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Computing Hasse–Schmidt derivations and Weil restrictions over jets

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ABSTRACT

We give an explicit and compact description of the Hasse–Schmidt derivations, using Fitting ideals and symmetric tensor algebras. Finally we verify the localization conjecture [9].

1. Introduction

The higher order derivations were introduced by Hasse and Schmidt, and generalize the usual notion of derivations. We will give a new description of the higher order derivations, a description using Fitting ideals and symmetric tensor algebras. We obtain a compact presentation of the higher order derivations, naturally relating the order n derivations with those of order $n-1$. Finally we verify that the Hasse–Schmidt derivations behave well under localization, thereby showing that the localization conjecture [9] holds.

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Let R and A' be two (commutative) A-algebras. A higher order derivation [5], of order n, is a sequence $\partial_n = (d_0, \dots, d_n)$ of A-linear maps $d_p: R \longrightarrow A'$, where d_0 is an algebra homomorphism, and where

$$d_p(xy) = \sum_{i+j=p} d_i(x) d_j(y),$$

for all $1 \le p \le n$. A higher order derivation is also called Hasse–Schmidt derivation, and works relating these to ordinary derivations can be found in e.g. [7,3].

We let $\mathrm{Der}^n_{R/A}(A')$ denote the set of all higher order derivations from R to A'. By composition one obtains that the set of higher order derivations form a functor $\mathrm{Der}^n_{R/A}(-)$, from the category of A-algebras to sets. One can construct the representing object $\mathrm{HS}^n_{R/A}$ by forming the polynomial ring over R, with n variables for each element in R, modulo all the expected relations, see e.g. [10].

We will give a different, and a more compact, description of the representing object $\mathrm{HS}^n_{R/A}$. It is well-known that Hasse–Schmidt derivations are equivalently described by jets and arc spaces. An A-algebra homomorphism

$$u: R \longrightarrow A'[t]/\big(t^{n+1}\big) = A' \bigotimes\nolimits_A A[t]/\big(t^{n+1}\big)$$

encodes the same information as a higher order derivation from R to A'. In other words the higher order derivations are given as the functor $\mathrm{Hom}_{A\text{-alg}}(R, E)$, with co-domain the arc of jets $E = A[t]/(t^{n+1})$.

The functors $\mathrm{Hom}_{A\text{-alg}}(R, E)$ were described explicitly using Fitting ideals in [8], and the results in the present paper are obtained by specializing to $E = A[t]/(t^{n+1})$. With this specific co-domain $E = A[t]/(t^{n+1})$ we will write down explicitly the representing object, and use the explicit presentation to extract new information as well as answer questions about the Hasse–Schmidt derivations.

Our explicit presentation is as follows. Write the A-algebra $R = A[x, y, \dots, z]/(f_1, \dots, f_m)$, that is as a quotient of a polynomial ring modulo some relations (for notational simplicity we present the result only using finitely many variables and relations). Then the Hasse–Schmidt derivations of order n are represented by $\mathrm{HS}^n_{R/A}$ which is the algebra

$$\mathrm{HS}^{n-1}_{R/A}[d_n x, d_n y, \dots, d_n z]/(d_n f_1, \dots, d_n f_m),$$

where $d_n x, d_n y, \dots, d_n z$ are variables over $\mathrm{HS}^{n-1}_{R/A}$, and where $d_n f$ is the n'th order derivation of $f \in A[x, y, \dots, z]$. In particular we recover that $\mathrm{HS}^1_{R/A}$ equals the symmetric tensor algebra of the R-module of differentials $\Omega^1_{R/A}$.

As the Weil restriction of $A[t]/(t^{n+1})$-algebras is closely related, we also write explicit presentations for these rings. The Weil restrictions are important objects within algebraic geometry, and it might be of interest to see explicit presentations of these algebras as well.

In the last section we focus on Hasse–Schmidt derivations and localizations. We show that the functor of Hasse–Schmidt derivations $\mathrm{HS}^{\infty}_{R/A}$ is representable, and that it behaves well under localizations. We give two proofs for that claim. One proof is obtained by identifying the Hasse–Schmidt derivations as the direct limit of the Hasse–Schmidt derivations of finite order, of which we have an explicit description. The other proof is based on identifying the Hasse–Schmidt derivations with the infinite tensor product. The question concerning the localization property for the Hasse–Schmidt derivations was conjectured in [9], apparently motivated by [1].

2. Hasse–Schmidt derivations

We will first give a description of the ring of Hasse–Schmidt derivations of order n, and thereafter relate that ring with the Hasse–Schmidt derivations of order $n-1$. All rings and algebras considered are commutative and with unit.

2.1. Parameterizing algebra homomorphisms

As a prelude to our description of Hasse–Schmidt derivations, we will consider the following functor. Let R and E be two A-algebras. Then we have the functor $\underline{\mathrm{Hom}}_A(R,E)$ from the category of A-algebras to sets, sending an A-algebra A' to the set

$$\underline{\mathrm{Hom}}_A(R,E)\big(A'\big) = \mathrm{Hom}_{A\text{-alg}}\big(R, E \bigotimes_A A'\big).$$

2.2. Symmetric algebra

Let E be an A-algebra that is free of finite rank. It is well-known, and readily checked, that the symmetric tensor algebra $\mathrm{S}_A(V \bigotimes_A E^{\star})$ represents the functor $\underline{\mathrm{Hom}}_A(\mathrm{S}_A(V),E)$, where $E^{\star}$ denotes the dual module $E^{\star} = \mathrm{Hom}_A(E,A)$, and where V is any A-module. We denote by

$$u: \mathrm{S}_A(V) \longrightarrow E \bigotimes_A \mathrm{S}_A\big(V \bigotimes_A E^{\star}\big) \tag{2.2.1}$$

the universal element, given explicitly below.

2.3. Universal map and grading

We fix a basis $e_0, \dots, e_n$ for E, and let $e_0^{\star}, \dots, e_n^{\star}$ denote its dual basis. Then the universal element u (2.2.1) is the A-algebra homomorphism determined by sending elements $x \in V$ to

$$u(x) = \sum_{i=0}^{n} e_i \otimes x \otimes e_i^{\star}.$$

Having the basis of E fixed we also get an induced grading on the A-module $\mathrm{S}_A(V \bigotimes_A E^\star)$. We let elements of the form $x \otimes e_i^\star$ in $V \bigotimes_A E^\star$ have degree

$$\deg(x \otimes e_i^\star) = i \quad \text{for } i = 0, \dots, n.$$

Definition 2.4. Let $d: \mathrm{S}_A(V) \longrightarrow \mathrm{S}_A(V \bigotimes_A E^\star)$ denote the A-algebra homomorphism determined by sending $x \in V$ to $d(x) = \sum_{i=0}^n x \otimes e_i^\star$. For each integer $0 \le k \le n$ we define the map

$$d_k: \mathrm{S}_A(V) \longrightarrow \mathrm{S}_A\big(V \textstyle\bigotimes_A E^\star\big)$$

by letting $d_k(x) = (d(x))_k$ denote the degree k-part of the A-algebra homomorphism d.

2.5. Notation

We let E_n denote the A-algebra $E_n = A[\epsilon]/(\epsilon^{n+1})$. The basis we fix is given by powers of the variable ϵ, so $e_0, \dots, e_n$ with $e_i = \epsilon^i$ is an A-module basis of E_n. Note that with this convention we have that the products of the basis elements are given as

$$e_i \cdot e_j = \begin{cases} e_{i+j} & \text{if } i + j \le n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.1}$$

Lemma 2.6. *Let $E_n = A[\epsilon]/(\epsilon^{n+1})$, where the basis is given by the powers of the variable ϵ. For monomials $x = x_1 \otimes \cdots \otimes x_m$ in $\mathrm{S}_A(V)$ we have that*

$$d_k(x) = \sum_{i_1 + \cdots + i_m = k} (x_1 \otimes e_{i_1}^\star) \cdots (x_m \otimes e_{i_m}^\star). \tag{2.6.1}$$

In particular, for any $x \in \mathrm{S}_A(V)$ we get that

$$u(x) = \sum_{k=0}^n e_k \otimes d_k(x), \tag{2.6.2}$$

where u is the universal map (2.2.1).

Proof. The expression (2.6.1) follows if we prove that Eq. (2.6.2) holds. As u is A-linear, it suffices to show the statement for monomial elements $x = x_1 \otimes \cdots \otimes x_m$ of $\mathrm{S}_A(V)$. Then, by definition, we have that

$$u(x) = \left(\sum_{k=0}^n e_k \otimes x_1 \otimes e_k^\star \right) \cdots \left(\sum_{k=0}^n e_k \otimes x_m \otimes e_k^\star \right).$$

Expanding the product gives

$$u(x) = \sum_{\substack{0\leq k_i\leq n\\ i=1,\ldots,m}} e_{k_1}\cdots e_{k_m} \otimes \left(x_1 \otimes e^{\star}_{k_1}\right)\cdots\left(x_m \otimes e^{\star}_{k_m}\right).$$

Now, using the product on E_n, displayed in (2.5.1), we get that the k'th component of $u(x)$, written out with respect to the basis $e_0, \ldots, e_n$, is

$$\sum_{i_1+\cdots+i_m=k} \left(x_1 \otimes e^{\star}_{i_1}\right)\cdots\left(x_m \otimes e^{\star}_{i_m}\right),$$

proving the claim. □

Lemma 2.7. *The sequence $\partial_n = (d_0, \ldots, d_n)$ is a higher order derivation, of length n, from $\mathrm{S}_A(V)$ to $\mathrm{S}_A(V \bigotimes_A E^{\star}_n)$. In particular we have that $d_0 \colon \mathrm{S}_A(V) \longrightarrow \mathrm{S}_A(V \bigotimes_A E^{\star}_n)$ is an A-algebra homomorphism, and $d_1 \colon \mathrm{S}_A(V) \longrightarrow \mathrm{S}_A(V \bigotimes_A E^{\star})$ is an A-linear derivation.*

Proof. The universal map $u \colon \mathrm{S}_A(V) \longrightarrow E_n \bigotimes_A \mathrm{S}_A(V \bigotimes E^{\star}_n)$ in (2.2.1) is an A-algebra homomorphism. The co-domain is simply $B[\epsilon]/(\epsilon^{n+1})$, where $B = \mathrm{S}_A(V \bigotimes_A E^{\star}_n)$. By Lemma 2.6 we have $u(x)$ expressed in terms of the basis $e_0, \ldots, e_n$ as $u(x) = \sum_{k=0}^{n} d_k(x)e_k$. This is equivalent with ∂_n being a higher derivation of order n, over A, see e.g. [5]. □

Example 2.8. For $x \in V$, let $x_i = x \otimes e^{\star}_i$, for $i \geq 0$, so that $\deg(x_i) = i$. For any monomial $x \otimes y \otimes \cdots \otimes z$, we consider the product

$$(x_0 + x_1 + \cdots)(y_0 + y_1 + \cdots)\cdots(z_0 + z_1 + \cdots).$$

We have that $d_k(x \otimes y \otimes \cdots \otimes z)$ equals the degree k term of the expansion of the product above. In particular with $x \otimes x \otimes y = x^2y$ we get that $d_0(x^2y) = x_0^2y_0$, that $d_1(x^2y) = 2x_1y_0 + x_0^2y_1$, and that

$$d_2\left(x^2y\right) = 2x_2x_0y_0 + x_1^2y_0 + 2x_0x_1y_1 + x_0^2y_2.$$

Definition 2.9. For any ideal $I \subseteq \mathrm{S}_A(V)$ we let $\partial_n I \subseteq \mathrm{S}_A(V \otimes_A E^{\star}_n)$ denote the ideal generated by

$$\partial_n I = \left\{d_0(f), \ldots, d_n(f) \mid f \in I\right\}.$$

Proposition 2.10. *Let $E_n = A[\epsilon]/(\epsilon^n)$, and let $R = \mathrm{S}_A(V)/I$ be an A-algebra. Then the functor $\underline{\mathrm{Hom}}_A(R, E_n)$ is represented by*

$$\mathrm{HS}^n_{R/A} := \mathrm{S}_A\left(V \bigotimes_A E^{\star}_n\right)/\partial_n I.$$

The universal element is induced by u (2.2.1).

Proof. We have (see e.g. [8]) that the representing object is given as the quotient algebra

$$\mathrm{S}_A\big(V \bigotimes_A E_n^\star\big)/\mathrm{Fitt}\big(u(I)\big),$$

where $\mathrm{Fitt}(u(I))$ is the $(n-1)$-th Fitting ideal of the $\mathrm{S}_A(V \bigotimes_A E_n^\star)$ cokernel module $u(I) \subseteq E \bigotimes_A \mathrm{S}_A(V \bigotimes_A E_n^\star)$. Let $f \in I$ be an element. By Lemma 2.6 we have that $u(f) = \sum_{k=0}^n d_k e_k$. We then get that

$$\mathrm{Fitt}\big(u(f)\big) = \big(d_0(f), \dots, d_n(f)\big),$$

and the claim follows. □

2.11. Hasse–Schmidt derivations

A Hasse–Schmidt derivation, of order n on an A-algebra R, is an A-algebra homomorphism

$$\partial_n : R \longrightarrow R[\epsilon]/\big(\epsilon^{n+1}\big)$$

that decomposed $\partial_n = (d_0, \dots, d_n)$ is such that $d_0 = \mathrm{id}$. The Hasse–Schmidt derivation becomes in a natural way a functor $\mathrm{Der}_{R/A}(-)$ from the category of A-algebras to sets.

Corollary 2.12. *Let R be an A-algebra. Then the functor of Hasse–Schmidt derivations $\mathrm{Der}_{R/A}(-)$ is represented by the pair $(\mathrm{HS}^n_{R/A}, \partial_n^R)$, where ∂_n^R is induced by the sequence ∂_n.*

Proof. By the usual properties of the tensor product we have that an A-algebra homomorphism $R \longrightarrow R \bigotimes_A A'$ is a pair (ι, ∂) of A-algebra homomorphisms $\iota : R \longrightarrow R$ and $\partial : R \longrightarrow A'$. As the A-algebra homomorphism $\iota : R \longrightarrow R$ is assumed to be the identity, we see that a Hasse–Schmidt derivation is nothing but an A-algebra homomorphism $R \longrightarrow A'[\epsilon]/(\epsilon^{n+1})$. □

2.13. Iterations

We have a canonical morphism $E_n \longrightarrow E_{n-1}$. Having the basis of E_n fixed, we get an identification $E_n = E_n^\star$. Hence we have an induced map $E_n^\star \longrightarrow E_{n-1}^\star$ of A-modules. We then have a, non-canonical, induced A-algebra homomorphism

$$p_n^V : \mathrm{S}_A\big(V \bigotimes_A E_n^\star\big) \longrightarrow \mathrm{S}_A\big(V \bigotimes_A E_{n-1}^\star\big).$$

Let $d_k^n : \mathrm{S}_A(V) \longrightarrow \mathrm{S}_A(V \bigotimes_A E_n^\star)$ denote the degree operator in Definition 2.4, where we now have added the superscript to keep track of n. Then, for every $0 \leq k \leq n-1$ we have that $p_n^V \circ d_k^n = d_k^{n-1}$. Thus we have that

$$p_n^V \circ \partial_n = \partial_{n-1}. \tag{2.13.1}$$

Proposition 2.14. *Write an A-algebra R as a quotient of a polynomial ring $R = A[x_\alpha]/(f_\beta)_{\alpha\in\mathscr{A},\beta\in\mathscr{B}}$, and set $A := \mathrm{HS}^{-1}_{R/A}$. Then we have, for each integer $n \geq 0$, that $\mathrm{HS}^n_{R/A}$ is the quotient of the polynomial ring in the variables $\{d_n x_\alpha\}_{\alpha\in\mathscr{A}}$ over $\mathrm{HS}^{n-1}_{R/A}$, modulo the ideal generated by $\{d_n f_\beta\}_{\beta\in\mathscr{B}}$; that is*

$$\mathrm{HS}^n_{R/A} = \mathrm{HS}^{n-1}_{R/A}[d_n x_\alpha]/(d_n f_\beta)_{\alpha\in\mathscr{A},\,\beta\in\mathscr{B}}.$$

In particular we have that $\mathrm{HS}^0_{R/A} = R$, and that $\mathrm{HS}^1_{R/A} = \mathrm{S}_R(\Omega^1_{R/A})$.

Proof. With the basis of $E^\star_n$ fixed, we get an isomorphism $V = V_i$, where V_i is the A-module generated by tensors of the form $x \otimes e^\star_i$, where $x \in V$, for each $i = 0, \dots, n$. Then we get an induced A-module isomorphism $V \bigotimes_A E^\star = \bigoplus_{i=0}^n V_i$, and an isomorphism of A-algebras

$$\mathrm{S}_A\big(V \bigotimes\nolimits_A E^\star\big) = \bigotimes_{i=0}^{n} \mathrm{S}_A(V_i). \tag{2.14.1}$$

Let I be an ideal in $S_A(V)$, and for each integer $0 \leq k \leq n$ we let $\partial^k_n I \subseteq \mathrm{S}_A(V \bigotimes_A E^\star_n)$ denote the ideal generated by the $k+1$-first components of $\partial_n I$. That is $\partial^k_n I$ is the ideal generated by $(d_0 f, d_1 f, \dots, d_k f)$, with $f \in I$. Under the isomorphism (2.14.1) we have that

$$\mathrm{S}_A\big(V \bigotimes\nolimits_A E^\star_n\big)/\partial^k_n I = \left(\bigotimes_{i=0}^{k} \mathrm{S}_A(V_i)\right) \Big/ \partial^k_n I \bigotimes_{i=k+1}^{n} \mathrm{S}_A(V_i).$$

From the equality (2.13.1) combined with Proposition 2.10, it follows that

$$\bigotimes_{i=0}^{k} \mathrm{S}_A(V_i)/\partial^k_n I = \mathrm{HS}^k_{R/A},$$

where $R = \mathrm{S}_A(V)/I$. The result then follows by letting V be a free A-module, and choosing $k = n-1$. $\square$

Example 2.15. Consider the A-algebra $R = A[x,y]/(x^2 y)$, and identify $x = d_0 x$ and $y = d_0 y$. We get

$$\mathrm{HS}^1_{R/A} = R[d_1 x, d_1 y]/\big(2xy \cdot d_1 x + x^2 \cdot d_1 y\big) = \mathrm{S}_R\big(\Omega^1_{R/A}\big).$$

And we have that $\mathrm{HS}^2_{R/A}$ equals the quotient of the polynomial ring $\mathrm{HS}^1_{R/A}[d_2 x, d_2 y]$ modulo the ideal generated by $d_2(x^2 y)$ which we computed in Example 2.8 as

$$y \cdot (d_1 x)^2 + 2x \cdot d_1 x \cdot d_1 y + 2xy \cdot d_2 x + x^2 \cdot d_2 y.$$

Remark 2.16. In the previous example, note that $d_2(x^2y)$ is not expressed in terms of d_2x and d_2y. The expression for $\mathrm{HS}^1_{R/A}$ as symmetric tensor algebra of a module, appears to be coincidental, and not the first order step of a more general pattern.

3. Weil restriction

The Weil restriction of a homomorphism $E_n \longrightarrow R$ is closely related to the Hasse–Schmidt derivations considered in the previous section. We will include a similar description of these Weil restrictions.

3.1. Weil restriction

In algebraic geometry the Weil restrictions appear naturally as fibers of morphism between moduli spaces [4, 4. Variantes], and are special instances of Hom-stacks [6]. For a thorough description of their basic properties we refer to [2].

Let $E \longrightarrow R$ be an A-algebra homomorphism. The Weil restriction, that we denote by $\mathfrak{R}_{E/A}(R)$, is the functor from the category of A-algebras to sets, that takes an A-algebra A' to the set

$$\mathfrak{R}_{E/A}(R)\big(A'\big) = \mathrm{Hom}_{A\text{-alg}}\big(R, E \bigotimes_A A'\big).$$

Proposition 3.2. *Let R be an A-algebra, and let $E_n = A[\epsilon]/(\epsilon^{n+1})$. Then $\mathrm{HS}^n_{R/A}$ represents the Weil restriction $\mathfrak{R}_{E_n/A}(R \bigotimes_A E_n)$. The universal element is obtained by extension of scalars of the universal map (2.2.1).*

Proof. An A-algebra homomorphism $R \longrightarrow A'$ is equivalent with an A-algebra homomorphism $R \bigotimes_A A' \longrightarrow A'$ being the identity on A', from which the result follows. $\square$

3.3. Extended grading

We will introduce some notation before we continue with the general situation with E_n-algebras in general. Let V be an A-module, and let E be a free A-module of finite rank. We fix a basis $e_0, \ldots, e_n$ for the A-module E, and similarly we fix the dual basis for the dual module $E^\star$. For each integer $0 \le k \le n$ we have the A-linear map $d^k_+ : V \longrightarrow V \bigotimes_A E^\star$ sending $x \in V$ to

$$d^k_+(x) = \sum_{i=0}^{n-k} x \otimes e^\star_{i+k}.$$

The map is simply a shift of the map d described in Definition 2.4. Together these maps give an A-module map

$$\big(d^0_+, d^1_+, \ldots, d^n_+\big) : \bigoplus_{i=0}^{n} V \longrightarrow V \bigotimes_A E^\star.$$

The induced A-algebra homomorphism is denoted

$$d_+\colon \bigotimes_{i=0}^{n} \mathrm{S}_A(V) \longrightarrow \mathrm{S}_A\big(V \otimes_A E^\star\big).$$

Recall that we in Section 2.3, introduced a grading on $\mathrm{S}_A(V \otimes_A E^\star)$ induced by the basis. Similar to Definition 2.4 we next extend the graded operators d_k.

Definition 3.4. For each integer $0 \leq k \leq n$ we define the map

$$D_k^n\colon \bigotimes_{i=0}^{n} \mathrm{S}_A(V) \longrightarrow \mathrm{S}_A\big(V \otimes_A E_n^\star\big)$$

by letting $D_k^n(F) = (d_+(F))_k$ denote the degree k-part of the A-algebra homomorphism d_+.

Remark 3.5. It is clear that we have $q_n^V \circ D_k^n = D_k^{n-1}$, for each integer $0 \leq k \leq n-1$. We will therefore in the sequel skip the reference to E_n in the notation of the degree map, and simply write D_k instead of D_k^n.

Lemma 3.6. *Let $E_n = A[\epsilon]/(\epsilon^{n+1})$, and identify the $\mathrm{S}_A(V)$-modules $\mathrm{S}_A(V) \otimes_A E_n = \bigotimes_{i=0}^n \mathrm{S}_A(V)$. For any $F \in \mathrm{S}_A(V) \otimes_A E_n$ we have that*

$$u_n(F) = \sum_{k=0}^{n} D_k(F) e_k,$$

where $u_n\colon \mathrm{S}_A(V) \otimes_A E_n \longrightarrow E_n \otimes_A \mathrm{S}_A(V \otimes_A E_n^\star)$ is the universal map in Proposition 3.2. Moreover, write $F = \sum_{i=0}^n F_i e_i$, with $F_i \in \mathrm{S}_A(V)$, then we have that

$$D_k(F) = D_k\left(\sum_{i=0}^{n} F_i e_i\right) = \sum_{j=0}^{k} d_j(F_{k-j}),$$

where $(d_0, \ldots, d_n) = \partial_n$ is the higher order derivation introduced earlier.

Proof. As $\mathrm{S}_A(V) \otimes_A E_n = \bigoplus_{i=0}^n \mathrm{S}_A(V) e_i$, we get that the universal map

$$u_n\colon \bigoplus_{i=0}^{n} \mathrm{S}_A(V) e_i \longrightarrow \bigoplus_{i=0}^{n} \mathrm{S}_A\big(V \otimes_A E^\star\big) e_i$$

is $n+1$-copies $u_n = (u, \ldots, u)$ of the universal map (2.2.1). We therefore have that $u_n(F) = \sum_{i=0}^n u(F_i) e_i$. Now, using Lemma 2.6 together with the relations (2.5.1), we get that

$$\sum_{i=0}^{n} u(F_i) e_i = \sum_{i=0}^{n} \left(\sum_{k=0}^{n} d_k(F_i) e_k\right) e_i = \sum_{k=0}^{n} \left(\sum_{j=0}^{k} d_j(F_{k-j})\right) e_k.$$

We then have that $\sum_{j=0}^{k} d_j(F_{k-j})e_k$ is the degree k part of $u_n(F)$, and we have proven the statements of the lemma. □

Definition 3.7. If $I \subseteq \mathrm{S}_A(V) \bigotimes_A E_n$ is an ideal, we define the ideal $\Delta_n I \subseteq \mathrm{S}_A(V \bigotimes_A E_n^\star)$ as the ideal generated by $D_k(f)$, with $k = 0, \ldots, n$, for each $f \in I$.

Proposition 3.8. *Let $E_n \longrightarrow R$ be an A-algebra homomorphism. Write R as a quotient $R = \mathrm{S}_A(V) \bigotimes_A E_n/I$. Then the Weil restriction $\mathfrak{R}_{E_n/A}(R)$ is represented by the A-algebra*

$$\mathrm{R}_{E_n/A}(R) := \mathrm{S}_A\big(V \bigotimes_A E_n^\star\big)/\Delta_n I.$$

The universal element is induced by u_n.

Proof. We have by Proposition 3.2, together with Proposition 2.10 that $\mathrm{S}_A(V \bigotimes_A E_n^\star)$ represents $\mathfrak{R}_{E_n/A}(\mathrm{S}_A(V) \bigotimes_A E_n)$. We need only to verify that $\Delta_n I$ is the $(n-1)$-th Fitting ideal of the $\mathrm{S}_A(V \bigotimes_A E_n^\star)$-module $u_n(I)$, where $u_n \colon \mathrm{S}_A(V) \bigotimes_A E_n \longrightarrow \mathrm{S}_A(V \bigotimes_A E_n^\star)$ is the universal map. By Lemma 3.6 we have that $u_n(F)$, for any element F, is $\sum_{k=0}^{n} D_k(F)e_k$, and the result follows. □

Example 3.9. As an example, consider the polynomial ring in two variables x and y over E_1, that is the A-algebra $A[x, y, \epsilon]/(\epsilon^2)$. Let V be a free A-module of rank 2, so $A[x, y, \epsilon]/(\epsilon^2)$ equals $\mathrm{S}_A(V) \bigotimes_A E_1$. Let $F = F_0 + F_1 \cdot \epsilon$ be the element

$$F = xy + x^2 y \cdot \epsilon.$$

Identify, furthermore, $d_0 x = x$ and $d_0 y = y$, so that $\mathrm{S}_A(V \bigotimes_A E_1^\star)$ is the polynomial ring $A[x, y, d_1 x, d_1 y]$. We have that $D_0(F) = F_0 = xy$, and that

$$D_1(F) = d_0 F_1 + d_1 F_0 = x^2 y + 2xy d_1 x + x^2 d_1 y.$$

Then the Weil restriction of $E_1 \longrightarrow R = A[x, y, \epsilon]/(\epsilon^2, xy + x^2 y \epsilon)$ is the A-algebra

$$\mathrm{R}_{E_1/A}(R) = A[x, y, d_1 x, d_1 y]/\big(xy, x^2 y + 2xy d_1 x + x^2 d_1 y\big).$$

4. Localization of Hasse–Schmidt derivations

We end this note by verifying that the Hasse–Schmidt derivations behave well under localization.

Proposition 4.1. *For any A-algebra R, and any multiplicatively closed subset $S \subseteq R$ we have that*

$$\mathrm{HS}^n_{S^{-1}R/A} = \mathrm{HS}^n_{R/A} \bigotimes_R S^{-1} R.$$

In order words, the Hasse–Schmidt derivations of order n, commute with localization.

Proof. As tensor product commutes with direct limit it suffices to show the proposition in the particular case with $S = \{1, f, f^2, \dots,\}$, that is the multiplicatively closed set generated by an element $f \in R$. Then $S^{-1}R = R[t]/(F)$, where $F = ft - 1$. Take a presentation $R = A[x_\alpha]/(f_\beta)_{\alpha\in\mathscr{A},\beta\in\mathscr{B}}$, so $S^{-1}R = A[x_\alpha, t]/(f_\beta, F)_{\alpha\in\mathscr{A},\beta\in\mathscr{B}}$. By iterative use of Proposition 2.14 we get that the ring $\mathrm{HS}^n_{S^{-1}R/A}$ has the following presentation

$$S^{-1}R[d_p x_\alpha, d_p t]/(d_p f_\beta, d_p F) \quad \text{where } p = 1, \dots, n,\ \alpha \in \mathscr{A},\ \beta \in \mathscr{B}.$$

Again, using $S^{-1}R = R[t]/(F)$ and the presentation given in Proposition 2.14, we see that we can re-arrange the presentation of $\mathrm{HS}^n_{S^{-1}R/A}$ as

$$\big(\mathrm{HS}^n_{R/A} \bigotimes_R S^{-1}R[d_1 t, \dots d_n t]\big)/(d_1 F, \dots, d_n F). \tag{4.1.1}$$

A property of a higher order derivation $(d_0, \dots, d_n)$ is that for each $1 \le p \le n$ we have that $d_p(xy) = \sum_{i+j=p} d_i(x) d_j(y)$. Applying this property, and that the map d_p is linear, to $F = ft - 1$ gives us for each $p = 1, \dots, n$ that $d_p(ft - 1) = \sum_{i=0}^{p} d_i f d_{p-i} t$. We have that $t = f^{-1}$, so in the quotient ring (4.1.1) we have the identity

$$d_p t = t(-t \cdot d_p f - d_1 t \cdot d_{p-1} f - \cdots - d_{p-1} t \cdot d_1 f). \tag{4.1.2}$$

Since $f \in R$, we have that $d_p f \in \mathrm{HS}^n_{R/A}$ for all $p = 1, \dots, n$. We claim that we can eliminate the variables $d_1 t, \dots, d_n t$, starting with the lowest. We have, for $p = 1$, that $d_1 t = -t^2 \cdot d_1 f$, and therefore we can eliminate the variable $d_1 t$, expressing it as an element of $\mathrm{HS}^n_{R/A} \bigotimes_R S^{-1}R$. By induction we carry on this elimination. In the elimination process we see that for each $p = 1, \dots, n$, all the terms on the right hand side of Eq. (4.1.2) are elements of the ring $\mathrm{HS}^n_{R/A} \bigotimes_R S^{-1}R$. That is, the two rings $\mathrm{HS}^n_{S^{-1}R/A}$ and $\mathrm{HS}^n_{R/A} \bigotimes_R S^{-1}R$ are naturally isomorphic. □

4.2. Hasse–Schmidt derivations

Having the A-algebra R fixed, a Hasse–Schmidt derivation is an A-algebra homomorphism $R \longrightarrow A[\![t]\!]$, where $A[\![t]\!]$ denotes the formal power series ring in one variable t over A. This notion is naturally made functorial in the following way. For any A-algebra A' we consider the set of A-algebra homomorphisms

$$\mathrm{HS}^\infty_{R/A}\big(A'\big) = \mathrm{Hom}_{A\text{-alg}}\big(R, A'[\![t]\!]\big).$$

For any element φ in $\mathrm{HS}^\infty_{R/A} A'$, and any A-algebra homomorphism $A' \longrightarrow A''$, we compose the map $\varphi \otimes 1$ with the natural map

$$A'[\![t]\!] \bigotimes_A A'' \longrightarrow A''[\![t]\!].$$

Then $\mathrm{HS}^\infty_{R/A}(-)$ becomes a functor.

Remark 4.3. Note that the functor $\mathrm{HS}^{\infty}_{R/A}$ is not the inverse limit of the Hasse–Schmidt derivations of finite order. There exists no natural map $\mathrm{HS}^{n+1}_{R/A} \longrightarrow \mathrm{HS}^{n}_{R/A}$, so the notion of inverse limit does not naturally arise.

4.4. Direct limit

The natural map corresponding to truncating an order $(n+1)$-derivation, that is sending $(d_0, d_1, \ldots, d_{n+1})$ to $(d_0, d_1, \ldots, d_n)$ gives an A-algebra homomorphism $\mathrm{HS}^{n}_{R/A} \longrightarrow \mathrm{HS}^{n+1}_{R/A}$. In fact, truncating the universal derivation corresponds to the natural map

$$\varphi_n \colon \mathrm{HS}^{n}_{R/A} \longrightarrow \mathrm{HS}^{n}_{R/A}[d_{n+1}x_\alpha]/(d_{n+1}f_\beta) = \mathrm{HS}^{n+1}_{R/A}$$

described in Proposition 2.14. Any morphism from $\mathrm{HS}^{n+1}_{R/A}$ corresponds to a derivation of length $n+1$. Composing that given morphism with the natural map $\mathrm{HS}^{n}_{R/A} \longrightarrow \mathrm{HS}^{n+1}_{R/A}$ corresponds to truncating that particular derivation.

Proposition 4.5. *For any A-algebra R we have that the direct limit*

$$\lim_{n\to\infty} \{\mathrm{HS}^{n}_{R/A}, \varphi_n\}$$

is the A-algebra representing the Hasse–Schmidt derivations $\mathrm{HS}^{\infty}_{R/A}$. In particular we have that $\mathrm{HS}^{\infty}_{R/A}$ commutes with localization.

Proof. Let H denote the direct limit $\lim_{n\to\infty}\{\mathrm{HS}^{n}_{R/A}\}$, and let A' be an A-algebra. Then, by definition, an A'-valued point of H is a collection of A-algebra homomorphisms $u_n \colon \mathrm{HS}^{n}_{R/A} \longrightarrow A'$ such that $u_n = u_{n+1} \circ \varphi_n$. By the defining properties of $\mathrm{HS}^{n}_{R/A}$ we have that each u_n corresponds to an A-algebra homomorphism $\delta_n \colon R \longrightarrow A' \bigotimes_A A[t]/(t^n)$, where δ_{n+1} composed with the projection $A'[t]/(t^{n+1}) \longrightarrow A'[t]/(t^n)$ equals δ_n, for all n. In other words we have commutative diagrams

$$\begin{tikzcd} & R \arrow[dl, "\delta_{n+1}"'] \arrow[d, "\delta_n"] \arrow[dr, "\delta_{n-1}"] & \\ A'[t]/(t^{n+1}) \arrow[r] & A'[t]/(t^n) \arrow[r] & A'[t]/(t^{n-1}) \end{tikzcd} \tag{4.5.1}$$

That is an A-algebra homomorphism from R to the inverse limit of the horizontal arrows of the diagram (4.5.1) above, that is $A'[\![t]\!] = \lim_{\leftarrow}\{A'[t]/(t^n)\}$. Thus any A'-valued point of H gives naturally an A' valued point of the Hasse–Schmidt derivations $\mathrm{HS}^{\infty}_{R/A}$. But also conversely; an A'-valued point of $\mathrm{HS}^{\infty}_{R/A}$ is given by a diagram (4.5.1), which gives an A'-valued point of H. The last statement about localization follows as direct limits commute with tensor product, combined with Proposition 4.1. $\square$

Remark 4.6. The localization conjecture was stated by Traves [9], who verified it for monomial rings. The conjecture was stated for finite type algebras, but holds without finiteness assumptions.

Remark 4.7. One could be tempted to look at the set

$$F(A') = \mathrm{Hom}_{A\text{-alg}}\bigl(R, A[\![t]\!] \bigotimes_A A'\bigr),$$

of A-algebra homomorphisms from R to the ring $A[\![t]\!] \bigotimes_A A'$, for any given A-algebra A'. That does not appear to be a good functor to consider, for example when R is the polynomial ring $A[X]$, then the functor F is not representable: Assume conversely, that F is represented by the pair (H, u), where $u: A[X] \longrightarrow A[\![t]\!] \bigotimes_A H$ is the universal element. Then

$$u(X) = \sum_{i=1}^{N} f_i \bigotimes_A x_i$$

is a finite sum, with $f_i \in A[\![t]\!]$ and $x_i \in H$. Let $F \in A[\![t]\!]$ be a power series which is not a linear combination of the f_i's, that is an element $F \neq \sum_{i=1}^{N} a_i f_i$, with $a_i \in A$. Such elements exist. Then we have an A-valued point of F, namely the A-algebra homomorphism $A[X] \longrightarrow A[\![t]\!]$ determined by sending $X \mapsto F$. However, by our assumption on F the element will not be a specialization of $u(X)$, and in particular there exists no A-algebra homomorphism $\varphi: H \longrightarrow A$ that specializes to the one constructed. So, the functor F is not representable.

4.8. Infinite tensor products and Hasse–Schmidt derivations

In this last section, we will take a closer look at the A-algebra that represents $\mathrm{HS}^{\infty}_{R/A}$, and in particular give another proof of the localization conjecture.

Lemma 4.9. *Let E_i be finitely generated and projective A-modules ($i \in \mathscr{I}$). Denote by ${E_i}^\star = \mathrm{Hom}_A(E_i, A)$ their duals, and $E = \prod_i E_i$ the direct product. For any A-module V we have natural isomorphisms*

$$\mathrm{Hom}_A(V, E) = \prod_{i \in \mathscr{I}} \bigl(V \bigotimes_A {E_i}^\star\bigr)^\star = \Bigl(\bigoplus_{i \in \mathscr{I}} V \bigotimes_A {E_i}^\star\Bigr)^\star.$$

Proof. Since E is the direct product, we have that $\mathrm{Hom}_A(V, E)$ equals the direct product $\prod_i \mathrm{Hom}_A(V, E_i)$. As E_i is finitely generated and projective, we have $\mathrm{Hom}_A(V, E_i) = \mathrm{Hom}_A(V \bigotimes_A {E_i}^\star, A)$, proving the first identity. The second identity follows from the general fact that the dual of a direct sum, is the direct product of the duals of all the components in the direct sum. $\square$

Proposition 4.10. *Let V be an A-module, and let $\mathrm{S}_A(V)$ denote the symmetric tensor algebra. Then the A-algebra $\mathrm{S}_A(\bigoplus_{i\geq 1} V)$ represents the functor $\mathrm{HS}^{\infty}_{\mathrm{S}_A(V)/A}$. The universal family*

$$u: \mathrm{S}_A(V) \longrightarrow \mathrm{S}_A\left(\bigoplus_{i\geq 1} V\right)[\![t]\!],$$

is the algebra homomorphism $u = (u_1, u_2, \ldots,)$ that on each degree i is the map u_i induced by identifying V as the degree i component of $\bigoplus_{i\geq 1} V$.

Proof. Let A' be an A-algebra, and let $\varphi': \mathrm{S}_A(V) \longrightarrow A'[\![t]\!]$ be an A'-valued point. Such an A-algebra homomorphism is the same as an A'-module map $\varphi: V \bigotimes_A A' \longrightarrow \prod_{i\geq 1} A'$. By Lemma 4.9 such a map equals a collection of A'-module maps $\{\varphi_i: V \bigotimes_A A' \longrightarrow A'\}$. That is, a collection of A-algebra homomorphisms $u'_i: \mathrm{S}_A(V) \bigotimes_A A' \longrightarrow A'$, or equivalently an A-algebra homomorphism

$$u': \mathrm{S}_A\left(\bigoplus_{i\geq 1} V\right) \longrightarrow A'.$$

It is clear that such an element is the specialization of the morphism u described in the proposition. So, u is the universal element, and $\mathrm{S}_A(\bigoplus_{i\geq 1} V)$ is the representing object. $\square$

Corollary 4.11. *For any A-algebra R, the functor $\mathrm{HS}^{\infty}_{R/A}$ is represented by the infinite tensor product*

$$\bigotimes\nolimits_A^{\infty} R = \lim_{n\to\infty} \left(\bigotimes\nolimits_A^{n} R\right).$$

Proof. Write $R = \mathrm{S}_A(V)/I$ for some ideal I, and some A-module V. Write $H = \mathrm{S}_A(\bigoplus_{i\geq 1} V)$, then H is the infinite tensor product $H = \bigotimes_A^{\infty} \mathrm{S}_A(V)$. Let $u: \mathrm{S}_A(V) \longrightarrow H[\![t]\!]$ denote the universal map described in the proposition. It is clear that $H/u(f)$, for all $f \in I$, will be the representing object of $\mathrm{HS}^{\infty}_{R/A}$, where the map induced by u will be the universal element. We have furthermore that the map $u = (u_1, u_2, \ldots)$, where $u_i: \mathrm{S}_A(V) \longrightarrow H$ is the i'th co-projection map that identifies $\mathrm{S}_A(V)$ with the i factor of H. It follows that $\mathrm{S}_A(V)/u_i(f) = \mathrm{S}_A(V)/f$, and that $H/u(I) = \bigotimes_A^{\infty} R$. $\square$

Corollary 4.12. *The Hasse–Schmidt derivations commute with localization, that is*

$$\mathrm{HS}^{\infty}_{R/A} \bigotimes\nolimits_R S^{-1}R = \mathrm{HS}^{\infty}_{S^{-1}R/A},$$

for any multiplicatively closed subset $S \subseteq R$.

Proof. The direct product of rings is the direct product of the underlying modules, and the algebra structure is naturally induced. Since tensor product commutes with direct limit of modules, we get that $(\bigotimes_A^\infty R) \bigotimes_R S^{-1}R$ is the direct limit

$$\lim_{n\to\infty} \left(\bigotimes_A^n R \bigotimes_R S^{-1}R\right) = \lim_{n\to\infty} \left(\bigotimes_A^n S^{-1}R\right).$$

Hence the localization $\mathrm{HS}_{R/A}^\infty \bigotimes_R S^{-1}R$ equals the infinite tensor product $\bigotimes_A^\infty S^{-1}R$, which by the corollary above is $\mathrm{HS}_{S^{-1}R/A}^\infty$. □

References

[1] William C. Brown, Wei-eihn Kuan, Ideals and higher derivations in commutative rings, Canad. J. Math. 24 (1972) 400–415.
[2] Siegfried Bosch, Werner Lütkebohmert, Michel Raynaud, Néron Models, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) (Results in Mathematics and Related Areas (3)), vol. 21, Springer-Verlag, Berlin, 1990.
[3] M. Fernández-Lebrón, L. Narváez-Macarro, Hasse–Schmidt derivations and coefficient fields in positive characteristics, J. Algebra 265 (1) (2003) 200–210.
[4] Alexander Grothendieck, Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert, Séminaire Bourbaki, vol. 6, Soc. Math. France, Paris, 1995, pp. 249–276, Exp. No. 221.
[5] Hideyuki Matsumura, Commutative Ring Theory, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1986, translated from Japanese by M. Reid.
[6] Martin C. Olsson, $\underline{\mathrm{Hom}}$-stacks and restriction of scalars, Duke Math. J. 134 (1) (2006) 139–164.
[7] Sadi Abu Saymeh, On Hasse–Schmidt higher derivations, Osaka J. Math. 23 (2) (1986) 503–508.
[8] Roy Mikael Skjelnes, Weil restrictions and the Quot scheme, arXiv:1111.4814.
[9] William N. Traves, Localization of the Hasse–Schmidt algebra, Canad. Math. Bull. 46 (2) (2003) 304–309.
[10] Paul Vojta, Jets via Hasse–Schmidt derivations, in: Diophantine Geometry, in: CRM Series, vol. 4, Ed. Norm, Pisa, 2007, pp. 335–361.