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Normal sections, class sizes and solvability of finite groups

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ABSTRACT

If G is a finite group, we show that any normal subgroup of G which has exactly three G -conjugacy class sizes is solvable. Thus, we give an extension for normal subgroups of the classical N. Itô's theorem which asserts that those finite groups having three class sizes are solvable, and particularly, a new proof of it is provided. In order to do this, we investigate the structure of a normal section N/K of G such that every element in N lying outside of K has the same G -class size.

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1. Introduction

Let G be a finite group and N be a normal subgroup of G . Recent research works have shown that the set of sizes of those conjugacy classes of G contained in N , also called G -class sizes of N and denoted by $cs_G(N)$, exerts a strong influence on the structure of N . In general, from the set $cs_G(N)$, very little information on the class sizes of N can be obtained. Nevertheless, it is surprising that the G -class sizes of normal subgroups still seem to keep control on their structure. This has emerged as a new useful technique to obtain information regarding normal subgroups, and moreover, this approach has the advantage of enabling to argue by induction on the order of N .

In [3] it is proved that every normal subgroup of G having two G -class sizes is nilpotent, and in fact, such subgroup is either abelian or the direct product of a p -group for a prime p and a central subgroup of G . This result obviously extends the celebrated N. Itô's theorem on the nilpotency of

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groups with exactly two class sizes. However, while Itô's proof is quite elementary, the proof of the above extension is much deeper.

The solvability of groups with three conjugacy class sizes is a more complex problem. N. Itô proved in [14] that such groups are solvable by appealing to Feit–Thompson's theorem and some deep classification theorems by M. Suzuki. The result was simplified years later by J. Rebmann [15] for F -groups (those groups which do not have any two centralizers of noncentral elements such that one properly contains the other). Afterwards, A.R. Camina showed in [6] by using different techniques from those employed in [15], that if G is not an F -group and has three class sizes, then G is the direct product of an abelian subgroup and a subgroup whose order involves no more than two primes. In 2009, S. Dolfi and E. Jabara completely determined their structure and their proof was based on the solvability of this kind of groups [7].

In [1], we obtained the solvability and the structure of a normal subgroup N of G with three G -class sizes just when $\text{cs}_G(N) = \{1, m, n\}$ and m does not divide n . The proof of this result sets up a generalization and a subsequent classification of the concept of F -group for normal subgroups. In [4] the authors asked whether every normal subgroup having three G -class sizes should be solvable or not and thus, the current open problem is reduced to the case of a normal subgroup N with $\text{cs}_G(N) = \{1, m, n\}$ such that m does divide n . In this latter paper, the solvability of N is proved in a particular case and distinct techniques from those in [1] are required. For instance, Theorem B of [4] (see Theorem 6) plays a crucial role so as to get the solvability in the general case.

In order to prove our main goal, Theorem A, we will use the structure of “CP-groups”, that is, the groups in which every element has prime power order. These groups also appear in a natural way in the main results of [3] and [4].

Theorem A. *If N is a normal subgroup of a finite group G and $|\text{cs}_G(N)| = 3$, then N is solvable.*

Recently [2], it has been determined the solvability and the structure of those normal subgroups of G having at most two p -regular G -class sizes (a p -regular class is a class of an element whose order is not divisible by p), where p is a prime number. This turns out to be a key result to progress in the proof of Theorem A. We would like to remark that in all the cited papers on G -classes [1–4] the Classification of the Finite Simple Groups is required.

Another central tool is our analysis of the structure of normal sections in a group G involving certain hypotheses on the G -class sizes. In Theorem B, which has interest on its own, we put forward an extension of an M.I. Isaacs' result (see [11]) about groups having a proper normal subgroup N such that all of the conjugacy classes of G lying outside of N have equal size. Extending Isaacs' definition, we give the following

Definition. We say that a normal section N/K of a group G , that is, $K, N \trianglelefteq G$ with $K \subseteq N$, satisfies condition (*) over G when N is a nonabelian normal subgroup of G such that all the G -conjugacy classes in N lying outside of K have equal size.

Theorem B. *Let N/K be a normal section satisfying (*) over G .*

- (i) *If $\mathbf{Z}(N) \not\subseteq K$, then N/K is a p -group for some prime p and N/K is either abelian or has exponent p .*
- (ii) *If $\mathbf{Z}(N) \subseteq K$, then either N/K is cyclic or is a CP-group. If N/K is not a CP-group, then N has abelian Hall π -subgroups and a normal π -complement, where π is the set of prime divisors of $|N/K|$.*

If x is any element of a group G , we denote by x^G the conjugacy class of x in G and by $|x^G|$ the G -conjugacy class size of x . This is also called the index of x in G . If p is a prime and n is an integer, we use n_p to denote the p -part of n , and $\pi(n)$ is the set of primes dividing n , and $\pi(G)$ is $\pi(|G|)$. On the other hand, if H is a group and q a prime, then we will denote by H_q a Sylow q -subgroup of H , and analogously for a set of primes π , H_π denotes a Hall π -subgroup of H . For the rest of the notation, we will follow [12].

2. Preliminaries

In this section we will set out all those results on G -class sizes and on CP-groups that we are going to use, some of which have been pointed out in the introduction.

Theorem 1. *Suppose that N is a normal subgroup of a group G and that the size of any G -conjugacy class contained in N is 1 or m , for some integer m . Then N is nilpotent.*

Proof. See Theorem 8 of [3]. \square

Theorem 2. *Let N be a normal subgroup of a finite group G . Let p be a prime number and suppose that the G -conjugacy class of every p -regular element of N has size 1 or m for some fixed integer m . Then N has abelian p -complements or $N = RP \times A$, where R and P are a Sylow r -subgroup for some prime r and a Sylow p -subgroup of N respectively, and A is a central group of G .*

Proof. This is the main theorem of [2]. \square

Theorem 3. *Let N be a normal subgroup of a finite group G . Assume that $cs_G(N) = \{1, m, n\}$, with $m < n$ and m does not divide n . Then N is solvable.*

Proof. This is the main result in [1]. \square

Theorem 4. *Suppose that N is a normal solvable subgroup of a group G and suppose that m divides s for every $s \in cs_G(N)$, $s \neq 1$. If $g \in N$ and $|g^G| = m$, then $g \in \mathbf{F}(N)$.*

Proof. See Theorem 5 of [4]. \square

Theorem 5. *Suppose that N is a normal nonsolvable subgroup of a group G and suppose that an integer m divides $|x^G|$ for every $x \in N \setminus \mathbf{Z}(N)$. Then m divides $|\mathbf{Z}(N)|$.*

Proof. This is Theorem 7 of [4]. \square

Theorem 6. *If N is a nonabelian normal subgroup of a finite group G and $|cs_G(N)| = 3$, then $\mathbf{Z}(N) \subset \mathbf{F}(N)$.*

Proof. This is Theorem B of [4]. \square

We will also make use of a nice result due to Isaacs about conjugacy class sizes in groups having an abelian normal subgroup, which is the following.

Lemma 7. *Let $N \trianglelefteq G$, where G is an arbitrary finite group and N is abelian. Let x be a noncentral element of G , and let $y = [t, x]$ for some element $t \in N$. Then $|C_G(y)| > |C_G(x)|$, and so the G -class of y is smaller than that of x .*

Proof. See Lemma 1 of [13]. \square

The structure of finite solvable CP-groups was given by G. Higman in [9]. The structure of nonsolvable CP-groups formerly appeared in [5] within the framework of locally finite groups, and was given later by H. Heineken in [8] together with the classification of the simple CP-groups. As we have said in the introduction we will make use of the following results.

Theorem 8. *If G is a finite, nonsolvable CP-group, then there are normal subgroups B, C of G such that $1 \subseteq B \subseteq C \subseteq G$ and B is a 2-group, C/B is nonabelian and simple, and G/C is a p -group for some prime p and cyclic or generalized quaternion.*

Proof. This is the main part of Proposition 2 of [8]. \square

Theorem 9. *If G is a finite nonabelian simple CP-group, then G is isomorphic to one of the following groups: $L_2(q)$, for $q = 5, 7, 8, 9, 17$, $L_3(4)$, $Sz(8)$ or $Sz(32)$.*

Proof. See Proposition 3 of [8]. \square

3. Normal sections and partitions

Before developing the properties of normal sections that satisfy conditions on class sizes we present some results on “relative partitions”, which were introduced by Isaacs in [11].

Definition 10. Suppose that $N \trianglelefteq G$ and $G = N \cup (\bigcup_i H_i)$ where $H_i \subset G$ are subgroups satisfying $H_i \cap H_j \subseteq N$ when $i \neq j$. In this situation we say that G is partitioned relative to N .

Proposition 11. *Let $N \trianglelefteq G$ and suppose that G is partitioned relative to N and that G/N is abelian. Let p be a prime divisor of $|G : N|$ and suppose that G has a normal Sylow p -subgroup. Then G/N is an elementary abelian p -group.*

Proof. This is Proposition 4 of [11]. \square

Moreover, we will use the following elementary result.

Lemma 12. *Let U be an abelian group which acts nontrivially on a group K of relatively prime order. Suppose $Z \subseteq U$ and that $C_K(u) = C_K(U)$ for every $u \in U \setminus Z$. Then U/Z is cyclic.*

Proof. See for instance Lemma 6 of [11]. \square

We recall from the introduction that a normal section N/K of a group G satisfies condition (*) over G when N is a nonabelian normal subgroup of G and all the G -conjugacy classes in N lying outside of K have equal size. The following lemma shows the relevance of studying partitions when we are dealing with a normal section satisfying the property (*).

Lemma 13. *If N/K satisfies (*) over G and $Z(N) \subseteq K$, then N is partitioned relative to K .*

Proof. Observe that $C_G(C_G(x)) \subseteq C_G(x) \subset G$ for all $x \in N \setminus K$. We denote by $C_x = C_N(C_G(x)) \subseteq C_N(x)$ with $x \in N \setminus K$. Since $C_x \subseteq Z(C_G(x))$, we have that C_x is abelian. If $C_x = N$, then $N \subseteq C_G(x)$ and so $x \in Z(N) \subseteq K$, a contradiction. Therefore, $C_x \subset N$. Now let $y, z \in N \setminus K$ such that $C_y \cap C_z \not\subseteq K$ and let $u \in C_y \cap C_z \setminus K$. Hence $u \in C_y \subseteq Z(C_G(y))$ and, as a consequence, $C_G(y) = C_G(u)$. Analogously we have $C_G(z) = C_G(u)$ and $C_y = C_N(C_G(y)) = C_N(C_G(z)) = C_z$. Thus if $C_y \neq C_z$, then $C_y \cap C_z \subseteq K$ and so N is partitioned relative to K . \square

Now we are ready to exploit these properties to get the main results on normal sections.

Lemma 14. *Suppose that N/K satisfies (*) over G . Let $x \in N \setminus K$.*

- (i) *If $xK \in N/K$ is not a p -element, then there exists a Sylow p -subgroup P of $C_N(x)$ such that $P \subseteq Z(C_G(x))$.*
- (ii) *If the order of $xK \in N/K$ is divisible by two distinct primes, then $C_N(x) \subseteq Z(C_G(x))$ and in particular, $C_N(x)$ is abelian.*

Proof. (i) Let $y \in \mathbf{C}_N(x)$ be a p -element. Then $(xy)^{o(y)} = x^{o(y)} \notin K$ and $\mathbf{C}_G(xy) = \mathbf{C}_G((xy)^{o(y)}) = \mathbf{C}_G(x^{o(y)}) = \mathbf{C}_G(x)$. Thus $xy \in \mathbf{Z}(\mathbf{C}_G(x))$, which implies that $y \in \mathbf{Z}(\mathbf{C}_G(x))$. Therefore, if $P \in \text{Syl}_p(\mathbf{C}_N(x))$, then $P \subseteq \mathbf{Z}(\mathbf{C}_G(x))$.

(ii) If $o(xK)$ is divisible by two distinct primes p and q , then by applying (i) to p and q (since xK is neither a p -element nor a q -element), we obtain that, for every prime r and for every $R \in \text{Syl}_r(\mathbf{C}_N(x))$, R lies in the center of $\mathbf{C}_G(x)$. So (ii) is proved. \square

Proposition 15. Let N/K be a normal section satisfying (*) over G and suppose that N/K has elements of order p^2 , for a certain prime p . Let $P \in \text{Syl}_p(N)$ and let $A = \langle x \in P \mid x^p \notin K \rangle$. Then

- (i) A is abelian and $P \cap K \subseteq A \trianglelefteq \mathbf{N}_G(P)$.
- (ii) If $\bar{A} = A/(P \cap K)$ and $\bar{P} = P/(P \cap K)$, for every $a \in P$ such that $a^p \notin K$ we have $\mathbf{C}_P(a) = A$ and $\mathbf{C}_{\bar{P}}(\bar{a}) = \bar{A}$.

Proof. (i) Notice that A is nontrivial by the hypotheses. We first show that $P \cap K \subseteq A$. Let $a \in P$ such that $a^p \notin K$ and let $n \in P \cap K$. Then $an \in P \setminus (P \cap K)$. If $(an)^p \in K$, then $aK = anK$, so $a^pK = (an)^pK = K$, that is $a^p \in K$, a contradiction. Therefore, $(an)^p \notin K$ and thus $an \in A$. So $n \in A$ and hence $P \cap K \subseteq A$, as wanted. On the other hand, since P and K are $\mathbf{N}_G(P)$ -invariant, it clearly follows that A also is.

We show now that A is abelian. Let $Z = \mathbf{Z}(A)$ and assume $Z \subset A$. We consider an element $1 \neq \bar{w} \in \mathbf{Z}(A/Z)$ such that $o(\bar{w}) = p$. We have that $w^p \in Z$ and $[w, A] \subseteq Z$. Let $a \in P$ such that $a^p \notin K$. Then $1 = [w^p, a] = [w, a]^p = [w, a^p]$ and then $w \in \mathbf{C}_G(a^p)$. However, $\mathbf{C}_G(a^p) = \mathbf{C}_G(a)$ because N/K satisfies (*), so $w \in \mathbf{C}_G(a)$, for every $a \in P$ such that $a^p \notin K$. This implies that $w \in Z$, a contradiction, which means that A is abelian.

(ii) Let $a \in P$ such that $a^p \notin K$ and write $M = P \cap K$. By (i) we know that $A \subseteq \mathbf{C}_P(a)$, so $\bar{A} \subseteq \mathbf{C}_{\bar{P}}(\bar{a})$. Conversely, if $\bar{x} \in \mathbf{C}_{\bar{P}}(\bar{a})$, then $xMaM = aMxM$ and $(xa)^pM = x^pMa^pM$. Now we distinguish two cases: whether $(xa)^p \in K$ or not. If $(xa)^p \notin K$, then $xa \in A$ and so $x \in A$. In this case, we trivially have $\bar{x} \in \bar{A}$. If $(xa)^p \in K \cap P = M$, then $M = x^pMa^pM$. If $x^p \in K$, then $a^p \in K$, which is a contradiction. So $x^p \notin K$ and hence $x \in A$. In both cases we have proved $\mathbf{C}_{\bar{P}}(\bar{a}) = \bar{A}$ and certainly $\mathbf{C}_P(a) = A$, as desired. \square

Lemma 16. Let N/K satisfy (*) over G and $\mathbf{Z}(N) \subseteq K$. Let $xK \in N/K$ whose order is not a prime number and let p be a prime divisor of the order of xK . If P is a Sylow p -subgroup of K , then P is abelian, K has a normal p -complement and $xK \in \mathbf{C}_N(P)K/K$.

Proof. Let $P \in \text{Syl}_p(K)$ and note that by the Frattini argument we have $N = \mathbf{N}_N(P)K$ and then we may write $x = x'k$ with $x' \in \mathbf{N}_N(P)$ and $k \in K$. Notice that the orders of xK and $x'K$ are the same. Set $T = \langle P, x' \rangle$. As we can see in the proof of Lemma 13, we have $N = K \cup (\bigcup_{y \in N} C_y)$, where $C_y = \mathbf{C}_N(\mathbf{C}_G(y))$ and $C_y \cap C_z \subseteq K$ if $C_y \neq C_z$ with $y, z \in N$. Then

$$T = (T \cap K) \cup \left(\bigcup_{y \in N} (C_y \cap T) \right).$$

We know that $C_y \cap T$ is abelian. If T is nonabelian, then $T \neq C_y \cap T$ for every $y \in N$. So T is partitioned relative to $T \cap K$. Moreover $T/(T \cap K) \cong TK/K = \langle x' \rangle K/K$ is abelian. We have that p divides $|T/(T \cap K)|$ and $\langle P, x'_p \rangle \in \text{Syl}_p(T)$ and $\langle P, x_p \rangle \trianglelefteq T$, where x'_p is the p -part of x' . By Proposition 11, we deduce that $T/(T \cap K)$ is an elementary abelian p -group, and so $x'K$ would have order p , which is a contradiction. Therefore, T must be abelian and this means that $x' \in \mathbf{C}_N(P)$, so $xK \in \mathbf{C}_N(P)K/K$. Now, if we take $n \in \mathbf{N}_K(P)$ and $a = x'n$, then $aK = x'K = xK \in N/K$. By reasoning with a instead of x' , we obtain $a \in \mathbf{C}_N(P)$ and consequently, $n \in \mathbf{C}_K(P)$. This proves the equality $\mathbf{N}_K(P) = \mathbf{C}_K(P)$ and by Burnside's Theorem, we conclude that K has a normal p -complement and abelian Sylow p -subgroups. \square

4. Proof of Theorem B

Proof of Theorem B. (i) Assume $\mathbf{Z}(N) \not\subseteq K$. We show first that N/K is a p -group for some prime p . Let $z \in \mathbf{Z}(N) \setminus K$ be a p -element for some prime p and suppose that there exists $n \in N \setminus K$ which is a p' -element. Observe that $nz \notin K$ and it easily follows that $\mathbf{C}_G(nz) = \mathbf{C}_G(z)$. Since nzK has composite order in N/K , by Lemma 14 (ii), we get that $N = \mathbf{C}_N(z) = \mathbf{C}_N(nz)$ is abelian, a contradiction. Therefore, such element n does not exist and thus, N/K is a p -group.

Assume now that N/K does not have exponent p and we will show that N/K is abelian. Let P be a Sylow p -subgroup of N and consider the subgroup $A = \langle a \in P \mid a^p \notin K \rangle$ defined in Proposition 15. We claim that every p -element of N lies in some N -conjugate of A . Let y be a p -element of N and choose P_1 to be a Sylow p -subgroup of G such that $\mathbf{C}_{P_1}(y)$ is a Sylow p -subgroup of $\mathbf{C}_G(y)$. Then $y \in P_1 \cap N = P^n$, for some $n \in N$. Suppose that $y \notin A^n$ and take a generator of A^n , that is, $a \in P^n$ such that $a^p \notin K$. Moreover, we know by Proposition 15 that A^n is abelian and normal in P_1 . Note that y is not central in P_1 , since y does not centralize A^n by Proposition 15. Hence, Lemma 7 yields that $|\mathbf{C}_{P_1}([a, y])| > |\mathbf{C}_{P_1}(y)|$. This implies that $[a, y] \in K$, because of condition (*). If we denote $\overline{P^n} = P^n/P^n \cap K$, then we have proved that $\bar{y} \in \mathbf{C}_{\overline{P^n}}(\bar{a})$. However, by Proposition 15 we have $\mathbf{C}_{\overline{P^n}}(\bar{a}) = \overline{A^n}$ and consequently, $y \in A^n$, a contradiction. Therefore, the claim is proved and this yields

$$N = PK = \bigcup_{n \in N} A^n K.$$

This forces that $N = AK$ and hence N/K is abelian, as desired.

(ii) Suppose that $\mathbf{Z}(N) \subseteq K$ and that N/K is not a CP-group and let $xK \in N/K$ of composite order and $p, q \in \pi := \pi(N/K)$, with $p \neq q$, such that pq divides $o(xK)$. We will prove that N/K is cyclic and that N has a normal π -complement and abelian Hall π -subgroups. By Lemma 14 (ii), we know that $\mathbf{C}_N(x)$ is abelian and $\mathbf{C}_N(x) \subseteq \mathbf{Z}(\mathbf{C}_G(x))$. Furthermore, observe that we can assume that x is a π -element. The proof is divided into several steps.

Step 1. If $z \in N \setminus K$ and $\mathbf{C}_G(x) \neq \mathbf{C}_G(z)$, then $\mathbf{C}_N(z) \cap \mathbf{C}_N(x) \subseteq \mathbf{Z}(G) \cap N$.

Let $a \in \mathbf{C}_N(z) \cap \mathbf{C}_N(x) \setminus \mathbf{Z}(G) \cap N$. Then $a \in \mathbf{Z}(\mathbf{C}_G(x))$. So $\mathbf{C}_G(x) = \mathbf{C}_G(a)$ and $z \in \mathbf{C}_N(a) = \mathbf{C}_N(x) \subseteq \mathbf{Z}(\mathbf{C}_G(x))$. Hence $\mathbf{C}_G(x) = \mathbf{C}_G(z)$, which is a contradiction.

Step 2. We have $|x^G| = |\bar{x}^{\bar{G}}|$, where $\bar{G} = G/\mathbf{Z}(G)$.

Observe that q divides $o(\bar{x})$, then $\overline{\mathbf{C}_G(x)} \subseteq \mathbf{C}_{\bar{G}}(\bar{x}) \subseteq \mathbf{C}_{\bar{G}}(\bar{x}_q)$. Assume that \bar{y} is an r -element of $\mathbf{C}_{\bar{G}}(\bar{x}_q)$, for some prime $r \neq q$. Then $y^{o(\bar{y})} \in \mathbf{Z}(G)$. Since $[y, x_q] \in \mathbf{Z}(G)$, we obtain $1 = [y^{o(\bar{y})}, x_q] = [y, x_q]^{o(\bar{y})} = [y, x_q^{o(\bar{y})}]$. So $y \in \mathbf{C}_G(x_q^{o(\bar{y})}) = \mathbf{C}_G(x)$ and thus $\bar{y} \in \overline{\mathbf{C}_G(x)}$. Therefore $|\mathbf{C}_{\bar{G}}(\bar{x}_q)|_r \leq |\overline{\mathbf{C}_G(x)}|_r$. On the other hand, $|\overline{\mathbf{C}_G(x)}|_r$ divides $|\mathbf{C}_{\bar{G}}(\bar{x})|_r$ and $|\mathbf{C}_{\bar{G}}(\bar{x})|_r$ divides $|\mathbf{C}_{\bar{G}}(\bar{x}_q)|_r$. We get $|\overline{\mathbf{C}_G(x)}|_r = |\mathbf{C}_{\bar{G}}(\bar{x})|_r$, for every prime $r \neq q$. By arguing similarly for the prime p , which also divides $o(\bar{x})$, we have $|\overline{\mathbf{C}_G(x)}|_s = |\mathbf{C}_{\bar{G}}(\bar{x})|_s$ for every prime $s \neq p$ and this leads to the equality $|\bar{x}^{\bar{G}}| = |x^G|$.

Step 3. If $g \in N \setminus K$, then $\bar{x}^{\bar{N}} \cap \overline{\mathbf{C}_N(g)} \neq \emptyset$. As a consequence, $N = \mathbf{C}_N(x)K$ and N/K is abelian.

Suppose that there exists $g \in N \setminus K$ such that $\bar{x}^{\bar{N}} \cap \overline{\mathbf{C}_N(g)} = \emptyset$. We consider the action of $\mathbf{C}_N(x)$ on the set g^N . As $\mathbf{C}_G(x) \neq \mathbf{C}_G(g^n)$ for every $n \in N$, by applying Step 1, if $y \in \mathbf{C}_N(x)$ stabilizes some element in the set g^N , that is, if $(g^n)^y = g^n$ for some $n \in N$, then $y \in \mathbf{Z}(G) \cap N$. Thereby, the size of each orbit, which is $|\mathbf{C}_N(x)|/|\mathbf{Z}(G) \cap N|$, divides $|g^N|$, and this certainly divides $|g^G| = |x^G| = |\bar{x}^{\bar{G}}|$. On the other hand, observe that $\overline{\mathbf{C}_N(x)} \cong \mathbf{C}_N(x)/(\mathbf{Z}(G) \cap N)$ acts on $\bar{x}^{\bar{G}} \setminus (\bar{x}^{\bar{G}} \cap \overline{\mathbf{C}_N(x)})$ by conjugation. We see that this action has no fixed point. Assume that there is $\bar{t} \in \overline{\mathbf{C}_N(x)}$ such that $\bar{x}^{\bar{a}\bar{t}} = \bar{x}^{\bar{a}}$ for some

$\bar{x}^{\bar{\alpha}} \in \bar{x}^{\bar{G}} \setminus (\bar{x}^{\bar{G}} \cap \overline{\mathbf{C}_N(x)})$. Hence $\bar{t} \in \mathbf{C}_{\bar{N}}(\bar{x}^{\bar{\alpha}}) = \overline{\mathbf{C}_N(x^{\alpha})}$. So $t \in \mathbf{C}_N(x) \cap \mathbf{C}_N(x^{\alpha}) \subseteq \mathbf{Z}(G) \cap N$, by Step 1. This shows that there are not fixed points for each $1 \neq \bar{t}$. This implies again that $|\overline{\mathbf{C}_N(x)}|$ divides

$$|\bar{x}^{\bar{G}} \setminus (\bar{x}^{\bar{G}} \cap \overline{\mathbf{C}_N(x)})| = |\bar{x}^{\bar{G}}| - |\bar{x}^{\bar{G}} \cap \overline{\mathbf{C}_N(x)}|.$$

We conclude that $|\overline{\mathbf{C}_N(x)}|$ divides $|\bar{x}^{\bar{G}} \cap \overline{\mathbf{C}_N(x)}|$ and as $0 < |\bar{x}^{\bar{G}} \cap \overline{\mathbf{C}_N(x)}| < |\overline{\mathbf{C}_N(x)}|$, we get a contradiction. Thus, the first claim of the step is proved. As a result we can write $N = \bigcup_{t \in N} \mathbf{C}_N(x^t)K$ and this forces that $N = \mathbf{C}_N(x)K$.

Step 4. We can write $N = US$, where U is a Hall π -subgroup of N and S is a normal π -complement of N . Moreover, $Z := K \cap U$ is an abelian Hall π -subgroup of K .

As N/K is abelian, we can take $yK \in N/K$ whose order is divisible by every $p \in \pi$. By applying Lemma 16 to yK for every $p \in \pi$, we obtain that if $P \in \text{Syl}_p(K)$, then P is abelian and K has a normal p -complement. Consequently, K has a normal π -complement S and K/S has a normal p -complement for every $p \in \pi$. This implies that K/S is abelian. Moreover, note that N is π -separable and we can take U as a Hall π -subgroup of N . Then $Z = U \cap K$ is an abelian Hall π -subgroup of K and $N = US$ with $S \trianglelefteq N$.

Step 5. U is abelian.

As N is a π -separable group, by the Frattini argument we may write $N = \mathbf{N}_N(Z)K$. Let $yK \in N/K$ such as in Step 4. Then we can write $y = y'a$, with $y' \in \mathbf{N}_N(Z)$ and $a \in K$. Since Z is abelian, this certainly implies that $y' \in \mathbf{N}_N(P)$ for every $P \in \text{Syl}_p(Z)$ and $p \in \pi$. By arguing as in the proof of Lemma 16, we easily get $y' \in \mathbf{C}_N(P)$, and this property holds for every Sylow subgroup of Z . Therefore, $Z \subseteq \mathbf{C}_N(y')$. On the other hand, by Step 3 we know that $N = \mathbf{C}_N(y')K$ and thus, $N = \mathbf{C}_N(y')S$, where S denotes the π -complement of N , and $|N : \mathbf{C}_N(y')|$ is a π' -number. Then $U^\beta \subseteq \mathbf{C}_N(y')$ for some $\beta \in N$. Therefore, U^β and U are abelian, since $\mathbf{C}_N(y')$ is abelian by Lemma 14 (ii).

Step 6. N/K is cyclic.

We know that U is abelian and then U must act nontrivially on S ; otherwise $U \subseteq \mathbf{Z}(N)$, which contradicts the fact that $\mathbf{Z}(N) \subseteq K$. We claim that $\mathbf{C}_S(u) = \mathbf{C}_S(U)$ for every $u \in U \setminus Z$. Let $u \in U \setminus Z$. Obviously $\mathbf{C}_S(U) \subseteq \mathbf{C}_S(u)$. Since $U/Z \cong N/K$ is not a CP-group, we can take $x \in U$ such that the order of xZ , or equivalently the order of xK , is composite. As U is abelian, then $x, u \in \mathbf{C}_G(x) \cap \mathbf{C}_G(u)$, and by Step 1, we have $\mathbf{C}_G(x) = \mathbf{C}_G(u)$. Therefore, $u \in U \subseteq \mathbf{C}_N(x) = \mathbf{C}_N(u)$, which moreover is abelian. Thus $\mathbf{C}_N(u) \subseteq \mathbf{C}_N(U)$, and the claim is proved. We apply Lemma 12 and as a result $U/Z \cong N/K$ is cyclic. Furthermore, N has abelian Hall π -subgroups and a normal π -complement by Steps 4 and 5, so the proof is complete. \square

5. Proof of Theorem A

Proof of Theorem A. By Theorem 3, we can assume that $\text{cs}_G(N) = \{1, m, mn\}$ for certain positive integers $m, n > 1$. We will argue by induction on the order of N and assume that N is nonsolvable in order to get a contradiction. Notice that if K is a normal subgroup of G such that $K \subset N$, then $|\text{cs}_G(K)| \leq 3$ and, by induction and Theorem 1, it follows that K is solvable. In particular, N must be a perfect group. The proof of Theorem A is divided into several steps.

Step 1. If $z \in N$ and $|z^G| = m$, then $z \in \mathbf{F}(N)$.

We first claim that $\mathbf{Z}(N) = \mathbf{Z}(G) \cap N$. Suppose that there exists $w \in \mathbf{Z}(N) \setminus \mathbf{Z}(G)$. By considering the primary decomposition of w , it is clear that there exists a prime r such that the r -part of w is not

central in G , whence there is no loss in assuming w to be an r -element. Now, let us consider $N \leq C_G(w)$ and distinguish two possibilities depending on index of w , which may be m or mn . Assume first that $|w^G| = m$. Then, for any r' -element $x \in N$ we have $C_G(wx) = C_G(w) \cap C_G(x) \subseteq C_G(w)$. It follows that the index in $C_G(w)$ of such x may be 1 or n . If all these indexes were equal to 1, then $N \subseteq C_G(w) \subseteq C_G(x)$ for every r' -element $x \in N$, so N would have a central r -complement and would be solvable, a contradiction. Therefore, both possibilities for the indexes really occur, so the hypotheses of Theorem 2 are satisfied. As a result, we get that N has nilpotent r -complements, which certainly contradicts again the non-solvability of N . So this case is finished. Suppose now that $|w^G| = mn$. In this case, for any r' -element $x \in N$ we have $C_G(wx) = C_G(w) \cap C_G(x) = C_G(w)$, and it follows that $N \subseteq C_G(w) \subseteq C_G(x)$, that is, $x \in Z(N)$ and this leads again to a contradiction. Thus, we have shown that $Z(N) \subseteq Z(G) \cap N$ and the other containment is trivial, so the claim is proven.

Let z be an element such that $|z^G| = m$ and we show that z belongs to $F(N)$. First, we note that if there exists a prime q dividing the order of N such that $Z(N)_q \subset O_q(N)$ and $q \notin \pi(m)$, then $O_q(N) \subseteq C_G(z)$, that is, $z \in C_N(O_q(N)) \subset N$. By the observation at the beginning of the proof, this subgroup is solvable, so by Theorem 4, we get $z \in F(C_N(O_q(N))) \subseteq F(N)$. Therefore, we can assume that $Z(N)_q = O_q(N)$ for any prime q dividing $|N|$ and $q \notin \pi(m)$.

We can assume that z is an r -element for some prime r by minimality of the index of z . We claim that $z \in F(N)$ or $O_q(N) = Z(N)_q$ for every $q \in \pi(m)$ and $q \neq r$. Let $q \in \pi(m)$, $q \neq r$, and suppose that $Z(N)_q \subset O_q(N)$. Let Q be a Sylow q -subgroup of $C_G(z)$ and let us consider the action of the direct product $Q \times \langle z \rangle$ on $O_q(N)$. We show that $C_{O_q(N)}(Q) \subseteq C_{O_q(N)}(z)$. If $x \in C_{O_q(N)}(Q)$ and x is noncentral in G , then $\langle Q, x \rangle \subseteq C_G(x)$. Since $|C_G(x)|_q \leq |C_G(z)|_q$, we have $x \in Q$ and $x \in Q \cap O_q(N) \subseteq C_{O_q(N)}(z)$. By Thompson's $P \times Q$ -Lemma, we obtain $z \in C_N(O_q(N))$. Notice that $C_N(O_q(N))$ is solvable and, again by Theorem 4, we conclude that $z \in F(C_N(O_q(N))) \subseteq F(N)$.

Thus, we can assume that $O_q(N) = Z(N)_q$ for every prime $q \neq r$. If every r' -element of N has G -class size 1 or mn , then N would be solvable by Theorem 2, which is not possible. Hence, there exists an s -element $y \in N$ with $s \neq r$ such that $|y^G| = m$. Now, by reasoning as in the above paragraph with y in place of z we have either $y \in F(N)$ or $O_q(N) = Z(N)_q$ for every $q \neq s$. However, the former case cannot happen since y would belong to $O_s(N) = Z(N)_s \subseteq Z(G)$, a contradiction. Hence, $O_q(N) = Z(N)_q$ for every $q \neq s$, and in particular $O_r(N) = Z(N)_r$, so we conclude that $F(N) = Z(N)$, which contradicts Theorem 6.

Step 2. Let B be the subgroup defined by $B/F(N) = O_2(N/F(N))$. Then N/B is simple, $N/F(N)$ is a CP-group and B is the solvable radical of N .

By Step 1, we have that $N/F(N)$ is a normal section of G satisfying (*), and by Theorem B (ii) it follows that $N/F(N)$ is a CP-group. By applying Theorem 8 we know that there exist two normal subgroups in N , say B and C , such that $F(N) \subseteq B \subseteq C \subseteq N$, and $B/F(N)$ is a 2-group, C/B is nonabelian simple and N/C is a p -group. As N is perfect, then $C = N$. Also, as N/B is simple nonabelian, then clearly B is the solvable radical of N .

Step 3. If q is a prime dividing $|N|$ such that N/B has a (possibly trivial) cyclic Sylow q -subgroup, then $Z(N)_q \subset O_q(N)$.

First, note that any prime satisfying the hypotheses must be an odd prime. Let $q \in \pi(N)$ such that $q \notin \pi(N/B)$ and suppose that the equality $O_q(N) = Z(N)_q$ holds. Then N would have a central Sylow q -subgroup, a contradiction, and so this case is proved. Therefore, we assume that $q \in \pi(N/B)$ and that N/B has a cyclic Sylow q -subgroup. By applying Step 1, we can write the formula

$$|N| = |F(N)| + mnl$$

for some positive integer l , or equivalently,

$$|N/F(N)| = 1 + mnl/|F(N)|.$$

As q divides $|N/\mathbf{F}(N)|$, we deduce that $(mn)_q$ divides $|\mathbf{F}(N)|_q$, and in particular, n_q divides $|\mathbf{O}_q(N)|$. Now, if $\mathbf{O}_q(N) = \mathbf{Z}(N)_q$, then $N/\mathbf{Z}(N)$ has cyclic Sylow q -subgroups, and hence N has abelian Sylow q -subgroups. Since N is perfect, by applying Taunt's Theorem (see for instance 17.7 of [10]) we obtain that q does not divide $|\mathbf{Z}(N)|$, and on the other hand, Theorem 5 asserts that q neither divides m . However, note that q must divide n ; otherwise q does not divide any G -class size of N , and consequently, N would have a central Sylow q -subgroup, which is not possible. As a consequence, q divides $|\mathbf{O}_q(N)| = |\mathbf{Z}(N)|_q$, a contradiction. Then $\mathbf{Z}(N)_q \subset \mathbf{O}_q(N)$ as wanted.

Step 4. Assume that s and t are two distinct primes dividing $|N/B|$ such that $\mathbf{Z}(N)_r \subset \mathbf{O}_r(N)$ for $r \in \pi = \{s, t\}$. Then for every π' -element $w \in N \setminus B$ we have $\mathbf{C}_N(w)_\pi = \mathbf{Z}(N)_\pi$.

Let $w \in N \setminus B$ be a π' -element and note that $|w^G| = mn$ by Step 1. Furthermore, by considering the primary decomposition of w , it is clear that at least one component of w must have class size mn , and as a result, they have the same centralizer, that is, there is no loss if we assume that the order of w is a power of some prime in π' . Then, for every π -element $x \in \mathbf{C}_N(w)$ we have $\mathbf{C}_G(xw) = \mathbf{C}_G(x) \cap \mathbf{C}_G(w) = \mathbf{C}_G(w)$, so $\mathbf{C}_G(w) \subseteq \mathbf{C}_G(x)$, which means that $x \in \mathbf{Z}(\mathbf{C}_N(w))$. This proves that we can write $\mathbf{C}_N(w) = S_w \times T_w$, where S_w and T_w are a π -complement and a π -Hall subgroup of $\mathbf{C}_N(w)$, respectively. We will show that T_w is central in N .

Assume not and take some $x \in T_w \setminus \mathbf{Z}(N)$ to be a q -element for some prime $q \in \pi$. If $|x^G| = mn$ then we can argue as above to obtain $\mathbf{C}_G(w) = \mathbf{C}_G(x)$, and we can apply Thompson's $P \times Q$ -Lemma to get $w \in \mathbf{C}_N(\mathbf{O}_q(N))$. However, this subgroup is normal in G , proper in N , and hence by induction, it must be contained in B , the solvable radical of N . This contradicts the fact that $w \notin B$. So, we can assume that $|x^G| = m$ and then, by Step 1, we know that $x \in \mathbf{O}_q(N)$. Let us consider the subgroup $\mathbf{C}_N(x) \trianglelefteq \mathbf{C}_G(x)$. Observe that $\mathbf{O}_s(N) \subseteq \mathbf{C}_N(x)$ for every prime $s \neq q$ and also it is easy to see that every q' -element of $\mathbf{C}_N(x)$ may have class size 1 or n in $\mathbf{C}_G(x)$. We show that not every q' -element of $\mathbf{C}_N(x)$ has class size 1 in $\mathbf{C}_G(x)$, that is, is central in $\mathbf{C}_N(x)$. If this happens, we have in particular $w \in \mathbf{Z}(\mathbf{C}_N(x))$ and if we choose $r \in \pi \setminus \{q\}$, we have $w \in \mathbf{C}_N(\mathbf{O}_r(N)) \subseteq B$, a contradiction. Thus, really both possibilities, 1 and n , appear as class sizes in $\mathbf{C}_G(x)$ of the q' -elements in $\mathbf{C}_N(x)$. We can apply then Theorem 2 and obtain that $\mathbf{C}_N(x)$ has nilpotent q -complements. Since w and $\mathbf{O}_r(N)$ both belong to certain q -complement of $\mathbf{C}_N(x)$, we conclude that $w \in \mathbf{C}_N(\mathbf{O}_r(N))$, which provides again a contradiction.

Step 5. Final contradiction.

As N/B is a simple CP-group, and taking into account Theorem 9, exactly eight possibilities appear: $L_2(q)$ for $q \in \{5, 7, 8, 9, 17\}$, $L_3(4)$, $\text{Sz}(8)$ and $\text{Sz}(32)$. We will do a case-by-case analysis to see that none of them can occur.

Case 1. Let $N/B \cong L_2(8)$ or $L_2(17)$.

Recall that $|L_2(8)| = 2^3 \cdot 3^2 \cdot 7$ and $|L_2(17)| = 2^4 \cdot 3^2 \cdot 17$, and that both groups have cyclic Sylow 3-subgroups of order 9. Then for any element $xB \in N/B$ of order 9, we can apply Lemma 16 to N/B and get $xB \in \mathbf{C}_N(\mathbf{O}_3(N))B/B$, since $\mathbf{O}_3(N) \in \text{Syl}_3(B)$. This implies that $|N : \mathbf{C}_N(\mathbf{O}_3(N))B|$ is a $\{2, s\}$ -number, where $s = 7$ or $s = 17$ respectively for each one of the groups. As N is perfect, we deduce that $N = \mathbf{C}_N(\mathbf{O}_3(N))B$. However, $\mathbf{C}_N(\mathbf{O}_3(N))$ is a proper subgroup of N by Step 3 and is normal in G , so by induction, it is solvable. As a result N would be solvable too, a contradiction.

Case 2. Let $N/B \cong L_2(7)$, $\text{Sz}(8)$, $L_3(4)$ or $\text{Sz}(32)$.

In all cases, N/B has an element of order 4, say xB . Then, we can apply Lemma 16 to obtain $xB \in \mathbf{C}_N(P)B/B$, where P is a Sylow 2-subgroup of B (and P is abelian). Hence we can assume that $x \in \mathbf{C}_N(P)$. This implies that

$$x\mathbf{F}(N) \in \mathbf{C}_{N/\mathbf{F}(N)}(\mathbf{P}\mathbf{F}(N)/\mathbf{F}(N)) = \mathbf{C}_{N/\mathbf{F}(N)}(B/\mathbf{F}(N)) = S/\mathbf{F}(N),$$

for certain normal subgroup S of G . If S is proper in N , it follows by induction that S is solvable, and as a consequence $S \subseteq B$, a contradiction. Therefore, $S = N$, which means that $B/\mathbf{F}(N)$ is central in $N/\mathbf{F}(N)$. As N is perfect, then $B = \mathbf{F}(N)$, so $N/\mathbf{F}(N)$ is simple.

For each one of the groups considered, by applying [Step 3](#), we can find a set π of odd primes dividing $|N/B|$ with $|\pi| = 2$ such that $\mathbf{Z}(N)_r \subset \mathbf{O}_r(N)$ for every $r \in \pi$ (for example, choose $\{3, 5\}$, $\{3, 7\}$, $\{5, 7\}$ or $\{31, 41\}$ for each one of the groups, respectively). We will show that if P is a Sylow 2-subgroup of N , then $P/\mathbf{O}_2(N)$ acts fixed point freely on $\mathbf{O}_q(N)/\mathbf{Z}(N)_q$ (for every $q \in \pi$). Indeed, if $y\mathbf{O}_2(N) \in P/\mathbf{O}_2(N)$ and $h\mathbf{Z}(N)_q \in \mathbf{O}_q(N)/\mathbf{Z}(N)_q$ such that the commutator $[h, y] \in \mathbf{Z}(N)_q \subseteq \mathbf{Z}(N)$, then we have $[h^{o(h)}, y] = [h, y]^{o(h)} = 1$ and analogously, $[h, y^{o(y)}] = [h, y]^{o(y)} = 1$. Since h is a q -element and y is a 2-element and $q \neq 2$, we obtain $[h, y] = 1$, that is, $h \in \mathbf{C}_N(y)_\pi$. However, by [Step 4](#) this subgroup is exactly equal to $\mathbf{Z}(N)_\pi$, so $h \in \mathbf{Z}(N)_q$. Consequently, $P/\mathbf{O}_2(N)$ must be cyclic or generalized quaternion. Nevertheless, this group is isomorphic to a Sylow 2-subgroup of $N/\mathbf{F}(N)$, which is simple. It is well known that simple groups do not possess such Sylow 2-subgroups.

Case 3. Let $N/B \cong A_6$.

We remark that N/B has elements of order 4 and that $\mathbf{Z}(N)_5 \subset \mathbf{O}_5(N)$ by [Step 3](#). If $\mathbf{Z}(N)_3 \subset \mathbf{O}_3(N)$ then we can argue as in [Case 2](#) to get a contradiction. Thus, for the rest of the case we will assume that $\mathbf{O}_3(N) = \mathbf{Z}(N)_3$.

If $B/\mathbf{F}(N)$ is nontrivial and $P_3 \in \text{Syl}_3(N)$, then the fact that $N/\mathbf{F}(N)$ is a CP-group implies that $P_3\mathbf{F}(N)/\mathbf{F}(N)$ acts on $B/\mathbf{F}(N)$ fixed point freely. As a consequence $P_3/\mathbf{Z}(N)_3 \cong P_3\mathbf{F}(N)/\mathbf{F}(N)$ should be cyclic, which clearly is a contradiction.

Hence, we will assume that $B = \mathbf{F}(N)$. Suppose first that there exists a prime $q \notin \{2, 3, 5\}$ dividing $|N|$ and notice that $\mathbf{Z}(N)_q \subset \mathbf{O}_q(N)$ by [Step 3](#). Then for every 2-element $w \in N \setminus B$, by using [Steps 3 and 4](#), we have $\mathbf{C}_N(w)_{\{5, q\}} = \mathbf{Z}(N)_{\{5, q\}}$. So, if we consider the action of $P_2/\mathbf{O}_2(N)$ on $\mathbf{O}_5(N)/\mathbf{Z}(N)_5$ where $P_2 \in \text{Syl}_2(N)$, this is fixed point free, and similarly to [Case 2](#), this leads to a contradiction.

Thus, we can assume that no other prime distinct from 2, 3 and 5 divides $|N|$ and we have $\mathbf{F}(N) = \mathbf{O}_2(N) \times \mathbf{O}_5(N) \times \mathbf{Z}(N)_3$. On the other hand, note that N/B has an element xB of order 4, and by applying [Lemma 16](#), we have that $xB \in \mathbf{C}_N(\mathbf{O}_2(N))B/B$. Now, if $\mathbf{C}_N(\mathbf{O}_2(N))$ were a proper subgroup of N , then it would be solvable by induction, so $\mathbf{C}_N(\mathbf{O}_2(N)) \subseteq B$, again a contradiction since $x \notin B$. So $\mathbf{O}_2(N) = \mathbf{Z}(N)_2$ and consequently, $\mathbf{F}(N) = \mathbf{Z}(N)_2 \times \mathbf{Z}(N)_3 \times \mathbf{O}_5(N)$. Then we have

$$\mathbf{F}(N)/\mathbf{O}_5(N) \subseteq \mathbf{Z}(N/\mathbf{O}_5(N)) \subseteq N/\mathbf{O}_5(N).$$

However,

$$\frac{N/\mathbf{O}_5(N)}{\mathbf{F}(N)/\mathbf{O}_5(N)} \cong N/\mathbf{F}(N)$$

is simple, so we infer that $\mathbf{F}(N)/\mathbf{O}_5(N) = \mathbf{Z}(N/\mathbf{O}_5(N))$. Thus, we have proved that $N/\mathbf{O}_5(N)$ is a quasi-simple group associated to the simple group A_6 . It is known then that the order of its center, which coincides with $|\mathbf{Z}(N)_{5'}|$, must divide the order of the Schur multiplier of A_6 , which is equal to 6. On the other hand, every class size of $N/\mathbf{F}(N)$ (which are $\{1, 40, 45, 72, 90\}$) divides some class size of N , and consequently, divides mn too, and from this property we deduce that 360 divides mn . Moreover, the equation

$$|N| = |\mathbf{F}(N)| + mnt, \quad t \in \mathbb{Z}^+$$

yields that $360 = |N/\mathbf{F}(N)| = 1 + mnt/|\mathbf{F}(N)|$. As a consequence, $mn_{\{2, 3\}}$ divides $|\mathbf{F}(N)|_{\{2, 3\}} = 6$, which contradicts the fact that 360 divides mn .

Case 4. Let $N/B \cong A_5$.

We remark that there are no elements of order 4 in N/B and that $Z(N)_3 \subset O_3(N)$ and $Z(N)_5 \subset O_5(N)$ by Step 3. We will construct a suitable subgroup of $B/F(N)$, isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, acting on $O_5(N)/Z_5(N)$ (or similarly on the corresponding group for the prime 3). We will see that this action will provide a contradiction. Take $P \in \text{Syl}_2(N)$ and we divide the proof into several parts.

We first show that there exists an element $x \in P \setminus B$ such that $o(xF(N)) = 4$. As a consequence, $x \in A := \langle y \in P \mid y^2 \notin F(N) \rangle$ and $F(N) \subset B$. Suppose that $xF(N)$ has order 2 for every element $x \in P \setminus B$, and let us take $x, y \in P \setminus B$ so that we construct the Sylow 2-subgroup $\langle xB \rangle \times \langle yB \rangle = \{B, xB, yB, xyB\} = PB/B$. As $xy \in P \setminus B$, we are assuming that $xyF(N)$ has also order 2. This implies that $S/F(N) = \langle xF(N) \rangle \times \langle yF(N) \rangle$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. On the other hand, $S/F(N)$ acts fixed point freely on $O_5(N)/Z(N)_5$, since Step 4 provides the equality $C_N(w)_{\{3,5\}} = Z(N)_{\{3,5\}}$ for every $w \in \{x, y, xy\}$. This is clearly a contradiction, because $S/F(N)$ is neither a generalized quaternion group nor a cyclic group. Therefore, there exists an element $x \in P \setminus B$ with $o(xF(N)) = 4$ as desired. In particular, $x \in A = \langle y \in P \mid y^2 \notin F(N) \rangle$ (defined and studied in Proposition 15). Also, the fact that N/B does not have elements of order 4 trivially implies that $F(N) \subset B$.

We prove now that there exists an element $a \in N_N(P)$ such that $aF(N)$ has order 3. As $N/B \cong A_5$ the normalizer of every Sylow 2-subgroup of N/B has order 12. In fact, if $P \in \text{Syl}_2(N)$, then $|N_{N/B}(PB/B)| = 12$. Observe that

$$N_{N/B}(PB/B) = N_N(P)B/B \cong N_N(P)/(N_N(P) \cap B),$$

which implies that 3 divides the order of

$$N_N(P)/(N_N(P) \cap F(N)) \cong F(N)N_N(P)/F(N).$$

Then we can choose an element $a \in N_N(P)$ such that $aF(N)$ has order 3.

We claim that the elements x and x^a in P are distinct and commute and in particular, $\langle xB \rangle \times \langle x^aB \rangle \in \text{Syl}_2(N/B)$. It is trivial that $x^a \in P \setminus B$, since B is normal in N and also $x^aF(N)$ has order 4. If $xB = x^aB$, then aB centralizes xB , which implies that xaB has order 12 and this is not possible because N/B is a CP-group. Therefore, x and x^a are two distinct elements lying in A , and both trivially commute because A is abelian by Proposition 15. We certainly deduce that $\langle xB \rangle \times \langle x^aB \rangle \in \text{Syl}_2(N/B)$, as wanted.

We are ready to get the final contradiction. We assert that $(x^a)^2F(N) \neq x^2F(N)$; otherwise $x^2aF(N)$ would have order 6, contradicting the fact that $N/F(N)$ is a CP-group. Now we consider the subgroup $\langle x^2F(N) \rangle \times \langle (x^a)^2F(N) \rangle$, isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Notice that since x^2 is not in $F(N)$, then the G -conjugacy class of x^2 has size mn . Furthermore, the fact that $C_G(x) \subseteq C_G(x^2)$ leads to $C_G(x) = C_G(x^2)$, and in particular, $C_N(x) = C_N(x^2)$. Then, by using Step 4, we have $C_N(x^2)_{\{3,5\}} = Z(N)_{\{3,5\}}$ and hence, the action of $\langle x^2F(N) \rangle$ on $O_5(N)/Z(N)_5$ would be fixed point free. A similar argument holds for $\langle (x^a)^2F(N) \rangle$ and for $\langle (xx^a)^2F(N) \rangle$ since x^2 and $(xx^a)^2$ do not belong to $F(N)$. We conclude that the action of $\langle x^2F(N) \rangle \times \langle (x^a)^2F(N) \rangle$ on $O_5(N)/Z(N)_5$ has no fixed points, which provides the final contradiction. \square

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