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Degenerate cyclotomic Hecke algebras and higher level Heisenberg categorification

Marco Mackaay [a,*], Alistair Savage [b]

[a] *Departamento de Matemática, Universidade do Algarve and CAMGSD, Instituto Superior Técnico, Portugal*

[b] *Department of Mathematics and Statistics, University of Ottawa, Canada*

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ABSTRACT

We associate a monoidal category $\mathcal{H}^\lambda$ to each dominant integral weight λ of $\widehat{\mathfrak{sl}}_p$ or $\mathfrak{sl}_\infty$. These categories, defined in terms of planar diagrams, act naturally on categories of modules for the degenerate cyclotomic Hecke algebras associated to λ. We show that, in the $\mathfrak{sl}_\infty$ case, the level d Heisenberg algebra embeds into the Grothendieck ring of $\mathcal{H}^\lambda$, where d is the level of λ. The categories $\mathcal{H}^\lambda$ can be viewed as a graphical calculus describing induction and restriction functors between categories of modules for degenerate cyclotomic Hecke algebras, together with their natural transformations. As an application of this tool, we prove a new result concerning centralizers for degenerate cyclotomic Hecke algebras.

Contents

* Corresponding author.
E-mail addresses: mmackaay@ualg.pt (M. Mackaay), alistair.savage@uottawa.ca (A. Savage).
URL: http://alistairsavage.ca (A. Savage).

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1. Introduction

The Heisenberg algebra plays a fundamental role in many areas of mathematics and physics. The universal enveloping algebra of the infinite-dimensional Heisenberg Lie algebra is the associative algebra with generators $p_n^{\pm}$, $n \in \mathbb{N}_+$, and c, and relations

$$p_n^+ p_m^+ = p_m^+ p_n^+, \quad p_n^- p_m^- = p_m^- p_n^-, \quad p_n^+ p_m^- = p_m^- p_n^+ + \delta_{n,m} c, \quad p_n^{\pm} c = c p_n^{\pm}, \quad n, m \in \mathbb{N}_+.$$

(See Section 4.1 for a more detailed treatment.) On any irreducible representation, the central element c acts by a constant. For a positive integer d, the (associative) Heisenberg algebra $\mathfrak{h}_d$ of level d is the quotient of this algebra by the ideal generated by $c - d$.

In [16], Khovanov introduced a diagrammatic monoidal category $\mathcal{H}$ that acts naturally on categories of modules for symmetric groups. He proved that the Grothendieck ring of $\mathcal{H}$ contains the level one Heisenberg algebra $\mathfrak{h}_1$ and conjectured that the two are actually equal. This work has inspired an active area of research into Heisenberg categorification. Replacing group algebras of symmetric groups by Hecke algebras of type A led to the q-deformed categorification of [21], while replacing them by wreath product algebras led to categorifications of (quantum) lattice Heisenberg algebras in [10,25].

One remarkable feature of Khovanov's category is that degenerate affine Hecke algebras H_n appear naturally in the endomorphism spaces of certain objects. The group algebras of symmetric groups are level one cyclotomic quotients of these degenerate affine Hecke algebras, wherein the polynomial generators of H_n are mapped to the Jucys–Murphy elements. It is thus natural to conjecture that suitably modified versions of Khovanov's category should act on categories of modules for more general degenerate cyclotomic Hecke algebras. These modified categories should categorify higher level Heisenberg algebras and should encode much of the representation theory of degenerate affine Hecke algebras and their cyclotomic quotients.

On the other hand, cyclotomic Hecke algebras and their degenerate versions have appeared in the context of categorification before. In particular, Ariki's categorification theorem ([1,5]) relates the representation theory of these algebras to highest weight irreducible representations of affine Lie algebras of type A. In addition, Brundan–Kleshchev ([3,4]) and Rouquier ([24]) have related cyclotomic Hecke algebras to the quiver Hecke algebras appearing in the Khovanov–Lauda–Rouquier categorification of quantum $\mathfrak{sl}_n$ ([17,24]).

In the current paper we define a family of diagrammatic $\Bbbk$-linear monoidal categories $\mathcal{H}^\lambda$ depending on a dominant integral weight λ of $\mathfrak{sl}_\infty$ (when $\Bbbk$ is of characteristic zero) or $\widehat{\mathfrak{sl}}_p$ (when $\Bbbk$ is of characteristic $p > 0$). When λ is a fundamental weight (i.e. is of level one), the category $\mathcal{H}^\lambda$ reduces to Khovanov's Heisenberg category $\mathcal{H}$ (see Remark 2.10). One of our main results is Theorem 4.4, which states that, if $\Bbbk$ is a field of characteristic zero, then we have an injective ring homomorphism

$$\mathfrak{h}_d \hookrightarrow K_0(\mathcal{H}^\lambda),$$

where $K_0(\mathcal{H}^\lambda)$ denotes the Grothendieck ring of $\mathcal{H}^\lambda$. We conjecture that this homomorphism is also surjective. In level one, this reduces to Khovanov's conjecture [16, Conj. 1].

Our next main result is Theorem 6.7, which states that the category $\mathcal{H}^\lambda$ acts naturally on categories of modules for the degenerate cyclotomic Hecke algebras H_n^λ, $n \in \mathbb{N}$, corresponding to λ. The action arises from a collection of functors from $\mathcal{H}^\lambda$ to categories of bimodules over the H_n^λ. If $\Bbbk$ is a field of characteristic zero, we prove in Theorem 7.6 that these functors are full. Thus, the categories $\mathcal{H}^\lambda$ give a graphical calculus for bimodules over degenerate cyclotomic Hecke algebras. As an application of the resulting computational tool, we prove in Corollary 7.7 that the centralizer of H_n^λ in H_{n+k}^λ under the natural inclusion of algebras $H_n^\lambda \otimes H_k \hookrightarrow H_{n+k}^\lambda$ is generated by H_k and the center of H_n^λ. This result, which seems to be new, is a generalization of a result of Olshanski on centralizers for group algebras of symmetric groups.

The constructions of the current paper suggest a number of applications and further research directions. For example, we expect truncations of the $\mathcal{H}^\lambda$ to be related to categorified quantum groups. There should also exist q-deformations of the categories $\mathcal{H}^\lambda$. In addition, it would be interesting to examine the traces of the $\mathcal{H}^\lambda$ and use them to construct diagrammatic pairings on spaces related to symmetric functions. We expand on these ideas for further research in Section 8.

We note that higher level Heisenberg algebras are also categorified by the categories $\mathcal{H}_B$ of [25] for suitable choices of the Frobenius algebra B on which these categories depend. The approach of the current paper is quite different, but there are relations between the two. First, there is a natural filtration on the categories $\mathcal{H}^\lambda$ introduced here. The associated graded category is closely related to some special cases of the categories of [25]. See Proposition 3.4. In addition, we expect that the approaches of [25] and the current paper can be unified, as we explain in Section 8.2. One aspect of the categories introduced in the current paper that differs substantially from previous Heisenberg categories is the introduction of *dual dots* in planar diagrams. These dots, which correspond algebraically to the duals of the elements x_i of degenerate cyclotomic Hecke algebras under the natural trace form on those algebras, behave quite differently than the usual dots do. This makes some computations substantially more difficult, requiring new techniques relying on a filtration on the category. See Remark 2.1.

The paper is organized as follows. In Section 2 we introduce our main object of study, the diagrammatic categories $\mathcal{H}^\lambda$, and study their morphism spaces in detail. In Section 3 we define a filtration on these morphism spaces. This filtration is used in Section 4 to show that the higher level Heisenberg algebra maps to the Grothendieck ring of $\mathcal{H}^\lambda$.

In Section 5 we recall the definition of the degenerate cyclotomic Hecke algebras and prove various results concerning them. We use these results in Section 6 to define the action of the category $\mathcal{H}^\lambda$ on the category of modules for these algebras and use the action to prove injectivity of the map $\mathfrak{h}_d \to K_0(\mathcal{H}^\lambda)$. Then, in Section 7, we prove some properties of the action. In particular, we prove that the functors involved are full in the characteristic zero case, allowing us to deduce new facts about centralizers of degenerate cyclotomic Hecke algebras. Finally, in Section 8 we discuss some further directions of research naturally suggested by the current paper.

Notation and conventions. Throughout the paper, $\Bbbk$ will denote a commutative ring. In some places, we will assume that $\Bbbk$ is a field of characteristic zero. For the benefit of the reader, we will specify the assumptions on $\Bbbk$ at the beginning of each section. We let I denote the image of $\mathbb{Z}$ in $\Bbbk$ under the natural ring homomorphism $n \mapsto n \cdot 1$. We identify I with $\mathbb{Z}/p\mathbb{Z}$, where $p \geq 0$ is the characteristic of $\Bbbk$. We let $\mathbb{N}$ denote the set of nonnegative integers, and let $\mathbb{N}_+$ denote the set of positive integers.

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Hidden details. For the interested reader, the tex file of the arXiv version of this paper includes hidden details of some straightforward computations and arguments that are omitted in the pdf file. These details can be displayed by switching the `details` toggle to true in the tex file and recompiling.

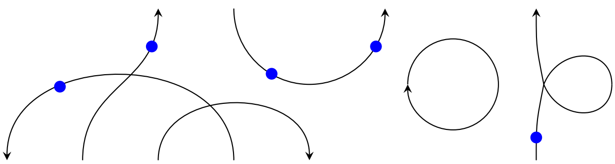
2. The diagrammatic category

Throughout this section, $\Bbbk$ is an arbitrary commutative ring. Let $\mathfrak{g} = \widehat{\mathfrak{sl}}_p$ if the characteristic p of $\Bbbk$ is greater than zero, and let $\mathfrak{g} = \mathfrak{sl}_\infty$ if $p = 0$. Recall that $I \cong \mathbb{Z}/p\mathbb{Z}$ is the canonical image of $\mathbb{Z}$ in $\Bbbk$. For $i \in I$, let ω_i denote the i-th fundamental weight of $\mathfrak{g}$, and let $P_+ = \bigoplus_{i \in I} \mathbb{N}\omega_i$ denote the dominant weight lattice of $\mathfrak{g}$. Fix $\lambda = \sum_{i \in I} \lambda_i \omega_i \in P_+$, $\lambda \neq 0$, and set $d = \sum_i \lambda_i$.

2.1. Definition

Let $\tilde{\mathcal{H}}^\lambda$ be the additive $\Bbbk$-linear strict monoidal category defined as follows. The set of objects is generated by two objects Q_+ and Q_-. Thus each object is a formal direct sum of objects of the form $\mathsf{Q}_\epsilon := \mathsf{Q}_{\epsilon_1} \cdots \mathsf{Q}_{\epsilon_k}$, where $\epsilon = (\epsilon_1, \dots, \epsilon_k)$ is a sequence whose entries are either $+$ or $-$ and we denote the tensor product by juxtaposition. We denote the unit object by $\mathbf{1} := \mathsf{Q}_\varnothing$.

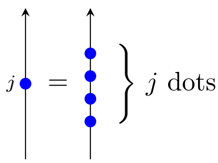
The space of morphisms between two objects Q_ϵ and $Q_{\epsilon'}$ is the $\Bbbk$-algebra generated by suitable planar diagrams modulo local relations. The diagrams consist of oriented compact one-manifolds immersed into the plane strip $\mathbb{R} \times [0,1]$, modulo isotopies fixing the boundary, and modulo certain local relations. Strands are allowed to carry dots. Only double intersections are allowed and the dots do not lie on them. The dots are allowed to move freely along the strands of the diagram as long as they do not cross double points. The endpoints of the diagrams are considered to be located at $\{1,\dots,m\} \times \{0\}$ and $\{1,\dots,k\} \times \{1\}$, where m and k are the lengths of ϵ and ϵ' respectively. (This is for the purposes of composition—we will not always draw our diagrams with these positions apparent.) The orientation of the one-manifold at the endpoints must agree with the signs in ϵ and ϵ'. For example



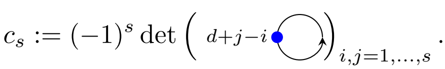
is an element of $\mathrm{Hom}_{\tilde{\mathcal{H}}^\lambda}(\mathsf{Q}_{-+++-+}, \mathsf{Q}_{+-++})$.

Composition of morphisms is given by vertical stacking of diagrams (followed by rescaling of the vertical axis). The monoidal structure on morphisms is given by horizontal juxtaposition of diagrams.

An endomorphism of $\mathbf{1}$ is a diagram with no endpoints. A dot labeled $j \in \mathbb{N}$ will denote a strand with j dots:

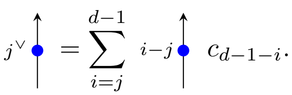


Define $c_0 = 1$ and, for any $1 \le s \le d$, let



(2.1)

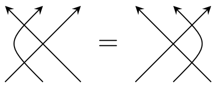
Then, for $j \in \{0,\dots,d-1\}$, define the *dual dots*

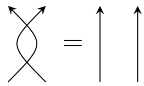


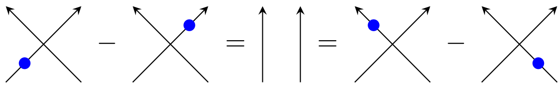
(2.2)

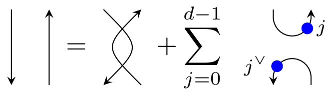
(The reason for this definition of the dual dots will become apparent in Lemma 2.5 below.)

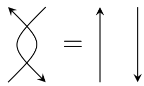
The local relations are as follows:

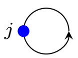
 (2.3)

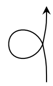
 (2.4)

 (2.5)

 (2.6)

 (2.7)


$$= \begin{cases} 0 & \text{if } j < d-1, \\ 1 & \text{if } j = d-1, \\ \sum_{i \in I} i\lambda_i & \text{if } j = d. \end{cases} \quad (2.8)$$

 $= 0$ (2.9)

We allow the dots to move freely along strands, in particular over cups and caps. (Note that the second equality of (2.5) follows from the first, together with (2.4). However, we include it for ease of reference.)

Remark 2.1. Note that the dual dots do *not* satisfy relation (2.5) in general. In fact, the formulas for sliding a dual dot through a crossing are rather complicated. This introduces a computational difficulty into the graphical calculus. We will handle this difficulty by using a natural filtration on the morphism spaces of $\tilde{\mathcal{H}}^\lambda$ and performing some computations in the associated graded category. See Section 3.

We let $\mathcal{H}^\lambda$ be the idempotent completion (sometimes called the Karoubi envelope) of $\tilde{\mathcal{H}}^\lambda$. Thus, the objects of $\mathcal{H}^\lambda$ are direct sums of pairs (Q_ϵ, e), where $e\colon \mathsf{Q}_\epsilon \to \mathsf{Q}_\epsilon$ is an idempotent morphism. Morphisms $(\mathsf{Q}_\epsilon, e) \to (\mathsf{Q}_{\epsilon'}, e')$ in $\mathcal{H}^\lambda$ are morphisms $\alpha\colon \mathsf{Q}_\epsilon \to \mathsf{Q}_{\epsilon'}$ in $\tilde{\mathcal{H}}^\lambda$ such that $fe = f = e'f$. Equivalently,

$$\mathrm{Hom}_{\mathcal{H}^\lambda}((\mathsf{Q}_\epsilon, e), (\mathsf{Q}_{\epsilon'}, e')) = e' \,\mathrm{Hom}_{\tilde{\mathcal{H}}^\lambda}(\mathsf{Q}_\epsilon, \mathsf{Q}_{\epsilon'}) e.$$

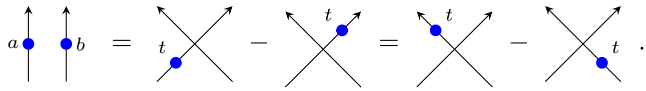
The composition and monoidal structure of $\mathcal{H}^\lambda$ are induced by those of $\tilde{\mathcal{H}}^\lambda$. Identifying Q_ϵ with $(\mathsf{Q}_\epsilon, \mathrm{id}_{\mathsf{Q}_\epsilon})$, we view $\tilde{\mathcal{H}}^\lambda$ as a full subcategory of $\mathcal{H}^\lambda$.

2.2. Morphism spaces

We now investigate the morphism spaces of $\mathcal{H}^\lambda$ in some detail.

Lemma 2.2. *We have*

$$\sum_{a+b=t-1}$$



(2.10)

In (2.10), and throughout the paper, the notation $\sum_{a+b=t-1}$ means that we sum over all $a, b \in \mathbb{N}$ satisfying the condition $a + b = t - 1$.

Proof. The equations in (2.10) are obtained by applying the ones in (2.5) t times. $\square$

Lemma 2.3. *Suppose a_k, $k \in \mathbb{Z}$, are elements of a commutative ring such that*

$$a_{-1} = 1, \quad \textit{and} \quad a_k = 0 \quad \textit{for } k < -1.$$

Then, for $m \geq 1$, we have

$$\sum_{s=0}^{m} (-1)^s a_{m-1-s} \det(a_{j-i})_{i,j=1,\dots,s} = 0.$$

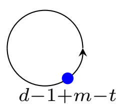
Proof. If we compute the determinant $\det(a_{j-i})_{i,j=1,\dots,m}$ by repeatedly expanding along the first column, we get

$$\begin{aligned}
\det(a_{j-i})_{i,j=1,\dots,m} &= a_0 \det(a_{j-i})_{i,j=1,\dots,m-1} - \det \begin{pmatrix} a_1 & a_2 & \cdots & a_{m-1} \\ 1 & a_0 & \cdots & a_{m-3} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_0 \end{pmatrix} \\
&= a_0 \det(a_{j-i})_{i,j=1,\dots,m-1} - a_1 \det(a_{j-i})_{i,j=1,\dots,m-2} + \cdots \\
&= \sum_{s=0}^{m-1} (-1)^{m-1-s} a_{m-1-s} \det(a_{j-i})_{i,j=1,\dots,s}.
\end{aligned}$$

The result follows. $\square$

Lemma 2.4. *For $m \in \mathbb{N}$, we have*

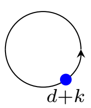
$$\sum_{t=0}^{m} c_t$$



$$= \delta_{m,0}. \qquad (2.11)$$

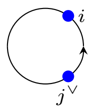
Proof. The case $m = 0$ follows immediately from (2.8). The case $m > 0$ follows from Lemma 2.3 with

$$a_k =$$



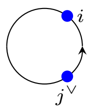
. $\square$

Lemma 2.5. *For any $i, j \in \{0, \dots, d-1\}$, we have*

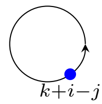


$$= \delta_{i,j}.$$

Proof. We have

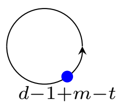


$$= \sum_{k=j}^{d-1} c_{d-1-k}$$



If $i \le j$, the result follows immediately from (2.8). Therefore, assume $i > j$. Setting $m = i - j$ and $t = d - 1 - k$ in the above sum, it remains to prove that

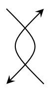
$$\sum_{t=0}^{d-1-j} c_t$$



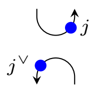
$$= 0. \qquad (2.12)$$

By (2.8), the terms in (2.12) with $t > m$ are equal to zero. Since $i \le d-1$, we have $m \le d - 1 - j$. Therefore, the result follows from Lemma 2.4. □

Lemma 2.6. *The equality (2.6) is a decomposition of the identity morphism of $\mathsf{Q}_-\mathsf{Q}_+$ into $d+1$ orthogonal idempotents. More precisely, the morphisms*



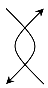
and



$$, \quad j = 0, \dots, d-1,$$

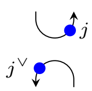
are orthogonal idempotents.

Proof. By (2.7),



$$(2.13)$$

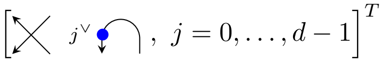
is an idempotent. It also follows immediately from Lemma 2.5 that the



$$, \quad j = 0, \dots, d-1, \qquad (2.14)$$

are orthogonal idempotents. Finally, since, by (2.6), we have that (2.13) is equal to the identity minus the sum of the elements in (2.14), it follows that (2.13) is orthogonal to each of the idempotents in (2.14). □

Corollary 2.7. *In $\tilde{\mathcal{H}}^\lambda$, we have an isomorphism*



$$: \mathsf{Q}_-\mathsf{Q}_+ \cong \mathsf{Q}_+\mathsf{Q}_- \oplus \mathbf{1}^{\oplus d}.$$

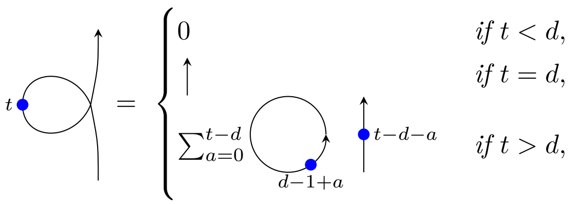
Proof. This follows immediately from Lemma 2.6. □

Corollary 2.8. *Every object of $\tilde{\mathcal{H}}^\lambda$ is isomorphic to a direct sum of objects of the form $\mathsf{Q}_+^n\mathsf{Q}_-^m$, $n, m \in \mathbb{N}$.*

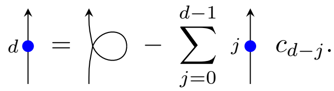
Proof. By definition, objects of $\tilde{\mathcal{H}}^\lambda$ are sums of products of Q_+ and Q_-. The result then follows from Corollary 2.7 by induction. □

We now deduce some other consequences of the local relations in $\mathcal{H}^\lambda$.

Lemma 2.9. *We have*

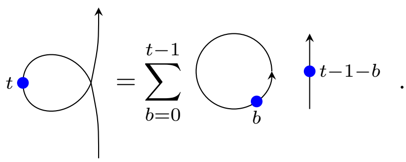


(2.15)



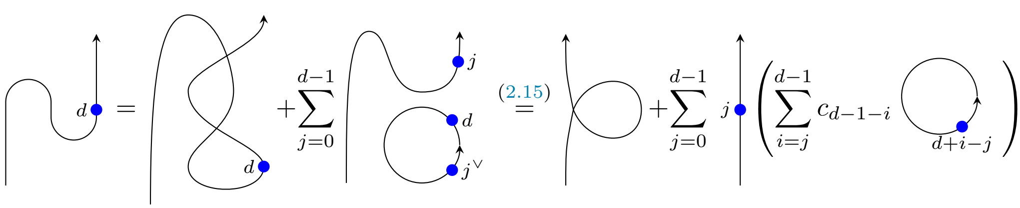
(2.16)

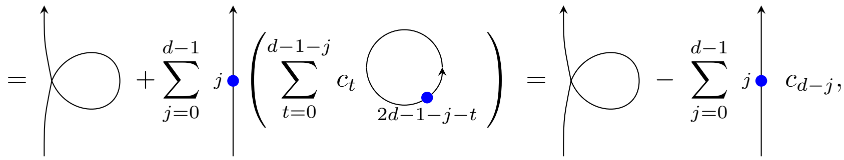
Proof. By (2.10) and (2.9), we have



Then (2.15) follows from (2.8).

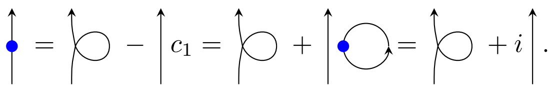
Now, to all terms in (2.6), add an upward pointing strand to the left, then a right cap joining the tops of the two leftmost strands, and a right cup with d dots joining the bottoms of the two rightmost strands. This gives





where, in the third equality, we have made the substitution $t = d - 1 - i$, and the last equality follows from Lemma 2.4 with $m = d - j$. □

Remark 2.10. If $\lambda = \omega_i$ is a fundamental weight, (2.16) implies that

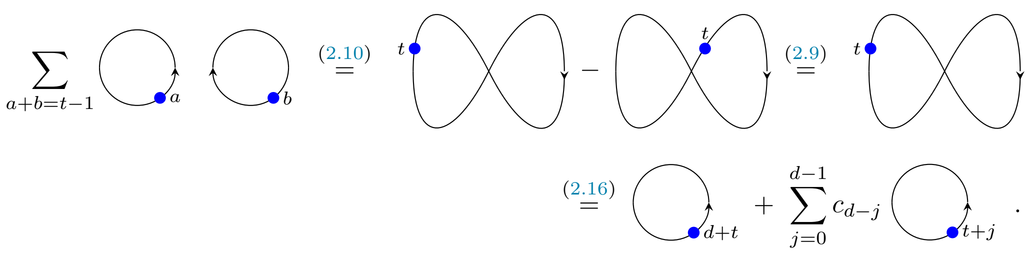


Thus, all dots can be eliminated. In particular, the category $\tilde{\mathcal{H}}^{\omega_0}$ reduces to the category $\mathcal{H}'$ defined by Khovanov in [16]. By Proposition 2.17 below, it then follows that $\tilde{\mathcal{H}}^{\omega_i}$ is isomorphic to Khovanov's category for all $i \in I$.

In the sequel we will use the term *bubble* for clockwise or counterclockwise circles with or without dots.

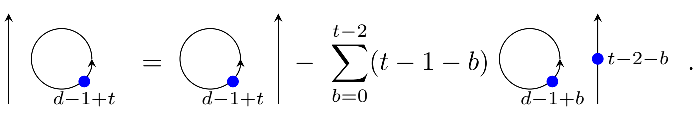
Lemma 2.11 *(Bubble reduction). Any clockwise bubble is equal to a linear combination of counterclockwise bubbles.*

Proof. For any $t \in \mathbb{N}_+$, we have



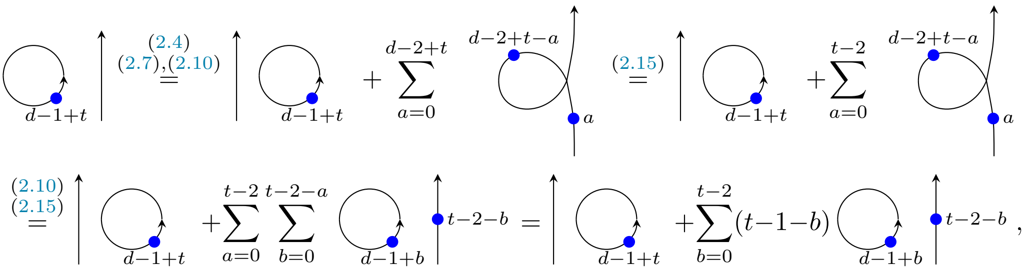
Now, for $s \in \mathbb{N}$, take $t = d + s$. By (2.8), the above allows us to write a clockwise bubble with s dots in terms of counterclockwise bubbles and clockwise bubbles with fewer than s dots. The result then follows by induction. □

Lemma 2.12 *(Bubble slide). For $t \geq 0$, we have*



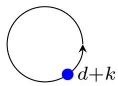
(When $t \leq 1$, we interpret the sum as being empty.)

Proof. We have



where the last equality follows by counting, in the penultimate expression, how many times we get the term corresponding to a fixed b. The term with $b = 0$ appears once for all $a = 0, \dots, t-2$, the term with $b = 1$ appears once for all $a = 0, \dots, t-3$, etc. $\square$

Let $\Pi := \Bbbk[y_1, y_2, y_3, \dots]$ be a polynomial algebra in countably many variables and consider the homomorphism of algebras

$$\psi_0 \colon \Pi \to \mathrm{End}_{\tilde{\mathcal{H}}^\lambda}(\mathbf{1}), \quad y_k \mapsto$$  . (2.17)

Proposition 2.13. *The homomorphism ψ_0 is an isomorphism.*

Proof. It is clear from the above results, that any closed diagram can be converted into a linear combination of non-nested dotted counterclockwise bubbles. Thus ψ_0 is surjective. We will prove it is injective in Proposition 7.3. $\square$

Let H_m be the degenerate affine Hecke algebra of rank m (see Definition 5.1). By (2.3), (2.4), and (2.5), there exists a well-defined homomorphism

$$H_m \to \mathrm{End}_{\tilde{\mathcal{H}}^\lambda}(\mathsf{Q}_+^m) \tag{2.18}$$

determined by sending s_i to the diagram with only one crossing, between the i-th and $(i+1)$-st strand, and x_j to a dot on the j-th strand of the identity endomorphism. *It is important to note that here we number strands from right to left.* By placing bubbles in the far right region of a diagram, we obtain a homomorphism of algebras

$$\psi_m \colon H_m \otimes \Pi \to \mathrm{End}_{\tilde{\mathcal{H}}^\lambda}(\mathsf{Q}_+^m). \tag{2.19}$$

Proposition 2.14. *The homomorphism ψ_m is an isomorphism.*

Proof. Just as in [16, Prop. 4], it is straightforward to show that ψ_m is surjective, because any diagram in $\mathrm{End}_{\tilde{\mathcal{H}}^\lambda}(\mathsf{Q}_+^m)$ is equal to a linear combination of permutation diagrams with dots at the top and dotted counterclockwise bubbles in the far right region. We use (2.16) to eliminate right curls (the dots of Khovanov's Heisenberg category). We will prove ψ_m is injective in Proposition 7.4. □

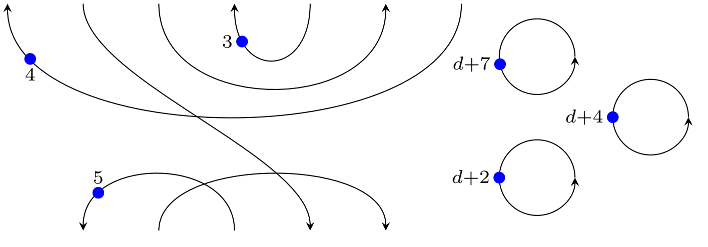
Following [16, Prop. 5], we can give an explicit basis of the hom-spaces in $\tilde{\mathcal{H}}^\lambda$.

Definition 2.15. For two sign sequences ϵ, ϵ', let $B(\epsilon, \epsilon')$ be the set of planar diagrams obtained in the following manner:

- We write the sequence ϵ at the bottom of the plan strip $\mathbb{R} \times [0, 1]$, and we write the sequence ϵ' at the top of this strip.
- We match the elements of ϵ and ϵ' by oriented segments embedded in the strip. The orientation of the endpoints of each segment must agree with the sign (i.e. be oriented up at a + sign and down at a − sign). No two segments can intersect more than once, and no self-intersection or triple intersections are allowed.
- We place some number (possibly zero) of dots on each segment near its out endpoint (i.e. between its out endpoint and any intersections with other intervals).
- In the rightmost region of the diagram, we draw a finite number of counterclockwise disjoint nonnested circles with at least $d + 1$ dots each.

The diagrams in $B(\epsilon, \epsilon')$ are parameterized by $k!$ possible matchings of the $2k$ oriented endpoints, a sequence of k nonnegative integers determining the number of dots on each interval, and by a finite sequence of nonnegative integers determining the number of counterclockwise circles with various numbers of dots.

Below is an example of an element of $B(-++--, +--+-+-)$.

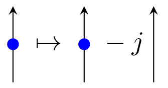


Proposition 2.16. *For any sign sequences ϵ, ϵ', the set $B(\epsilon, \epsilon')$ is a $\Bbbk$-basis of* $\mathrm{Hom}_{\mathcal{H}^\lambda}(\mathsf{Q}_\epsilon, \mathsf{Q}_{\epsilon'})$.

Proof. The proof is almost identical to that of [16, Prop. 5] and so will be omitted. (See also [21, Prop. 3.11].) □

Proposition 2.17. *Fix $j \in I$ and define $\mu = \sum_{i \in I} \mu_i \omega_i \in P_+$ by $\mu_i = \lambda_{i-j}$ for $i \in I$. Then the categories $\mathcal{H}^\lambda$ and $\mathcal{H}^\mu$ are isomorphic.*

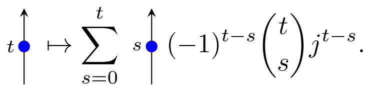
Proof. We claim that there is a functor $\Psi\colon \mathcal{H}^\lambda \to \mathcal{H}^\mu$ given by



(2.20)

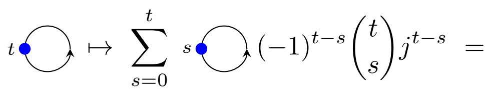
and leaving all caps, cups and crossings unchanged. It is clear that Ψ is invertible and so it is enough to show that it is well-defined by proving that it preserves the local relations. It preserves (2.3), (2.4), (2.7), and (2.9), because those relations do not involve dots (or dual dots). Furthermore, it is straightforward to check that relation (2.5) is also preserved.

As for relation (2.8), note that



(2.21)

Therefore, we have



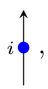
$$= \begin{cases} 0 & \text{if } t < d-1, \\ 1 & \text{if } t = d-1, \\ \sum_{i \in I} i\lambda_i & \text{if } t = d, \end{cases}$$

where the case $t = d$ follows from

$$\sum_{i \in I} i\mu_i - jd = \sum_{i \in I} i\lambda_{i-j} - j \sum_{i \in I} \lambda_i = \sum_{i \in I} (i+j)\lambda_i - j \sum_{i \in I} \lambda_i = \sum_{i \in I} i\lambda_i.$$

(The other cases are immediate.) This shows that (2.8) is preserved by Ψ.

It remains to prove that (2.6) is preserved. Let V be the free Π-module spanned by upwards pointing strands with at most $d-1$ dots. The Π-action is via placing bubbles to the right of the strand. We have a Π-bilinear pairing $V \times V \to \Pi$ given by the usual vertical composition in our category (i.e. stacking diagrams), followed by closing off to the left. If we choose the Π-basis



$, \quad i = 0, 1, \dots, d-1,$

then the matrix for the pairing is unitriangular. By Lemma 2.5, the dual dots are uniquely obtained by inverting this matrix. Since the pairing only involves (2.8), which we have already shown is invariant under the functor, it follows that the sum in (2.6) is preserved. Hence (2.6) is preserved under Ψ. $\square$

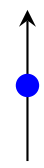
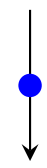
In light of Proposition 2.17, it is natural to ask if the categories $\mathcal{H}^\lambda$ and $\mathcal{H}^\mu$ are isomorphic more generally for λ and μ both of level d. We do not currently know the answer to this question. The explicit dependence on λ appears only in the $j = d$ case of (2.8).[1]

3. A filtration and the associated graded category

In Section 3.1 we let $\Bbbk$ be an arbitrary commutative ring, while in Section 3.2 we assume that $\Bbbk$ is a field of characteristic zero.

3.1. Definition

We define a filtration on the hom-spaces of $\tilde{\mathcal{H}}^\lambda$ as follows. We define

deg  $= 1 - d$, deg  $= d - 1$, deg  $=$ deg  $= 1.$

Right cups/caps and crossings are assigned degree zero. This determines the degree of any planar diagram. Then, for $\mathsf{x}, \mathsf{y} \in \operatorname{Ob} \mathcal{H}^\lambda$ and $k \in \mathbb{Z}$, we define $\operatorname{Hom}^{\le k}_{\tilde{\mathcal{H}}^\lambda}(\mathsf{x}, \mathsf{y})$ to be the span of the diagrams of degree less than or equal to k. This yields a filtration

$$\cdots \subseteq \operatorname{Hom}^{\le k-1}_{\tilde{\mathcal{H}}^\lambda}(\mathsf{x}, \mathsf{y}) \subseteq \operatorname{Hom}^{\le k}_{\tilde{\mathcal{H}}^\lambda}(\mathsf{x}, \mathsf{y}) \subseteq \operatorname{Hom}^{\le k+1}_{\tilde{\mathcal{H}}^\lambda}(\mathsf{x}, \mathsf{y}) \subseteq \cdots$$

of $\operatorname{Hom}_{\tilde{\mathcal{H}}^\lambda}(\mathsf{x}, \mathsf{y})$. This filtration of the hom-spaces clearly respects composition:

$$\operatorname{Hom}^{\le k}_{\tilde{\mathcal{H}}^\lambda}(\mathsf{y}, \mathsf{z}) \times \operatorname{Hom}^{\le \ell}_{\tilde{\mathcal{H}}^\lambda}(\mathsf{x}, \mathsf{y}) \to \operatorname{Hom}^{\le k+\ell}_{\tilde{\mathcal{H}}^\lambda}(\mathsf{x}, \mathsf{z}), \quad \mathsf{x}, \mathsf{y}, \mathsf{z} \in \operatorname{Ob} \tilde{\mathcal{H}}^\lambda.$$

We thus have the associated graded category $\tilde{\mathcal{H}}^\lambda_{\mathrm{gr}}$ defined by $\operatorname{Ob} \tilde{\mathcal{H}}^\lambda_{\mathrm{gr}} = \operatorname{Ob} \tilde{\mathcal{H}}^\lambda$, and

$$\operatorname{Hom}_{\tilde{\mathcal{H}}^\lambda_{\mathrm{gr}}}(\mathsf{x}, \mathsf{y}) = \operatorname{gr} \operatorname{Hom}_{\tilde{\mathcal{H}}^\lambda}(\mathsf{x}, \mathsf{y}), \quad \text{for all } \mathsf{x}, \mathsf{y} \in \operatorname{Ob} \tilde{\mathcal{H}}^\lambda_{\mathrm{gr}},$$

where $\operatorname{gr} \operatorname{Hom}_{\tilde{\mathcal{H}}^\lambda}(\mathsf{x}, \mathsf{y})$ denotes the associated graded space. The composition in $\tilde{\mathcal{H}}^\lambda_{\mathrm{gr}}$ is induced from the one in $\tilde{\mathcal{H}}^\lambda$. We will use the symbol $\mathrel{\mathring{=}}$ to denote equality of morphisms in $\tilde{\mathcal{H}}^\lambda_{\mathrm{gr}}$. The morphism spaces of $\tilde{\mathcal{H}}^\lambda_{\mathrm{gr}}$ are of course graded:

$$\operatorname{Hom}_{\tilde{\mathcal{H}}^\lambda_{\mathrm{gr}}}(\mathsf{x}, \mathsf{y}) = \bigoplus_{k \in \mathbb{Z}} \operatorname{Hom}^k_{\tilde{\mathcal{H}}^\lambda_{\mathrm{gr}}}(\mathsf{x}, \mathsf{y}), \qquad \operatorname{Hom}^k_{\tilde{\mathcal{H}}^\lambda_{\mathrm{gr}}}(\mathsf{x}, \mathsf{y}) := \operatorname{Hom}^{\le k}_{\tilde{\mathcal{H}}^\lambda}(\mathsf{x}, \mathsf{y}) / \operatorname{Hom}^{\le k-1}_{\tilde{\mathcal{H}}^\lambda}(\mathsf{x}, \mathsf{y}).$$

Lemma 3.1. *If $d > 1$, then*

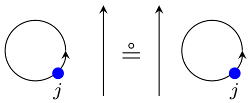
$$\operatorname{End}^{\le k}_{\tilde{\mathcal{H}}^\lambda}(\mathsf{Q}_+^n \mathsf{Q}_-^m) = 0 \quad \textit{for } k < 0, \quad \textit{and} \tag{3.1}$$

$$\operatorname{End}^{\le 0}_{\tilde{\mathcal{H}}^\lambda}(\mathsf{Q}_+^n \mathsf{Q}_-^m) \cong \Bbbk S_n \otimes \Bbbk S_m. \tag{3.2}$$

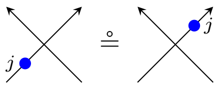
[1] After the current paper appeared, Brundan defined a Heisenberg category in [6] that omits the $j = d$ case of (2.8) and therefore depends only on the level d.

Proof. This follows immediately from Proposition 2.16. □

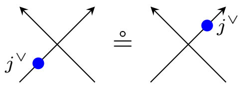
Lemma 3.2. *We have the following equalities in* $\widetilde{\mathcal{H}}^{\lambda}_{\mathrm{gr}}$*:*



$, \quad j \in \mathbb{N}, \qquad (3.3)$



$, \quad j \in \mathbb{N}, \qquad (3.4)$



$, \quad 0 \le j \le d-1. \qquad (3.5)$

Proof. For $j \le d$, (3.3) follows immediately from (2.8). For $j > d$, it follows from Lemma 2.12 after noting that the terms in the sum there have degree $t-2$, while a counterclockwise circle with $d-1+t$ dots has degree t.

The relation (3.4) follows from (2.10). Then (3.5) follows from (3.3) and (3.4). □

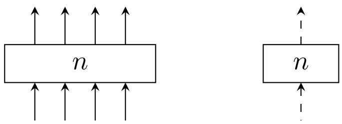
3.2. Symmetrizer relations

Through this subsection, we assume $\Bbbk$ is a field of characteristic zero.

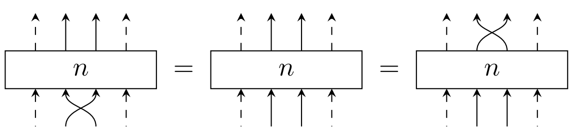
For $n \in \mathbb{N}_+$ and μ a partition of n, we have the corresponding minimal idempotent $e_\mu \in \Bbbk S_n \subseteq H_n$. In particular,

$$e_{(n)} = \frac{1}{n!} \sum_{w \in S_n} w$$

is the complete symmetrizer. Via the homomorphism ψ_n of (2.19), we view e_μ as an element of $\mathrm{End}_{\mathcal{H}^\lambda}(\mathsf{Q}_+^n)$ and, via adjunction, as an element of $\mathrm{End}_{\mathcal{H}^\lambda}(\mathsf{Q}_-^n)$. In the case $\mu = (n)$, we denote these idempotents by a white box labeled n across n strands. We sometimes use a dashed strand to denote multiple strands, when the number of strands is clear from the diagram.

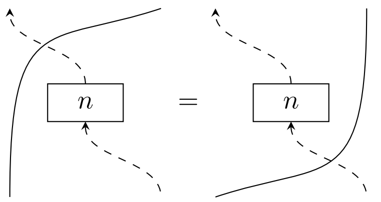


Symmetrizers absorb crossings:



(3.6)

It also follows from (2.3) that symmetrizers pass through crossings (for either orientation of the solid strand):



Corresponding to the idempotent e_μ, for μ a partition of n, we have the object

$$\mathsf{Q}_\pm^\mu := (\mathsf{Q}_\pm^n, e_\mu) \in \operatorname{Ob} \mathcal{H}^\lambda.$$

By convention, we set $\mathsf{Q}_\pm^\varnothing = \mathbf{1}$.

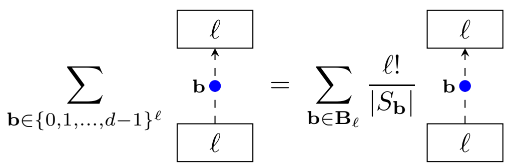
If $\mathbf{b} = (b_1, b_2, \dots, b_\ell) \in \{0, 1, \dots, d-1\}^\ell$, we define

$$S_\mathbf{b} = \{w \in S_\ell \mid b_{w(i)} = b_i \text{ for all } 1 \le i \le \ell\}.$$

For $\ell \in \mathbb{N}_+$, let

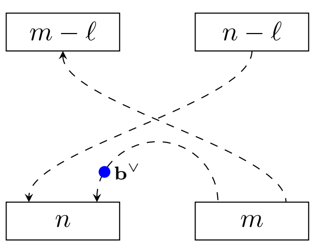
$$\mathbf{B}_\ell = \{(b_1, b_2, \dots, b_\ell) \mid 0 \le b_1 \le b_2 \le \cdots b_\ell \le d-1\}.$$

Note that



(3.7)

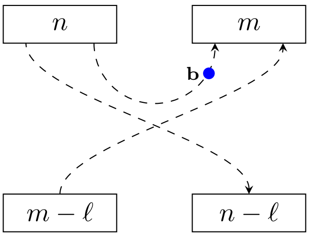
For $\mathbf{b} = (b_1, \dots, b_\ell) \in \{0, 1, \dots, d-1\}^\ell$, define

$\alpha_{\mathbf{b}^\vee} =$


$: \mathsf{Q}_-^{(n)} \otimes \mathsf{Q}_+^{(m)} \to \mathsf{Q}_+^{(m-\ell)} \otimes \mathsf{Q}_-^{(n-\ell)}$ (3.8)

and

$$\beta_{\mathbf{b}} =$$



$$: \mathsf{Q}_+^{(m-\ell)} \otimes \mathsf{Q}_-^{(n-\ell)} \to \mathsf{Q}_-^{(n)} \otimes \mathsf{Q}_+^{(m)}. \tag{3.9}$$

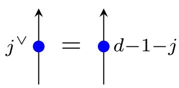
Lemma 3.3. *Suppose* $0 \le k, \ell \le \min\{m,n\}$. *Then, in* $\mathcal{H}_{\mathrm{gr}}^{\lambda}$, *we have*

$$\alpha_{\mathbf{c}^\vee} \circ \beta_{\mathbf{b}} \doteq \delta_{\mathbf{c},\mathbf{b}} \frac{|S_{\mathbf{b}}|(n-k)!(m-k)!}{n!m!} \mathrm{id}_{\mathsf{Q}_+^{(m-k)}\mathsf{Q}_-^{(n-k)}} \quad \textit{for all } \mathbf{b} \in \mathbf{B}_\ell,\ \mathbf{c} \in \mathbf{B}_k.$$

Proof. Using Lemma 2.5, the proof is almost identical to the analogous argument in the proof of [25, Th. 9.2] and so will be omitted. □

Proposition 3.4. *The category obtained from* $\tilde{\mathcal{H}}_{\mathrm{gr}}^{\lambda}$ *by imposing the extra local relation that* d *dots is equal to zero is isomorphic to the Heisenberg category* $\mathcal{H}'_B$ *of [25, §6] with* $B = \Bbbk[x]/(x^d)$ *and trace map* $\mathrm{tr}_B : B \to \Bbbk$ *given by* $\mathrm{tr}(x^j) = \delta_{j,d-1}$.

Proof. Let $\mathcal{C}$ be the category obtained from $\tilde{\mathcal{H}}_{\mathrm{gr}}^{\lambda}$ by imposing the extra relation that d dots is equal to zero. It follows from (2.1) that $c_s = 0$ in $\mathcal{C}$ for all $s \ge 1$. Thus, (2.2) gives



Therefore, the local relations in $\mathcal{C}$ become precisely the defining local relations of $\mathcal{H}'_B$ (see [25, (6.9)–(6.16)]), where the dot in $\mathcal{C}$ corresponds to a dot labeled x in $\mathcal{H}'_B$. □

4. The Grothendieck ring

Throughout this section we assume that $\Bbbk$ is a field of characteristic zero.

4.1. The Heisenberg algebra

Recall that $d = \sum_{i \in I} \lambda_i$ is a positive integer. Let $\mathrm{Sym}_{\mathbb{Q}}^+$ and $\mathrm{Sym}_{\mathbb{Q}}^-$ be two copies of the Hopf algebra of symmetric functions over $\mathbb{Q}$. Consider the Hopf pairing

$$\langle -,- \rangle_d : \mathrm{Sym}_{\mathbb{Q}}^- \times \mathrm{Sym}_{\mathbb{Q}}^+ \to \mathbb{Q}, \quad \langle p_n^-, p_m^+ \rangle_d = nd\delta_{n,m},$$

where $p_n^\pm$ denotes the n-th power sum in $\mathrm{Sym}^\pm$. (The Hopf pairing is uniquely determined by its values on these elements.) One can show that the pairing of two complete symmetric functions is an integer. (This follows, for example, by comparing the coefficients appearing in [28, Th. 5.3] to [28, (2.2)].) Thus we can restrict to obtain a bilinear form

$\mathrm{Sym}_{\mathbb{Z}}^- \times \mathrm{Sym}_{\mathbb{Z}}^+ \to \mathbb{Z}$, where $\mathrm{Sym}_{\mathbb{Z}}^\pm$ are copies of the Hopf algebra of symmetric functions over $\mathbb{Z}$.

The *level d Heisenberg algebra* is the Heisenberg double $\mathfrak{h}_d := \mathrm{Sym}_{\mathbb{Z}}^+ \# \mathrm{Sym}_{\mathbb{Z}}^-$. One can obtain a presentation by choosing any sets of generators of $\mathrm{Sym}_{\mathbb{Z}}^\pm$. Choosing the complete symmetric functions $h_n^\pm$, $n \in \mathbb{N}$, we obtain that $\mathfrak{h}_d$ is the unital associative $\mathbb{Z}$-algebra generated by $h_n^\pm$, $n \in \mathbb{N}$, subject to the relations $h_0^+ = h_0^- = 1$ and

$$h_n^+ h_m^+ = h_m^+ h_n^+, \quad h_n^- h_m^- = h_m^- h_n^-,$$
$$h_n^- h_m^+ = \sum_{r=0}^{\min\{m,n\}} \binom{d+r-1}{r} h_{m-r}^+ h_{n-r}^-, \quad n, m \in \mathbb{N}. \tag{4.1}$$

See [28, Th. 5.3] and [20, Prop. A.1]. (Readers not familiar with the Heisenberg double construction may take the presentation (4.1) as the definition of $\mathfrak{h}_d$.)

4.2. Decomposition of the identity

Fix $m, n \in \mathbb{N}$ and define $\beta_{\mathbf{b}}$ as in (3.9).

Lemma 4.1. *There exist*

$$\theta_{\mathbf{b}} \colon \mathsf{Q}_-^{(n)} \otimes \mathsf{Q}_+^{(m)} \to \mathsf{Q}_+^{(n-k)} \otimes \mathsf{Q}_-^{(n-k)}, \quad \mathbf{b} \in \mathbf{B}_k,\ 0 \le k \le \min\{m, n\},$$

such that

$$\theta_{\mathbf{c}} \circ \beta_{\mathbf{b}} = \delta_{\mathbf{c},\mathbf{b}} \operatorname{id}_{\mathsf{Q}_+^{(m-k)} \mathsf{Q}_-^{(n-k)}} \quad \textit{for all } \mathbf{b} \in \mathbf{B}_\ell,\ \mathbf{c} \in \mathbf{B}_k,\ 0 \le k, \ell \le \min\{m, n\}. \tag{4.2}$$

Proof. For all $\mathbf{b} \in \mathbf{B}_\ell$, define

$$\theta'_{\mathbf{b}} = \frac{n!m!}{|S_{\mathbf{b}}|(n-k)!(m-k)!} \alpha_{\mathbf{b}^\vee}.$$

By Lemma 3.3, we have

$$\theta'_{\mathbf{c}} \circ \beta_{\mathbf{b}} - \delta_{\mathbf{c},\mathbf{b}} \operatorname{id}_{\mathsf{Q}_+^{(m-k)} \mathsf{Q}_-^{(n-k)}} \in \mathrm{End}_{\mathcal{H}^\lambda}^{\le \deg \mathbf{b} - \deg \mathbf{c} - 1} \left(\mathsf{Q}_+^{(m-k)} \mathsf{Q}_-^{(n-k)} \right).$$

It follows from Lemma 3.1 that $\theta'_{\mathbf{c}} \circ \beta_{\mathbf{b}} = \delta_{\mathbf{c},\mathbf{b}} \operatorname{id}_{\mathsf{Q}_+^{(m-k)} \mathsf{Q}_-^{(n-k)}}$ if $\deg \mathbf{b} \le \deg \mathbf{c}$. The desired $\theta_{\mathbf{c}}$ then exist by the usual the usual process of inverting a unitriangular matrix. $\square$

Proposition 4.2. *In $\mathcal{H}^\lambda$, we have*

$$\operatorname{id}_{\mathsf{Q}_-^{(n)} \mathsf{Q}_+^{(m)}} = \sum_{k=0}^{\min\{m,n\}} \sum_{\mathbf{b} \in \mathbf{B}_k} \beta_{\mathbf{b}} \circ \theta_{\mathbf{b}}. \tag{4.3}$$

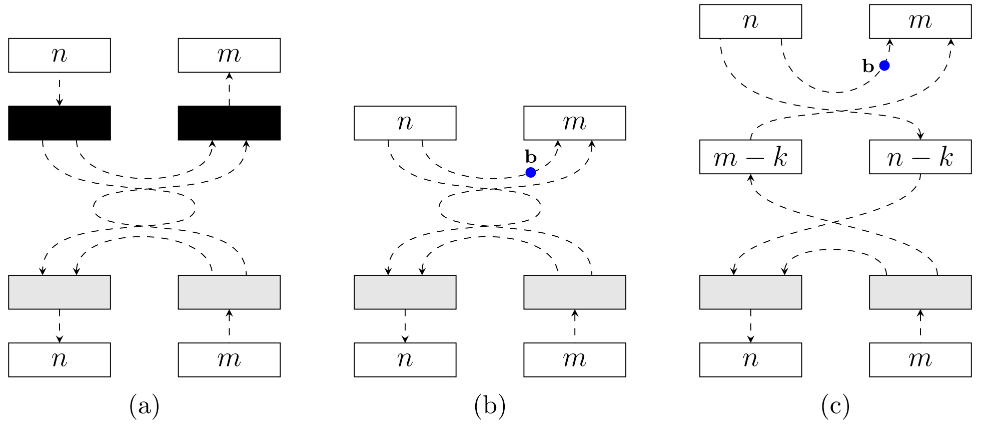


Fig. 1. Diagrams involved in the proof of Proposition 4.2.

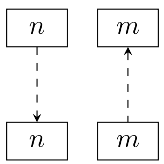
Furthermore,

$$\beta_{\mathbf{b}} \circ \theta_{\mathbf{b}}, \quad \mathbf{b} \in \mathbf{B}_k,\ 1 \le k \le \min\{m, n\},$$

are orthogonal idempotents in $\mathrm{End}_{\mathcal{H}^\lambda}(\mathsf{Q}_-^{(n)}\mathsf{Q}_+^{(m)})$.

Proof. Starting with

$$\mathrm{id}_{\mathsf{Q}_-^{(n)}\mathsf{Q}_+^{(m)}} =$$



,

we repeatedly use (2.6) to pull the upward strands left through the downward strands. This yields a linear combination of diagrams of the form depicted in Fig. 1(a), where

- the black boxes represent some linear combinations of diagrams involving crossings and dots, with dots only appearing on the strands involved in the cups, and
- where the gray boxes represent some linear combinations of diagrams involving crossings and dual dots.

Then, using (2.5) and the fact that crossings can be absorbed into symmetrizers, we can assume that the black boxes do not involve any crossings and the total number of dots on the cups weakly increases as we move down. Thus, we see that $\mathrm{id}_{\mathsf{Q}_-^{(n)}\mathsf{Q}_+^{(m)}}$ can be written as a linear combination of diagrams of the form depicted in Fig. 1(b), where $\mathbf{b} \in \mathbf{B}_k$ for some $0 \le k \le \min\{m, n\}$, and the gray boxes represent some linear combinations of diagrams involving crossings and dual dots. Then, using the fact that smaller symmetrizers can be absorbed into larger ones, and that symmetrizers slide through crossings, we see that

$\mathrm{id}_{\mathsf{Q}_-^{(n)}\mathsf{Q}_+^{(m)}}$ can be written as a linear combination of diagrams of the form depicted in Fig. 1(c), where $0 \le k \le \min\{m,n\}$, $\mathbf{b} \in \mathbf{B}_k$, and the gray boxes represent some linear combinations of diagrams involving crossings and dual dots.

The above shows that we can write

$$\mathrm{id}_{\mathsf{Q}_-^{(n)}\mathsf{Q}_+^{(m)}} = \sum_{k=0}^{\min\{m,n\}} \sum_{\mathbf{b}\in\mathbf{B}_k} \beta_{\mathbf{b}} \circ \theta''_{\mathbf{b}},$$

for some $\theta''_{\mathbf{b}} \in \mathrm{Hom}_{\mathcal{H}^\lambda}(\mathsf{Q}_-^{(n)}\mathsf{Q}_+^{(m)}, \mathsf{Q}_+^{(n-k)}\mathsf{Q}_-^{(n-k)})$, $\mathbf{b} \in \mathbf{B}_k$, $0 \le k \le \min\{m,n\}$. Composing on the left with $\theta_{\mathbf{c}}$ and using Lemma 4.1 then yields that $\theta_{\mathbf{b}} = \theta''_{\mathbf{b}}$ for all $\mathbf{b}$. The final statement about orthogonal idempotents follows immediately from (4.2). $\square$

4.3. *Categorification of the higher level Heisenberg algebra*

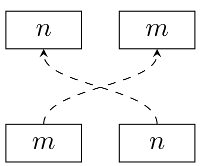
Proposition 4.3. *Suppose $n, m \in \mathbb{N}$. In $\mathcal{H}^\lambda$, we have*

$$\mathsf{Q}_+^{(n)}\mathsf{Q}_+^{(m)} \cong \mathsf{Q}_+^{(m)}\mathsf{Q}_+^{(n)}, \qquad \mathsf{Q}_-^{(n)}\mathsf{Q}_-^{(m)} \cong \mathsf{Q}_-^{(m)}\mathsf{Q}_-^{(n)}, \tag{4.4}$$

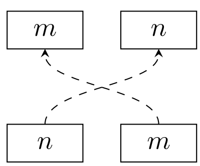
$$\mathsf{Q}_-^{(n)}\mathsf{Q}_+^{(m)} \cong \sum_{k=0}^{\min\{m,n\}} \left(\mathsf{Q}_+^{(m-k)}\mathsf{Q}_-^{(n-k)}\right)^{\oplus\binom{d+k-1}{k}}, \tag{4.5}$$

where, by convention, $\mathsf{Q}_\pm^{(0)} = \mathbf{1}$.

Proof. Using the fact that symmetrizers slide through crossings, it is straightforward to verify that the morphisms



and



are mutually inverse, giving the first isomorphism in (4.4). The second is similar, simply reversing orientations of strands.

The isomorphism (4.5) follows immediately from Proposition 4.2, after noting that $|\mathbf{B}_k|$ is the dimension of the k-th symmetric power of a vector space of dimension d, and is therefore equal to $\binom{d+k-1}{k}$. $\square$

Let $K_0(\mathcal{H}^\lambda)$ be the split Grothendieck group of $\mathcal{H}^\lambda$. The monoidal structure on $\mathcal{H}^\lambda$ endows $K_0(\mathcal{H}^\lambda)$ with the structure of a ring.

Theorem 4.4. *We have an injective ring homomorphism*

$$\mathfrak{h}_d \hookrightarrow K_0(\mathcal{H}^\lambda), \quad s_\mu^\pm \mapsto \mathsf{Q}_\pm^\mu, \quad \mu \text{ a partition}, \tag{4.6}$$

where $s_\mu^\pm$ denotes the Schur function in $\mathrm{Sym}^\pm$ *corresponding to the partition μ. In particular, this ring homomorphism maps $h_n^\pm$ to $\mathsf{Q}_\pm^{(n)}$ for $n \in \mathbb{N}_+$.*

Proof. Recall that $s_{(n)}^\pm = h_n^\pm$. Comparing (4.1) and Proposition 4.3, we see that

$$\mathfrak{h}_d \to K_0(\mathcal{H}^\lambda), \quad h_n^\pm \mapsto \mathsf{Q}_\pm^{(n)}, \quad n \in \mathbb{N}_+$$

is a well-defined ring homomorphism. We will show in Proposition 6.8 that it is injective. An argument analogous to the one in the proof of [20, Th. 4.5], based on the fact that the expression for $\mathsf{Q}_\pm^\mu$ as a linear combination of products of $\mathsf{Q}_\pm^{(n)}$ is given by the Giambelli rules in the ring of symmetric functions for expressing the Schur functions in terms of the complete homogeneous symmetric functions, then shows that (4.6) holds for all partitions μ. $\square$

Conjecture 4.5. *The map* (4.6) *is an isomorphism.*

If $d = 1$, Conjecture 4.5 reduces to [16, Conj. 1]. In [25, Th. 10.5], it is possible to prove the analogue of Conjecture 4.5 because of the presence of a nontrivial grading coming from a Frobenius algebra. This is also the case in [10, Th. 1]. The difficulty in the setting of the current paper is that we only have a filtration, and not a grading.

5. Degenerate cyclotomic Hecke algebras

Our next goal is to define an action of the category $\mathcal{H}^\lambda$ on categories of modules for degenerate cyclotomic Hecke algebras. In this section, we recall these algebras and prove some results concerning them that we will need to show our action is well defined. The action will then be defined in Section 6.

Throughout this section, $\Bbbk$ is an arbitrary commutative ring. We will cite [19] for various basic results. Even though that reference works over a field of characteristic zero, one can check that, for all of the results we cite, the proofs go through in the more general setting of an arbitrary commutative ring.

5.1. Definitions

Definition 5.1 *(Degenerate affine Hecke algebra H_n).* Let $n \in \mathbb{N}$. The *degenerate affine Hecke algebra H_n* is the $\Bbbk$-algebra generated by the elements $s_1, \dots, s_{n-1}$ and $x_1, \dots, x_n$, subject to the relations

$$s_i^2 = 1, \tag{5.1}$$

$$s_i s_j = s_j s_i, \quad |i - j| > 1, \tag{5.2}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \tag{5.3}$$

$$s_j x_i = x_i s_j, \quad i \neq j, j+1, \tag{5.4}$$

$$s_i x_i = x_{i+1} s_i - 1, \tag{5.5}$$

$$x_i x_j = x_j x_i. \tag{5.6}$$

By convention, $H_0 = \Bbbk$ and $H_1 = \Bbbk[x_1]$.

Any result for H_n in the rest of the paper is understood to hold for all $n \in \mathbb{N}$ unless explicitly stated otherwise.

Note that in H_n we also have, for $1 \le i \le n$ and $t \ge 1$:

$$s_i x_{i+1} = x_i s_i + 1, \tag{5.7}$$

$$s_i x_i^t = x_{i+1}^t s_i - \sum_{a+b=t-1} x_i^a x_{i+1}^b, \tag{5.8}$$

$$s_i x_{i+1}^t = x_i^t s_i + \sum_{a+b=t-1} x_i^a x_{i+1}^b. \tag{5.9}$$

Recall the weight λ as defined at the beginning of Section 2.

Definition 5.2 *(Cyclotomic quotient H_n^λ).* Let I_n^λ be the *cyclotomic ideal* of H_n, which is the ideal generated by $\prod_{i \in I}(x_1 - i)^{\lambda_i}$. The *cyclotomic quotient* of H_n corresponding to λ is defined to be

$$H_n^\lambda := H_n / I_n^\lambda.$$

By convention, $I_0^\lambda = \{0\}$ and so $H_0^\lambda = H_0$.

One can define a cyclotomic quotient H_n^f for any monic polynomial $f \in \Bbbk[x_1, \dots, x_n]$ by replacing I_n^λ in Definition 5.2 with the ideal generated by f. However, as explained in [19, §7.1], it is sufficient to consider the cases where $f = \prod_{i \in I}(x_1 - i)^{\lambda_i}$ for some $\lambda \in P_+$. For this reason, we focus on such quotients in the current paper.

Note that $H_n \subseteq H_{n+1}$, for any $n \in \mathbb{N}$. Composing with the projection onto H_{n+1}^λ gives a homomorphism of algebras $H_n \to H_{n+1}^\lambda$, whose kernel is exactly I_n^λ. Therefore, we get an embedding of H_n^λ into H_{n+1}^λ. Via this embedding, we will view H_n^λ as a subalgebra of H_{n+1}^λ from now on.

Lemma 5.3 *([19, Lem. 7.6.1(i)]). We have that H_{n+1}^λ is a free left H_n^λ-module with basis*

$$\{s_n \cdots s_j x_j^a \mid 0 \le a < d,\ 1 \le j \le n+1\}, \tag{5.10}$$

and a free right H_n^λ-module with basis

$$\{x_j^a s_j \cdots s_n \mid 0 \le a < d,\ 1 \le j \le n+1\}. \tag{5.11}$$

By convention, $s_n \cdots s_j x_j^a$ and $x_j^a s_j \cdots s_n$ are equal to x_{n+1}^a when $j = n+1$.

Lemma 5.4 *([19, Lem. 7.6.1(ii,iii)]). As $(H_n^\lambda, H_n^\lambda)$-bimodules, we have*

$$H_{n+1}^\lambda = H_n^\lambda s_n H_n^\lambda \oplus \bigoplus_{j=0}^{d-1} H_n^\lambda x_{n+1}^j, \tag{5.12}$$

$$H_n^\lambda s_n H_n^\lambda \cong H_n^\lambda \otimes_{H_{n-1}^\lambda} H_n^\lambda, \quad u s_n v \mapsto u \otimes v, \tag{5.13}$$

$$H_n^\lambda x_{n+1}^a \cong H_n^\lambda, \quad u x_{n+1}^a \mapsto u, \quad 0 \le a < d. \tag{5.14}$$

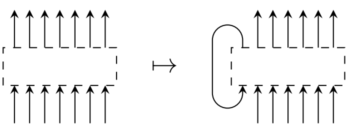
5.2. The trace map

Lemma 5.5 *([19, Lem. 7.7.2]). The algebra H_{n+1}^λ is a Frobenius extension of H_n^λ, with the nondegenerate trace*

$$\mathrm{tr}_{n+1} \colon H_{n+1}^\lambda \to H_n^\lambda$$

being defined by composing the projection onto $H_n^\lambda x_{n+1}^{d-1}$ in (5.12) with the isomorphism $H_n^\lambda x_{n+1}^{d-1} \cong H_n^\lambda$ in (5.14).

At the end of this section we will show that the trace is not symmetric. Diagrammatically, the map tr_{n+1} corresponds to the operation of taking a diagram on n upward strands and closing off the leftmost strand to the left:



Lemma 5.6. *We have*

$$\mathrm{tr}_{n+1}(x_{n+1} z) = \mathrm{tr}_{n+1}(z x_{n+1}) \quad \text{for all } z \in H_{n+1}^\lambda.$$

Proof. Fix $z \in H_{n+1}^\lambda$. By (5.12), we have

$$z = \sum_{j=0}^{\ell} h_j' s_n h_j'' + \sum_{k=0}^{d-1} h_k x_{n+1}^k,$$

for some $h_j', h_j'', h_k \in H_n^\lambda$, $\ell \in \mathbb{N}$. Then, since x_{n+1} commutes with H_n^λ, we have

$$\begin{aligned} \mathrm{tr}_{n+1}(x_{n+1} z) - \mathrm{tr}_{n+1}(z x_{n+1}) &= \mathrm{tr}_{n+1}(x_{n+1} z - z x_{n+1}) \\ &= \mathrm{tr}_{n+1}\left(\sum_{j=0}^{\ell} h_j'(x_{n+1} s_n - s_n x_{n+1}) h_j'' \right) \overset{(5.5)}{\underset{(5.7)}{=}} \mathrm{tr}_{n+1}\left(\sum_{j=0}^{\ell} h_j'(s_n x_n - x_n s_n) h_j'' \right) = 0, \end{aligned}$$

where the last equality follows from the fact that $h'_j(s_n x_n - x_n s_n)h''_j \subseteq H_n^\lambda s_n H_n^\lambda$ for all j. □

As noted in [19, Cor. 7.7.4], we can use Lemma 5.5 recursively to see that H_{n+1}^λ is a Frobenius algebra over $\Bbbk$, with nondegenerate trace being defined by composing the projection onto $\Bbbk x_1^{d-1} \cdots x_{n+1}^{d-1}$ with the isomorphism of vector spaces $\Bbbk x_1^{d-1} \cdots x_{n+1}^{d-1} \cong \Bbbk$. However, we will not need this fact here.

The trace induces an H_n^λ-valued bilinear form on H_{n+1}^λ, defined by

$$\langle u, v\rangle_{n+1} := \mathrm{tr}_{n+1}(uv). \tag{5.15}$$

Our next goal is to find a basis for H_{n+1}^λ as an H_n^λ-module that is left dual to the basis (5.10) with respect to the form (5.15). For $n \in \mathbb{N}_+$ and $k \in \{0, \dots, d-1\}$, define

$$y_{n,k} := \sum_{t=k}^{d-1} (-1)^{d-1-t} x_n^{t-k} \det\left(\mathrm{tr}_n(x_n^{d+j-i})\right)_{i,j=1,\dots,d-1-t}, \tag{5.16}$$

where we take the determinant above to be equal to one when $t = d-1$. Under the action to be defined in Section 6, the elements $y_{n,k}$ will correspond to the dual dots in the category $\mathcal{H}^\lambda$.

Lemma 5.7. *The set*

$$\{s_n \cdots s_i y_{i,k} \mid i = 1, \dots, n+1,\ k = 0, \dots, d-1\} \tag{5.17}$$

is a basis of H_{n+1}^λ as a left H_n^λ-module. Here we adopt the convention that, for $i = n+1$, the notation $s_n \cdots s_i y_{i,k}$ means $y_{n+1,k}$ for $k = 0, \dots, d-1$.

Proof. This follows from the fact that the transition matrix between the basis (5.10) and the set

$$\{s_n \cdots s_i y_{i,d-1-k} \mid i = 1, \dots, n+1,\ k = 0, \dots, d-1\}$$

is triangular unipotent by (5.16). □

Lemma 5.8. *We have*

$$\mathrm{tr}_{n+1}(gs_n) = \mathrm{tr}_{n+1}(s_n g) = 0 \quad \textit{for all} \quad g \in \bigoplus_{m=0}^{d-1} H_n^\lambda x_{n+1}^m.$$

Proof. Let $g = \sum_{m=0}^{d-1} a_m x_{n+1}^m$, for some $a_m \in H_n^\lambda$. We prove $\mathrm{tr}_{n+1}(s_n g) = 0$, the proof of $\mathrm{tr}_{n+1}(gs_n) = 0$ being similar. Note that x_{n+1} commutes with the a_m, $m \in \{0, \dots, m-1\}$, because the latter all belong to H_n^λ by assumption. So we have

$$s_n g = \sum_{m=0}^{d-1} s_n a_m x_{n+1}^m = \sum_{m=0}^{d-1} s_n x_{n+1}^m a_m \overset{(5.9)}{=} \sum_{m=0}^{d-1} \left(x_n^m s_n a_m + \sum_{a+b=m-1} x_n^a x_{n+1}^b a_m \right)$$
$$= \sum_{m=0}^{d-1} \left(x_n^m s_n a_m + \sum_{a+b=m-1} x_n^a a_m x_{n+1}^b \right) \in H_n^\lambda s_n H_n^\lambda \oplus \bigoplus_{j=0}^{d-2} H_n^\lambda x_{n+1}^j.$$

Therefore the projection onto $H_n^\lambda x_{n+1}^{d-1}$ is zero and the result follows. $\square$

Lemma 5.9. *We have*

$$s_n H_{n-1}^\lambda s_{n-1} H_{n-1}^\lambda s_n \subseteq H_n^\lambda s_n H_n^\lambda, \quad \textit{and}$$
$$s_n H_{n-1}^\lambda x_n^a s_n \subseteq H_n^\lambda s_n H_n^\lambda \oplus \bigoplus_{0 \le b \le a} H_n^\lambda x_{n+1}^b, \quad 0 \le a \le d-1.$$

Proof. Since s_n commutes with H_{n-1}^λ, the first inclusion follows from the braid relation (5.3). The second inclusion follows from the fact that $H_{n-1}^\lambda x_n^a s_n = x_n^a s_n H_{n-1}^\lambda$, and that

$$s_n x_n^a s_n = x_{n+1}^a - \sum_{u+v=a-1} x_n^u x_{n+1}^v s_n = x_{n+1}^a - \sum_{u+v=a-1} \left(x_n^u s_n x_n^v + \sum_{s+t=v-1} x_n^{u+s} x_{n+1}^t \right)$$

by using (5.8) twice. $\square$

Corollary 5.10. *For any $y \in H_n^\lambda$ we have*

$$\mathrm{tr}_{n+1}(s_n y s_n) = \mathrm{tr}_n(y).$$

Proof. If $\mathrm{tr}_n(y) = 0$, then $y \in H_{n-1}^\lambda s_{n-1} H_{n-1}^\lambda \oplus \bigoplus_{j=0}^{d-2} H_{n-1}^\lambda x_n^j$. Therefore, Lemma 5.9 implies

$$s_n y s_n \in H_n^\lambda s_n H_n^\lambda \oplus \bigoplus_{j=0}^{d-2} H_n^\lambda x_{n+1}^b,$$

so $\mathrm{tr}_{n+1}(s_n y s_n) = 0$.

The only remaining case to prove is when $y = y' x_n^{d-1}$, for some $y' \in H_{n-1}^\lambda$. In this case we have

$$s_n y s_n = s_n y' x_n^{d-1} s_n \overset{(5.9)}{=} y' x_{n+1}^{d-1} - \sum_{a+b=d-2} y' s_n x_n^a x_{n+1}^b$$
$$\overset{(5.9)}{=} y' x_{n+1}^{d-1} - \sum_{a+b=d-2} \left(y' x_n^b s_n x_n^a + \sum_{s+t=b-1} y' x_n^{a+s} x_{n+1}^t \right).$$

Therefore,

$$\operatorname{tr}_n(y) = y' = \operatorname{tr}_{n+1}(s_n y s_n). \quad \square$$

Proposition 5.11. *The basis*

$$\{s_n \cdots s_i y_{i,a} \mid i = 1, \dots, n+1,\ a = 0, \dots, d-1\}$$

is left dual to the basis (5.11) *with respect to* $\langle -, - \rangle_{n+1}$*. More precisely,*

$$\langle s_n \cdots s_i y_{i,a}, x_j^b s_j \cdots s_n \rangle_{n+1} = \delta_{i,j}\delta_{a,b}, \quad i, j \in \{1, \dots, n+1\}, \quad a, b \in \{0, \dots, d-1\}.$$

Proof. We first show that

$$\langle y_{n+1,a}, x_{n+1}^b \rangle_{n+1} = \delta_{a,b} \quad \text{for all } n \in \mathbb{N},\ a, b \in \{0, \dots, d-1\}. \tag{5.18}$$

We have

$$\langle y_{n+1,a}, x_{n+1}^b \rangle_{n+1} = \sum_{t=a}^{d-1} (-1)^{d-1-t} \operatorname{tr}_{n+1}(x_{n+1}^{t+b-a}) \det \big(\operatorname{tr}_{n+1}(x_{n+1}^{d+j-i}) \big)_{i,j=1,\dots,d-1-t}.$$

If $b \leq a$, then $\operatorname{tr}_{n+1}(x_{n+1}^{t+b-a}) = 0$ unless $a = b$ and $t = d-1$, and so (5.18) follows immediately. Now suppose $b > a$. Setting $m = b - a$ and $s = d - 1 - t$ in the above sum, it suffices to prove that

$$\sum_{s=0}^{d-1-a} (-1)^s \operatorname{tr}_{n+1}(x_{n+1}^{d-1+m-s}) \det \big(\operatorname{tr}_{n+1}(x_{n+1}^{d+j-i}) \big)_{i,j=1,\dots,s} = 0.$$

By (5.12) and the definition of the trace in Lemma 5.5, the terms with $s > m$ are equal to zero. Since $b \leq d-1$, we have $m \leq d-1-a$. Therefore, it is enough to prove

$$\sum_{s=0}^{m} (-1)^s \operatorname{tr}_{n+1}(x_{n+1}^{d-1+m-s}) \det \big(\operatorname{tr}_{n+1}(x_{n+1}^{d+j-i}) \big)_{i,j=1,\dots,s} = 0. \tag{5.19}$$

But (5.19) follows from Lemma 2.3 with $a_k = \operatorname{tr}_{n+1}(x_{n+1}^{d+k})$, considered as elements of the center of H_n^λ. This completes the proof of (5.18).

The proof of the proposition will now proceed by induction with respect to $n \in \mathbb{N}$. The $n = 0$ case follows immediately from (5.18). Now suppose that $\{s_{n-1} \cdots s_i y_{i,a} \mid i = 1, \dots, n,\ a = 0, \dots, d-1\}$ is left dual to $\{x_i^a s_i \cdots s_{n-1} \mid i = 1, \dots, n,\ a = 0, \dots, d-1\}$ with respect to $\langle -, - \rangle_n$, and consider the bases $\{s_n \cdots s_i y_{i,a} \mid i = 1, \dots, n+1,\ a = 0, \dots, d-1\}$ and $\{x_i^a s_i \cdots s_n \mid i = 1, \dots, n+1,\ a = 0, \dots, d-1\}$.

By Corollary 5.10 we have

$$\langle s_n \cdots s_i y_{i,a}, x_j^b s_j \cdots s_n \rangle_{n+1} = \langle s_{n-1} \cdots s_i y_{i,a}, x_j^b s_j \cdots s_{n-1} \rangle_n,$$

for $1 \leq i,j \leq n$. By induction the latter is equal to $\delta_{i,j}\delta_{a,b}$. It remains to show the equalities

$$\begin{aligned} &\langle y_{n+1,a}, x_j^b s_j \cdots s_n \rangle_{n+1} = 0 \quad \text{and} \\ &\langle s_n \cdots s_j y_{j,a}, x_{n+1}^b \rangle_{n+1} = 0, \quad 1 \leq j \leq n,\ 0 \leq a,b \leq d-1. \end{aligned}$$

The first one follows from

$$\langle y_{n+1,a}, x_j^b s_j \cdots s_n \rangle_{n+1} = \mathrm{tr}_{n+1}(y_{n+1,a} x_j^b s_j \cdots s_n) = 0,$$

where the last equality holds by Lemma 5.8, with $g = y_{n+1,a} x_j^b s_j \cdots s_{n-1}$. Similarly, we have

$$\langle s_n \cdots s_j y_{j,a}, x_{n+1}^b \rangle_{n+1} = \mathrm{tr}_{n+1}(s_n \cdots s_j y_{j,a} x_{n+1}^b) = 0,$$

where the last equality holds by Lemma 5.8, with $g = s_{n-1} \cdots s_j y_{j,a} x_{n+1}^b$. □

Lemma 5.12. *For $n \geq 1$, we have*

$$\mathrm{tr}_n(x_n^d) = \sum_{i \in I} i\lambda_i.$$

Proof. We prove the result by induction on n. For $n = 1$, note that the equation $\prod_i (x_1 - i)_i^\lambda = 0$ allows us to write x_1^d as a polynomial in x_1 of degree at most $d-1$. Then $\mathrm{tr}_1(x_1^d)$ is the coefficient of x_1^{d-1} in this polynomial. This coefficient is clearly $\sum_{i \in I} i\lambda_i$.

Now, for $n \geq 1$, by Corollary 5.10 and (5.8), we have

$$\begin{aligned} \mathrm{tr}_n(x_n^d) &= \mathrm{tr}_{n+1}(s_n x_n^d s_n) \\ &= \mathrm{tr}_{n+1}\left(x_{n+1}^d - \sum_{a=0}^{d-1} x_n^{d-1-a} x_{n+1}^a s_n \right) \\ &= \mathrm{tr}_{n+1}\left(x_{n+1}^d - \sum_{a=0}^{d-1} x_n^{d-1-a} \left(s_n x_n^a + \sum_{b=0}^{a-1} x_n^{a-1-b} x_{n+1}^b \right)\right) \\ &= \mathrm{tr}_{n+1}\left(x_{n+1}^d\right) - \sum_{a=0}^{d-1} \mathrm{tr}_{n+1}\left(x_n^{d-1-a} s_n x_n^a\right) - \sum_{a=0}^{d-1}\sum_{b=0}^{a-1} \mathrm{tr}_{n+1}\left(x_n^{d-b-2} x_{n+1}^b\right) \\ &= \mathrm{tr}_{n+1}\left(x_{n+1}^d\right). \end{aligned}$$

This completes the proof of the inductive step. □

Note that the trace is not symmetric in general. For example, by (5.8), (5.9), and the fact that the trace is an $(H_n^\lambda, H_n^\lambda)$-bimodule map, we get

$$\mathrm{tr}_{n+1}\left(s_n x_{n+1}^{d+1} s_{n-1}\right) = \left(x_n + \sum_{i\in I} i\lambda_i\right) s_{n-1}$$

and

$$\mathrm{tr}_{n+1}\left(x_{n+1}^{d+1} s_{n-1} s_n\right) = \mathrm{tr}_{n+1}\left(s_{n-1} x_{n+1}^{d+1} s_n\right) = s_{n-1}\left(x_n + \sum_{i\in I} i\lambda_i\right).$$

But

$$\left(x_n + \sum_{i\in I} i\lambda_i\right) s_{n-1} \neq s_{n-1}\left(x_n + \sum_{i\in I} i\lambda_i\right)$$

because $x_n s_{n-1} = s_{n-1}x_{n-1} + 1 \neq s_{n-1}x_n$. If $s_{n-1}x_{n-1} + 1 = s_{n-1}x_n$ were true, multiplying all terms by s_{n-1} would give $x_{n-1} + s_{n-1} = x_n$, which contradicts (5.12). Note that the full trace map

$$\mathrm{tr}_1 \circ \mathrm{tr}_2 \circ \cdots \circ \mathrm{tr}_n \colon H_n^\lambda \to \Bbbk$$

is symmetric. See [2, Th. A.2].

5.3. Biadjointness of induction and restriction

We use the notation in [16,21] for bimodules, so ${}_n(n+1)$ denotes H_{n+1}^λ, viewed as an $(H_n^\lambda, H_{n+1}^\lambda)$-bimodule, and $(n+1)_n$ denotes H_{n+1}^λ viewed as an $(H_{n+1}^\lambda, H_n^\lambda)$-bimodule. We use juxtaposition to denote the tensor product, so that, for example, $(n+1)_n(n+1)$ is the $(H_{n+1}^\lambda, H_{n+1}^\lambda)$-bimodule $H_{n+1}^\lambda \otimes_{H_n^\lambda} H_{n+1}^\lambda$.

Proposition 5.13. *The maps*

$$\varepsilon_{\mathrm{R}} \colon (n+1)_n(n+1) \to (n+1), \quad \varepsilon_{\mathrm{R}}(a\otimes b) = ab, \quad a \in (n+1)_n,\ b \in {}_n(n+1),$$
$$\eta_{\mathrm{R}} \colon (n) \hookrightarrow {}_n(n+1)_n, \quad \eta_{\mathrm{R}}(a) = a, \quad a \in (n),$$
$$\varepsilon_{\mathrm{L}} = \mathrm{tr}_{n+1} \colon {}_n(n+1)_n \to (n),$$
$$\eta_{\mathrm{L}} \colon (n+1) \to (n+1)_n(n+1),$$
$$\eta_{\mathrm{L}}(a) = a \sum_{\substack{i=1,\dots,n+1\\ k=0,\dots,d-1}} x_i^k s_i \cdots s_{n-1}s_n \otimes s_n s_{n-1}\cdots s_i y_{i,k}, \quad a \in (n+1),$$

are bimodule homomorphisms and satisfy the relations

$$(\varepsilon_{\mathrm{R}} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \eta_{\mathrm{R}}) = \mathrm{id}, \quad (\mathrm{id} \otimes \varepsilon_{\mathrm{R}}) \circ (\eta_{\mathrm{R}} \otimes \mathrm{id}) = \mathrm{id}, \tag{5.20}$$
$$(\varepsilon_{\mathrm{L}} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \eta_{\mathrm{L}}) = \mathrm{id}, \quad (\mathrm{id} \otimes \varepsilon_{\mathrm{L}}) \circ (\eta_{\mathrm{L}} \otimes \mathrm{id}) = \mathrm{id}. \tag{5.21}$$

In particular, $(n+1)_n$ is both left and right adjoint to ${}_n(n+1)$ in the 2-category of bimodules over rings.

Proof. The map ε_{R} is multiplication and η_{R} is the inclusion $H_n^\lambda \hookrightarrow H_{n+1}^\lambda$. Thus, both are clearly bimodule homomorphisms. The relations (5.20) are standard. They are the usual unit-counit equations making induction left adjoint to restriction. Similarly, that the maps ε_{L} and η_{L} are bimodule homomorphisms and satisfy equations (5.21) follows from the standard proof that induction is right adjoint to restriction for Frobenius extensions. See, for example, [14, §1.3]. $\square$

6. Action on categories of modules for degenerate cyclotomic Hecke algebras

In this section, $\Bbbk$ is an arbitrary commutative ring until Section 6.4, where we assume it is a field of characteristic zero.

6.1. Definition of the action functors

Definition 6.1 *(The bimodule categories $\mathcal{B}_n^\lambda$).* For $n \in \mathbb{N}$, we let $\mathcal{B}_n^\lambda$ be the category defined as follows. The objects of $\mathcal{B}_n^\lambda$ are direct sums of direct summands of bimodules of the form

$$(n_k)_{m_k}(n_{k-1})_{m_{k-1}} \cdots {}_{m_2}(n_1)_{m_1}(n) \tag{6.1}$$

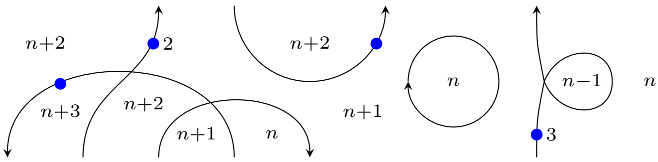
with $k \in \mathbb{N}$ and $\{m_i, m_{i-1}\} = \{n_{i-1}, n_{i-1}-1\}$ for all $i = 1, \dots, k+1$, where we adopt the convention thats $n_0 = m_0 = n$ and $m_{k+1} = n_k$. (When the n_k do not agree, direct sums of bimodules are formal.) Morphisms in $\mathcal{B}_n^\lambda$ are bimodule homomorphisms (where the only morphism between bimodules of the form (6.1) for distinct n_k is the zero morphism).

For each $n \in \mathbb{N}$, we will define a functor $\mathbf{F}_n \colon \tilde{\mathcal{H}}^\lambda \to \mathcal{B}_n^\lambda$. We define the functor $\mathbf{F}_n$ on objects as follows. The object Q_+ is sent to the bimodule $(k+1)_k$, and the object Q_- is sent to the bimodule ${}_k(k+1)$, where k is uniquely determined by the fact that our functor should respect the tensor product and be a functor to the category $\mathcal{B}_n^\lambda$ (i.e. the right action should be by H_n^λ). For example,

$$\mathbf{F}_n(\mathsf{Q}_{++-+--+}) = (n+1)_n(n)_{n-1}(n)_n(n)_{n-1}(n)_n(n+1)_{n+1}(n+1)_n$$

We define the $\mathbf{F}_n$ to be zero on any object for which the indices k become negative. For example, $\mathbf{F}_1(\mathsf{Q}_{--}) = 0$.

We now define the functor on morphisms. Consider a morphism in $\tilde{\mathcal{H}}^\lambda$ consisting of a single planar diagram. We label the rightmost region of the diagram by n. Then, we label all other regions of the diagram by integers such that, as we move from right to left across the diagram, labels increase by one when we cross upward pointing strands and decrease by one when we cross downward pointing strands. It is clear that there is a unique way to label the regions of the diagram in this way. For instance, the following diagram is labeled as indicated.



Each diagram is a composition of dots, crossings, cups and caps. Thus, we define the functors $\mathbf{F}_n$, $n \in \mathbb{N}$, on such atoms. Since, on these pieces, the functor $\mathbf{F}_n$ will be independent of n, we will drop the index and describe the functor $\mathbf{F} = \bigoplus_{n\in\mathbb{N}} \mathbf{F}_n$.

We define $\mathbf{F}$ on cups and caps by

$\mapsto \varepsilon_{\mathrm{R}},$ (6.2)

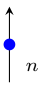
$\mapsto \eta_{\mathrm{R}},$ (6.3)

$\mapsto \varepsilon_{\mathrm{L}},$ (6.4)

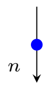
$\mapsto \eta_{\mathrm{L}},$ (6.5)

where ε_{R}, η_{R}, ε_{L}, and η_{L} are the bimodule homomorphisms of Proposition 5.13.

We define $\mathbf{F}$ on dots by



$\mapsto {}^r x_{n+1},$ (6.6)

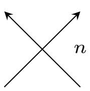


$\mapsto {}^\ell x_{n+1},$ (6.7)

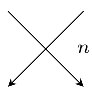
where, for $a \in H^\lambda_{n+1}$, ${}^r a$ and ${}^\ell a$ denote the maps of right and left multiplication by a:

$$
\begin{aligned}
{}^r a &\colon H^\lambda_{n+1} \to H^\lambda_{n+1}, \quad b \mapsto ba, \\
{}^\ell a &\colon H^\lambda_{n+1} \to H^\lambda_{n+1}, \quad b \mapsto ab.
\end{aligned}
$$

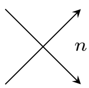
We define $\mathbf{F}$ on crossings as follows:



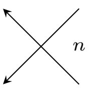
$\mapsto {}^r s_{n+1},$ (6.8)



$\mapsto {}^\ell s_{n-1},$ (6.9)



$$\mapsto \Big((n)_{n-1}(n) \to {}_n(n+1)_n,\ a \otimes a' \mapsto a s_n a' \Big), \tag{6.10}$$



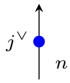
$$\mapsto \Big({}_n(n+1)_n \to (n)_{n-1}(n), \quad s_n \mapsto 1_n \otimes 1_n,\ x^j_{n+1} \mapsto 0,\ j = 0, \dots, d-1 \Big). \tag{6.11}$$

In the remainder of this section, we prove that the functors $\mathbf{F}_n$ are well-defined.

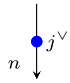
6.2. Cyclicity

We first prove some results that imply that the functors $\mathbf{F}_n$ respect isotopy invariance.

Lemma 6.2. *Under the functor* $\mathbf{F}$, *we have*



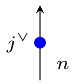
$$\mapsto {}^r y_{n+1,j}, \qquad (6.12)$$



$$\mapsto {}^\ell y_{n+1,j}, \qquad (6.13)$$

for $j \in \{0, \dots, d-1\}$.

Proof. To prove (6.12), we compute

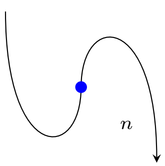
$\mathbf{F}\Bigg($  $\Bigg)$

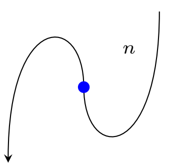
$$\overset{(2.2)}{=} \sum_{i=j}^{d-1} {}^r x_{n+1}^{i-j} \mathbf{F}(c_{d-1-i})$$

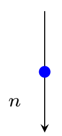
$$\overset{(2.1)}{=} \sum_{i=j}^{d-1} (-1)^{d-1-i} \left({}^r x_{n+1}^{i-j}\right) \det\left(\mathrm{tr}_{n+1}(x_{n+1}^{d+b-a})\right)_{a,b=1,\dots,d-1-i} \overset{(5.16)}{=} {}^r y_{n+1,j}.$$

The proof of (6.13) is analogous. □

Lemma 6.3. *The images of the diagrams*



, 

, *and* 

under the functor $\mathbf{F}$ *are equal.*

Proof. The image under $\mathbf{F}$ of the leftmost diagram is the map

$$\begin{aligned} {}_n(n+1) \cong (n)_n(n+1) &\xrightarrow{\eta_{\mathrm{R}} \otimes \mathrm{id}} {}_n(n+1)_n(n+1) \\ &\xrightarrow{{}^r x_{n+1} \otimes \mathrm{id}} {}_n(n+1)_n(n+1) \xrightarrow{\varepsilon_{\mathrm{R}}} {}_n(n+1), \end{aligned}$$

$$a \mapsto 1_n \otimes a \mapsto 1_{n+1} \otimes a \mapsto x_{n+1} \otimes a \mapsto x_{n+1} = {}^\ell x_{n+1}(a).$$

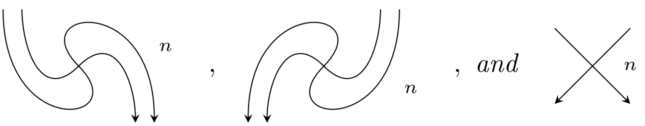
The image under $\mathbf{F}$ of the second diagram is the map

$${}_n(n+1) \xrightarrow{\eta_{\mathrm{L}}} {}_n(n+1)_n(n+1) \xrightarrow{{}^r x_{n+1} \otimes \mathrm{id}} {}_n(n+1)_n(n+1) \xrightarrow{\varepsilon_{\mathrm{L}} \otimes \mathrm{id}} (n)_n(n+1) \cong {}_n(n+1),$$

$$
\begin{aligned}
a &\mapsto \sum_{\substack{i=1,\ldots,n+1\\k=0,\ldots,d-1}} x_i^k s_i \cdots s_{n-1} s_n \otimes s_n s_{n-1} \cdots s_i y_{i,k} a \\
&\mapsto \sum_{\substack{i=1,\ldots,n+1\\k=0,\ldots,d-1}} x_i^k s_i \cdots s_{n-1} s_n x_{n+1} \otimes s_n s_{n-1} \cdots s_i y_{i,k} a \\
&\mapsto \sum_{\substack{i=1,\ldots,n+1\\k=0,\ldots,d-1}} \mathrm{tr}_{n+1}(x_i^k s_i \cdots s_{n-1} s_n x_{n+1}) s_n s_{n-1} \cdots s_i y_{i,k} a \\
&= \sum_{\substack{i=1,\ldots,n+1\\k=0,\ldots,d-1}} \mathrm{tr}_{n+1}(x_{n+1} x_i^k s_i \cdots s_{n-1} s_n) s_n s_{n-1} \cdots s_i y_{i,k} a = x_{n+1} a = {}^{\ell}x_{n+1}(a),
\end{aligned}
$$

where we have used Lemma 5.6 in the first equality, and Proposition 5.11 in the second equality. □

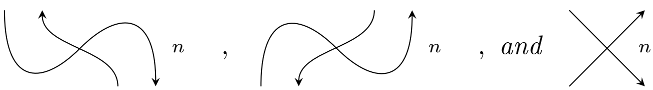
Lemma 6.4. *The images of the diagrams*



under the functor **F** *are equal.*

Proof. The proof of this lemma is completely analogous to that of [25, Lem. 7.1] and is therefore omitted. □

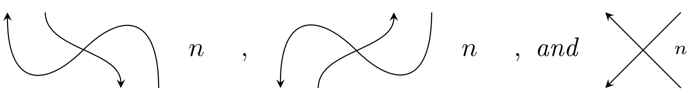
Lemma 6.5. *The images of the diagrams*



under the functor **F** *are equal.*

Proof. This follows from a straightforward computation, which will be omitted. (See also [25, Lem. 7.2].) □

Lemma 6.6. *The images of the diagrams*



under the functor **F** *are equal.*

Proof. By Lemma 5.4, ${}_n(n{+}1)_n$ is generated, as an $(H_n^\lambda, H_n^\lambda)$-bimodule, by s_n and x_{n+1}^j, $j = 0, \dots, d-1$. Thus, it suffices to compute the images of these elements. The image under $\mathbf{F}$ of the first diagram is the composition

$$ {}_n(n{+}1)_n \cong (n)_n(n{+}1)_n \xrightarrow{\eta_{\mathrm{L}} \otimes \mathrm{id}} (n)_{n-1}(n{+}1)_n \xrightarrow{\mathrm{id} \otimes {}^\ell s_n} (n)_{n-1}(n{+}1)_n \xrightarrow{\mathrm{id} \otimes \varepsilon_{\mathrm{L}}} (n)_{n-1}(n). $$

For $j \in \{0, \dots, d-1\}$, we have

$$ \begin{aligned} x_{n+1}^j &\mapsto \sum_{\substack{i=1,\dots,n \\ k=0,\dots,d-1}} x_i^k s_i \cdots s_{n-2} s_{n-1} \otimes s_{n-1} s_{n-2} \cdots s_i y_{i,k} x_{n+1}^j \\ &\mapsto \sum_{\substack{i=1,\dots,n \\ k=0,\dots,d-1}} x_i^k s_i \cdots s_{n-2} s_{n-1} \otimes s_n s_{n-1} s_{n-2} \cdots s_i y_{i,k} x_{n+1}^j \mapsto 0, \end{aligned} $$

and

$$ \begin{aligned} s_n &\mapsto \sum_{\substack{i=1,\dots,n \\ k=0,\dots,d-1}} x_i^k s_i \cdots s_{n-2} s_{n-1} \otimes s_{n-1} s_{n-2} \cdots s_i y_{i,k} s_n \\ &\mapsto \sum_{\substack{i=1,\dots,n \\ k=0,\dots,d-1}} x_i^k s_i \cdots s_{n-2} s_{n-1} \otimes s_n s_{n-1} s_{n-2} \cdots s_i y_{i,k} s_n \mapsto 1_n \otimes 1_n, \end{aligned} $$

where we used Proposition 5.11 in the last map.

The image under $\mathbf{F}$ of the second diagram is the composition

$$ {}_n(n{+}1)_n \cong {}_n(n{+}1)_n(n) \xrightarrow{\mathrm{id} \otimes \eta_{\mathrm{L}}} {}_n(n{+}1)_{n-1}(n) \xrightarrow{{}^r s_n \otimes \mathrm{id}} {}_n(n{+}1)_{n-1}(n) \xrightarrow{\varepsilon_{\mathrm{L}} \otimes \mathrm{id}} (n)_{n-1}(n). $$

For $j = \{0, \dots, d-1\}$, we have

$$ \begin{aligned} x_{n+1}^j &\mapsto x_{n+1}^j \sum_{\substack{i=1,\dots,n \\ k=0,\dots,d-1}} x_i^k s_i \cdots s_{n-2} s_{n-1} \otimes s_{n-1} s_{n-2} \cdots s_i y_{i,k} \\ &\mapsto x_{n+1}^j \sum_{\substack{i=1,\dots,n \\ k=0,\dots,d-1}} x_i^k s_i \cdots s_{n-2} s_{n-1} s_n \otimes s_{n-1} s_{n-2} \cdots s_i y_{i,k} \mapsto 0, \end{aligned} $$

and

$$ \begin{aligned} s_n &\mapsto s_n \sum_{\substack{i=1,\dots,n \\ k=0,\dots,d-1}} x_i^k s_i \cdots s_{n-2} s_{n-1} \otimes s_{n-1} s_{n-2} \cdots s_i y_{i,k} \\ &\mapsto s_n \sum_{\substack{i=1,\dots,n \\ k=0,\dots,d-1}} x_i^k s_i \cdots s_{n-2} s_{n-1} s_n \otimes s_{n-1} s_{n-2} \cdots s_i y_{i,k} \mapsto 1_n \otimes 1_n, \end{aligned} $$

where, in the last map, we used the fact that

$$\operatorname{tr}_{n+1}\left(s_n x_i^k s_i \cdots s_{n-2} s_{n-1} s_n\right) = \operatorname{tr}_n\left(x_i^k s_i \cdots s_{n-2} s_{n-1}\right)$$

by Corollary 5.10, and that

$$\sum_{\substack{i=1,\dots,n\\k=0,\dots,d-1}} \operatorname{tr}_n\left(x_i^k s_i \cdots s_{n-2} s_{n-1}\right) \otimes_{H_{n-1}^\lambda} s_{n-1} s_{n-2} \cdots s_i y_{i,k}$$

$$= 1_n \otimes_{H_{n-1}^\lambda} \sum_{\substack{i=1,\dots,n\\k=0,\dots,d-1}} \operatorname{tr}_n\left(x_i^k s_i \cdots s_{n-2} s_{n-1}\right) s_{n-1} s_{n-2} \cdots s_i y_{i,k} = 1_n \otimes_{H_{n-1}^\lambda} 1_n$$

by Proposition 5.11. □

6.3. The action functors are well-defined

We now prove one of our main results: that the functors $\mathbf{F}_n$ are well defined. We assume some knowledge of cyclic biadjointness and its relation to planar diagrammatics for bimodules. We refer the reader to [15] for an overview of this topic.

Theorem 6.7. *The above maps give a well-defined additive functor*

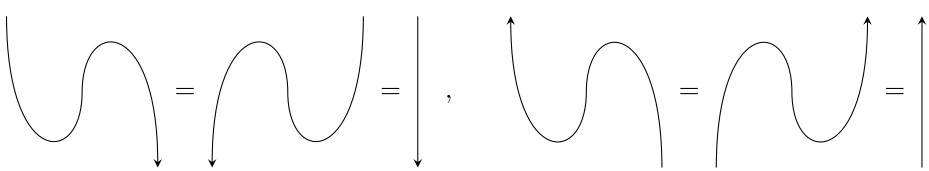
$$\mathbf{F}_n \colon \mathcal{H}^\lambda \to \mathcal{B}_n^\lambda$$

for each $n \in \mathbb{N}$ and hence define an action of $\mathcal{H}^\lambda$ on $\bigoplus_{n\in\mathbb{N}} H_n^\lambda$-mod.

Proof. The action comes from the standard action of bimodules on categories of modules, via the tensor product. See, for example, [25, (7.1)].

By definition, the category $\mathcal{B}_n^\lambda$ is idempotent complete for all $n \in \mathbb{N}$. Thus, any functor from $\tilde{\mathcal{H}}^\lambda$ to $\mathcal{B}_n^\lambda$ naturally induces a functor $\mathcal{H}^\lambda \to \mathcal{B}^\lambda$. Therefore, it suffices to consider the category $\tilde{\mathcal{H}}^\lambda$.

By Proposition 5.13, the images $(n+1)_n$ and $_n(n+1)$ of the objects Q_+ and Q_- under $\mathbf{F}$ are biadjoint. Thus, the zigzag identities



are preserved by $\mathbf{F}$. The fact that $\mathbf{F}$ preserves invariance under local isotopies then follows from Lemmas 6.3, 6.4, 6.5, and 6.6.

It remains to show that **F** preserves the relations (2.3)–(2.9). The fact that **F** preserves relations (2.3), (2.4), and (2.5) follows immediately from (5.3), (5.1), and (5.5), respectively.

If the rightmost region is labeled n, then the image under **F** of the left side of (2.9) is the map

$$(n+1)_n \xrightarrow{\eta_{\mathrm{R}}} {}_{n+1}(n+2)_n \xrightarrow{{}^r s_{n+1}} {}_{n+1}(n+2)_n \xrightarrow{\varepsilon_{\mathrm{L}}} (n+1)_n,$$
$$a \mapsto a \mapsto a s_{n+1} \mapsto 0.$$

Thus, **F** preserves (2.9).

If the outside region is labeled n, then the image under **F** of the left side of (2.8) is the map

$$(n) \xrightarrow{\eta_{\mathrm{R}}} {}_n(n+1)_n \xrightarrow{{}^\ell\left(x_{n+1}^j\right)} {}_n(n+1)_n \xrightarrow{\varepsilon_{\mathrm{L}}} (n),$$
$$a \mapsto a \mapsto x_{n+1}^j a = a x_{n+1}^j \mapsto \begin{cases} 0 & \text{if } j < d-1, \\ a & \text{if } j = d-1, \\ a \sum_{i \in I} i\lambda_i & \text{if } j = d, \end{cases}$$

where the last case follows from Lemma 5.12. Thus, **F** preserves (2.8).

Now, if the rightmost region is labeled n, then the image under **F** of the first term on the right side of (2.6) (i.e. the double crossing) is the map

$${}_n(n+1)_n \xrightarrow{(6.11)} (n)_{n-1}(n) \xrightarrow{(6.10)} {}_n(n+1)_n.$$

This map is uniquely determined by the images of s_n and x_{n+1}^k, $k = 0, \dots, d-1$. We compute

$$s_n \mapsto 1_n \otimes 1_n \mapsto s_n \quad \text{and} \quad x_{n+1}^k \mapsto 0.$$

On the other hand, the image under **F** of the sum in (2.6) of diagrams over j, is (using (6.13))

$$\sum_{j=0}^{d-1} {}^r x_{n+1}^j \circ \eta_{\mathrm{R}} \circ \varepsilon_{\mathrm{L}} \circ {}^\ell y_{n+1,j}.$$

This acts as

$$s_n \mapsto \sum_{j=0}^{d-1} \mathrm{tr}_{n+1}(y_{n+1,j} s_n) x_{n+1}^j = 0, \quad x_{n+1}^k \mapsto \sum_{j=0}^{d-1} \mathrm{tr}_{n+1}\left(y_{n+1,j} x_{n+1}^k\right) x_{n+1}^j = x_{n+1}^k,$$

where the equalities follow from Proposition 5.11. It follows that **F** preserves (2.6).

Finally, if the rightmost region is labeled n, then the image under $\mathbf{F}$ of the left side of (2.7) is the map

$$(n)_{n-1}(n) \xrightarrow{(6.10)} {}_n(n+1)_n \xrightarrow{(6.11)} (n)_{n-1}(n),$$
$$a \otimes b \mapsto as_n b \mapsto a \otimes b.$$

Thus, $\mathbf{F}$ preserves (2.7). □

6.4. Categorification of Fock space

We assume in this subsection that $\Bbbk$ is a field of characteristic zero. For $n \in \mathbb{N}$, let $K_0(H_n^\lambda\text{-pmod})$ be the split Grothendieck group of the category H_n^λ-pmod of finitely-generated projective H_n^λ-modules. By Theorems 4.4 and 6.7, we have ring homomorphisms

$$\mathfrak{h}_d \to K_0(\mathcal{H}^\lambda) \xrightarrow{\bigoplus_{n\in\mathbb{N}} K_0(\mathbf{F}_n)} \mathrm{End}\left(\bigoplus_{n\in\mathbb{N}} K_0(H_n^\lambda\text{-pmod})\right),$$

yielding an action of $\mathfrak{h}_d$ on $\bigoplus_{n\in\mathbb{N}} K_0(H_n^\lambda\text{-pmod})$. (Note that we do not use the injectivity statement in Theorem 4.4, which we have yet to prove.)

If $\Bbbk_0$ denotes the trivial one-dimensional H_0^λ-module, then, for any partition μ, we have

$$\mathbf{F}_0([\mathsf{Q}_-^\mu]) \cdot [\Bbbk_0] = 0, \qquad \mathbf{F}_0([\mathsf{Q}_+^\mu]) \cdot [\Bbbk_0] = [H_n^\lambda e_\mu].$$

It follows from the Stone–von Neumann Theorem that the submodule of $\bigoplus_{n\in\mathbb{N}} K_0(H_n^\lambda\text{-pmod})$ generated by $[\Bbbk_0]$ is the Fock space representation of $\mathfrak{h}_d$. (See [29, Th. 2.11(c)] for a version of the Stone–von Neumann Theorem in the general setting of Heisenberg doubles.)

Proposition 6.8. *The ring homomorphism $\mathfrak{h}_d \to K_0(\mathcal{H}^\lambda)$ of Theorem 4.4 is injective.*

Proof. This follows from the above discussion and the fact that the Fock space representation is faithful. (See, for example, [29, Th. 2.11(d)] for this statement in the setting of Heisenberg doubles.) □

The action of i-induction and i-restriction on $\bigoplus_{n\in\mathbb{N}} K_0(H_n^\lambda\text{-pmod})$ realizes the irreducible highest weight module $V(\lambda)$ of $\mathfrak{g}$ of highest weight λ. (See, for example, [19, Ch. 9].) In general, the $\mathfrak{h}_d$-module $\bigoplus_{n\in\mathbb{N}} K_0(H_n^\lambda\text{-pmod})$ is a direct sum of Fock space representations. This corresponds to the fact that the restriction of $V(\lambda)$ to the principal Heisenberg subalgebra of $\mathfrak{g}$ is a direct sum of Fock spaces. If λ is of level one, then $V(\lambda)$ remains irreducible as a module over the Heisenberg algebra, and we are in the setting of [23]. See Section 8.1 for further comments in this direction. In general, the projective modules $H_n^\lambda e_\mu$ are not indecomposable. An explicit description of the indecomposable projective H_n^λ-modules is not currently known for arbitrary level. (See [8] for the level two case.)

7. Properties of the action functors

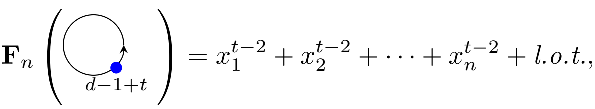
In this section, $\Bbbk$ is an arbitrary commutative ring until after Proposition 7.1. At that point, we assume $\Bbbk$ is a field of characteristic zero for the remainder of the section.

Note that $\mathbf{F}_n$ maps closed diagrams, which are endomorphisms of the identity object $\mathbf{1}$, to $\mathrm{End}_{\mathcal{B}_n^\lambda}(n)$, the algebra of bimodule endomorphisms of (n). We have a natural isomorphism of algebras

$$Z(H_n^\lambda) \xrightarrow{\cong} \mathrm{End}_{\mathcal{B}_n^\lambda}(n), \quad a \mapsto {}^\ell a,$$

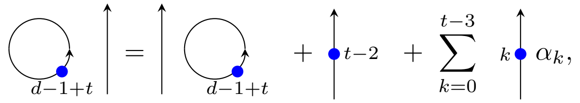
where we recall that ${}^\ell a$ denotes left multiplication by a. In what follows, we will often identify $Z(H_n^\lambda)$ and $\mathrm{End}_{\mathcal{B}_n^\lambda}(n)$ via this isomorphism.

Proposition 7.1. *For all $n \in \mathbb{N}$ and $t \geq 3$, we have*

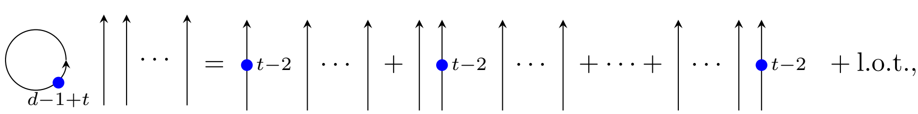


where "l.o.t." is polynomial in $x_1, \dots, x_n$ of degree less than or equal to $t-3$. In other words, $\mathbf{F}_n$ maps a counterclockwise circle with $d-1+t$ dots, $t \geq 3$, to the $(t-2)$-nd power sum in the generators $x_1, \dots, x_n$, up to terms of lower degree.

Proof. It follows from Lemma 2.12 that, for $t \geq 3$,



where $\alpha_k \in \mathrm{End}_{\mathcal{H}^\lambda}(\mathbf{1})$, $k \in \{0, \dots, t-3\}$, is a linear combination of counterclockwise circles with at most $d-1+t-2$ dots. It follows that



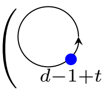
(7.1)

where "l.o.t." denotes a linear combination of lower order terms—diagrams consisting of upwards pointing strands carrying fewer than $t-2$ total dots, with a closed diagram in the rightmost region.

Now, consider the commutative diagram

$$\begin{tikzcd} \mathrm{End}_{\mathcal{H}^\lambda} \mathbf{1} \arrow[r, "\mathbf{F}_n"] \arrow[d] & \mathrm{End}_{\mathcal{B}_n^\lambda}(n) \arrow[d, hook, "\iota"] \\ \mathrm{End}_{\mathcal{H}^\lambda} \mathsf{Q}_+^n \arrow[r, "\mathbf{F}_0"] & \mathrm{End}_{\mathcal{B}_0^\lambda}(n)_0 \end{tikzcd}$$

where the leftmost vertical arrow is given by horizontal composition on the right by the identity morphism of Q_+^n, and ι is the natural inclusion. (Diagrammatically, the leftmost vertical arrow takes a closed diagram and places n upwards pointing arrows to its right.) It follows from (7.1) that

$\iota \circ \mathbf{F}_n \Bigg($ 

$\Bigg) = x_1^{t-2} + x_2^{t-2} + \cdots + x_n^{t-2} + \text{l.o.t.},$

where "l.o.t." is polynomial in $x_1, \dots, x_n$ of degree less than or equal to $t-3$. □

For the remainder of the section, we assume that $\Bbbk$ is a field of characteristic zero.

Corollary 7.2. *For all $n \in \mathbb{N}$, the functor $\mathbf{F}_n$ induces a surjective homomorphism of algebras* $\mathrm{End}_{\mathcal{H}^\lambda} \mathbf{1} \twoheadrightarrow \mathrm{End}_{\mathcal{B}_n^\lambda}(n) \cong Z(H_n^\lambda)$.

Proof. Since the power sums generate the ring of symmetric polynomials over a field of characteristic zero, the result follows from Proposition 7.1 and [7, Th. 1], which states that $Z(H_n^\lambda)$ consists of all symmetric polynomials in the $x_1, \dots, x_n$. □

Proposition 7.3. *The homomorphism ψ_0 of* (2.17) *is injective.*

Proof. Define a grading on Π by setting $\deg y_i = i$ for all $i \in \mathbb{N}_+$. Suppose $f \in \Pi$ is a nonzero element of top degree a (i.e. f is a sum of a nonzero element of degree a and elements of lower degree). Then $\mathbf{F}_n(\psi_0(f)) \in Z(H_n^\lambda)$ and so, by [7, Th. 1], $\mathbf{F}_n(\psi_0(f))$ is a symmetric polynomial in $x_1, \dots, x_n$. By Proposition 7.1, the top degree of the monomials appearing in $\mathbf{F}_n(\psi_0(f))$ is a. Then, by [7, Th. 3.2], $\mathbf{F}_n(\psi_0(f))$ is nonzero for sufficiently large n. It follows that $\psi_0(f) \neq 0$. □

Consider the $\Bbbk$-algebra

$$(H_{n+k}^\lambda)^{H_n^\lambda} := \{a \in H_{n+k}^\lambda \mid ha = ah \text{ for all } h \in H_n^\lambda\}.$$

This algebra is canonically isomorphic to the endomorphism ring of the bimodule ${}_n(n+k)$, and, therefore, to the endomorphism ring of the restriction functor, via the map that assigns to $a \in (H^\lambda_{n+k})^{H^\lambda_n}$ the endomorphism ${}^\ell a$. Likewise, the opposite algebra of $(H^\lambda_{n+k})^{H^\lambda_n}$ is canonically isomorphic to the endomorphism ring of the bimodule $(n+k)_n$ and, therefore, to that of the induction functor, via the map that assigns to $a \in (H^\lambda_{n+k})^{H^\lambda_n}$ the endomorphism ${}^r a$.

Proposition 7.4. *The homomorphism ψ_m of (2.19) is injective.*

Proof. Fix $j \in I$ such that $\lambda_j \neq 0$. Define $\mu = \sum_{i\in I} \mu_i \omega_i \in P_+$ by $\mu_i = \lambda_{i-j}$ for $i \in I$. Thus $\mu_0 \neq 0$. We have an isomorphism of algebras $H^\lambda_n \cong H^\mu_n$ that fixes $s_1, \dots, s_{n-1}$ and maps x_i to $x_i - j$ for $i \in \{1, 2, \dots, n\}$. We also have a surjective homomorphism of algebras $H^\mu_n \twoheadrightarrow \Bbbk S_n$ given by mapping x_i to the i-th Jucys–Murphy element.

For $n \in \mathbb{N}$, consider the composition of algebra homomorphisms

$$H_m \otimes \Pi \xrightarrow{\psi_m} \mathrm{End}_{\mathcal{H}^\lambda}(\mathsf{Q}^m_+) \xrightarrow{\mathbf{F}_n} \mathrm{End}(m+n)_n \hookrightarrow \left(H^\lambda_{m+n}\right)^{\mathrm{op}} \cong \left(H^\mu_{m+n}\right)^{\mathrm{op}} \twoheadrightarrow (\Bbbk S_{m+n})^{\mathrm{op}}.$$

This composition is precisely the homomorphism $\psi'_{m,n}$ of [16, §4]. It is shown there that these maps are asymptotically injective, in the sense that if z is some nonzero element of $H_m \otimes \Pi$, then $\psi'_{m,n}(z) \neq 0$ for sufficiently large n. It follows that ψ_m is injective. $\square$

Proposition 7.5. *For all $n, k \in \mathbb{N}$, the functor $\mathbf{F}_n$ induces a surjective homomorphism of algebras*

$$\mathrm{End}_{\mathcal{H}^\lambda} \mathsf{Q}^k_+ \twoheadrightarrow \mathrm{End}_{\mathcal{B}^\lambda_n}(n+k)_n.$$

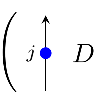
Proof. We proceed by induction on k. For $k = 0$, the result is Corollary 7.2.

Let $m = n + k$. Assume that the result holds for some $m \geq n$. We must prove that ${}^r a$ is in the image of $\mathbf{F}_n$ for all $a \in (H^\lambda_{m+1})^{H^\lambda_n}$. Now, it follows from the bimodule decomposition (5.12) that

$$(H^\lambda_{m+1})^{H^\lambda_n} = \left(H^\lambda_m s_m H^\lambda_m\right)^{H^\lambda_n} \oplus \bigoplus_{j=0}^{d-1} (H^\lambda_m)^{H^\lambda_n} x^j_{m+1}.$$

Therefore, it suffices to consider the cases where $a \in (H^\lambda_m)^{H^\lambda_n} x^j_{m+1}$ for some $j \in \{0, \dots, d-1\}$ and $a \in \left(H^\lambda_m s_m H^\lambda_m\right)^{H^\lambda_n}$.

First suppose that $a = a' x^j_{m+1} = x^j_{m+1} a'$ for some $a' \in (H^\lambda_m)^{H^\lambda_n}$ and $j \in \{0, \dots, d-1\}$. By the inductive hypothesis, there exists a $D \in \mathrm{End}_{\mathcal{H}^\lambda} \mathsf{Q}^{m-n}_+$ such that $\mathbf{F}_n(D) = a'$. Then

$\mathbf{F}_n \Big($  $\Big) = {}^\ell a.$

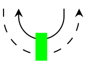
Now suppose $a \in \left(H_m^\lambda s_m H_m^\lambda\right)^{H_n^\lambda}$. By (5.13), a is the image under the map

$$(m)_{m-1}(m) \to {}_m(m+1)_m, \quad u \otimes v \mapsto u s_m v,$$

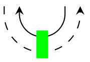
of some element $a' \in (m)_{m-1}(m)$ satisfying $ba' = a'b$ for all $b \in H_n^\lambda$. Consider the bimodule homomorphism

$$^\ell a' = {}^r a' \colon (n) \to {}_n(m)_{m-1}(m)_n.$$

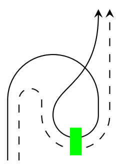
By adjunction and the induction hypothesis, we have an element of $\mathrm{Hom}_{\mathcal{H}^\lambda}(\mathbf{1}, \mathsf{Q}_-^{m-n}\mathsf{Q}_+\mathsf{Q}_-\mathsf{Q}_+^{m-n})$, which we will depict as

 $: \mathbf{1} \to \mathsf{Q}_-^{m-n}\mathsf{Q}_+\mathsf{Q}_-\mathsf{Q}_+^{m-n},$

(where the dashed line represents $m-n$ lines) such that

$\mathbf{F}_n$ () $= {}^\ell a'.$

It follows that

$\mathbf{F}_n$ () $= {}^\ell a.$

This completes the proof. $\square$

Theorem 7.6. *For all $n \in \mathbb{N}$, the functor $\mathbf{F}_n$ is full.*

Proof. It follows from Corollary 2.7 that every object of $\mathcal{H}^\lambda$ is a direct sum of objects of the form $\mathsf{Q}_-^k\mathsf{Q}_+^\ell$ for some $k, \ell \in \mathbb{N}$. By adjunction, we have

$$\mathrm{Hom}_{\mathcal{H}^\lambda}(\mathsf{Q}_-^k\mathsf{Q}_+^\ell, \mathsf{Q}_-^a\mathsf{Q}_+^b) \cong \mathrm{Hom}_{\mathcal{H}^\lambda}(\mathsf{Q}_+^\ell, \mathsf{Q}_+^k\mathsf{Q}_-^a\mathsf{Q}_+^b).$$

Thus, it suffices to consider hom-spaces with domain Q_+^ℓ, $\ell \in \mathbb{N}$. Again, by adjunction, we have

$$\mathrm{Hom}_{\mathcal{H}^\lambda}(\mathsf{Q}_+^\ell, \mathsf{Q}_-^a\mathsf{Q}_+^b) \cong \mathrm{Hom}_{\mathcal{H}^\lambda}(\mathsf{Q}_+^{\ell+a}, \mathsf{Q}_+^b).$$

Since this hom-space is zero unless $\ell+a = b$, the theorem follows from Proposition 7.5. $\square$

It follows from Theorem 7.6 that the category $\mathcal{H}^\lambda$ yields a graphical calculus for the representation theory of the H_n^λ, $n \in \mathbb{N}$. In the remainder of this section, we use this fact to deduce algebraic properties of these algebras.

There is an natural inclusion of algebras $H_n \otimes H_k \hookrightarrow H_{n+k}$. This induces an inclusion of algebras

$$H_n^\lambda \otimes H_k \hookrightarrow H_{n+k}^\lambda, \tag{7.2}$$

Corollary 7.7. *Under the inclusion* (7.2)*, the centralizer of H_n^λ in H_{n+k}^λ is generated by H_k and the center of H_n^λ.*

Proof. The opposite algebra of the centralizer of H_n^λ in H_{n+k}^λ is precisely $\mathrm{End}\big((n+k)_n\big)$. Thus, by Theorem 7.6 and Proposition 2.14, the centralizer of H_n^λ in H_{n+k}^λ is generated by $\mathbf{F}_n(\psi_k(H_k \otimes \Pi))$. Since $\mathbf{F}_n(\psi_k(\Pi))$ is the center of H_n^λ and $\mathbf{F}_n(\psi_k(H_k))$ is the image of H_k in H_{n+k}^λ under the inclusion (7.2), the result follows. $\square$

The level one case of Corollary 7.7, describing centralizers for group algebras of symmetric groups was proved by Olshanski (see [22], or [13, Th. 3.2.6]). Corollary 7.7, which is a generalization of Olshanski's result, seems to be new. The analogue for the degenerate affine Hecke algebra (before taking cyclotomic quotients) is mentioned in [9, p. 577].

Corollary 7.8. *The four bimodule maps ε_{R}, η_{R}, ε_{L}, and η_{L} of Proposition 5.13 turn the induction and restriction functors $\mathrm{Ind}_{H_n^\lambda}^{H_{n+1}^\lambda}$ and $\mathrm{Res}_{H_n^\lambda}^{H_{n+1}^\lambda}$ into a cyclic biadjoint pair.*

Proof. This follows immediately from the isotopy invariance in $\mathcal{H}^\lambda$ and Theorem 7.6. $\square$

8. Further directions

The results of the current paper suggest some natural interesting directions of future research. In this final section, we briefly outline some of these directions.

8.1. Truncations and categorified quantum groups

In [23], it is shown that a certain truncation of a 2-category version of Khovanov's Heisenberg category is equivalent to a truncation of the Khovanov–Lauda categorified quantum group of type A_∞, introduced in [18]. (See also the related work [24].) This equivalence is a categorification of the principal realization of the basic representation (i.e. the irreducible representation of highest weight ω_0) of $\mathfrak{sl}_\infty$. It yields an explicit action of categorified quantum groups on categories of modules for symmetric groups.

Replacing Khovanov's Heisenberg category by the more general categories $\mathcal{H}^\lambda$ of the current paper should yield a generalization of the results of [23]. The 2-category version $\mathscr{H}^\lambda$ of $\mathcal{H}^\lambda$ has objects labeled by integers. For $k, \ell \in \mathbb{Z}$, the category $\mathrm{Mor}_{\mathscr{H}^\lambda}(k, \ell)$ of morphisms from k to ℓ is the full subcategory of $\mathcal{H}^\lambda$ on direct sums of summands of objects $\mathsf{Q}_+^m\mathsf{Q}_-^n$ with $\ell - k = m - n$. The horizontal composition in $\mathscr{H}^\lambda$ comes from the monoidal structure of $\mathcal{H}^\lambda$. One can truncate $\mathscr{H}^\lambda$ by killing all morphisms factoring through negative integers. Then taking the idempotent completion as defined in [23, Def. 5.1] yields a 2-category $\mathscr{H}^{\lambda,\mathrm{tr}}$. We expect that the 2-category $\mathscr{H}^{\lambda,\mathrm{tr}}$ is equivalent to the truncation of the Khovanov–Lauda categorified quantum group obtained by killing all morphisms factoring through weights not appearing in the irreducible highest weight representation of $\mathfrak{g}$ of highest weight λ. This would yield actions of categorified quantum groups on categories of modules for degenerate cyclotomic Hecke algebras as in [4].

8.2. Wreath product algebras

In [25], a general Heisenberg category $\mathcal{H}_B$ was introduced that depends on a graded Frobenius superalgebra B. When B is the ground field, this category reduces to the Heisenberg category of Khovanov. The inspiration behind the definition of the categories $\mathcal{H}_B$ is the passage from the group algebra of the symmetric group to the wreath product algebras $B^{\otimes n} \rtimes S_n$. Just as the degenerate affine Hecke algebra appears naturally in the endomorphism algebra of Q_+^n in Khovanov's category, the endomorphism algebra of Q_+^n in $\mathcal{H}_B$ contains an algebra denoted D_n in [25, Def. 8.12]. These algebras have been studied in [26], where they are called *affine wreath product algebras*, and denoted $\mathcal{A}_n(B)$. In particular, cyclotomic quotients $\mathcal{A}_n^{\mathbf{C}}(B)$ of these algebras are defined [26, §6], and it is shown that level one cyclotomic quotients are isomorphic to $B^{\otimes n} \rtimes S_n$ (see [26, Cor. 6.13]). On the other hand, taking $B = \Bbbk$ yields H_n^λ, where λ depends on the quotient parameter $\mathbf{C}$.

$$\begin{tikzcd}
& \mathcal{A}_n^{\mathbf{C}}(B) \arrow[dl, rightsquigarrow, "B=\Bbbk"'] \arrow[dr, "\text{level one}"] & \\
H_n^\lambda \arrow[dr, "\text{level one}"'] & & B^{\otimes n} \rtimes S_n \arrow[dl, rightsquigarrow, "B=\Bbbk"] \\
& \Bbbk S_n &
\end{tikzcd}$$

We expect that examination of the representation theory of the algebras $\mathcal{A}_n(B)$, together with their cyclotomic quotients, should lead to even more general Heisenberg categories $\mathcal{H}_B^{\mathbf{C}}$ that specialize to both the categories $\mathcal{H}^\lambda$ of the current paper and the categories $\mathcal{H}_B$ of [25].[2] One of the advantages of the presence of the graded Frobenius superalgebra B is that, provided it is not concentrated in degree zero, it allows one to use grading arguments to prove the analogue of Conjecture 4.5.

[2] Since the writing of the current paper, these categories have been defined in [27].

8.3. Deformations

A q-deformed version of Khovanov's Heisenberg algebra was introduced in [21]. This deformation corresponds to replacing group algebras of symmetric groups with Iwahori–Hecke algebras of type A. We expect that the results of the current paper can also be q-deformed by replacing cyclotomic degenerate Hecke algebras by cyclotomic Hecke algebras (otherwise known as Ariki–Koike algebras). This would lead to a higher level generalization of the results of [21].

8.4. Traces and diagrammatic pairings

In [12,11], the traces of Khovanov's Heisenberg category and the q-deformed version introduced in [21] have been related to W-algebras and the elliptic Hall algebra. The trace decategorification has also been used in [20] to categorify the Jack inner pairing on symmetric functions. We expect that these results have higher level versions based on the categories $\mathcal{H}^\lambda$ of the current paper or the more general categories $\mathcal{H}^\lambda_B$ mentioned above.

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