New simple groups with a BN-pair

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We show that there are simple groups with a spherical BN-pair of rank 2 which are non-Moufang and hence not of algebraic origin.

1. Introduction

It was shown in [7,5] that any group with a *split* BN-pair of rank at least 2 is essentially a simple algebraic group. While it was known that there are non-algebraic groups with a spherical BN-pair (see e.g. [9] or [6]), none of the examples of such groups were known to be abstractly simple. Thus one might wonder whether any simple group with a BN-pair is algebraic. We here show that this is not the case: there exist *simple non-algebraic* groups with a (non-split) BN-pair of rank 2. The result relies on the construction of very homogeneous generalized polygons by the second author given in [4] and a result of Lascar's [3] which can be applied to these generalized polygons. It implies that a certain subgroup of the automorphism group has a simple quotient. Our main task will be to show that in this case the automorphism group itself is a simple group. The existence

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of a BN-pair for this group was already established in [4]. While these results in the background are model theoretic, the proof here is essentially geometric.

We try to keep the paper accessible to the non-model theorist. To this end, we will give the model theoretic definitions and tools adjusted to this specific situation rather than going into the general model theoretic context.

It would be interesting to prove similar results for the higher rank case or for non-spherical buildings. Note that spherical buildings of rank at least 3 always arise from standard BN-pairs in (essentially simple) algebraic groups. However, also in this case there are non-algebraic groups acting on these with a BN-pair. Whether or not these groups could be simple, is still open.

Note that Caprace [2] (and independently the second author) proved that the group of type-preserving automorphisms of any irreducible semi-regular thick right-angled building is abstractly simple.

2. Generalized polygons and spherical BN-pairs of rank 2

Recall that a generalized n-gon is a bipartite graph Γ of diameter n and girth $2n$. It is called *thick* if all valencies are at least 3. We say that a group G acts *strongly transitively* on Γ if G acts transitively on the set of $2n$-cycles of Γ with chosen starting point of fixed type or, equivalently, if for any simple path $\gamma = (x_0, \ldots, x_n)$ in Γ the pointwise stabilizer G_γ acts transitively on $D_1(x_n) \setminus \{x_{n-1}\}$ where $D_i(x)$ denotes the set of elements of Γ at distance i from x.

We need the following fact due to Tits which will serve us as a definition:

Theorem/Definition 2.1. *A group G has a spherical BN-pair of rank 2 if and only if there is a generalized n-gon Γ and a strongly transitive action of G on Γ.*

The generalized n-gons constructed in [4] are *almost strongly minimal* structures and their automorphism groups have a BN-pair. In fact, the automorphism groups of these n-gons even act transitively on ordered $(2n+2)$-cycles starting in a fixed class of vertices:

Remark 2.2. (See [11].) If Γ is a generalized n-gon, then a group G acts transitively on the set of ordered $(2n + 2)$-cycles if and only if G acts transitively on the set of ordered $2n$-cycles and the stabilizer of a $2n$-cycle $(x_0, \ldots, x_{2n-1}, x_{2n} = x_0)$ acts transitively on the set $(D_1(x_1) \setminus \{x_0, x_2\}) \times (D_1(x_2) \setminus \{x_1, x_3\})$.

Recall that a generalized n-gon satisfies the *Moufang condition* if for any simple path $\gamma = (x_0, \ldots, x_n)$ in Γ the stabilizer $\bigcap_{i=1}^{n-1} G_{D_1(x_i)}$ acts transitively on $D_1(x_n) \setminus \{x_{n-1}\}$.

By the classification of Moufang polygons due to Tits and Weiss [10] any Moufang polygon arises from the standard BN-pair of an essentially simple algebraic group. As explained below, the examples of [4] do not satisfy the Moufang condition.

For more background on generalized polygons we refer the reader to [11].

3. Construction of very homogeneous generalized n-gons

We first recall the construction of the very homogeneous generalized n-gons given in [4] (see also [8], Sec. 10.4). Fix $n \geq 3$. For a finite graph A we define

$$\delta(A) = (n-1)|A| - (n-2)e(A)$$

where $e(A)$ denotes the number of edges between vertices of A. If A, B are finite subgraphs of a given graph, we write AB for $A \cup B$. Similarly if b is a single point we write Ab for $A \cup \{b\}$. For finite subgraphs A, B of a given graph we put

$$\delta(A/B) = \delta(AB) - \delta(B).$$

If A, B are disjoint we have

$$\delta(AB) = \delta(A) + \delta(B) - (n-2)e(A, B),$$

where $e(A, B)$ denotes the number of edges between vertices of A and vertices of B. Therefore for disjoint A, B we have $\delta(A/B) = \delta(A) - (n-2)e(A, B)$.

For graphs $A \subseteq B$ with A finite, we say that A is *strongly embedded into* B, and write $A \leq B$ if $\delta(B') \geq \delta(A)$ for all $A \subseteq B' \subseteq B, B'$ finite.

Definition 3.1. Suppose A and B are disjoint finite subgraphs of a given graph. Then B is called 0-*algebraic* over A if $\delta(B/A) = 0$ and $\delta(B'/A) > 0$ for any proper non-empty subset $B' \subset B$. The set B is called 0-*minimally algebraic* over A if there is no proper subset A' of A such that B is 0-algebraic over A'.

Remark 3.2.

1. If B is 0-algebraic over A, then there is a unique $A' \subseteq A$ such that B is 0-minimally algebraic over A', namely $A' = \{a \in A : e(a, B) \geq 1\}$.
2. If B is 0-algebraic over A, then clearly

$$|B|(n-1) = (n-2)\big(e(B) + e(B, A)\big).$$

3. For a path $\{a = x_0, x_1, \ldots, x_m, x_{m+1} = b\}$ of length $m+1$, the set $\{x_1, \ldots, x_m\}$ is 0-minimally algebraic over $\{a, b\}$ if and only if $m = n - 2$.

We fix a function μ from the set of pairs (A, B) where B is 0-minimally algebraic over A into the natural numbers with the following properties:

1. If (A, B) and (A', B') have the same isomorphism type then $\mu(A, B) = \mu(A', B')$.
2. If $A = \{a, b\}$ and B consists of a path of length $n - 3$ connecting a and b, (so AB is a path of length $n - 1$), then $\mu(A, B) = 1$; otherwise $\mu(A, B) \geq \max\{\delta(A), n\}$.

Definition 3.3. Let $\mathbb{K}^\mu$ be the class of all finite graphs C, bipartite with respect to a predicate P and satisfying the following conditions:

1. The graph C contains no $2m$-cycle for $m < n$.
2. If $B \subseteq C$ contains a $2m$-cycle for $m > n$, then $\delta(B) \geq 2n + 2$.
3. If B is a 0-minimally algebraic set over A and $A, B \subset C$, then the number of copies of B over A inside C is at most $\mu(A, B)$.

The following was shown in [4]:

Theorem 3.4. *(See [4], Thm. 4.6.) There is a countable generalized n-gon Γ_n such that every $C \in \mathbb{K}^\mu$ can be strongly embedded into Γ_n and any isomorphism between $A, B \leq \Gamma_n$ extends to an automorphism of Γ_n.*

In particular, the automorphism group of Γ_n acts transitively on the set of ordered $(2n + 2)$-cycles. $\square$

The last statement follows from the fact that by Definition 3.3.2 any $(2n+2)$-cycle is strongly embedded into Γ_n. Moreover we have the following:

Remark 3.5. Using Theorem 3.11 of [4], one sees easily any set $A \subseteq \Gamma_n$ with $\delta(A) \leq 2n+1$ is strongly embedded into Γ_n. The description given there also implies that for $k \leq n$ and any path $\gamma = (x_0, \ldots, x_k)$ in Γ_n the stabilizer $\mathrm{Aut}_\gamma(\Gamma_n)$ acts $(n+3-k)$-transitively on $D_1(x_k) \setminus \{x_{k-1}\}$.

The main result of the paper now is the following:

Theorem 3.6. *The automorphism group $\mathrm{Aut}(\Gamma_n)$ of Γ_n is a simple group.*

To prove this we will invoke Lascar's result [3]. For this we need to introduce some more terminology. In order to keep the model theoretic notions as accessible as possible we will use the following definition of algebraic closure $\mathrm{acl}(A)$ of a set $A \subset \Gamma_n$. (For the general definition of algebraic closure we refer the reader to [8], Ch. 5.6.)

Definition 3.7. (See [12].)

1. For finite subsets $A, B \subset \Gamma_n$ we define

$$d(A) := \min\big\{\delta\big(A'\big) : A \subseteq A' \subseteq \Gamma_n, A' \text{ finite}\big\}$$

and $d(B/A) = d(BA) - d(A)$.

2. We say that $A \subseteq \Gamma_n$ is $\leq$-closed if $\delta(A) = d(A)$, or, equivalently, if A is strong in Γ_n. The $\leq$-*closure* $\mathrm{cl}(A)$ of A is defined as

$$\mathrm{cl}(A) := \bigcap \{B \leq \Gamma_n : A \subset B\}.$$

Thus $\mathrm{cl}(A)$ is the smallest strong subset of Γ_n containing A.

3. For $A \subseteq \Gamma_n$ we define the *algebraic closure* $\mathrm{acl}(A)$ of A as

$$\mathrm{acl}(A) = \big\{x \in \Gamma_n : d(x/A_0) = 0 \text{ for some finite } A_0 \subseteq A\big\}.$$

Note that if A is finite, then so is $\mathrm{cl}(A)$ since the δ-value of any non-empty set is positive and hence can decrease only finitely many times. However, $\mathrm{acl}(A)$ is not finite in general, see Lemma 5.3. For every finite set $B \subset \Gamma_n$, we have $\mathrm{cl}(B) \subset \mathrm{acl}(B)$.

The following was proved in [4] where for vertices $a, b \in \Gamma_n$ we let $\mathrm{dist}(a, b)$ denote the graph theoretic distance between a, b in Γ_n:

Theorem 3.8. *For any $x \in \Gamma_n$ we have $\Gamma_n \subset \mathrm{acl}(D_1(x) \cup \{y_1, y_2, y_3\})$ where $\mathrm{dist}(x, y_1) = \mathrm{dist}(x, y_3) = n$ and (y_1, y_2, y_3) is a path of length 2. Furthermore, the set $D_1(x)$ is strongly minimal, i.e. for any finite set $C \subset D_1(x)$ and $z_1, z_2 \in D_1(x) \setminus \mathrm{acl}(C)$ there is an automorphism fixing $C \cup \{y_1, y_2, y_3\}$ and taking z_1 to z_2. Hence Γ_n is almost strongly minimal over $A_0 = \{x, y_1, y_2, y_3\}$.*

For the general definition of a strongly minimal set see [8], Sec. 5.7.

Lemma 3.9. *With $A_0 = \{x, y_1, y_2, y_3\}$ as in Theorem 3.8, we have $\mathrm{acl}(A_0) = \gamma = (x = x_0, \ldots, x_{n-1} = y_2, x_n = y_1, x_{n+1} = y_3)$ where $(x_0, \ldots, x_n)$ is a path of length n with an additional neighbour $x_{n+1} = y_3$ added to $y_2 = x_{n-1}$. In particular, $\mathrm{acl}(\gamma) = \gamma$.*

Proof. Note that $\delta(A_0) = \delta(\gamma) = 2n$. It suffices to show that there are no 0-minimally algebraic sets over γ. This follows from Lemma 3.12 in [4] since there is no set B with $\delta(B) = 2n$ properly containing γ. □

4. Lascar's theorem

Lascar's theorem refers to the group of strong automorphisms of an almost strongly minimal structure. In light of [1], Lemma 5.4, we may here use the following definition:

Definition 4.1.

1. An automorphism of Γ_n is called *strong* over A if it fixes $\mathrm{acl}(A)$ pointwise. We let $\mathrm{Autf}_A(\Gamma_n)$ denote the group of all automorphisms strong over A. We drop the subscript in the case where A is the empty set.

2. An automorphism $\beta \in \mathrm{Aut}(\Gamma_n)$ is called *bounded* if there exists a finite set $A \subset \Gamma_n$ such that $x^\beta \in \mathrm{acl}(xA)$ for all $x \in \Gamma_n$ where x^β denotes the image of $x \in \Gamma_n$ under β. In this case we say that β is bounded over A. Let $\mathrm{Bdd}(\Gamma_n)$ be the set of all bounded automorphisms of Γ_n.

With the notation from Theorem 3.8 and Lemma 3.9, using the fact that $\gamma = \mathrm{acl}(\gamma) = \mathrm{acl}(A_0)$, we have $\mathrm{Autf}_\gamma(\Gamma_n) = \mathrm{Aut}_\gamma(\Gamma_n)$. Note that the *exchange property* for algebraic closure in almost strongly minimal structures implies that $\mathrm{Bdd}_\gamma(\Gamma_n)$ is a (normal) subgroup of $\mathrm{Aut}_\gamma(\Gamma_n)$: namely, $x \in \mathrm{acl}(yA) \setminus \mathrm{acl}(A)$ implies $y \in \mathrm{acl}(xA)$ for all sets $A \subseteq \Gamma_n$ with $A_0 \subseteq A$ and $x, y \in \Gamma_n$ (see e.g. [8], Thm. 5.7.5).

Since we saw that Γ_n is almost strongly minimal over $\gamma = \mathrm{acl}(\gamma)$ the main theorem of [3] applied to Γ_n now yields:

Theorem 4.2 *(Lascar). The group* $\mathrm{Aut}_\gamma(\Gamma_n)/\mathrm{Bdd}_\gamma(\Gamma_n)$ *is simple.*

In order to prove Theorem 3.6 we will first show in Proposition 6.3 that Γ_n does not allow any non-trivial bounded automorphisms (whether or not we fix γ) and finally that the simplicity of $\mathrm{Aut}_\gamma(\Gamma_n)$ implies that of $\mathrm{Aut}(\Gamma_n)$.

5. 0-Minimally algebraic sets

In this section we investigate some properties of 0-minimally algebraic sets in Γ_n.

Lemma 5.1. *Let A be a finite $\leq$-closed set.*

1. *If D is 0-minimally algebraic over $A_0 \subset A$, then either D is 0-algebraic over A or $D \subset A$.*
2. *If D_1, D_2 are 0-algebraic over A, they are equal or disjoint.*

Proof. 1: Let $D_0 = D \cap A$. Since D is 0-minimally algebraic over A_0, we have $\delta(D/A) \leq \delta(D/D_0 A_0) \leq 0$. Since A is $\leq$-closed, it follows that either $D_0 = \emptyset$ and D is 0-algebraic over A or $D_0 = D \subseteq A$.

2: This follows from part 1 and the fact that if A is $\leq$-closed and D is 0-algebraic over A, then also AD is $\leq$-closed. $\square$

Definition 5.2. A *base set* is a set $A_0 = \{s_0, s_1, s_2, s_3\}$ of vertices with

$$\mathrm{dist}(s_i, s_{i+1}) = \mathrm{dist}(s_3, s_0) = n, \quad 0 \leq i \leq 3$$

and such that

$$\text{dist}(s_0, s_2) = \text{dist}(s_1, s_3) \in \{n - 1, n\}$$

depending on whether n is even or odd.

Note that if n is even, then there are base sets of two different types of vertices.

Lemma 5.3. *Let $A_0 = \{s_0, s_1, s_2, s_3\}$ be a base set. Then for any $\ell \geq 2$, any simple cycle $C_\ell = \{c_0, c_1, \ldots, c_{4\ell(n-2)} = c_0\}$ of length $4\ell(n-2)$ with additional edges between $c_{i(n-2)}$ and s_{i^*} ($i^* \equiv i \mod 4$), $i = 0, \ldots, 4\ell - 1$ is 0-minimally algebraic over A_0.*

Proof. Since $\delta(C_\ell) = 4\ell(n-2)$ and $e(C_\ell, A) = 4\ell$ we have

$$\delta(C_\ell/A_0) = \delta(C_\ell) - (n-2)e(C_\ell, A_0) = 0.$$

It is left to show that $\delta(D/A_0) > 0$ for any proper subset D of C_ℓ. It clearly suffices to prove this for connected subsets of C_ℓ. But any such subset D is a simple path. If D has length r, then $\delta(D) = (n-1) + r$. Since $e(D, A_0) \leq 1 + r/(n-2)$ we have $\delta(D/A_0) \geq (n-1) + r - (n-2)(1 + r/(n-2)) = 1$. $\square$

Note that since $\mu(C_\ell, A_0) \geq 1$ and Γ_n is ω-saturated, such cycles exist in Γ_n for any $\ell \geq 2$.

Corollary 5.4. *Let A be a base set if n is odd and the union of two base sets of different type if n is even. Let $b \in \Gamma_n$ be such that $d(Ab) = d(A) + n - 1$. Then for any finite $\leq$-closed set B containing Ab, there is a set D not contained in B which is 0-algebraic over B and with $e(D, b) = 1$.*

Proof. Note that $d(b/A) = n - 1$ implies $\text{dist}(b, a) \geq n - 1$ for all $a \in A$. If n is even let $A_0 \subset A$ be the base set of the same type as b (otherwise $A_0 = A$). For $\ell \geq 2$ let C_ℓ be as above with one of the edges between $c_{i(n-2)}$ to s_{i^*} replaced by an edge to b. Then the same proof shows that C_ℓ is 0-minimally algebraic over $A_0 b$. Any finite $\leq$-closed set B contains only finitely many of these C_ℓ. Hence for some ℓ the set C_ℓ is as required. $\square$

6. Proof of the main theorem

The main step towards proving simplicity of $\text{Aut}(\Gamma_n)$ is to prove that there is no non-trivial bounded automorphism of Γ_n. If an automorphism β is bounded over a finite set A, then clearly it is also bounded over any set B containing A. Therefore we may assume that β is bounded over a finite set A which is $\leq$-closed and contains a base set of each type (in case n is even).

Lemma 6.1. *Suppose β is a bounded automorphism over the $\leq$-closed set A. If $A \leq Ab \leq M$ and $d(b/A) = n - 1$, then b is fixed by β.*

Proof. Suppose $b \neq b^{\beta}$. Since β is bounded, we have $b^{\beta} \in \mathrm{acl}(bA)$ and hence

$$d\big(bb^{\beta}A\big) = d(bA) \leq \delta(bA) \leq \delta(A) + n - 1 = d(bA).$$

Put $B := \mathrm{cl}(AbA^{\beta}b^{\beta})$. Since $A^{\beta}b^{\beta} \subset \mathrm{acl}(Ab)$, we also have

$$d(B) = \delta(B) = \delta(A) + n - 1.$$

With $B_0 := B \setminus A$ we have

$$n - 1 = \delta(B/A) = \delta(B_0/A) = \delta(B_0) - (n-2)e(B_0, A). \tag{$*$}$$

By Corollary 5.4 we find a set D which is 0-algebraic over B and $e(b, D) = 1$ and such that D, D^{β} are disjoint from B. Since by assumption $b \neq b^{\beta}$, it follows that $D \cap D^{\beta} = \emptyset$ and hence $e(B_0, DD^{\beta}) = 2$. We then have

$$\delta\big(B_0/DD^{\beta}A\big) = \delta(B_0) - (n-2)e\big(B_0, DD^{\beta}A\big)$$

and hence $\delta(B_0/DD^{\beta}A) = (n-1) - (n-2)e(B_0, DD^{\beta})$ by $(*)$. Since $n \geq 3$ we have $\delta(B_0/DD^{\beta}A) \leq 0$. This implies $B_0 \subseteq \mathrm{cl}(DD^{\beta}A) \subseteq \mathrm{acl}(DD^{\beta}A) = \mathrm{acl}(DA)$ and so in particular $b \in \mathrm{acl}(DA)$.

On the other hand, since AbD is $\leq$-closed, we have $d(b/DA) = \delta(b/DA) = 1$ and so $b \notin \mathrm{acl}(DA)$, a contradiction. $\square$

Lemma 6.2. *Suppose β is bounded over $A \leq \Gamma_n$ and fixes all $b \in \Gamma_n$ with $d(b/A) \geq k$. Let $b \in \Gamma_n$ with $d(b/A) = k$ and $c \in D_1(b)$. Then β fixes c.*

Proof. If $d(c/A) \geq k$, then the result holds by assumption. Hence we may assume $d(c/A) = k - 1$ and so $d(b/Ac) > 0$, i.e. $b \notin \mathrm{acl}(cA)$. If $c^{\beta} \neq c$, then $e(b, cc^{\beta}A) = 2$ and $b \in \mathrm{acl}(cc^{\beta}A) = \mathrm{acl}(cA)$, a contradiction. $\square$

Proposition 6.3. *There is no non-trivial bounded automorphism.*

Proof. Suppose $\beta \in \mathrm{Aut}(\Gamma_n)$ is bounded over $A \leq \Gamma_n$. Then β fixes any $c \in \Gamma_n$ with $d(c/A) = n - 1$ by Lemma 6.1. Now assume inductively that β fixes any $b \in \Gamma_n$ with $d(b/A) > k$ and let $c \in \Gamma_n$ with $d(c/A) = k \geq 0$. By Lemma 6.2 it suffices to show that there is $b \in D_1(c)$ such that $d(b/A) = k+1$. To find such an element b let $E = \mathrm{cl}(cA)$ and let b be a neighbour of c such that $\delta(bE) = \delta(E) + 1$. By the properties of Γ_n given in Theorem 3.4 there is a copy b' of b over E strongly embedded into Γ_n. Thus, $d(b'/E) = 1$ and hence $d(b'/A) = k + 1$. $\square$

By Theorem 4.2 we now have:

Corollary 6.4. $\mathrm{Aut}_{\gamma}(\Gamma_n)$ *is a simple group.*

[Theorem 3.6](#) now follows from:

Proposition 6.5. *If* $\mathrm{Aut}_\gamma(\Gamma_n)$ *is simple, then so is* $\mathrm{Aut}(\Gamma_n)$.

Proof. Write $\gamma = (x_0, \ldots, x_n, x_{n+1})$ where $(x_0, \ldots, x_n)$ is a path of length n and $x_{n+1} \in D_1(x_{n-1}) \setminus \{x_n, x_{n-2}\}$. For $0 \leq i \leq n+1$ let $\gamma_i = (x_0, \ldots, x_i)$. Then $\delta(\gamma_i) = n - 1 + i$, so each γ_i is strongly embedded into Γ_n by [Remark 3.5](#).

We now prove for $i = 0, \ldots, n$, if $\mathrm{Aut}_{\gamma_{i+1}}(\Gamma_n)$ is simple, then so is $\mathrm{Aut}_{\gamma_i}(\Gamma_n)$. Note that for $i = 0, \ldots, n-1$, by [Remark 3.5](#) and the homogeneity of Γ_n for strong subsets the group $\mathrm{Aut}_{\gamma_i}(\Gamma_n)$ acts 3-transitively on the set of neighbours of x_i different from x_{i-1}. For the same reason, $\mathrm{Aut}_{\gamma_n}(\Gamma_n)$ acts 3-transitively on the set of neighbours of x_{n-1} different from x_n, x_{n-2}. Therefore $\mathrm{Aut}_{\gamma_{i+1}}(\Gamma_n)$ is a maximal subgroup of $\mathrm{Aut}_{\gamma_i}(\Gamma_n)$.

Now let $1 \neq N \lhd \mathrm{Aut}_{\gamma_i}(\Gamma_n)$. Since any normal subgroup of a 3-transitive group acts 2-transitively, we have $N \cap \mathrm{Aut}_{\gamma_{i+1}}(\Gamma_n) \neq 1$. Since $\mathrm{Aut}_{\gamma_{i+1}}(\Gamma_n)$ is simple, maximal and not normal in $\mathrm{Aut}_{\gamma_i}(\Gamma_n)$, this shows that $N = \mathrm{Aut}_{\gamma_i}(\Gamma_n)$. This shows inductively that $\mathrm{Aut}_{\gamma_0}(\Gamma_n) = \mathrm{Aut}_{x_0}(\Gamma_n)$ is simple.

It is left to show that $\mathrm{Aut}(\Gamma_n)$ is simple given that $\mathrm{Aut}_{x_0}(\Gamma_n)$ is. Since $\mathrm{Aut}(\Gamma_n)$ has a BN-pair, we know that $\mathrm{Aut}_{x_0}(\Gamma_n)$ is a maximal subgroup and any normal subgroup of $\mathrm{Aut}(\Gamma_n)$ acts transitively on the set of vertices of a given type. Suppose there is a normal subgroup $1 \neq N \lhd \mathrm{Aut}(\Gamma_n)$ with $N \cap \mathrm{Aut}_{x_0}(\Gamma_n) = 1$. Then N must act regularly on vertices of the same type as x_0. We show that this is impossible: choose $z_0 \in \Gamma_n$ with $\mathrm{dist}(z_0, x_0) = 2$ and let $g \in N$ with $x_0^g = z_0$. Let (x_0, z_1, z_0) be a path of length 2 and let $a_1, a_2 \in D_1(x_0) \setminus \{z_1\}$ and such that $a_i^g \neq z_1$, $i = 1, 2$. Let $b_i = a_i^g \in D_1(z_0)$. Then for $i = 1, 2$ we have $\delta(a_i, x_0, z_1, z_0, b_1, b_2) = n + 4 \leq 2n + 1$. By [Remark 3.5](#) there exists some $h \in \mathrm{Aut}(\Gamma_n)$ fixing $(x_0, z_1, z_0, b_1, b_2)$ with $a_2^h = a_1$. Then $g^h \in N$ with $x_0^{g^h} = z_0$, but $a_1^{g^h} = b_2 \neq b_1$ showing that N is not regular. $\square$

This finishes the proof of [Theorem 3.6](#). That these BN-pairs are not of algebraic origin follows from the fact that in contrast to the examples studied here the classification of Moufang polygons by Tits and Weiss [10] implies that in the algebraic case no point stabilizer G_x acts 6-transitively on $D_1(x)$, see [Remark 3.5](#).

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