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The character degree simplicial complex of a finite group

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A B S T R A C T

The character degree graph $\Gamma(G)$ of a finite group G has long been studied as a means of understanding the structural properties of G. For example, a result of Manz and Pálfy states that the character degree graph of a finite solvable group has at most two connected components. In this paper, we introduce the character degree simplicial complex $\mathcal{G}(G)$ of a finite group G. We provide examples justifying the study of this simplicial complex as opposed to $\Gamma(G)$, and prove an analogue of Manz's Theorem on the number of connected components that is dependent upon the dimension of $\mathcal{G}(G)$.

1. Introduction

A valuable way of studying the structure of a finite group G is through the study of its irreducible characters. The value that the character takes on the identity of the group is called the *degree* of the character, and in fact, much can be said about the structure of the group by simply examining the irreducible character degrees of the

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group. We write $\mathrm{Irr}(G)$ for the set of irreducible characters of a group G and we write $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$.

Historically, two different graphs have been associated with the set $\mathrm{cd}(G)$. The first is called the *character degree graph* of $\mathrm{cd}(G)$, denoted by $\Gamma(G)$. The vertices of this graph are the members of the set $\mathrm{cd}(G) \setminus \{1\}$, and there is an edge connecting two vertices if the corresponding irreducible character degrees have a nontrivial common divisor. The second graph associated with $\mathrm{cd}(G)$ is the *prime vertex graph*, denoted $\Delta(G)$, which has the primes dividing some member of $\mathrm{cd}(G)$ as its vertices, and there is an edge between two vertices if there is a member of $\mathrm{cd}(G)$ divisible by the two associated primes.

A plethora of research has been conducted about the graph theoretic properties that must be satisfied by both $\Gamma(G)$ and $\Delta(G)$ when G is a finite group. Perhaps the first result in this area is the following theorem of Manz [5].

Theorem 1.1 *(Manz). Suppose that G is a finite solvable group. Then the number of connected components of $\Gamma(G)$ is at most 2.*

Theorem 1.1 demonstrates that not all sets of integers can appear as $\Gamma(G)$ when G is a finite solvable group. Pálfy showed independently in [6] that when G is a finite solvable group, $\Delta(G)$ also has at most two connected components. These results have inspired many works in this area that seek to classify other constraints that can be placed upon both $\Gamma(G)$ and $\Delta(G)$ when G is a finite solvable group; see [4] for a thorough description.

It has long been suggested that one should really consider the *character degree simplicial complex* and the *prime vertex simplicial complex* of a finite group G instead of studying $\Gamma(G)$ or $\Delta(G)$, yet this proposal has long been overlooked. In this paper, we introduce the character degree simplicial complex of a finite group G, denoted $\mathcal{G}(G)$, and we provide results demonstrating the necessity of this definition. We do not examine the prime vertex graph or the prime vertex simplicial complex any further in this work.

Although not entirely algebraic in nature, the novelty of this area of study requires a certain amount of topological background. We provide the topological necessities for this paper in Section 2. Subsections 2.1, 2.2, and 2.3 are purely topological, providing the definitions and machinery that will be necessary in order to study our applications. Subsection 2.4 introduces the common divisor and character degree simplicial complex in more detail, while also providing the applications of the first three subsections of Section 2 to our context. Subsection 2.5 discusses an analogue of Theorem 1.1 that becomes nontrivial with the introduction of the character degree simplicial complex.

In Section 3, we will investigate the analogue of Theorem 1.1 discussed in Subsection 2.5. More specifically, we will obtain a bound on the rank of $\Pi_1(\mathcal{G}(G))$ in terms of the dimension of $\mathcal{G}(G)$. The primary result of Section 3 is the following.

Theorem A. *Suppose G is a finite solvable group with $\mathcal{G}(G)$ connected and $\dim(\mathcal{G}(G)) = n$. Then $\mathrm{rk}(\Pi_1(\mathcal{G}(G))) \leq n^2 + n - 1$.*

Finally, we will present in Section 4 some applications of Theorem A, examples of groups where $\mathcal{G}(G)$ has interesting properties, and directions for further research.

2. Calculations

2.1. Topological definitions

We begin with some basic definitions, mostly following the conventions and notations of [7].

Definition 2.1. An abstract simplicial complex K is a pair (V, S), where V is a finite set (whose elements are called vertices) and S is a set of nonempty finite subsets of V (called simplices) such that all singleton subsets of V are in S, and, if $\sigma \in S$ and $\sigma' \subseteq \sigma$, then $\sigma' \in S$.

If $K = (V, S)$ is an abstract simplicial complex and $\sigma \in S$ contains $n+1$ elements, then we say that σ has *dimension* n, or alternatively, that σ is an n*-simplex*. The *dimension* of the simplicial complex K is the dimension of its largest simplex. Finally, suppose that n is a nonnegative integer. We define the n*-skeleton* of K to be the abstract simplicial complex $K^n = (V, S^n)$, where $S^n \subseteq S$ is the collection of all simplices of dimension at most n.

Let $K = (V, S)$ be an abstract simplicial complex. An *edge* of K is an ordered pair of vertices $\langle v_0, v_1 \rangle$ with $v_0, v_1 \in V$ and where v_0 and v_1 are part of some simplex in S. Note that this allows us to have edges where $v_0 = v_1$. In the ordered pair $\langle v_0, v_1 \rangle$, we call v_0 the *start vertex* and we call v_1 the *end vertex*. More generally, we have the following definition.

Definition 2.2. An edge path $\mathcal{P}$ of an abstract simplicial complex K is a finite nonempty sequence of edges $e_1 e_2 \dots e_n$ of K such that the end vertex of e_i is the start vertex of e_{i+1} for all $1 \leq i < n$.

For an edge path $\mathcal{P} = e_1 e_2 \dots e_n$, we call the start vertex of e_1 the *start vertex* of $\mathcal{P}$ and we call the end vertex of e_n the *end vertex* of $\mathcal{P}$. Note that if the paths $\mathcal{P}_1$ and $\mathcal{P}_2$ are such that the end vertex of $\mathcal{P}_1$ is equal to the start vertex of $\mathcal{P}_2$, then these paths can be combined via concatenation, written $\mathcal{P}_1 * \mathcal{P}_2$.

Definition 2.3. An edge path $\mathcal{P}$ is called a loop if the start vertex and end vertex of $\mathcal{P}$ are the same.

We now introduce the notion of a homotopy between two edge paths in an abstract simplicial complex. In general,it is unconventional to allow the notion of the empty path, but we make an exception for the following definition.

Definition 2.4. For an abstract simplicial complex K, two edge paths $\mathcal{P}$ and $\mathcal{P}'$ of K are simply equivalent if there exists vertices v_0, v_1, and v_2 belonging to some simplex of K so that the unordered pair $\{\mathcal{P}, \mathcal{P}'\}$ is equal to $\Big\{\mathcal{P}_1\langle v_0, v_2\rangle\mathcal{P}_2, \mathcal{P}_1\langle v_0, v_1\rangle\langle v_1, v_2\rangle\mathcal{P}_2\Big\}$, where $\mathcal{P}_1$ is a possibly empty edge path in K with end vertex v_0 and $\mathcal{P}_2$ is a possibly empty edge path in K with start vertex v_2.

Two edge paths $\mathcal{P}$ and $\mathcal{P}'$ are said to be *homotopy equivalent* or *homotopic*, denoted $\mathcal{P} \sim \mathcal{P}'$, if there is a finite sequence of edge paths $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n$ such that $\mathcal{P} = \mathcal{P}_1$ and $\mathcal{P}' = \mathcal{P}_n$ and we have that $\mathcal{P}_i$ is simply equivalent to $\mathcal{P}_{i+1}$ for all $1 \leq i < n$. A loop with terminal vertex v_0 is said to *trivial* if it is homotopic to the loop $\langle v_0, v_0\rangle$.

An abstract simplicial complex $K = (V, S)$ is called *connected* if for every pair of distinct vertices $v_0, v_1 \in V$ there exists a path $\mathcal{P}$ having start vertex v_0 and end vertex v_1. Having established the notions of a connected abstract simplicial complex and of a homotopy between two paths or loops, we can define the *edge path group* of a connected abstract simplicial complex.

Definition 2.5. Suppose $K = (V, S)$ is a connected abstract simplicial complex with $v_0 \in V$. The edge path group of K with basepoint v_0, denoted $E(K, v_0)$, is the group of equivalence classes of loops with terminal vertex v_0, where the equivalence classes are homotopy classes of loops. If $\mathcal{L}$ is a loop in K with terminal vertex v_0, we write $[\mathcal{L}]$ to denote the equivalence class of $\mathcal{L}$ in $E(K, v_0)$.

The group operation in the edge path group is concatenation of paths, and we will also use $*$ to denote this operation. Theorem 16 of Chapter 3.6 of [7] states that if K is a connected abstract simplicial complex and v_0 and v_1 are members of V, then $E(K, v_0) \cong E(K, v_1)$. This means that the edge path group of K does not depend on the chosen basepoint, so we can simply write $E(K)$ for the edge path group of K. It is also well known that if K is a finite abstract simplicial complex, then K has a geometric realization, denoted $|K|$, and that $E(K) \cong \Pi_1(|K|)$. We will henceforth refer to the edge path group of a finite simplicial complex K as the fundamental group of K, and we will write $\Pi_1(K)$ as opposed to $E(K)$.

One last definition of importance to us is the following.

Definition 2.6. Suppose K is a finite connected abstract simplicial complex. The rank of $\Pi_1(K)$ is the size of a minimal generating set for $\Pi_1(K)$.

2.2. Topological lemmas

In this subsection, we prove a handful of technical lemmas that we will need to prove Theorem A.

Lemma 2.7. *Suppose that σ is an m-simplex and τ is an n-simplex. Assume that $K = (V, S)$ is a connected abstract simplicial complex with $V = \sigma \cup \tau$ and $\sigma \cap \tau = \emptyset$. Finally,* *suppose that $a, b \in \sigma$ and that b is connected by an edge to $x \in \tau$. Then $\Pi_1(K, a)$ is generated by loops of the form $\langle a, b\rangle\langle b, x\rangle\langle x, y\rangle\langle y, c\rangle\langle c, a\rangle$, where $c \in \sigma$ and $y \in \tau$.*

Proof. We compute $\Pi_1(K, a)$ for $a \in \sigma$. We first show that $\Pi_1(K, a)$ is generated by loops of the form $\langle a, b'\rangle\langle b', x'\rangle\langle x', y\rangle\langle y, c\rangle\langle c, a\rangle$, where $b', c \in \sigma$ and $x', y \in \tau$. We will then establish that for $b \in \sigma$ and $x \in \tau$ connected by an edge, we can generate $\Pi_1(K, a)$ by loops where b and x are fixed, and only c and y are allowed to vary.

Consider a loop $\mathcal{L} = e_1 e_2 \dots e_n$ in K based at a that includes vertices from both σ and τ. Note that a loop containing vertices only from σ must be trivial, and therefore all such loops automatically belong to the group generated by any nonempty set. It follows that we may assume that the loop $\mathcal{L}$ contains at least one edge with start vertex in σ and end vertex in τ. Since $\mathcal{L}$ is based at $a \in \sigma$ and $\sigma \cap \tau = \emptyset$, it also follows that there is an edge from τ back to σ. Let i represent the smallest index for which e_i is an edge from σ to τ, and let j represent the smallest index for which e_j is an edge from τ to σ; note that $i < j$. Now let $\mathcal{P}_1$ be the (possibly empty) collection of edges preceding e_i in $\mathcal{L}$. We define $\mathcal{P}_2$ to be the (possibly empty) path consisting of the edges between e_i and e_j, and finally, let $\mathcal{P}_3$ be the (possibly empty) path consisting of the edges following e_j in $\mathcal{L}$. According to this decomposition, we have that $\mathcal{L} = \mathcal{P}_1 * e_i * \mathcal{P}_2 * e_j * \mathcal{P}_3$.

Write b' for the end vertex of $\mathcal{P}_1$, choosing $b' = a$ if $\mathcal{P}_1$ is the empty path. Write x' for the end vertex of e_i, and write y for the end vertex of $\mathcal{P}_2$, choosing $y = x'$ if $\mathcal{P}_2$ is the empty path. Write c for the end vertex of e_j. If $\mathcal{P}_3$ is empty, then $c = a$, and we can replace the empty path of $\mathcal{P}_3$ by the path $\mathcal{P}_3 = \langle a, a\rangle$.

Since a and b' belong to σ and σ is a simplex, the path $\mathcal{P}_1$ is homotopic to the path $\langle a, b'\rangle$. Similarly, x' and y belong to τ and τ is a simplex, so $\mathcal{P}_2$ is homotopic to the path $\langle x', y\rangle$. Combining these homotopies, we have that

$$\mathcal{L} \sim \langle a, b'\rangle * e_i * \langle x', y\rangle * e_j * \mathcal{P}_3 = \langle a, b'\rangle\langle b', x'\rangle\langle x', y\rangle\langle y, c\rangle * \mathcal{P}_3 \ .$$

Now $\mathcal{P}_3$ has start vertex $c \in \sigma$, and as σ is a simplex, we deduce that c is connected by an edge to a. Thus $\mathcal{P}_3$ is simply equivalent to $\langle c, a\rangle * \langle a, c\rangle * \mathcal{P}_3$. Letting $\mathcal{L}_1 = \langle a, b'\rangle\langle b', x'\rangle\langle x', y\rangle\langle y, c\rangle\langle c, a\rangle$, this shows that

$$\mathcal{L} \sim \mathcal{L}_1 * \langle a, c\rangle * \mathcal{P}_3 \ .$$

In other words, we have shown that an arbitrary loop $\mathcal{L}$ is homotopic to the concatenation of $\mathcal{L}_1$, a loop of the desired form, and another loop, namely $\langle a, c\rangle * \mathcal{P}_3$. Recall that $\mathcal{L} = e_1 e_2 \dots e_n$. We now claim that either $\langle a, c\rangle * \mathcal{P}_3$ consists of fewer edges than $\mathcal{L}$ or is trivial. To see this, first suppose that $\mathcal{P}_3$ was initially nonempty. Then since e_i and e_j were nonempty, $\mathcal{P}_3$ consists of at most $n-2$ edges, and therefore $\langle a, c\rangle * \mathcal{P}_3$ consists of at most $n-1$ edges. In this case, we have that $\langle a, c\rangle * \mathcal{P}_3$ consists of fewer edges than $\mathcal{L}$, and we can apply an inductive argument. Second, suppose that $\mathcal{P}_3$ was initially empty. Then we had $c = a$ and altered $\mathcal{P}_3 = \langle a, a\rangle$. Then $\langle a, c\rangle * \mathcal{P}_3$ is precisely the loop $\langle a, a\rangle\langle a, a\rangle$.

This loop is obviously simply equivalent to the loop $\langle a,a\rangle$, so we have the result in this case as well. This establishes the first claim that $\Pi_1(K,a)$ is generated by loops having the same form as $\mathcal{L}_1$.

Now let $b \in \sigma$ and $x \in \tau$, where b and x are connected by an edge. Suppose that $\mathcal{L}_0 = \langle a,b'\rangle\langle b',x'\rangle\langle x',y'\rangle\langle y',c'\rangle\langle c',a\rangle$ is a loop with b' and c' in σ and x' and y' in τ. We will show that $\mathcal{L}_0$ can be generated by loops of the desired form with initial edges $\langle a,b\rangle\langle b,x\rangle$. Let $\mathcal{L}_1 = \langle a,b\rangle\langle b,x\rangle\langle x,x'\rangle\langle x',b'\rangle\langle b',a\rangle$ and $\mathcal{L}_2 = \langle a,b\rangle\langle b,x\rangle\langle x,y'\rangle\langle y',c'\rangle\langle c',a\rangle$, and consider $\mathcal{L}_1^{-1} * \mathcal{L}_2$. Since the path $\langle x,b\rangle\langle b,a\rangle\langle a,b\rangle\langle b,x\rangle$ is homotopic to the constant path at x, we have that

$$\mathcal{L}_1^{-1} * \mathcal{L}_2 \sim \langle a,b'\rangle\langle b',x'\rangle\langle x',x\rangle\langle x,y'\rangle\langle y',c'\rangle\langle c',a\rangle \ . \tag{1}$$

Now x, x', and y' are all contained in the simplex τ, so the path $\langle x',x\rangle\langle x,y'\rangle$ is simply equivalent to the path $\langle x',y'\rangle$. Making this substitution in Equation (1), we find that $\mathcal{L}_1^{-1} * \mathcal{L}_2$ is homotopic to $\mathcal{L}_0$. The loops $\mathcal{L}_1$ and $\mathcal{L}_2$ are of the proper form and begin with the path $\langle a,b\rangle\langle b,x\rangle$, and we have shown that $\mathcal{L}_0$ belongs to the group generated by $\mathcal{L}_1^{-1}$ and $\mathcal{L}_2$. Since the group generated by $\mathcal{L}_1^{-1}$ and $\mathcal{L}_2$ is the same as the group generated by $\mathcal{L}_1$ and $\mathcal{L}_2$, the result holds. □

Corollary 2.8. *Suppose that σ is an m-simplex and τ is an n-simplex. Assume that $K = (V,S)$ is a connected simplicial complex where $V = \sigma \cup \tau$ and $\sigma \cap \tau = \emptyset$. Then the rank of $\Pi_1(K)$ is at most $nm + n + m$.*

Proof. Fix $a \in \sigma$. By Lemma 2.7, we know that if $b \in \sigma$ and $x \in \tau$ are connected by an edge, then $\Pi_1(K,a)$ is generated by loops of the form $\langle a,b\rangle\langle b,x\rangle\langle x,y\rangle\langle y,c\rangle\langle c,a\rangle$ for $c \in \sigma$ and $y \in \tau$. As b and x can be taken to be fixed, we can bound the rank of $\Pi_1(K,a)$ by counting the maximum number of loops of this form. Since $y \in \tau$ and τ has dimension n, we know that there are at most $n+1$ possible choices for y. Similarly, $c \in \sigma$ and σ has dimension m, so there are at most $m+1$ possible choices for c. This means that there are $(n+1)(m+1) = nm + n + m + 1$ total loops of this form. However, one of these loops corresponds to choosing $y = x$ and $c = b$, and this loop is trivial. Hence there are a maximum of $nm + n + m$ nontrivial loops of the form described in Lemma 2.7, establishing the claim. □

2.3. Induced maps

We now introduce a few more topological concepts of importance and derive a crucial theorem.

Definition 2.9. Suppose that $K = (V,S)$ and $K' = (V',S')$ are abstract simplicial complexes. A map $f : V \to V'$ is a simplicial map if it sends the vertices of each simplex of K to the vertices of some simplex of K'. That is, f is a simplicial map if for all $\sigma \in S$, we have that $f(\sigma) \in S'$.

Suppose that K and K' are connected abstract simplicial complexes and $f : K \to K'$ is a simplicial map. Since the definition of an edge between two vertices is defined in terms of simplices and the definition of homotopic paths is defined in terms of simplices, it follows that f preserves edges, loops, and homotopies between loops. Thus there is an induced map, denoted by $f_\star$, on the fundamental groups of K and K'. That is, we have an induced map $f_\star : \Pi_1(K) \to \Pi_1(K')$ defined by $f_\star([\mathcal{L}]) = [f(\mathcal{L})]$. Many intuitive and basic properties are satisfied by simplicial maps and their corresponding induced maps. For a fairly complete list of these properties and their proofs, see [2].

Lemma 2.10. *Suppose that $K = (V, S)$ is an abstract simplicial complex. Suppose that $\{x, y\} \subseteq V$, and that for all $\sigma \in S$ with $x \in \sigma$, we have that $\sigma \cup \{y\} \in S$. Let $V' = V \setminus \{x\}$, let $S' = \{\sigma \setminus \{x\} \mid \sigma \in S\}$, and let $K' = (V', S')$. Then the map $r : V \to V'$ via $r(x) = y$ and $r(z) = z$ for all $z \neq x$ in V is a simplicial map from K to K'.*

Proof. Say $\sigma \subseteq S$ is a simplex, so that σ is a set of vertices of V. If $x \notin \sigma$, then r is the identity on the members of σ, and every member of σ belongs to $V \setminus \{x\} = V'$; thus $r(\sigma) = \sigma$ is still a simplex in S'. Now suppose that $x \in \sigma$. We must argue that $r(\sigma) = (\sigma \setminus \{x\}) \cup \{y\}$ is a simplex in S'. By hypothesis, we know that $\sigma \cup \{y\}$ is a simplex in S. As all subsets of simplices are simplices, this means that $(\sigma \setminus \{x\}) \cup \{y\}$ is a simplex in S. That is $(\sigma \setminus \{x\}) \cup \{y\}$ is a simplex not involving x, so $(\sigma \setminus \{x\}) \cup \{y\}$ is a simplex in S'. $\square$

Using the properties of induced maps and Lemma 2.10, we obtain the following very important theorem.

Theorem 2.11. *Suppose that $K = (V, S)$ is a connected abstract simplicial complex. Suppose that $\{x, y\} \subseteq V$, and that for all $\sigma \in S$ with $x \in \sigma$, we have that $\sigma \cup \{y\} \in S$. Let $V' = V \setminus \{x\}$, let $S' = \{\sigma \setminus \{x\} \mid \sigma \in S\}$, and let $K' = (V', S')$. Then the induced inclusion map $i_\star : \Pi_1(K') \to \Pi_1(K)$ is an isomorphism.*

Proof. First, we show that if $a, b \in V'$ and $\mathcal{P}$ is a path in K from a to b that $\mathcal{P}$ is homotopic to a path in K' from a to b. We proceed by induction by the number of occurrences of x in $\mathcal{P}$. If $\mathcal{P}$ does not involve x, then $\mathcal{P}$ is a path in K', and the base case of our induction holds. Now suppose that at least one edge in $\mathcal{P}$ involves x, and write $\mathcal{P} = \mathcal{P}_1 * e_i e_j * \mathcal{P}_2$, where $e_i e_j = \langle c, x\rangle\langle x, d\rangle$ for some $c, d \in V$. Since $\langle c, x\rangle$ is an edge, c and x are contained in some simplex in S. By hypothesis, c, x, and y are contained in a simplex in S; therefore $\langle c, x\rangle$ is simply equivalent to the path $\langle c, y\rangle\langle y, x\rangle$. Similarly, d and x are contained in a simplex in S, so d, x, and y are contained in a simplex in S. Thus $\langle x, d\rangle$ is simply equivalent to $\langle x, y\rangle\langle y, d\rangle$. Combining these simple homotopies, we obtain $\langle c, x\rangle\langle x, d\rangle \sim \langle c, y\rangle\langle y, x\rangle\langle x, y\rangle\langle y, d\rangle$. Since $\langle y, x\rangle\langle x, y\rangle$ is homotopic to the constant path at y, we see that $\langle c, x\rangle\langle x, d\rangle$ is homotopic to the path $\langle c, y\rangle\langle y, d\rangle$. That is, $\mathcal{P} = \mathcal{P}_1 * e_i e_j * \mathcal{P}_2$ is homotopic to $\mathcal{P}_0 = \mathcal{P}_1 * \langle c, y\rangle\langle y, d\rangle * \mathcal{P}_2$. Now $\mathcal{P}_0$ involves fewer occurrences of x, so by induction, $\mathcal{P}_0$ is homotopic to a path contained in K'. As $\mathcal{P}$ and $\mathcal{P}_0$ are homotopic, the claim holds by the transitivity of homotopies, and it follows that K' is connected.

In fact, we have shown more with this argument; let $i : K' \to K$ be the inclusion map from K' to K. If we compute $\Pi_1(K')$ and $\Pi_1(K)$ with basepoint y, then all loops in K' and K are paths having terminal vertices in V'. Thus the above argument shows that every homotopy class of loops in $\Pi_1(K, y)$ contains a loop only involving vertices in K', showing that the induced inclusion map $i_\star : \Pi_1(K') \to \Pi_1(K)$ is surjective.

To finish the proof, we must show that $i_\star$ is also injective. Define $r : V \to V'$ via $r(x) = y$ and $r(z) = z$ for all $z \neq x$ in V. Now r is a simplicial map by Lemma 2.10, and we therefore have induced maps $i_\star : \Pi_1(K') \to \Pi_1(K)$ and $r_\star : \Pi_1(K) \to \Pi_1(K')$. It is clear that the composition $r \circ i$ is the identity map on K', and therefore the composition of the induced maps $r_\star \circ i_\star : \Pi_1(K') \to \Pi_1(K')$ is also the identity map. Suppose that $\mathcal{L}$ and $\mathcal{L}'$ are loops in K' with $i_\star([\mathcal{L}]) = i_\star([\mathcal{L}'])$. Then $r_\star(i_\star([\mathcal{L}])) = r_\star(i_\star([\mathcal{L}']))$. Since $r_\star \circ i_\star = (r \circ i)_\star$ is the identity map on K', we have that $[\mathcal{L}] = [\mathcal{L}']$, showing that $i_\star$ is injective. $\square$

A stronger version of Theorem 2.11 is in fact true; if K and K' are abstract simplicial complexes as in Theorem 2.11, then K' is a deformation retract of K. This stronger version of Theorem 2.11 can be proven using the geometric realizations of K and K'. As such an argument is not truly algebraic in nature and is not necessary for the work presented here, we omit the proof.

2.4. The common divisor and character degree simplicial complexes

Definition 2.12. The common divisor (abstract) simplicial complex of a set of integers X has vertex set X and simplices all subsets $X_0 \subseteq X$ satisfying $\gcd(X_0) > 1$. We denote this simplicial complex by $\mathcal{G}(X)$.

Suppose that X is a set of integers. If $x \in X$, we write $\pi(x)$ for the set of primes dividing x. If $k \neq 0$ is an integer, we write X_k to denote the set of integers in X divisible by k and $X_{k'}$ to denote the set of integers in X prime to k. Note that when k is a prime number, X is a disjoint union of X_k and $X_{k'}$.

To keep notation less cluttered, we will abbreviate $\Pi_1(\mathcal{G}(X))$ by $\Pi_1(X)$. Another definition of great importance to us is the following.

Definition 2.13. Suppose that X is a set of integers. We say that $x \in X$ is prime divisor maximal, abbreviated pd-maximal, if $\pi(x)$ is a maximal element in $\mathcal{X} = \{\pi(x) \mid x \in X\}$.

For a set of integers X, the collection of pd-maximal elements of X can be partitioned by an equivalence relation, where two pd-maximal elements are equivalent if they have the same set of prime divisors. In the remainder of this paper, we occasionally make use of this equivalence relation by selecting one representative from each equivalence class.

With these definitions in hand, we now apply some of the results in previous subsections to the common divisor simplicial complex of a set of integers X.

Lemma 2.14. *Suppose that X is a set of integers and that $\{x, y\} \subseteq X$ with $\pi(x) \neq \emptyset$ and $\pi(x) \subseteq \pi(y)$. Then the map $r : X \to (X \setminus \{x\})$ via $r(x) = y$ and $r(z) = z$ for all $z \in (X \setminus \{x\})$ is a simplicial map from $\mathcal{G}(X)$ to $\mathcal{G}(X \setminus \{x\})$.*

Proof. We begin by showing that whenever σ is a simplex in $\mathcal{G}(X)$ containing x then $\sigma \cup \{y\}$ is also a simplex in $\mathcal{G}(X)$. Let p be a prime dividing all members of the simplex σ. Since $x \in \sigma$, we see that p divides x, and as $\pi(x) \subseteq \pi(y)$, we conclude that p also divides y. Therefore p divides all members of the set $\sigma \cup \{y\}$, and $\sigma \cup \{y\}$ is a simplex in $\mathcal{G}(X)$. It follows by Lemma 2.10 that r is a simplicial map from $\mathcal{G}(X)$ to $\mathcal{G}(X \setminus \{x\})$. $\square$

The next result can be viewed as a corollary of Theorem 2.11.

Corollary 2.15. *Suppose that X is a finite set of integers with $\mathcal{G}(X)$ connected and $\Omega \subseteq X$ contains a representative of every class of pd-maximal elements in X. Then the induced inclusion map $i_\star : \Pi_1(\Omega) \to \Pi_1(X)$ is an isomorphism.*

Proof. We begin by noting that since $\mathcal{G}(X)$ is connected, we either have that $1 \notin X$ or else $X = \{1\}$. In the latter situation, $\Omega = X$, and the result follows trivially. We may therefore assume that $1 \notin X$, so that $\pi(x) \neq \emptyset$ for all $x \in X$.

Let $Y = X \setminus \Omega$, so that X is the disjoint union of Ω and Y; let $x \in Y$. Since Ω contains a representative of all pd-maximal elements of X, we know that there exists $y \in \Omega$ with $\pi(x) \subseteq \pi(y)$. By Lemma 2.14, we may apply Theorem 2.11 to conclude that $i_\star : \Pi_1(X \setminus \{x\}) \to \Pi_1(X)$ is an isomorphism. More generally, suppose that $|Y| = n$. Write $Y = \{x_i\}_{i=1}^n$ and define $X_i = X_{i-1} \setminus \{x_i\}$ for $1 \leq i \leq n$, viewing X as X_0. Applying Theorem 2.11 n times, we have isomorphisms $i_\star : \Pi_1(X_i) \to \Pi_1(X_{i-1})$ for all $1 \leq i \leq n$. That is, we have a chain of isomorphisms from $\Pi_1(\Omega)$ to $\Pi_1(X_{n-1})$ and so on all the way until $\Pi_1(X)$, and each one of these isomorphisms is induced by the corresponding inclusion map. As including first into an intermediate subspace and then including into the full space is the same as directly including into the full space, we see that the composition of these inclusion maps is equal to the map $i_\star : \Pi_1(\Omega) \to \Pi_1(X)$. Thus $i_\star : \Pi_1(\Omega) \to \Pi_1(X)$ is an isomorphism, as claimed. $\square$

Finally, we introduce the main application of this topological framework to our context.

For a finite group G, recall that we denote the irreducible characters of G by $\mathrm{Irr}(G)$ and we write $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$. Since $1 \in \mathrm{cd}(G)$, $\mathcal{G}(\mathrm{cd}(G))$ is never connected unless G is abelian. We write $\mathrm{cd}(G)^* = \mathrm{cd}(G) \setminus \{1\}$.

Definition 2.16. The character degree simplicial complex of a finite group G is the common divisor simplicial complex on the set $\mathrm{cd}(G)^*$. We use the shortened $\mathcal{G}(G)$ to denote $\mathcal{G}(\mathrm{cd}(G)^*)$ and $\Pi_1(G)$ for $\Pi_1(\mathcal{G}(G))$.

2.5. An analogue of Theorem 1.1

For a topological space X, the number of connected components of X is the rank of the 0-dimensional homology group and the 0-dimensional homotopy group of X; we denote these groups by $H_0(X)$ and $\Pi_0(X)$, respectively. One way of restating Theorem 1.1 is that if G is a solvable group, then the rank of $H_0(\Gamma(G))$ and the rank of $\Pi_0(\Gamma(G))$ is at most 2. As the number of connected components of $\Gamma(G)$ and the number of connected components of $\mathcal{G}(G)$ is the same, we see that Theorem 1.1 states that the number of connected components of $\mathcal{G}(G)$ is at most 2.

One way we could seek to generalize Manz's Theorem 1.1 is to ask if there are bounds on the ranks of $H_n(\Gamma(G))$ for $n > 0$. If we continue to study $\Gamma(G)$, however, this is not a very interesting question. Lemma 2.34 (b) of [2] tells us that if X is a simplicial complex of dimension n, then $H_k(X) = 0$ for all $k > n$. Since $\Gamma(G)$ is a graph, it has dimension 1, and therefore $H_k(\Gamma(G)) = 0$ for all $k \geq 2$. If $k = 1$, it is easy to show that there is no bound on the rank of $H_1(\Gamma(G))$. Let P be a nonabelian group of order p^3, and let G_n be the direct product of P with itself n times, so that the irreducible character degrees of G_n are the prime powers p^i for $0 \leq i \leq n$. It is easy to see that $\Gamma(G)$ is a complete graph with n vertices, which has fundamental group isomorphic to the free product of $\dfrac{(n-1)(n-2)}{2}$ copies of $\mathbb{Z}$. For a topological space X, $H_1(X)$ is the abelianization of $\Pi_1(X)$, and therefore these groups have equal rank. As the rank of $\Pi_1(\Gamma(G_n)) = \dfrac{(n-1)(n-2)}{2}$ for all $n \geq 2$, the rank of $H_1(\Gamma(G))$ is unbounded.

If we study $\mathcal{G}(G)$ as opposed to $\Gamma(G)$, however, the idea of studying $H_n(\mathcal{G}(G))$ or $\Pi_n(\mathcal{G}(G))$ is no longer trivial. For instance, the example built in the previous paragraph to show that the rank of $H_1(\Gamma(G))$ is unbounded no longer serves as an example to show that the rank of $H_1(\mathcal{G}(G))$ is unbounded. The remainder of this paper will be devoted to finding a bound on the rank of $\Pi_1(\mathcal{G}(G))$ in terms of the dimension of $\mathcal{G}(G)$.

3. A bound on $\mathrm{rk}(\Pi_1(G))$ in terms of $\dim(\mathcal{G}(G))$

Perhaps the only work to previously address the full character degree simplicial complex of a finite solvable group is that of Benjamin [1], although it should be noted that Benjamin did not think about her work this way. The main result of [1] can be interpreted in terms of the character degree simplicial complex as follows.

Theorem 3.1 *(Benjamin). Let G be a nonabelian finite solvable group with* $\dim(\mathcal{G}(G)) = n$. *Then*

$$|\mathrm{cd}(G)| \leq \begin{cases} 3 & \textit{if } n = 0 \\ 6 & \textit{if } n = 1 \\ 9 & \textit{if } n = 2 \\ n^2 + n + 2 & \textit{if } n \geq 3 \end{cases} .$$

It is straightforward to see that if G is a nonabelian finite solvable group with $\mathcal{G}(G)$ connected of dimension n, then Theorem 3.1 implies that a bound exists on the rank of $\Pi_1(G)$ in terms of n; when $n \geq 3$, this bound is on the order of n^4. A natural next question is to find the best possible bound on the rank of $\Pi_1(G)$ in terms of $\dim(\mathcal{G}(G))$. Our main theorem in this section demonstrates that we can improve the bound implied by Benjamin's result to a quadratic bound in terms of the dimension of $\mathcal{G}(G)$, and this theorem is independent of Benjamin's work.

We introduce two more pieces of notation. Suppose that $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$. We denote the collection of irreducible characters of G lying over θ by $\mathrm{Irr}(G|\theta)$, and similarly, we write $\mathrm{cd}(G|\theta) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G|\theta)\}$. We also remind the reader that an element $y \in \mathrm{cd}(G)$ is pd-maximal if $\pi(y)$ is maximal in the set $\mathcal{X} = \{\pi(x) \mid x \in \mathrm{cd}(G)\}$.

Our first result in this section determines the structure of $\mathcal{G}(G)$ in the case where G has a nonabelian quotient that is a p-group for some prime p, and this result does not require that the group G be solvable.

Lemma 3.2. *Suppose G is a finite group and that $K \lhd G$ is maximal so that G/K is nonabelian. If G/K is a p-group for some prime p, then every pd-maximal character degree is divisible by p. In particular, $\mathcal{G}(G)$ is connected and $\Pi_1(G)$ is trivial.*

Proof. To establish this result, we prove that if $x \in \mathrm{cd}(G)^*$ is prime to p, then x is connected by an edge to a member of $\mathrm{cd}(G)^*$ divisible by p and that x is not a pd-maximal element of $\mathrm{cd}(G)$. As G/K is solvable, we may apply Lemma 12.3 of [3] to conclude that G/K has a unique nontrivial irreducible character degree. Write f for this unique nontrivial irreducible character degree. Let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = x$, where x is prime to p, and let $\theta \in \mathrm{Irr}(K)$ be a constituent of χ_K. By Problem 6.7 of [3], we know that $\dfrac{\chi(1)}{\theta(1)}$ divides $|G : K|$. Since $\chi(1)$ is prime to p and $|G : K|$ is a p-power, we find that $\dfrac{\chi(1)}{\theta(1)}$ must equal 1, so that χ restricts irreducibly to K. Now $\chi \in \mathrm{Irr}(G)$ is an extension of θ, so we see that θ is invariant in G, and by Gallagher's Theorem, we have that $\mathrm{Irr}(G|\theta) = \{\chi\psi \mid \psi \in \mathrm{Irr}(G/K)\}$. We conclude that $\mathrm{cd}(G|\theta) = \{\theta(1)a \mid a \in \mathrm{cd}(G/K)\} = \{x, xf\}$.

Clearly x and xf are connected by an edge in $\mathcal{G}(G)$. This shows that every member of $\mathrm{cd}(G)^*$ that is prime to p is connected to a member of $\mathrm{cd}(G)^*$ divisible by p. The members of $\mathrm{cd}(G)^*$ divisible by p form a simplex, and are therefore connected, showing that $\mathcal{G}(G)$ is connected. We also see that since $f > 1$ is a p-power and x is prime to p that $\pi(x)$ is a proper subset of $\pi(xf)$, which means that x is not pd-maximal. If $\Omega \subseteq \mathrm{cd}(G)^*$ is the collection of pd-maximal elements, it follows that $\mathcal{G}(\Omega)$ forms a simplex, and is therefore contractible and has trivial fundamental group. By Corollary 2.15, the map $i_\star : \Pi_1(\Omega) \to \Pi_1(G)$ is an isomorphism, and therefore $\Pi_1(G)$ is also trivial. $\square$

Corollary 3.3. *Suppose that G is a finite group and that G has a nonabelian nilpotent factor group. Then $\mathcal{G}(G)$ is connected and $\Pi_1(G)$ is trivial.*

Proof. If G has a nonabelian nilpotent factor group, it follows that there exists $K \lhd G$ with G/K a p-group for some prime p and where K is maximal so that G/K is nonabelian. The result then follows by Lemma 3.2. □

We now proceed to our main result, the proof of Theorem A, which we restate here for convenience. We recall that if X is any set of integers and $k \neq 0$ is an integer, we write X_k and $X_{k'}$ for the set of integers in X divisible by k and prime to k, respectively.

Theorem 3.4. *Suppose G is a finite solvable group with $\mathcal{G}(G)$ connected and $\dim(\mathcal{G}(G)) = n$. Then $\mathrm{rk}(\Pi_1(G)) \leq n^2 + n - 1$.*

Proof. Suppose that $K \lhd G$ is maximal so that G/K is nonabelian. By Lemma 12.3 of [3], we know that G/K has a unique nontrivial irreducible character degree, which we denote by f. If G/K is a p-group for some prime p, then Lemma 3.2 gives us that $\Pi_1(G) = 1$. Then $\mathrm{rk}(\Pi_1(G)) = 0$, which is certainly less than the proposed bound. We may therefore assume by Lemma 12.3 of [3] that G/K is a Frobenius group with Frobenius kernel N/K. Furthermore, this lemma states that $|G : N| = f$ and that N/K is an elementary abelian p-group for some prime p.

Let $\Omega \subseteq \mathrm{cd}(G)$ be a collection of representatives of the pd-maximal elements of $\mathrm{cd}(G)$, so that $i_\star : \Pi_1(\Omega) \to \Pi_1(G)$ is an isomorphism by Corollary 2.15. To bound the rank of $\Pi_1(G)$ in terms of $\dim(\mathcal{G}(G))$, we first aim to determine the structure of $\mathcal{G}(\Omega)$. Since $\dim(\mathcal{G}(G)) = n$, we know that $\dim(\mathcal{G}(\Omega)) \leq n$, and we conclude that $|\Omega_p| \leq n + 1$. That is, Ω_p is a simplex of $\mathcal{G}(\Omega)$, and it follows that Ω_p has dimension at most n. We must therefore determine the structure of $\mathcal{G}(\Omega_{p'})$ to determine the structure of $\mathcal{G}(\Omega)$.

Let $x \in \Omega_{p'}$, and choose $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = x$. Suppose that $\psi \in \mathrm{Irr}(N)$ is a constituent of χ_N. By Theorem 12.4 of [3], either $|G : N|\psi(1) \in \mathrm{cd}(G)$ or else p divides $\psi(1)$. As $\psi(1)$ divides $\chi(1) = x$ and x is prime to p, we see that we must have $|G : N|\psi(1) = f\psi(1) \in \mathrm{cd}(G)$. Now $\chi(1)$ divides $|G : N|\psi(1)$ by Problem 6.7 of [3], and $|G : N|\psi(1) \in \mathrm{cd}(G)$. This means that $\pi(x) \subseteq \pi(|G : N|\psi(1))$, and by the pd-maximality of x, we must have $\pi(x) = \pi(|G : N|\psi(1)) = \pi(f\psi(1))$. In particular, this implies that $\pi(f) \subseteq \pi(x)$. Since $x \in \Omega_{p'}$ is arbitrary, we have that $\pi(f)$ is a subset of the set of prime divisors of all members of $\Omega_{p'}$; hence $\Omega_{p'}$ forms a simplex in $\mathcal{G}(\Omega)$. As the dimension of $\mathcal{G}(G)$ is n, it follows that $\Omega_{p'}$ is a simplex of dimension at most n.

In fact, we can do slightly better; if f is not pd-maximal, then we know that $f \in \mathrm{cd}(G)$ but $f \notin \Omega$, and therefore $\Omega_{p'}$ is a simplex of dimension at most $n-1$. We have just shown that if $x \in \Omega_{p'}$, then $\pi(f) \subseteq \pi(x)$. If f is pd-maximal, then we must have $\pi(f) = \pi(x)$, and as we have selected only one representative of each class of pd-maximal element, we have $|\Omega_{p'}| = 1$ in this case. That is, $\Omega_{p'} = \{x\}$, and in either case, we see that we actually have a bound of $n - 1$ on the dimension of $\Omega_{p'}$.

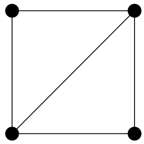

Fig. 1. A graph containing four vertices.

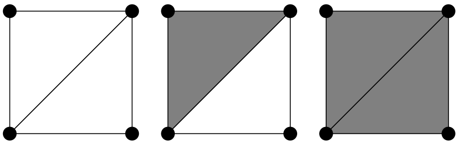

Fig. 2. Three simplicial complexes, each having Fig. 1 as its 1-skeleton.

Finally, we compute an upper bound on the rank of $\Pi_1(\Omega)$. Write σ for the simplex in $\mathcal{G}(\Omega)$ formed by $\Omega_{p'}$ and τ for the simplex formed by Ω_p. Of course, Ω is the disjoint union of σ and τ, where σ has dimension at most $(n-1)$ and τ has dimension at most n. It follows from Corollary 2.8 that $\Pi_1(\Omega)$ has rank at most $(n-1)n+n+(n-1) = n^2+n-1$, the bound we desire. $\square$

4. Applications and future research

4.1. Applications

We begin by presenting an example with two purposes. First, this example will demonstrate the need to consider the full structure of the character degree simplicial complex $\mathcal{G}(G)$ as opposed to just the character degree graph $\Gamma(G)$. Second, this example will show us how Theorem A can be used to exclude certain lists of positive integers as the irreducible character degrees of a finite solvable group.

Fig. 1 gives a graph containing four vertices. Fig. 2 shows the three different simplicial complexes that each have Fig. 1 as a 1-skeleton.

We first claim that Fig. 2(a) does not appear as the character degree simplicial complex of any finite solvable group. Suppose G is a finite solvable group having Fig. 2(a) as $\mathcal{G}(G)$. Then $\dim(\mathcal{G}(G)) = 1$ and $|\mathrm{cd}(G)| = 5$. Solvable groups with a connected character degree simplicial complex of dimension 1 have at most 6 vertices by Theorem 3.1, so we see that Theorem 3.1 does not exclude Fig. 2(a) as a character degree simplicial complex of a finite solvable group. However, the fundamental group of $\mathcal{G}(G)$ is isomorphic to the free product of two copies of $\mathbb{Z}$, and thus has rank 2. By Theorem A, the character degree simplicial complex of a finite solvable group of dimension 1 has rank at most 1, and therefore Fig. 2(a) cannot occur as the character degree simplicial complex of any finite solvable group.

To see that Fig. 2(c) does appear as the character degree simplicial complex of some finite solvable group, we construct an example. Suppose p and q are primes with $p \equiv 1$

(mod q). Let P be an extraspecial p-group of order p^3 of exponent p, so that P has the presentation $P = \langle a, b, c \mid a^p, b^p, c^p, [a, b] = c, [a, c], [b, c]\rangle$. Let σ be an automorphism of P that fixes a and sends b to b^n, where n is an element of order q in $U(\mathbb{Z}/p\mathbb{Z})$. When we consider the group $N = P \rtimes \langle\sigma\rangle$, it follows that N is a group of order p^3q with $\mathrm{cd}(N) = \{1, q, pq\}$. Finally, set $G = N \times P$. Then $\mathrm{cd}(G) = \{1, p, q, pq, p^2q\}$, and $\mathcal{G}(G)$ is Fig. 2(c).

Finally, we construct a more complicated example to show that Fig. 2(b) also appears as the character degree simplicial complex of a finite solvable group. Let A be an elementary abelian group of order p^3 where p is a prime congruent to 1 modulo 6. We then select $\sigma, \tau \in \mathrm{Aut}(A) \cong \mathrm{GL}_3(\mathbb{Z}/p\mathbb{Z})$ where $\sigma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $\tau = \begin{pmatrix} n^2 & 0 & 0 \\ 0 & n^3 & n^2\binom{n}{2} \\ 0 & 0 & n^4 \end{pmatrix}$, where n is an element of order 6 in $U(\mathbb{Z}/p\mathbb{Z})$. Last, set $G = A \rtimes \langle\sigma, \tau\rangle$. We claim that G is a solvable group with $|G| = 6p^4$ and $\mathrm{cd}(G) = \{1, 3, 6, 2p, 3p\}$.

First, note that for every positive integer m,

$$\sigma^m = \begin{pmatrix} 1 & m & \binom{m}{2} \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

and

$$\tau^m = \begin{pmatrix} n^{2m} & 0 & 0 \\ 0 & n^{3m} & n^2\binom{n}{2}n^{m-1}(1 + n + \ldots + n^{m-1}) \\ 0 & 0 & n^{4m} \end{pmatrix}. \tag{3}$$

One can verify from Equations (2) and (3) that σ has order p, τ has order 6, and $\sigma\tau = \tau\sigma^n$, or alternatively, $\tau^{-1}\sigma\tau = \sigma^n$. That is, $\langle\sigma, \tau\rangle$ is a Frobenius group of order $6p$. It follows that $P = A \rtimes \langle\sigma\rangle$ is a normal Sylow p-subgroup of G of order p^4 and that G modulo P is cyclic of order 6. We conclude both that G is solvable of order $6p^4$ and that $6 \in \mathrm{cd}(\langle\sigma, \tau\rangle) \subseteq \mathrm{cd}(G)$.

Let a, b, and c be generators for A that correspond to the basis vectors in rows 1, 2, and 3, respectively, of the matrices for σ and τ. Notice that $\langle c\rangle$ is normal in G and that $\langle c\rangle = \mathbf{Z}(P)$. Similarly, $\langle b, c\rangle \lhd G$ and modulo $\langle c\rangle$, $\langle b, c\rangle$ is the center of $P/\langle c\rangle$. Finally, note that the members of $\mathrm{cd}(G)$ are squarefree because A is an abelian normal subgroup of P of index p; this implies that $\mathrm{cd}(P) = \{1, p\}$, thus the p-part of every member of $\mathrm{cd}(G)$ is at most p.

Suppose $\chi \in \mathrm{Irr}(G)$ and $\chi(1) > 1$; let $N = P \cap \ker(\chi)$. As χ is nonlinear and G/P is abelian, we conclude that $N < P$. We will study the various options for N to show that $\chi(1) \in \{3, 6, 2p, 3p\}$ and that each of these numbers is in fact an irreducible character degree of G.

If $A \subseteq N$, then as $G/A \cong \langle\sigma, \tau\rangle$, we conclude that $\chi(1) = 6$. We may therefore assume that $A \nsubseteq N$. Next, suppose $\langle b, c\rangle \subseteq N$, so that $\chi \in \mathrm{Irr}(G/\langle b, c\rangle)$. As $A \nsubseteq N$, we can choose $\lambda \in \mathrm{Irr}(A/\langle b,c\rangle)$ nonprincipal with $\chi \in \mathrm{Irr}(G|\lambda)$. Since $a^\tau = a^{n^2}$, a has order p, and n^2 has order 3 in $U(\mathbb{Z}/p\mathbb{Z})$, we conclude that a is in a G-orbit of size 3. Now $A/\langle b,c\rangle$ is cyclic, hence the size of the G-orbit of a is equal to the size of the G-orbit of a nontrivial member of $\mathrm{Irr}(A/\langle b,c\rangle)$ by Theorem 6.32 of [3]. We conclude that λ is in a $G/\langle b,c\rangle$-orbit of size 3, and therefore 3 divides $\chi(1)$. This gives us that $\chi(1) \in \{3,6\}$, and to show that there exists an irreducible character of G of degree 3, note that λ is a linear character that extends to its stabilizer.

We may now assume that $\langle b,c\rangle \nsubseteq N$, and we consider the case where $\langle c\rangle \subseteq N$. Notationally, we will write T_θ for the stabilizer of θ in G. Let $\lambda \in \mathrm{Irr}(\langle b,c\rangle/\langle c\rangle)$ with $\chi \in \mathrm{Irr}(G|\lambda)$; note that λ is nonprincipal. Let $\eta \in \mathrm{Irr}(A)$ lie over λ and under χ. Since $\eta_{\langle b,c\rangle} = \lambda$, we have that $T_\eta \subseteq T_\lambda$. By considering the action of τ on $\langle b,c\rangle$ we see that, modulo $\langle c\rangle$, every nontrivial member of $\langle b,c\rangle$ is in a G-orbit of size 2. As $\langle b,c\rangle/\langle c\rangle$ is cyclic, we conclude that the G-orbit of λ has size 2. Hence 2 divides the size of the G-orbit of η. Additionally, we claim that p divides the size of the G-orbit of η. Because $\mathrm{Irr}(A/\langle c\rangle)$ is an elementary abelian group of order p^2 on which P/A acts nontrivially and P/A does act trivially on $\mathrm{Irr}(A/\langle b,c\rangle)$, we conclude that P/A must act nontrivially on every member of $\mathrm{Irr}(A/\langle c\rangle)$ that does not have $\langle b,c\rangle/\langle c\rangle$ in its kernel. Thus $2p$ divides the size of the G-orbit of η. Last, we claim that the G-orbit of η has size exactly equal to $2p$ and that $\chi(1) = 2p$. To see this, note that if $\chi(1) = 6p$, then as the G-orbit of λ has size 2, taking degrees gives us that $[\chi_{\langle b,c\rangle}, \lambda] = 3p$. Using Frobenius reciprocity and taking degrees, we conclude that $\lambda^G(1)$, which should be $|G : \langle b,c\rangle| = 6p^2$, is equal to at least $[\chi_{\langle b,c\rangle}, \lambda]\chi(1) = 18p^2$, which cannot hold. Since $2p$ must divide $\chi(1)$, the members of $\mathrm{cd}(G)$ are squarefree, and we have shown that $\chi(1)$ cannot be $6p$, we deduce that the G-orbit of η has size exactly $2p$ and that $\chi(1)$ must also be equal to $2p$.

Lastly, we assume that $N = 1$, and we let $\lambda \in \mathrm{Irr}(\langle c\rangle)$ with $\chi \in \mathrm{Irr}(G|\lambda)$, where λ is nonprincipal. Similar to the previous paragraph, we fix $\eta \in \langle b,c\rangle$ lying over λ and under χ. Using an argument analogous to the previous paragraph, we find that λ is in a G-orbit of size 3 and η is in a G-orbit of size divisible by $3p$. What remains to be shown is that η is in a G-orbit of size $3p$ and that $\chi(1) = 3p$. If η is in a G-orbit of size $6p$, then $A = T_\eta$. As A is abelian, η extends to A; it follows from Gallagher's Theorem, that there exists p distinct members of $\mathrm{Irr}(A|\eta)$. We also have a correspondence between $\mathrm{Irr}(A|\eta)$ and $\mathrm{Irr}(G|\eta)$ via induction, hence there are p members of $\mathrm{Irr}(G|\eta)$, each having degree $6p$. It follows that there are at least p irreducible characters of G of degree $6p$ lying over λ. Since λ is in a G-orbit of size 3, any irreducible character ψ of G of degree $6p$ lying over λ will satisfy $[\psi_{\langle c\rangle}, \lambda] = 2p$. Now $\lambda^G(1) = 6p^3$, and yet Frobenius reciprocity gives us that $\lambda^G(1) = \sum_{\psi \in \mathrm{Irr}(G|\lambda)} [\psi_{\langle c\rangle}, \lambda]\psi(1) \geq p(2p)(6p) = 12p^3$. This is a contradiction, hence we cannot have η in a G-orbit of size $6p$. Finally, we use this information to argue that $\chi(1) = 3p$. Now $A = T_\eta \cap P$, and as A is abelian, we know that η extends to A. Since $o(\eta)$ is a p-power, it follows that η extends to T_η, and we have a correspondence between $\mathrm{Irr}(T_\eta|\eta)$ and $\mathrm{Irr}(T_\eta/\langle b,c\rangle)$ by Gallagher's Theorem. It can be seen by looking at the matrix τ that the nontrivial members of $A/\langle b,c\rangle$ are in G-orbits of size 3, thus the group

T_η/A acts trivially on $A/\langle b,c\rangle$. That is, $T_\eta/\langle b,c\rangle$ is abelian, and therefore every member of $\mathrm{Irr}(T_\eta/\langle b,c\rangle)$ has degree 1. We conclude that every member of $\mathrm{cd}(G|\eta)$ has degree $3p$. In summary, we have created a solvable group G with $\mathrm{cd}(G) = \{1, 3, 6, 2p, 3p\}$ and $\mathcal{G}(G)$ is Fig. 2(b). This example has been verified with the prime $p = 7$ in Magma.

In conclusion, we have established that we can create examples of solvable groups G having $\mathcal{G}(G)$ as Fig. 2(b) and Fig. 2(c), but there is no solvable group G having $\mathcal{G}(G)$ as Fig. 2(a). This proves that the character degree simplicial complex is able to distinguish an ambiguity present in the character degree graph of a finite solvable group, and serves as justification that we should study the full structure provided by $\mathcal{G}(G)$ as opposed to $\Gamma(G)$ in this field of research.

4.2. Future research

When $\mathcal{G}(G)$ is connected, Theorem A gives a bound on the rank of $\Pi_1(G)$ in terms of the dimension of $\mathcal{G}(G)$ that is on the order of n^2, a significant improvement over the bound on the order of n^4 implied by Theorem 3.1. Theorem A is also independent of Benjamin's work, and can be used to show that certain simplicial complexes do not occur as the character degree simplicial complex of any finite solvable group. The question that remains is what is the best possible bound we can acquire on the rank of $\Pi_1(G)$ when G is a finite solvable group? Although Subsection 4.1 gives an example of a solvable group G having $\mathcal{G}(G)$ connected with $\Pi_1(G)$ of rank 1, the author is unaware of any examples of solvable groups having a connected character degree simplicial complex and a fundamental group not isomorphic to $\mathbb{Z}$ or the trivial group. That is, it may be the case that there is a universal bound on the rank of $\Pi_1(G)$ when $\mathcal{G}(G)$ is connected and G is a finite solvable group, independent of the dimension of $\mathcal{G}(G)$. It would be interesting, therefore, to either construct a sequence $\{G_i\}$ of finite solvable groups where the dimension of $\mathcal{G}(G_i)$ increases with i, as does the rank of $\Pi_1(G_i)$. Similarly, it would be interesting to prove that no such sequence exists, and that the rank of $\Pi_1(G)$ is independent of the dimension of $\mathcal{G}(G)$.

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