W_J$ might be involved otherwise.

*2.4.2. Coxeter elements for split groups*

Let $\{t_1, \ldots, t_n\}$ be the set of simple reflections associated to the basis $\Delta$ of the root system. Let $w = t_1 \cdots t_n$ be a Coxeter element. We claim that all the pieces of $\mathrm{X}(w)$ but one are empty: by Remark 2.10 applied to $J = \emptyset$, the quotient $U\backslash \mathrm{X}_x$ is non-empty if and only if there exists an $x$-distinguished subexpression of $w$ whose product is trivial. But the only subexpression of $w$ whose product is trivial is $(1, 1, \ldots, 1)$, and it is $x$-distinguished for $x = w_0$ only.

Now let $J$ be a subset of $S$ and let $x = w_J w_0$ be the element of minimal length in $W_J w_0$. Let $v \in W_J$ be such that there exists an $x$-distinguished subexpression of $w$ whose product is $v^x \in (W_J)^{w_0}$. Denote by $\widetilde{J} = \{t_{j_1}, \ldots, t_{j_m}\}$ the conjugate of $J$ by $w_0$. Then $\gamma_i = t_i$ forces $t_i \in \widetilde{J}$; furthermore, since $\gamma$ is $x$-distinguished then $\gamma_i = 1$ forces $t_i \notin \widetilde{J}$. We deduce that such a subexpression is unique and that $v = {}^x(t_{j_1} \cdots t_{j_m})$ is a Coxeter element of $W_J$.

For this subexpression, the cell $\Omega_\gamma$ is the ordered product of the groups $\mathbf{U}_i = u_{\gamma_1\cdots\gamma_i(-\alpha_i)}(?)$ where $? = \mathbb{F}$ is $\gamma_i \neq 1$ and $? = \mathbb{F}^\times$ otherwise. Note that when $i < j_b$ and $t_i \notin \widetilde{J}$, the groups $\mathbf{U}_i$ and $\mathbf{U}_{j_b}$ commute. Indeed, a positive combination of $\gamma_1\cdots\gamma_i(-\alpha_i) = t_{j_1}\cdots t_{j_a}(-\alpha_i)$ and $\gamma_1\cdots\gamma_{j_b}(-\alpha_{j_b}) = t_{j_1}\cdots t_{j_{b-1}}(\alpha_b)$ is never a root since a positive combination of $-\alpha_i \in S \smallsetminus \widetilde{J}$ and $t_{j_{a+1}}\cdots t_{j_{b-1}}(\alpha_b) \in \Phi_{\widetilde{J}}^+$ never is. Furthermore, $\mathbf{U}\cap {}^{vx}\mathbf{U} = \mathbf{L}_J \cap \mathbf{U} \cap {}^v\mathbf{U}$ and it is not difficult to show that this group commutes with the groups ${}^x\mathbf{U}_i$ whenever $t_i \notin \widetilde{J}$. As a consequence

$$\Upsilon_\gamma = \varpi_J\big({}^x\Omega \cdot \mathbf{U}\cap {}^{vx}\mathbf{U}\big) = \prod_{t_i \in S\smallsetminus \widetilde{J}} u_i\big(\mathbb{F}^\times\big).$$

We deduce that the morphism $U_J\backslash \mathrm{Z}_x = U_J\backslash \mathrm{X}(w) \longrightarrow \mathrm{X}_{\mathbf{L}_J}(v)$ has fibers isomorphic to $(\mathbb{G}_m)^{|S|-|J|}$. In [15], Lusztig actually constructs an isomorphism between $U_J\backslash \mathrm{X}(w)$ and $\mathrm{X}_{\mathbf{L}_J}(v) \times (\mathbb{G}_m)^{|S|-|J|}$, but which is not compatible with the action of $L_J$. However, he proves that the cohomology groups of these two varieties are isomorphic as $L_J$-modules [15, Corollary 2.10].

*2.4.3. $n$-th roots of $\pi$ for groups of type $A_n$*

Assume that $(\mathbf{G}, F)$ is a split group of type $A_n$. We denote by $t_1, \ldots, t_n$ the simple reflections of $W$ with the convention that there exists an isomorphism $W \simeq \mathfrak{S}_{n+1}$ sending the reflection $t_i$ to the transposition $(i, i+1)$. Let $J = \{t_1, \ldots, t_{n-1}\}$ and $w = t_1 t_2 \cdots t_{n-1} t_n t_{n-1}$ be an $n$-regular element. The $J$-reduced elements are of the form $x_i = t_n t_{n-1} \cdots t_i$ for $i = 1, \ldots, n+1$. If $i \neq 1, n$, then $x_i < x_i t_1 < x_i t_1 t_2 < \cdots < x_i w$ and therefore the only $x_i$-distinguished subexpression of $w$ is $(t_1, t_2, \ldots, t_n, t_{n-1})$. Since ${}^{x_i}w \notin W_J$, we deduce from Remark 2.10 that the pieces $\mathrm{X}_{x_i}$ are empty.

If $i = n$, then there are two $x_n$-distinguished subexpressions of $w$, namely $(t_1, t_2, \ldots, t_n, t_{n-1})$ and $(t_1, t_2, \ldots, t_n, 1)$. But only one will give an element of $W_J$, since ${}^{x_n}(t_1 \cdots t_n) \notin W_J$ whereas ${}^{x_n}w =$

$t_1 t_2 \cdots t_{n-1}$. By Example 2.4.1, the cohomology of $U\backslash \mathrm{X}_{x_n}$ is then, up to a shift, isomorphic to the cohomology of the Coxeter variety $\mathrm{X}_{\mathbf{L}_J}(t_1 \cdots t_{n-1})$.

If $i=1$ then $x_1 = w_J w_0$. In that case there are many distinguished subexpressions of $w$. However, only one has a product in $(W_J)^x = W_{\{t_2,\ldots,t_n\}}$. Indeed, that condition forces $\gamma_1$ to be 1 and therefore $\gamma = (1, t_2, \ldots, t_n, t_{n-1})$ is the only $x_1$-distinguished subexpression of $w$ whose product lies in $(W_J)^x$. For that subexpression, the Curtis–Deodhar cell ${}^x(\Omega_\gamma)$ is the product of $u_{\alpha_1+\cdots+\alpha_n}(\mathbb{G}_m)$ with some affine subspace of $\mathbf{L}_J \cap \mathbf{U}$. Since $\alpha_1 + \cdots + \alpha_n$ is the longest root, the group $\mathbf{L}_J \cap \mathbf{U}$ acts trivially on $\mathbf{U}_{\alpha_1+\cdots+\alpha_n}$ and we obtain $\Upsilon_\gamma = u_{\alpha_1+\cdots+\alpha_n}(\mathbb{G}_m) \simeq \mathbb{G}_m$.

As in the Coxeter case, the varieties $U_J\backslash \mathrm{X}_{x_1}$ and $\mathrm{X}_{\mathbf{L}_J}(t_1 t_2 \cdots t_{n-2} t_{n-1} t_{n-2}) \times \mathbb{G}_m$ can be shown to have the same cohomology (see [8, Proposition 8.17]) but are non-isomorphic as $L_J$-varieties. However, there is a good evidence that such an isomorphism should hold for some Galois coverings of X and $\mathbb{G}_m$. We shall make this statement precise in the next section (see Section 3.3 for an application to this example).

## 3. Lifting the decomposition to $\widetilde{\mathrm{X}}(I, \dot{w}F)$

Recall that one can associate to $\mathbf{I} \xrightarrow{\mathbf{b}} {}^F\mathbf{I}$ a variety $\widetilde{\mathrm{X}}(\mathbf{I}, \mathbf{b}F)$ together with a Galois covering $\pi_{\mathbf{b}} : \widetilde{\mathrm{X}}(\mathbf{I}, \mathbf{b}F) \longrightarrow \mathrm{X}(\mathbf{I}, \mathbf{b}F)$ with Galois group $\mathbf{L}_I^{t(\mathbf{b})F}$. Using this map one can pullback the previous constructions. More precisely, one can define the varieties $\widetilde{\mathrm{X}}_{\mathbf{x}} = \pi_{\mathbf{b}}^{-1}(\mathrm{X}_{\mathbf{x}})$ in order to obtain a partition of $\widetilde{\mathrm{X}}(\mathbf{I}, \mathbf{b}F)$ into locally closed $P_J \times \mathbf{L}_I^{t(\mathbf{b})F}$-subvarieties. Furthermore, we can lift the definition of $\mathrm{Z}_{\mathbf{x}}$ by considering the following cartesian diagram:

$$\begin{tikzcd} \widetilde{\mathrm{Z}}_{\mathbf{x}} \arrow[r, two heads, "/\mathbf{L}_I^{t(\mathbf{b})F}"] \arrow[d] & \mathrm{Z}_{\mathbf{x}} \arrow[d] \\ \widetilde{\mathrm{X}}_{\mathbf{x}} \arrow[r, two heads, "/\mathbf{L}_I^{t(\mathbf{b})F}"] & \mathrm{X}_{\mathbf{x}} \end{tikzcd} \tag{3.1}$$

For example, when $\mathbf{b} = \mathbf{w} \in \mathbf{W}$, we can identify $\mathbf{P}_I/\mathbf{U}_I$ with $\mathbf{L}_I$ to construct $\widetilde{\mathrm{Z}}_x$ explicitly by

$$\widetilde{\mathrm{Z}}_x = \left\{(p, m) \in \mathbf{P}_J \times {}^x\mathbf{L}_I \;\middle|\; (pm)^{-1\,F}(pm) \in \dot{x}\big(\mathbf{U}_I \dot{w}\, {}^F\mathbf{U}_I\big)^F \dot{x}^{-1}\right\}$$

where the action of $\mathbf{L}_J \cap {}^x\mathbf{P}_I$ is given by $(p, m) \cdot l = (pl, l^{-1}m)$ with the convention that $\mathbf{L}_J \cap {}^x\mathbf{U}_I$ acts trivially on $m$. With this description, the map $\widetilde{\mathrm{Z}}_x \longrightarrow \widetilde{\mathrm{X}}_x$ is then given by $(p, m) \longmapsto pm\dot{x}\mathbf{U}_I$. Unlike the case of $\mathrm{X}_{\mathbf{x}}$, it is unclear whether there always exists a precise relation between quotients of $\widetilde{\mathrm{X}}_{\mathbf{x}}$ and smaller parabolic Deligne–Lusztig varieties. We shall therefore restrict ourselves to the particular cases that we have already encountered in Section 2.4.

**Case 1.** If $v = xw^F x^{-1}$ lies in the parabolic subgroup $W_J$ then, as in Example 2.4.1, the cohomology of $U_J\backslash \widetilde{\mathrm{X}}_x$ is related to the cohomology of $\widetilde{\mathrm{X}}_{L_J}(K_x, \dot{v}F)$. In this situation $\mathbf{L}_{K_x}^{\dot{v}F} \simeq (\mathbf{L}_I \cap \mathbf{L}_J^x)^{\dot{w}F}$ is a split Levi subgroup of $\mathbf{L}_I^{\dot{w}F}$ so that one can modify $\widetilde{\mathrm{X}}_{L_J}(K_x, \dot{v}F)$ in order to obtain an action of $\mathbf{L}_I^{\dot{w}F}$ by Harish-Chandra restriction.

**Case 2.** If $w = sw'$ and $v = xw'^F x^{-1}$ lies in $W_J$, one can relate the varieties $U_J\backslash \widetilde{\mathrm{X}}_x$ and $\widetilde{\mathrm{X}}_{L_J}(K_x, \dot{v}F)$ (under some extra conditions on $s$ and $x$). The presence of $s$ is reflected by a Galois covering of $\mathbb{G}_m$ which explains the geometry of the fiber in Examples 2.4.2 and 2.4.3. This covering carries actions of $\mathbf{L}_I^{wF}$ and $\mathbf{L}_I^{w'F}$ giving rise to a natural isomorphism between large quotient of these groups as in [1] in the case of tori.

It turns out that this two rather specific cases are sufficient to study a large number of interesting Deligne–Lusztig varieties, namely the ones that are associated in [3] and [9] to principal $\Phi_d$-blocks when $2d$ is strictly bigger than the Coxeter number. This will be treated in subsequent papers (see [14] and [13]).

### 3.1. Case 1 – Fibers are affine spaces

We start under the assumptions of Example 2.4.1. We assume that $x$ and $w$ satisfy $xw^F x^{-1} \in W_J$. For simplicity, we shall also assume that this element is $W_{K_x}$-reduced, as it will always be the case in the examples.

**Proposition 3.2.** *Assume that $v = xw^F x^{-1}$ is a $W_{K_x}$-reduced element of $W_J$. Let $e = \dim(\mathbf{U}_J^x \cap {}^w\mathbf{U} \cap \mathbf{U}^-)$. Then there exists a group isomorphism $\mathbf{L}_I^{\dot{w}F} \simeq ({}^x\mathbf{L}_I)^{\dot{v}F}$ such that we have the following isomorphism in $D^b(\Lambda L_J \times (\mathbf{L}_I^{\dot{w}F} \rtimes \langle F\rangle)\text{-Mod})$:*

$$\mathrm{R}\Gamma_c(U_J\backslash\widetilde{\mathrm{X}}_x, \Lambda)[2e](-e) \simeq \mathrm{R}\Gamma_c\big(\widetilde{\mathrm{X}}_{\mathbf{L}_J}(K_x, \dot{v}F), \Lambda\big) \overset{\mathrm{L}}{\otimes}_{\Lambda\mathbf{P}_J\cap({}^x\mathbf{L}_I)^{\dot{v}F}} \Lambda\mathbf{L}_I^{\dot{w}F}.$$

**Proof.** Since $v = xw^F x^{-1}$, one can use Lang's Theorem to find an element $n \in N_{\mathbf{G}}(\mathbf{T})$ such that $\dot{v} = n\dot{w}^F n^{-1}$. Then the conjugation by $n$ induces an isomorphism $\mathbf{L}_I^{\dot{w}F} \simeq ({}^x\mathbf{L}_I)^{\dot{v}F}$. Moreover, the map $(p, m) \in \widetilde{\mathrm{Z}}_x \longmapsto (p, m\dot{x}n^{-1})$ induces an isomorphism

$$\widetilde{\mathrm{Z}}_x \simeq \big\{(p, m) \in \mathbf{P}_J \times {}^x\mathbf{L}_I \;\big|\; (pm)^{-1\,F}(pm) \in n\big(\mathbf{U}_I\dot{w}^F\mathbf{U}_I\big)^F n^{-1}\big\}$$

so that we can work with $n$ instead of $\dot{x}$. We shall relate the cohomology of this variety to the cohomology of $\widetilde{\mathrm{X}}_{\mathbf{L}_J}(K, \dot{v}F)$. For that purpose, we shall construct a morphism $\Psi : \widetilde{\mathrm{Z}}_x \longrightarrow \widetilde{\mathrm{X}}_{\mathbf{L}_J}(K, \dot{v}F) \times_{\mathbf{P}_J\cap({}^x\mathbf{L}_I)^{\dot{v}F}} \mathbf{L}_I^{\dot{w}F}$ which will factor through $\widetilde{\mathrm{Z}}_x \longrightarrow U_J\backslash\widetilde{\mathrm{Z}}_x/\mathbf{L}_J \cap {}^x\mathbf{P}_I$ and then study its fibers.

Let $(p, m) \in \widetilde{\mathrm{Z}}_x$. Since $p^{-1\,F}p$ lies in $x\mathbf{P}_I w^F\mathbf{P}_I{}^F x^{-1}$ one can proceed as in Example 2.4.1 to show that it also lies in the double coset $\mathbf{P}_{K_x} v^F \mathbf{P}_{K_x}$. If we write $p = ul \in \mathbf{U}_J\mathbf{L}_J$, we deduce that $l^{-1\,F}l \in (\mathbf{L}_J \cap {}^x\mathbf{P}_I)v^F(\mathbf{L}_J \cap {}^x\mathbf{P}_I)$. Therefore, there exists $l' \in \mathbf{L}_K = \mathbf{L}_J \cap {}^x\mathbf{L}_I$, unique up to multiplication on the right by $\mathbf{L}_K^{\dot{v}F}$ such that $(ll')^{-1\,F}(ll') \in (\mathbf{L}_J \cap {}^x\mathbf{U}_I)\dot{v}^F(\mathbf{L}_J \cap {}^x\mathbf{U}_I)$. As a consequence, any element of $\widetilde{\mathrm{Z}}_x/\mathbf{L}_J \cap {}^x\mathbf{P}_I$ can be written $[p; m]$ where $p = ul$ is such that $l$ yields an element of $\widetilde{\mathrm{X}}_{\mathbf{L}_I}(K_x, \dot{v}F)$. For such a representative, we have

$$p^{-1\,F}p = {}^{l^{-1}}\big(u^{-1\,F}u\big)\big(l^{-1\,F}l\big) \in \big(\mathbf{L}_J \cap {}^x\mathbf{U}_I\big) \cdot \mathbf{U}_J\dot{v}^F\big(\mathbf{L}_J \cap {}^x\mathbf{U}_I\big).$$

We can actually be more precise on the contribution of $\mathbf{U}_J$ in this decomposition. Indeed, we have seen in Example 2.4.1 that $x\mathbf{P}_I w^F\mathbf{P}_I{}^F x^{-1}v^{-1} \cap \mathbf{U}_J = (\mathbf{U}_J \cap {}^x\mathbf{U}) \cdot (\mathbf{U}_J \cap {}^{v^F x}\mathbf{U})$ and hence

$$p^{-1\,F}p \in \big(\mathbf{L}_J \cap {}^x\mathbf{U}_I\big) \cdot \big(\mathbf{U}_J \cap {}^x\mathbf{U}\big) \cdot \big(\mathbf{U}_J \cap {}^{v^F x}\mathbf{U}\big)\dot{v}^F\big(\mathbf{L}_J \cap {}^x\mathbf{U}_I\big).$$

Now, the condition $(p, m) \in \widetilde{\mathrm{Z}}_x$ can be written $p^{-1\,F}p \in m^{\dot{v}F}m^{-1}({}^x\mathbf{U}_I)\dot{v}^F({}^x\mathbf{U}_I)$ and we deduce that

$$m^{\dot{v}F}m^{-1} \in {}^x\mathbf{U}_I \cdot \big(\mathbf{U}_J \cap {}^x\mathbf{U}\big) \cdot \big(\mathbf{U}_J \cap {}^{v^F x}\mathbf{U}\big) \cdot {}^{v^F xF}\mathbf{U}_I.$$

We want to show that $m^{\dot{v}F}m^{-1} \in \mathbf{P}_J$. For that purpose, we can decompose the intersection $\mathbf{U}_J \cap {}^{v^F x}\mathbf{U}$ into $(\mathbf{U}_J \cap {}^{v^F xF}(\mathbf{L}_I \cap \mathbf{U})) \cdot (\mathbf{U}_J \cap {}^{v^F xF}\mathbf{U}_I)$ and we observe that $\mathbf{U}_J \cap {}^{v^F xF}(\mathbf{L}_I \cap \mathbf{U}) \subset {}^x\mathbf{U}$. Indeed, $x^{-1}vF(x) = w$ and by assumption $wF$ stabilises $\mathbf{L}_I \cap \mathbf{U}$. We deduce that

$$m^{\dot{v}F}m^{-1} \in {}^x\mathbf{U}_I \cdot \big(\mathbf{U}_J \cap {}^x\mathbf{U}\big) \cdot {}^{v^F xF}\mathbf{U}_I.$$

Note that ${}^x\mathbf{U}_I \cdot (\mathbf{U}_J \cap {}^x\mathbf{U})$ is contained in ${}^x\mathbf{P}_I$. In particular, the contribution of ${}^{v^F x^F}\mathbf{U}_I$ in the decomposition of $m^{\dot{v}F}m^{-1}$ should also lie in ${}^x\mathbf{P}_I$. Since $wF$ normalises $\mathbf{L}_I$, the intersection ${}^{v^F x^F}\mathbf{U}_I \cap {}^x\mathbf{P}_I$ is contained in ${}^x\mathbf{U}_I$. Finally, since $\mathbf{L}_I$ normalises $\mathbf{U}_I$ we deduce that $m^{\dot{v}F}m^{-1} \in \mathbf{U}_J \cap {}^x\mathbf{L}_I$.

Therefore there exists $u' \in \mathbf{U}_J \cap {}^x\mathbf{L}_I$, unique up to multiplication by $\mathbf{U}_J \cap ({}^x\mathbf{L}_I)^{\dot{v}F}$ on the right, such that $u'^{-1}m \in ({}^x\mathbf{L}_I)^{\dot{v}F}$. To summarise, we have shown that to any pair $(p,m) \in \widetilde{\mathrm{Z}}_x$ one can associate a pair $(p',m')$ such that

- $(p,m)$ and $(p',m')$ are in the same $\mathbf{P}_J \cap {}^x\mathbf{L}_I$-orbit, that is there exists $q \in \mathbf{P}_J \cap {}^x\mathbf{L}_I$ such that $p' = pq$ and $m' = q^{-1}p$;
- the image of $p'$ by the composition $\mathbf{P}_J \longrightarrow \mathbf{P}_J/\mathbf{U}_J \simeq \mathbf{L}_J \longrightarrow \mathbf{L}_J/(\mathbf{L}_J \cap {}^x\mathbf{U}_I)$ lies in $\widetilde{\mathrm{X}}_{\mathbf{L}_J}(K,\dot{v}F)$;
- $m' \in {}^x\mathbf{L}_I$ is invariant by $\dot{v}F$.

Moreover, if $(p'',m'')$ is any other pair satisfying the same conditions, then there exists $q' \in \mathbf{P}_J \cap ({}^x\mathbf{L}_I)^{\dot{v}F}$ such that $(p'',m'') = (p'q', q'^{-1}m')$ which means that $(p',m')$ is well-defined in $\mathbf{P}_J \times_{\mathbf{P}_J \cap ({}^x\mathbf{L}_I)^{\dot{v}F}} ({}^x\mathbf{L}_J)^{\dot{v}F}$. Let us define now the morphism $\Psi$ by

$$\Psi : (p,m) \in \widetilde{\mathrm{Z}}_x \longmapsto \big[\pi_J\big(p'\big)\big(\mathbf{L}_J \cap {}^x\mathbf{U}_I\big); m'\big] \in \widetilde{\mathrm{X}}(K,\dot{v}F) \times_{\mathbf{P}_J \cap ({}^x\mathbf{L}_I)^{\dot{v}F}} \big({}^x\mathbf{L}_J\big)^{\dot{v}F}$$

where the action of $\mathbf{P}_J \cap ({}^x\mathbf{L}_I)^{\dot{v}F}$ on $\widetilde{\mathrm{X}}_{\mathbf{L}_J}(K_x,\dot{v}F)$ is just the inflation of the action of $\mathbf{L}_{K_x}^{\dot{v}F} = \mathbf{L}_J \cap ({}^x\mathbf{L}_I)^{\dot{v}F}$. It is clearly surjective and equivariant for the actions of $P_J$ on the left and $({}^x\mathbf{L}_I)^{\dot{v}F}$ on the right. Furthermore, if $(p_1,m_1)$ and $(p_2,m_2)$ are in the same orbit under $\mathbf{L}_J \cap {}^x\mathbf{P}_I$, then $(p_1',m_1')$ and $(p_2',m_2')$ are in the same orbit under $\mathbf{P}_J \cap {}^x\mathbf{P}_I$. Let $q \in \mathbf{P}_J \cap {}^x\mathbf{P}_I$ be such that $(p_2',m_2') = (p_1'q, q^{-1}m_1')$ and write $q = ul \in (\mathbf{P}_J \cap {}^x\mathbf{U}_I)\cdot(\mathbf{P}_J \cap {}^x\mathbf{L}_I)$. Then $l = m_1' m_2'^{-1} \in ({}^x\mathbf{L}_I)^{\dot{v}F}$ so that $\Psi(p_1,m_1) = \Psi(p_2,m_2)$. In other words, $\Psi$ induces a morphism

$$\widetilde{\mathrm{Z}}_x/\mathbf{L}_J \cap {}^x\mathbf{P}_I \longrightarrow \widetilde{\mathrm{X}}_{\mathbf{L}_J}(K,\dot{v}F) \times_{\mathbf{P}_J \cap ({}^x\mathbf{L}_I)^{\dot{v}F}} \big({}^x\mathbf{L}_J\big)^{\dot{v}F}$$

which, in turn, yields a surjective equivariant morphism

$$U_J\backslash\widetilde{\mathrm{Z}}_x/\mathbf{L}_J \cap {}^x\mathbf{P}_I \longrightarrow \widetilde{\mathrm{X}}_{\mathbf{L}_J}(K,\dot{v}F) \times_{\mathbf{P}_J \cap ({}^x\mathbf{L}_I)^{\dot{v}F}} \big({}^x\mathbf{L}_J\big)^{\dot{v}F}.$$

To conclude, it remains to study the fibers of this morphism. Since $({}^x\mathbf{L}_J)^{\dot{v}F}$ acts freely on both varieties, we can rather look at the fibers of the map induced on the quotient varieties. Using the diagram (3.1), we can check that the latter coincides with the map $\mathfrak{Z}_x^v = U_J\backslash \mathrm{Z}_x/\mathbf{L}_J \cap {}^x\mathbf{P}_I \longrightarrow \mathrm{X}_{\mathbf{L}_J}(K_x, vF)$ which has affine fibers of dimension $r + \dim \mathbf{U}_J \cap {}^x\mathbf{U}$ (see Example 2.4.1). $\square$

### 3.2. Case 2 – Minimal degenerations

In this section we address the problem of computing the cohomology of the piece $\widetilde{\mathrm{X}}_x$ of $\widetilde{\mathrm{X}}(I,\dot{w}F)$ when $xw^Fx^{-1}$ is close to be an element of $W_J$. Namely, we shall consider the following situation: $w = sw' > w'$ where $s \in S$ and $v = xw^Fx^{-1} \in W_J$. Under some assumption on $s$ and $w'$ we will prove that the cohomology of $U_J\backslash \mathrm{X}_x$ and $\mathbb{G}_m \times \mathrm{X}_{\mathbf{L}_J}(K_x,vF)$ coincides. As we have seen in the examples, these two varieties are non-isomorphic in general. However, at the level of the varieties $\widetilde{\mathrm{X}}$ we shall construct a Galois covering $\widetilde{\mathbb{G}}_m \longrightarrow \mathbb{G}_m$ and a quasi-vector bundle

$$U_J\backslash\widetilde{\mathrm{X}}_x \rightsquigarrow \widetilde{\mathrm{X}}(K_x,\dot{v}F) \times_{\mathbf{P}_J \cap ({}^x\mathbf{L}_I)^{\dot{v}F}} \widetilde{\mathbb{G}}_m$$

such that $\widetilde{\mathbb{G}}_m/\mathbf{L}_I^{\dot{w}F} \simeq \mathbb{G}_m$. As a byproduct, we will relate the cohomology of $U_J\backslash\widetilde{\mathrm{X}}_x$ and $\mathbb{G}_m \times \widetilde{\mathrm{X}}(K_x,\dot{v}F)$ with coefficients in any unipotent local system.

Throughout this section, we will assume that $[\mathbf{G},\mathbf{G}]$ is simply connected. This is not a strong assumption since it has no effect on the unipotent part of the cohomology of a Deligne–Lusztig variety (see for example [1, Section 5.3]).

*3.2.1. Galois coverings of tori*

Let $\mathbf{I} \xrightarrow{\mathbf{b}} {}^F\mathbf{I}$, decomposed as $\mathbf{I} = \mathbf{I}_1 \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \xrightarrow{\mathbf{w}_2} \cdots \xrightarrow{\mathbf{w}_r} \mathbf{I}_{r+1} = {}^F\mathbf{I}$. Let us consider an element $\mathbf{c} \in B^+$ obtained by minimal degenerations of the $w_i's$: we assume that $\mathbf{c} = \mathbf{z}_1 \cdots \mathbf{z}_r$ where $z_i = \gamma_i w_i$ with $\gamma_i \in S \cup \{1\}$ and $\ell(\gamma_i w_i) \leqslant \ell(w)$. We will also assume that each $\gamma_i$ commutes with $I_i$ so that $\mathbf{cF}$ normalises $\mathbf{I}$. Following [1, Section 4], we set $\alpha_{\mathbf{b},\mathbf{c},i} = \alpha$ if $\gamma_i = s_\alpha$ or $\alpha_{\mathbf{b},\mathbf{c},i} = 0$ if $\gamma_i = 1$ and we define the following algebraic variety

$$\mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}} = \left\{ (l_1, \ldots, l_r) \in \mathbf{L}_{I_1} \times \cdots \times \mathbf{L}_{I_r} \;\middle|\; \begin{array}{l} l_i^{-1}({}^{\dot{w}_i} l_{i+1}) \in \operatorname{Im} \alpha^\vee_{\mathbf{b},\mathbf{c},i} \text{ if } 1 \leqslant i \leqslant r-1 \\ l_r^{-1}({}^{\dot{w}_r F} l_1) \in \operatorname{Im} \alpha^\vee_{\mathbf{b},\mathbf{c},r} \end{array} \right\}.$$

Note that the assumption on $\gamma_i$ ensures that the torus $\operatorname{Im}\alpha^\vee_{\mathbf{b},\mathbf{c},i}$ is central in $\mathbf{L}_{I_i}$, and therefore $\mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}}$ is an algebraic group.

Recall that $\mathbf{L}_I^{t(c)F}$ can be identified with $\mathbf{L}'^{\mathbf{c}F'}$ where $\mathbf{L} = \mathbf{L}_{I_1} \times \cdots \times \mathbf{L}_{I_r}$ and $\mathbf{c}F' : (l_1, \ldots, l_r) \longmapsto ({}^{\dot{z}_1} l_2, \ldots, {}^{\dot{z}_{r-1}} l_r, {}^{\dot{z}_r F} l_1)$. The condition $l_i^{-1}({}^{\dot{w}_i} l_{i+1}) \in \operatorname{Im}\alpha^\vee_{\mathbf{b},\mathbf{c},i}$ is equivalent to $l_i^{-1}({}^{\dot{z}_i} l_{i+1}) \in \operatorname{Im}\alpha^\vee_{\mathbf{b},\mathbf{c},i}$ so that we can replace $w_i$ by $z_i$ in the definition of $\mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}}$. In particular, the variety $\mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}}$ defines two Galois coverings of the torus $\prod \operatorname{Im}\alpha^\vee_{\mathbf{b},\mathbf{c},i}$, namely $\pi_{\mathbf{b}} : l \longmapsto l^{-1}\,{}^{\mathbf{b}F'}l$ and $\pi^{\mathbf{c}} : l \longmapsto ({}^{\mathbf{c}F'}l) l^{-1}$, with respective Galois groups $\mathbf{L}_I^{t(\mathbf{b})F}$ and $\mathbf{L}_I^{t(\mathbf{c})F}$. We will denote by $d = \ell(\mathbf{b}) - \ell(\mathbf{c})$ the dimension of this torus. Note that the induced action of $\mathbf{L}_I^{t(\mathbf{b})F}$ and $\mathbf{L}_I^{t(\mathbf{c})F}$ on $\mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}}$ is explicitly given by

$$\bigl(m, m'\bigr) \cdot (l_1, \ldots, l_r) = \bigl(m l_1 m'^{-1}, \bigl(m^{\dot{w}_1}\bigr) l_2 \bigl(m'^{-1}\bigr)^{\dot{z}_1}, \ldots, \bigl(m^{\dot{w}_1 \cdots \dot{w}_{r-1}}\bigr) l_r \bigl(m'^{-1}\bigr)^{\dot{z}_1 \cdots \dot{z}_{r-1}}\bigr)$$

for $m \in \mathbf{L}_I^{t(\mathbf{b})F}$ and $m' \in \mathbf{L}_I^{t(\mathbf{c})F}$.

Let $\mathbf{S}^\circ_{\mathbf{I},\mathbf{b},\mathbf{c}}$ be the identity component of $\mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}}$. Since $\mathbf{S}_{\emptyset,\mathbf{b},\mathbf{c}} = \mathbf{T}^r \cap \mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}}$ is a $d$-dimensional closed subvariety of $\mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}}$ (it is also a Galois covering of $\prod \operatorname{Im}\alpha^\vee_{\mathbf{b},\mathbf{c},i}$) it must contain the identity component $\mathbf{S}^\circ_{\mathbf{I},\mathbf{b},\mathbf{c}}$. This forces the stabiliser $N$ (resp. $N'$) of $\mathbf{S}^\circ_{\mathbf{I},\mathbf{b},\mathbf{c}}$ in $\mathbf{L}_I^{t(\mathbf{b})F}$ (resp. $\mathbf{L}_I^{t(\mathbf{c})F}$) to be contained in $\mathbf{T}$. In particular, we can readily extend the results in [1, Section 4.4.3] to obtain an explicit description of $N$ and $N'$ in terms of sublattices of the group of cocharacters of $\mathbf{T}$. For example one can check that $W_I$ acts trivially on these lattices so that $N$ and $N'$ are normal subgroups of $\mathbf{L}_I$.

It turns out that the covering $\mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}}$ will naturally appear in the quotient of the parabolic Deligne–Lusztig varieties that we will consider. The action of $\mathbf{L}_I^{t(\mathbf{b})F}$ and $\mathbf{L}_I^{t(\mathbf{c})F}$ yields canonical isomorphisms $\mathbf{L}_I^{t(\mathbf{b})F}/N \simeq \mathbf{L}_I^{t(\mathbf{c})F}/N' \simeq \mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}}/\mathbf{S}^\circ_{\mathbf{I},\mathbf{b},\mathbf{c}}$. Let us write $\mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}} = \mathbf{L}_I^{t(\mathbf{b})F} \times_N \mathbf{S}^\circ_{\mathbf{I},\mathbf{b},\mathbf{c}}$. The quotient of this variety by the action of $N$ (by left multiplication) is given by

$$N \backslash \mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}} \simeq \mathbf{L}_I^{t(\mathbf{b})F}/N \times \Bigl(\prod \operatorname{Im}\alpha^\vee_{\mathbf{b},\mathbf{c},i}\Bigr) \simeq \mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}}/\mathbf{S}^\circ_{\mathbf{I},\mathbf{b},\mathbf{c}} \times \Bigl(\prod \operatorname{Im}\alpha^\vee_{\mathbf{b},\mathbf{c},i}\Bigr).$$

On this quotient, $\mathbf{L}_I^{t(\mathbf{b})F}/N$ acts on the first factor only but the action of $\mathbf{L}_I^{t(\mathbf{c})F}$ is more complicated: an element $m \in \mathbf{L}_I^{t(\mathbf{c})F}$ acts on $\prod \operatorname{Im}\alpha^\vee_{\mathbf{b},\mathbf{c},i}$ by multiplication by $(m({}^{\gamma_1}m^{-1}), (m^{z_1})^{\gamma_2}((m^{-1})^{z_1}), \ldots, (m^{z_1\cdots z_{r-1}})^{\gamma_r}((m^{-1})^{z_1\cdots z_{r-1}}))$. This action can be extended to the connected group $\mathbf{L}_I$. Consequently, if the order of $\mathbf{L}_I^{t(\mathbf{c})F}$ is invertible in $\Lambda$, then the cohomology of $N\backslash\mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}}$ can be represented by a complex with a trivial action of $N'$ and we have

$$\mathrm{R}\Gamma_c(N\backslash\mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}}, \Lambda) \simeq \mathrm{R}\Gamma_c\bigl(N\backslash\mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}}/N', \Lambda\bigr) \simeq \Lambda\mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}}/\mathbf{S}^\circ_{\mathbf{I},\mathbf{b},\mathbf{c}} \overset{\mathrm{L}}{\otimes}_\Lambda \mathrm{R}\Gamma_c\bigl((\mathbb{G}_m)^d, \Lambda\bigr) \tag{3.3}$$

in $D^b(\Lambda\mathbf{L}_I^{t(\mathbf{b})F}/N \times \mathbf{L}_I^{t(\mathbf{c})F}/N'\text{-mod})$.

*3.2.2. The model $w = sw'$*

We start with the case $r = 1$, that is when $\mathbf{b} = \mathbf{w} \in \mathbf{W}$. Let $x$ be a $J$-reduced-$I$ element of $W$ and $s \in S$ be such that $w' = sw < w$ and $v = xsw^F x^{-1} \in W_J$. Recall from the previous section that if $s$ acts trivially on $\Phi_I$, then there exists normal subgroups $N$ of $\mathbf{L}_I^{\dot{w}F}$ and $N'$ of $\mathbf{L}_I^{\dot{w}'F}$ together with a canonical isomorphism $\mathbf{L}_I^{\dot{w}F}/N \simeq \mathbf{L}_I^{\dot{w}'F}/N'$. Using these small finite groups one can relate the cohomology of $U_J \backslash \widetilde{\mathrm{X}}_x$ to the cohomology of $\widetilde{\mathrm{X}}_{\mathbf{L}_J}(K_x, \dot{v}F)$:

**Proposition 3.4.** *Let $w$ be an $I$-reduced element of $W$ such that ${}^{wF}I = I$. Assume that $w$ can be decomposed into $w = sw'$ such that*

(i) $v = xw'^F x^{-1} \in W_J$ *with* $\ell(v) = \ell(w')$,
(ii) $s \in S$ *acts trivially on* $\Phi_I$,
(iii) ${}^x(W_I s) \cap W_J = 1$.

*Then there exists a group isomorphism $\mathbf{L}_I^{\dot{w}F}/N \simeq \mathbf{L}_I^{\dot{w}'F}/N'$ such that, if the order of $\mathbf{L}_I^{\dot{w}'F}$ is invertible in $\Lambda$, we have*

$$\mathrm{R}\Gamma_c(U_J \backslash \widetilde{\mathrm{X}}_x/N, \Lambda) \simeq \mathrm{R}\Gamma_c\big(\mathbb{G}_m \times \widetilde{\mathrm{X}}_{\mathbf{L}_J}(K_x, \dot{v}F), \Lambda\big) \overset{\mathrm{L}}{\otimes}_{\Lambda(\mathbf{P}_J \cap {}^x\mathbf{L}_I)^{\dot{v}F}} \Lambda\mathbf{L}_I^{\dot{w}'F}/N'$$

*in* $D^b(\Lambda L_J \times (\mathbf{L}_I^{\dot{w}F}/N \rtimes \langle F\rangle)\text{-mod})$.

**Proof.** Let $v = xw'^F x^{-1} \in W_J$ and let $n$ be a representative of $x$ in $N_{\mathbf{G}}(\mathbf{T})$ such that $\dot{v} = n\dot{w}'^F n^{-1}$. As in the proof of Proposition 3.2, we shall work with $n$ instead of $\dot{x}$ and identify the variety $\widetilde{\mathrm{Z}}_x$ with

$$\big\{(p,m) \in \mathbf{P}_J \times {}^x\mathbf{L}_I \;\big|\; (pm)^{-1\,F}(pm) \in n\big(\mathbf{U}_I \dot{w}^F \mathbf{U}_I\big)^F n^{-1}\big\}.$$

In order to compute the quotient by $U_J$, we need a precise condition on $u \in \mathbf{U}_J$, $l \in \mathbf{L}_J$ and $m$ for $(ul, m)$ to belong to this variety. We start by proving the following:

**Lemma 3.5.** *Under the assumptions of Proposition 3.4, if $(p,m)$ belongs to $\widetilde{\mathrm{Z}}_x$ then $m^{\dot{v}F}m^{-1}$ lies in $\mathbf{P}_J$.*

**Proof.** Since $sw'$ is $I$-reduced, $s \notin I$ and $\mathbf{U}_I\dot{s} \subset \mathbf{U}_{\alpha_s}\dot{s}\mathbf{U}_I$. Therefore we can write

$$\mathbf{U}_I\dot{s}\dot{w}'^F\mathbf{U}_I \subset \mathbf{U}_{\alpha_s}\dot{s}\mathbf{U}_I\dot{w}'^F\mathbf{U}_I.$$

Note that this inclusion is actually an equality: indeed, $w'^{-1}(\alpha_s) \in \Phi^+$ since $sw' > w'$ and $w'^{-1}(\alpha_s) \notin {}^F\Phi_I^+$ otherwise $-\alpha_s = sw'(w'^{-1}(\alpha_s))$ would be in $\Phi_I^+$ by assumption on $sw'$.

The double coset $\mathbf{U}_I\dot{w}'^F\mathbf{U}_I$ can also be simplified: for $a \in W$ we denote $N(a) = \{\alpha \in \Phi^+ \mid a^{-1}(\alpha) \in \Phi^-\}$. If $\ell(ab) = \ell(a) + \ell(b)$ then $N(ab) = N(a) \amalg aN(b)$. Using assumption (i) we can apply this to $xw' = v^F x$ in order to obtain

$$xN\big(w'\big) = N\big(xw'\big) \smallsetminus N(x) = \big(N(v) \amalg vN\big({}^F x\big)\big) \smallsetminus N(x).$$

Since $v \in W_J$ and $x$ is $J$-reduced, the sets $N(v)$ and $N(x)$ are disjoint. Moreover, $N(x)$ and $N({}^F x)$ have the same number of elements and hence $xN(w') = N(v)$. This proves that $\mathbf{U} \cap {}^{w'}\mathbf{U}^- = (\mathbf{U} \cap {}^v\mathbf{U}^-)^x \subset \mathbf{L}_J^x$. Since $w'F$ (like $wF$ by assumption (ii)) normalises $I$ we deduce that

$$\mathbf{U}_I\dot{w}'^F\mathbf{U}_I = \big(\mathbf{U}_I \cap \mathbf{L}_J^x\big)\dot{w}'^F\big(\big(\mathbf{U}_I \cap \mathbf{L}_J^x\big) \cdot \big(\mathbf{U}_I \cap \big(\mathbf{U}_J^-\big)^x\big) \cdot \big(\mathbf{U}_I \cap \mathbf{U}_J^x\big)\big). \tag{3.6}$$

Now let $p \in \mathbf{P}_J$ be an element of $mn\mathbf{U}_I\dot{s}\dot{w}'^F\mathbf{U}_I{}^F(mn)^{-1}$. There exists $l_s \in \mathbf{U}_{\alpha_s}\dot{s}$ such that $p \in mnl_s\mathbf{U}_I\dot{w}'^F\mathbf{U}_I{}^F(mn)^{-1}$. Since $\mathbf{L}_I$ normalises $\mathbf{U}_I$, we have $p \in (m^n l_s{}^{\dot{v}F}m^{-1})n\mathbf{U}_I\dot{w}'^F\mathbf{U}_I{}^F n^{-1}$. Now, by (3.6), the class $n\mathbf{U}_I\dot{w}'^F\mathbf{U}_I{}^F n^{-1}$ is contained in $\mathbf{P}_J^- \cdot \mathbf{P}_J$ and therefore $m^n l_s{}^{\dot{v}F}m^{-1} \in \mathbf{P}_J \cdot \mathbf{P}_J^-$. We claim that this forces $l_s \notin \mathbf{T}\dot{s}$. Otherwise ${}^x(\mathbf{L}_I s\mathbf{L}_I) = {}^x(\mathbf{L}_I s)$ would have a non-trivial intersection with $\mathbf{P}_J \cdot \mathbf{P}_J^-$, which is impossible by the Bruhat decomposition since ${}^x(W_I s)$ and $W_J$ are disjoint.

Let $\mathbf{T}_s$ be the image of $\alpha_s^\vee$. By a simple calculation in $\mathbf{G}_s = \langle \mathbf{U}_{\alpha_s}, \mathbf{U}_{-\alpha_s}\rangle$, we deduce that $l_s \in \mathbf{U}_{-\alpha_s}\mathbf{T}_s\mathbf{U}_{\alpha_s}$. Since $s$ acts trivially on $\Phi_I$, the group $\mathbf{L}_I$ normalises $\mathbf{U}_{\alpha_s}$ and $\mathbf{T}_s = \operatorname{Im}\alpha_s^\vee$. Moreover, ${}^x\mathbf{U}_{\alpha_s} \subset \mathbf{U}_J^-$ and therefore $m^{\dot{v}F}m^{-1} \in \mathbf{P}_J \cdot \mathbf{P}_J^-$. If we decompose $\mathbf{L}_I$ into $(\mathbf{B}_I, \mathbf{B}_I^-)$-orbits, we have, as $x$ is reduced-$I$

$$ {}^x(\mathbf{L}_I) \cap \big(\mathbf{P}_J \cdot \mathbf{P}_J^-\big) = \coprod_{v' \in W_{K_x}} {}^x\mathbf{B}_I v'{}^x\mathbf{B}_I^- = \big({}^x\mathbf{L}_I \cap \mathbf{P}_J\big) \cdot \big({}^x\mathbf{L}_I \cap \mathbf{U}_J^-\big). $$

We want to prove that the contribution of $\mathbf{U}_J^-$ on $m^{\dot{v}F}m^{-1}$ is trivial. Write $m^{\dot{v}F}m^{-1} = m'm''$ with $m' \in {}^x\mathbf{L}_I \cap \mathbf{P}_J$ and $m'' \in {}^x\mathbf{L}_I \cap \mathbf{U}_J^-$. Using (3.6) and the fact that $l_s \in \mathbf{U}_{\alpha_s}\mathbf{T}_s\mathbf{U}_{-\alpha_s}$, we see that there exists $l' \in ({}^x\mathbf{U}_I \cap \mathbf{L}_J)\dot{v}^F({}^x\mathbf{U}_I \cap \mathbf{L}_J)$ such that $p \in {}^x(\mathbf{U}_{-\alpha_s}\mathbf{T}_s)m'm''\mathbf{U}_{x(\alpha_s)}l'^F(({}^x\mathbf{U}_I \cap \mathbf{U}_J^-) \cdot ({}^x\mathbf{U}_I \cap \mathbf{U}_J))$. In this decomposition, ${}^x(\mathbf{U}_{-\alpha_s}\mathbf{T}_s)$, $m'$, $l'$ and ${}^F({}^x\mathbf{U}_I \cap \mathbf{U}_J)$ lie in $\mathbf{P}_J$, whereas $m''$, $\mathbf{U}_{x(\alpha_s)}$ and ${}^{l'F}({}^x\mathbf{U}_I \cap \mathbf{U}_J)$ lie in $\mathbf{U}_J^-$. Since $\mathbf{P}_J \cap \mathbf{U}_J^-$ is trivial, we deduce that $m'' \in {}^{l'F}({}^x\mathbf{U}_I \cap \mathbf{U}_J^-) \cdot \mathbf{U}_{x(\alpha_s)}$. Finally, since ${}^x\mathbf{U}_I \cap \mathbf{L}_J$ normalises ${}^x\mathbf{P}_I \cap \mathbf{U}_J^-$ and both $m''$ and $\mathbf{U}_{x(\alpha_s)}$ are contained in this group, we can conclude if we can show that $({}^x\mathbf{P}_I \cap \mathbf{U}_J^-) \cap {}^{vF}({}^x\mathbf{U}_I \cap \mathbf{U}_J^-) \subset {}^x\mathbf{U}_I$. But ${}^x\mathbf{P}_I \cap {}^{vF}({}^x\mathbf{U}_I) = {}^x(\mathbf{P}_I \cap {}^{w'F}\mathbf{U}_I) = {}^x(\mathbf{U}_I \cap {}^{w'F}\mathbf{U}_I)$ since $w'F$ normalises $\mathbf{L}_I$. $\square$

**Lemma 3.7.** *Under the assumptions of Proposition 3.4, let $m \in {}^x\mathbf{L}_I$ and $l \in ({}^x\mathbf{U}_I \cap \mathbf{L}_J)\dot{v}^F({}^x\mathbf{U}_I \cap \mathbf{L}_J)$. For $u \in \mathbf{U}_J$, the element $ul$ lies in $mn\mathbf{U}_I\dot{s}\dot{w}^F\mathbf{U}_I{}^F(mn)^{-1}$ if and only if there exist $\lambda \in \mathbb{F}^\times$, $m_1 \in {}^x\mathbf{L}_I \cap \mathbf{U}_J$ and $u_1 \in {}^F({}^x\mathbf{U}_I \cap \mathbf{U}_J)$ such that*

- $m^{\dot{v}F}m^{-1} = m_1 \cdot {}^n\alpha_s^\vee(\lambda)$,
- $u = {}^{mn}u_{-\alpha_s}(\lambda) \cdot m_1 \cdot {}^l u_1$.

**Proof.** We have already seen in the course of the proof of the previous lemma (see (3.6)) that $ul$ can be written $ul = ({}^{mn}l_s)(m^{\dot{v}F}m^{-1})l'u_2u_1$ with $l_s \in u_{\alpha_s}(\mathbb{F}^\times)\dot{s}$, $l' \in ({}^x\mathbf{U}_I \cap \mathbf{L}_J)\dot{v}^F({}^x\mathbf{U}_I \cap \mathbf{L}_J)$, $u_2 \in {}^F({}^x\mathbf{U}_I \cap \mathbf{U}_J^-)$ and $u_1 \in {}^F({}^x\mathbf{U}_I \cap \mathbf{U}_J)$. By a simple calculation in $\mathbf{G}_s = \langle \mathbf{U}_{\alpha_s}, \mathbf{U}_{-\alpha_s}\rangle$ we can decompose $l_s$ into $l_s = u_{-\alpha_s}(\lambda)\alpha_s^\vee(\lambda^{-1})u_{\alpha_s}(-\lambda)$ where $\lambda \in \mathbb{F}^\times$ is uniquely determined (note that we have chosen specific $u_\alpha$'s in Section 1). By the previous lemma $m^{\dot{v}F}m^{-1} = m_1m_2$ with $m_1 \in {}^x\mathbf{L}_I \cap \mathbf{U}_J$ and $m_2 \in {}^x\mathbf{L}_I \cap \mathbf{L}_J$. From the expression of $ul$ we obtain

$$ m_1^{-1}\big({}^{mn}u_{-\alpha_s}(-\lambda)\big)u^l u_1^{-1} = {}^{m_1^{-1}mn}\big(\alpha_s^\vee\big(\lambda^{-1}\big)u_{\alpha_s}(-\lambda)\big)m_2l'l^{-1l}u_2. $$

Since $\mathbf{L}_J$ (resp. $\mathbf{L}_I$) normalises $\mathbf{U}_J$ (resp. $\mathbf{U}_{\alpha_s}$ and $\mathbf{U}_{-\alpha_s}$) and $\mathbf{U}_{x(-\alpha_s)} \subset \mathbf{U}_J$, the left-hand side of this equality lies in $\mathbf{U}_J$ whereas the right-hand side lies in $\mathbf{P}_J^-$. Therefore it must be trivial and we obtain

- $u = {}^{mn}u_{-\alpha_s}(\lambda)m_1{}^l u_1$;
- ${}^{m_1^{-1}mn}(\alpha_s^\vee(\lambda^{-1}))m_2l'l^{-1} = 1$ and therefore $l = l'$ and $m_2 = {}^{m_1^{-1}mn}(\alpha_s^\vee(\lambda)) = {}^n\alpha_s^\vee(\lambda)$;
- ${}^{m_1^{-1}mn\alpha_s^\vee(\lambda^{-1})}(u_{\alpha_s}(-\lambda))^{m_2l}u_2 = 1$ and hence $u_2 = {}^{l^{-1}m_1^{-1}mn}(u_{\alpha_s}(-\lambda))$.

Conversely, one can readily check that if these relations are satisfied then $ul \in mn\mathbf{U}_I\dot{s}\dot{w}^F\mathbf{U}_I{}^F(mn)^{-1}$. $\square$

As a consequence of the lemmas, we can proceed as in the proof of Proposition 3.2 to show that any element of $\widetilde{Z}_x$ is in the $\mathbf{P}_J \cap {}^x\mathbf{L}_I$-orbit of an element $(p, m) = (ul, m)$ satisfying the following properties:

- $l^{-1F}l \in ({}^x\mathbf{U}_I \cap \mathbf{L}_J)\dot{v}^F({}^x\mathbf{U}_I \cap \mathbf{L}_J)$,
- $m^{\dot{v}F}m^{-1} = {}^n\alpha_s^\vee(\lambda)$,
- $(u^{-1F}u)^l = ({}^{mn}u_{-\alpha_s}(\lambda))({}^{l^{-1F}l}u_1)$,

for some $\lambda \in \mathbb{F}^\times$ and $u_1 \in {}^F({}^x\mathbf{U}_I \cap \mathbf{U}_J)$ both uniquely determined. Moreover, the elements of this form in the class of $(p, m)$ form a single $(\mathbf{P}_J \cap {}^x\mathbf{L}_I)^{\dot{v}F}$-orbit.

Recall from the previous section that to $w$ and $w'$ one can associate an algebraic group $\mathbf{S}_{\mathbf{I},\mathbf{w},\mathbf{w}'}$ above $\mathbb{G}_m$ defined by $\mathbf{S}_{\mathbf{I},\mathbf{w},\mathbf{w}'} = \{m \in \mathbf{L}_I \mid m^{-1\dot{w}F}m \in \mathbf{T}_s\}$. Using the special representatives of $\widetilde{\mathrm{Z}}_x/\mathbf{P}_J \cap {}^x\mathbf{L}_I$ mentioned above, we can define the following map

$$\Psi : [p; m] \in \widetilde{\mathrm{Z}}_x/\mathbf{P}_J \cap {}^x\mathbf{L}_I \longmapsto \big[l\big(\mathbf{L}_J \cap {}^x\mathbf{U}_I\big); m^{-1}\big] \in \widetilde{\mathrm{X}}(K_x, \dot{v}F) \times_{\mathbf{P}_J \cap ({}^x\mathbf{L}_I)^{\dot{v}F}} {}^n\mathbf{S}_{\mathbf{I},\mathbf{w},\mathbf{w}'}$$

where the action of $\mathbf{P}_J \cap ({}^x\mathbf{L}_I)^{\dot{v}F}$ on $\widetilde{\mathrm{X}}_{\mathbf{L}_J}(K_x, \dot{v}F)$ is just the inflation of the action of $\mathbf{L}_{K_x}^{\dot{v}F} = \mathbf{L}_J \cap ({}^x\mathbf{L}_I)^{\dot{v}F}$. It is clearly surjective and equivariant for the actions of $P_J$ and ${}^n(\mathbf{L}_I^{\dot{w}F})$. The quotient by $U_J$ (which acts trivially on $\widetilde{\mathrm{X}}_{\mathbf{L}_J}(K_x, \dot{v}F)$) gives rise to a surjective $L_J \times {}^n(\mathbf{L}_I^{\dot{w}F})$-equivariant morphism

$$U_J\backslash\widetilde{\mathrm{Z}}_x/\mathbf{P}_J \cap {}^x\mathbf{L}_I \longrightarrow \widetilde{\mathrm{X}}(K_x, \dot{v}F) \times_{\mathbf{P}_J \cap ({}^x\mathbf{L}_I)^{\dot{v}F}} {}^n\mathbf{S}_{\mathbf{I},\mathbf{w},\mathbf{w}'}. \tag{3.8}$$

Furthermore, any element $[U_J ull'; m]$ in the fiber of $[l(\mathbf{L}_J \cap {}^x\mathbf{U}_I); m^{-1}]$ is uniquely determined by an element $l' \in ({}^x\mathbf{U}_I \cap \mathbf{L}_J)$ and $u^{-1F}u$. Since the latter is determined by $u_1 \in {}^F({}^x\mathbf{U}_I \cap \mathbf{U}_J)$, we deduce that the fibers are affine spaces of dimension $\dim({}^x\mathbf{U}_I \cap \mathbf{P}_J)$. By comparing the dimensions, we obtain the following isomorphism in $D^b(L_J\text{-mod-}{}^n(\mathbf{L}_I^{\dot{w}F}))$:

$$\mathrm{R}\Gamma_c(U_J\backslash\widetilde{\mathrm{X}}_x, \Lambda) \simeq \mathrm{R}\Gamma_c\big(\widetilde{\mathrm{X}}(K_x, \dot{v}F), \Lambda\big) \overset{\mathrm{L}}{\otimes}_{\Lambda\mathbf{P}_J \cap ({}^x\mathbf{L}_I)^{\dot{v}F}} \mathrm{R}\Gamma_c\big({}^n\mathbf{S}_{\mathbf{I},\mathbf{w},\mathbf{w}'}, \Lambda\big),$$

and we conclude using (3.3), which gives the cohomology of $N\backslash\mathbf{S}_{\mathbf{I},\mathbf{w},\mathbf{w}'}$ with the action of $\mathbf{L}_I^{\dot{w}F}/N$ and $\mathbf{L}_I^{\dot{w}'F}/N'$. $\square$

**Remark 3.9.** In many cases we will use this lemma under the assumption that either $I = \emptyset$ or $x = w_0 w_J$. This extra condition makes the previous proof much simpler.

**Remark 3.10.** When $[\mathbf{G}, \mathbf{G}]$ is not simply connected, the coroot $\alpha_s^\vee$ might not be injective. In that case, the fibers of the morphism (3.8) are not necessarily affine spaces. To obtain an analogous statement, we need to change slightly the definition of $\mathbf{S}_{\mathbf{I},\mathbf{w},\mathbf{w}'}$ and consider instead $\{(m, \lambda) \in \mathbf{L}_I \times \mathbf{G}_m \mid m^{-1\dot{w}F}m = \alpha_s^\vee(\lambda)\}$.

*3.2.3. The main result*

More generally, one can combine Propositions 3.2 and 3.4 in order to obtain the following result for elements in the Braid monoid:

**Theorem 3.11.** *Let $\mathbf{I} \xrightarrow{\mathbf{b}} {}^F\mathbf{I}$ be decomposed as $\mathbf{I} = \mathbf{I}_1 \xrightarrow{\mathbf{w}_1} \mathbf{I}_2 \xrightarrow{\mathbf{w}_2} \cdots \xrightarrow{\mathbf{w}_r} \mathbf{I}_{r+1} = {}^F\mathbf{I}$ and $\mathbf{c} = \mathbf{z}_1 \cdots \mathbf{z}_r$ obtained by minimal degenerations of the $w_i$'s. More precisely, we assume that $z_i = \gamma_i w_i$ with $z_i \leqslant w_i$ and $\gamma_i \in S \cup \{1\}$. Let $\mathbf{x} = (x_1, \ldots, x_r)$ be an $r$-tuple of $J$-reduced-$I_i$ elements of $W$ of same length and put $x_{r+1} = {}^F x_1$. We assume that*

- *if $\gamma_i = 1$ then $v_i = x_i w_i x_{i+1}^{-1}$ is a $K_{x_i}$-reduced element of $W_J$;*
- *if $\gamma_i \in S$ then the following properties are satisfied:*
  - (i) *$v_j = x_i z_i x_{i+1}^{-1} \in W_J$ and $\ell(v_i) = \ell(z_i)$,*
  - (ii) *$\gamma_i$ acts trivially on $\Phi_{I_j}$,*
  - (iii) *${}^{x_i}(W_{I_i}s) \cap W_J = 1$.*

*Let us denote by*

- $e = \sum \dim(\mathbf{U}_J^{x_i} \cap {}^{z_i}\mathbf{U} \cap \mathbf{U}^-)$;
- $d = \#\{i = 1, \ldots, r \mid \gamma_i \in S\} = \dim \mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}}$;
- $\mathbf{v} = \mathbf{v_1} \cdots \mathbf{v_r} \in B_{W_J}^+$;
- *$N$ (resp. $N'$) the stabiliser of $\mathbf{S}^\circ_{\mathbf{I},\mathbf{b},\mathbf{c}}$ in $\mathbf{L}_I^{t(\mathbf{b})F}$ (resp. $\mathbf{L}_I^{t(\mathbf{c})F}$).*

*Then there exists a natural isomorphism $\mathbf{L}_I^{t(\mathbf{b})F}/N \simeq \mathbf{L}_I^{t(\mathbf{c})F}/N'$ such that if the order of $\mathbf{L}_I^{t(\mathbf{c})F}$ is invertible in $\Lambda$, the cohomology of the piece $\widetilde{\mathrm{X}}_\mathbf{x}$ of the Deligne–Lusztig variety $\widetilde{\mathrm{X}}(\mathbf{I}, \mathbf{b}F)$ satisfies*

$$\mathrm{R}\Gamma_c(U_J\backslash\widetilde{\mathrm{X}}_\mathbf{x}/N, \Lambda)[2e](-e) \simeq \mathrm{R}\Gamma_c\big((\mathbb{G}_m)^d \times \widetilde{\mathrm{X}}_{\mathbf{L}_J}(\mathbf{K}_{x_1}, \mathbf{v}F)\big) \overset{\mathrm{L}}{\otimes}_{\Lambda(\mathbf{P}_J\cap^{x_1}\mathbf{L}_I)^{t(\mathbf{v})F}} \Lambda\mathbf{L}_I^{t(\mathbf{c})F}/N'$$

*in $D^b(\Lambda L_J \times (\mathbf{L}_I^{t(\mathbf{b})F}/N \rtimes \langle F\rangle)$-mod).*

**Sketch of proof.** Recall that the piece $\widetilde{\mathrm{X}}_\mathbf{x}$ can be lifted up to a variety $\widetilde{\mathrm{Z}}_\mathbf{x}$ defined as the set of $2r$-tuples $(\mathbf{p}, \mathbf{m}) = (p_1, \ldots, p_r, m_1, \ldots, m_r) \in (\mathbf{P}_J)^r \times {}^{x_1}\mathbf{L}_{I_1} \times \cdots \times {}^{x_r}\mathbf{L}_{I_r}$ such that

$$(p_i m_i)^{-1} p_{i+1} m_{i+1} \in \dot{x}_i(\mathbf{U}_{I_i}\dot{w}_i\mathbf{U}_{i+1})\dot{x}_{i+1}^{-1}$$

and

$$(p_r m_r)^{-1}{}^F(p_1 m_1) \in \dot{x}_r\big(\mathbf{U}_{I_r}\dot{w}_r{}^F\mathbf{U}_{I_1}\big)^F\dot{x}_1^{-1}.$$

As in the proofs of Propositions 3.2 and 3.4 (see also Remark 2.11), we can find good representatives in the $\prod \mathbf{P}_J \cap {}^{x_i}\mathbf{L}_{I_i}$-orbit of $(\mathbf{p}, \mathbf{m})$, giving rise to a morphism

$$\Psi : \widetilde{\mathrm{Z}}_\mathbf{x} \longrightarrow \widetilde{\mathrm{X}}_{\mathbf{L}_J}(\mathbf{K}_{x_1}, \mathbf{v}F) \times_{(\mathbf{P}_J\cap^{x_1}\mathbf{L}_I)^{t(\mathbf{v})F}} \mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}}$$

which will factor via the quotient of $\widetilde{\mathrm{Z}}_\mathbf{x}$ by $U_J$ and $\prod \mathbf{P}_J \cap {}^{x_i}\mathbf{L}_{I_i}$ into a morphism whose fibers are affine spaces. Note that one can find $n_1 \in N_\mathbf{G}(\mathbf{T})$ such that $n_1 t(\mathbf{c})^F n_1^{-1} = t(\mathbf{v})$. The action of $({}^{x_1}\mathbf{L}_I)^{t(\mathbf{v})F}$ on $\mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}}$ is then given by the right action of $\mathbf{L}_I^{t(\mathbf{c})F} = (({}^{x_1}\mathbf{L}_I)^{t(\mathbf{v})F})^{n_1}$ on $\mathbf{S}_{\mathbf{I},\mathbf{b},\mathbf{c}}$. We conclude using (3.3). □

**Remark 3.12.** By definition, any unipotent character of $G$ appears in the cohomology of some Deligne–Lusztig variety. If $H$ is a normal subgroup of $\mathbf{G}$ contained in $T$, then $H$ acts trivially on $\mathbf{G}/\mathbf{B}$ and therefore any unipotent character of $G$ is trivial on $H$. This applies in particular to the subgroups $N$ and $N'$ of $\mathbf{L}_I^{t(\mathbf{b})F}$ and $\mathbf{L}_I^{t(\mathbf{c})F}$ so that they have the same unipotent characters. Now, the group $(\mathbf{P}_J)^{x_1} \cap \mathbf{L}_I$ is a parabolic subgroup of $\mathbf{L}_I$, stable by $t(\mathbf{c})F$, and it has $\mathbf{L}_J^{x_1} \cap \mathbf{L}_I = (\mathbf{L}_{K_{x_1}})^{x_1}$ as a rational Levi complement. Therefore any unipotent character $\chi$ of $\mathbf{L}_I^{t(\mathbf{c})F}$ (or equivalently of $\mathbf{L}_I^{t(\mathbf{b})F}$) has a Harish-Chandra restriction ${}^*\mathrm{R}_{K_{x_1}}^I\,\chi$ to $\mathbf{L}_{K_{x_1}}^{t(\mathbf{v})F}$ (after a suitable conjugation). With this notation, we obtain

$$\mathrm{R}\Gamma_c(\widetilde{\mathrm{X}}_\mathbf{x}, \overline{\mathbb{Q}}_\ell)_\chi^{U_J} \simeq \mathrm{R}\Gamma_c\big((\mathbb{G}_m)^d \times \widetilde{\mathrm{X}}_{\mathbf{L}_J}(\mathbf{K}_{x_1}, \mathbf{v}F), \overline{\mathbb{Q}}_\ell\big)_{{}^*\mathrm{R}_{K_{x_1}}^I\chi}[-2e](e).$$

In particular, if $\chi$ is the trivial character then

$$\mathrm{R}\Gamma_c(\mathrm{X}_\mathbf{x}, \overline{\mathbb{Q}}_\ell)^{U_J} \simeq \mathrm{R}\Gamma_c\big((\mathbb{G}_m)^d \times \mathrm{X}_{\mathbf{L}_J}(\mathbf{K}_{x_1}, \mathbf{v}F), \overline{\mathbb{Q}}_\ell\big)[-2e](e)$$

as expected.

### 3.3. Examples

We conclude by showing how Proposition 3.4 can solve the problems encountered in Section 2.4. As a new application, we determine the contribution of the principal series to the cohomology of a parabolic Deligne–Lusztig variety for a group of type $B_n$. Many other cases will be studied in subsequent papers (see [14] and [13]).

#### 3.3.1. $n$-th roots of $\pi$ for groups of type $A_n$

Recall from Section 2.4.3 that for $w = t_1 t_2 \cdots t_n t_{n-1} t_n$ one could decompose the variety $\mathrm{X}(w)$ into two pieces $\mathrm{X}_{x_n}$ and $\mathrm{X}_{x_1}$ with $x_n = t_n$ and $x_1 = t_n \cdots t_1$. However, one could not directly express the cohomology of the latter. Since $x(t_1 w)x^{-1} = t_1 \cdots t_{n-2} t_{n-1} t_{n-2} \in W_J$ one can now apply Proposition 3.4 to obtain

$$\mathrm{R}\Gamma_c(U_J \backslash \mathrm{X}_{x_1}, \overline{\mathbb{Q}}_\ell) \simeq \mathrm{R}\Gamma_c\big(\mathbb{G}_m \times \mathrm{X}_{\mathbf{L}_J}(t_1 \cdots t_{n-2} t_{n-1} t_{n-2}), \overline{\mathbb{Q}}_\ell\big)$$

in $D^b(\overline{\mathbb{Q}}_\ell \mathbf{L}_J \times \langle F \rangle\text{-mod})$.

#### 3.3.2. A new example in type $B_n$

Let $\mathbf{G}$ be a group of type $B_n$. We denote by $t_1, \ldots, t_n$ the simple reflections of $W$, with the convention that $t_2, \ldots, t_n$ generate a parabolic subgroup of type $A_{n-1}$. We will restrict our attention to the principal series of $\mathrm{Irr}\, G$, which is parametrised by the representations of the Weyl group $W$. Following [4], we will denote by $[\lambda; \mu]$ the unipotent character associated to the bipartition $(\lambda, \mu)$ of $n$, with the convention that $\mathrm{Id}_G = [n; -]$ and $\mathrm{St}_G = [-; 1^n]$.

For $n \geqslant 2$, we consider $w_n = t_n \cdots t_2 t_1 t_2$. It is an $I$-reduced element which normalises $I$ for $I = \{t_1\}$. Then one can use the previous method to determine the principal part of the cohomology of $\mathrm{X}(I, w_n)$, with coefficients in the trivial local system $\overline{\mathbb{Q}}_\ell$ or in the local system $\mathbf{St}$ associated to the Steinberg representation of $\mathbf{L}_I^{w_n F}$:

**Proposition 3.13.** *For $n \geqslant 2$, the contribution of the principal series to the cohomology of $\mathrm{X}(I, w_n)$ with coefficients in $\overline{\mathbb{Q}}_\ell$ or $\mathbf{St}$, together with the eigenvalues of $F$, is given by*

$$\mathrm{H}_c^{n+k}\big(\mathrm{X}(I, w_n), \overline{\mathbb{Q}}_\ell\big)_{\mathrm{pr}} = \begin{cases} q^k[k-1; 21^{n-k-1}] & \text{if } 1 \leqslant k \leqslant n-1, \\ q^{n+1}[n; -] & \text{if } k = n+2, \\ 0 & \text{otherwise} \end{cases}$$

*and*

$$\mathrm{H}_c^{n+k}\big(\mathrm{X}(I, w_n), \mathbf{St}\big)_{\mathrm{pr}} = \begin{cases} [-; 1^n] & \text{if } k = 1, \\ q^k[(k-1, 1); 1^{n-k}] & \text{if } 2 \leqslant k \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** We proceed by induction. When $n = 2$, [9, Corollary 8.27] applied to $v = 1$ forces $\mathrm{H}_c^\bullet(\mathrm{X}(I, w_2), \overline{\mathbb{Q}}_\ell)$ and $\mathrm{H}_c^\bullet(\mathrm{X}(I, w_2), \mathbf{St})$ to be $G \times \langle F \rangle$-submodules of $\mathrm{H}_c^\bullet(\mathrm{X}(w_2), \overline{\mathbb{Q}}_\ell)$. By [10, Theorem 4.3.4], the latter is multiplicity-free and hence the theorem can be deduced from [9, Corollary 8.41].

Assume that $n > 2$ and let $J = \{t_1, \ldots, t_{n-1}\}$. We want to compute the cohomology of $U_J \backslash \mathrm{X}(I, w_n)$. We first observe that any $J$-reduced-$I$ element of $W$ is either $x_i = t_n \cdots t_i$ or $y_i = t_n \cdots t_2 t_1 t_2 \cdots t_i$ for $i > 1$. We claim that $\mathrm{X}_{x_i}$ and $\mathrm{X}_{y_j}$ are empty if $i \neq 2$ and $j \neq n$. Indeed, if $i > 2$ then

$$W_J^{x_i} = W_J^{y_{i-1}} = \langle t_1, t_2, \ldots, t_{i-2}, t_{i-1} t_i t_{i-1}, t_{i+1}, \ldots, t_n \rangle.$$

If $\gamma$ is an $x_i$-distinguished subexpression of an element of $W_I w$ (that is, either $t_1 w$ or $w$) then the product of $\gamma$ is never in $W_J^{x_i}$. Otherwise $\gamma$ would contain neither $t_{i-1}$ nor $t_i$ which is impossible since $\gamma$ is distinguished. The case of $y_{i-1}$ is similar. We deduce that $\mathrm{X}(I, w_n) = \mathrm{X}_{x_2} \amalg \mathrm{X}_{y_n}$. Let us examine each of these two varieties:

- we have $x_2 w_n x_2^{-1} = t_{n-1} \cdots t_2 t_1 \in W_J$ and $K_{x_2} = J \cap {}^{x_2}(\Phi_I) = \emptyset$. We can therefore apply Proposition 3.2 and Remark 3.12 to obtain
$$\mathrm{R}\Gamma_c(U_J \backslash \mathrm{X}_{x_2}, \overline{\mathbb{Q}}_\ell) \simeq \mathrm{R}\Gamma_c\big(\mathrm{X}_{\mathbf{L}_J}(t_{n-1} \cdots t_2 t_1), \overline{\mathbb{Q}}_\ell\big)[-1]$$
and
$$\mathrm{R}\Gamma_c(U_J \backslash \mathrm{X}_{x_2}, \mathbf{St}) \simeq \mathrm{R}\Gamma_c\big(\mathrm{X}_{\mathbf{L}_J}(t_{n-1} \cdots t_2 t_1), \overline{\mathbb{Q}}_\ell\big)[-1]$$
since the Harish-Chandra restriction of $\mathrm{St}_{\mathbf{L}_I^{w_n F}}$ to $\mathbf{T}^{w_n F}$ is just the trivial character.
- $y_n = w_0 w_J$ acts trivially on $W_J$ and $y_n(t_n w_n) y_n^{-1} = w_{n-1}$. We have also $K_{y_n} = J \cap {}^{y_n}(\Phi_I) = I$. The assumptions of Proposition 3.4 are clearly satisfied and we obtain
$$\mathrm{R}\Gamma_c(U_J \backslash \mathrm{X}_{y_n}, \overline{\mathbb{Q}}_\ell) \simeq \mathrm{R}\Gamma_c\big(\mathbb{G}_m \times \mathrm{X}_{\mathbf{L}_J}(I, w_{n-1}), \overline{\mathbb{Q}}_\ell\big)$$
and
$$\mathrm{R}\Gamma_c(U_J \backslash \mathrm{X}_{y_n}, \mathbf{St}) \simeq \mathrm{R}\Gamma_c\big(\mathbb{G}_m \times \mathrm{X}_{\mathbf{L}_J}(I, w_{n-1}), \mathbf{St}\big).$$

The cohomology of $\mathrm{X}_{\mathbf{L}_J}(t_{n-1} \cdots t_2 t_1)$ has been computed in [15]. By induction, one can assume that the cohomology of $\mathrm{X}_{\mathbf{L}_J}(I, w_{n-1})$ is given by the theorem (since the unipotent part of the cohomology depends only on the isogeny class of the group). We observe that a character in the principal series different from Id or St cannot appear in both $\mathrm{H}_c^\bullet(\mathrm{X}_{\mathbf{L}_J}(t_{n-1} \cdots t_2 t_1), \overline{\mathbb{Q}}_\ell)$ and $\mathrm{H}_c^\bullet(\mathrm{X}_{\mathbf{L}_J}(I, w_{n-1}), \overline{\mathbb{Q}}_\ell)$ (resp. $\mathrm{H}_c^\bullet(\mathrm{X}_{\mathbf{L}_J}(t_{n-1} \cdots t_2 t_1), \overline{\mathbb{Q}}_\ell)$ and $\mathrm{H}_c^\bullet(\mathrm{X}_{\mathbf{L}_J}(I, w_{n-1}), \mathbf{St})$). Using the long exact sequences given by the decomposition of $U_J \backslash \mathrm{X}(I, w_n)$ and [9, Corollary 8.28.(v)], we can deduce explicitly each cohomology group of $U_J \backslash \mathrm{X}(I, w_n)$. To conclude, we observe that each of these cohomology groups is the Harish-Chandra restriction of the groups given in the theorem, corresponding to the characters of the principal series in the $\Phi_{2n-2}$-blocks of $\mathrm{Id}_G$ and $\mathrm{St}_G$. Finally, we know by [2] that these characters actually appear in the cohomology of $\mathrm{X}(I, w_n)$ since they already appear in the alternating sum. □

**Remark 3.14.** In order to deal with the series corresponding to the cuspidal unipotent character of $B_2$ we need extra information on the degree in which $B_{2,\mathrm{Id}}$ and $B_{2,\mathrm{St}}$ can appear.

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