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Cluster algebras of finite type via Coxeter elements and Demazure crystals of type A

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A R T I C L E I N F O

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A B S T R A C T

Let G be a simply connected simple algebraic group over $\mathbb{C}$, B and B_- be its two opposite Borel subgroups. For two elements u, v of the Weyl group W, it is known that the coordinate ring $\mathbb{C}[G^{u,v}]$ of the double Bruhat cell $G^{u,v} = BuB \cap B_-vB_-$ is isomorphic to a cluster algebra $\mathcal{A}(\mathbf{i})_{\mathbb{C}}$ [2, 12]. In the case $u = e$, $v = c^2$ (c is a Coxeter element), the algebra $\mathbb{C}[G^{e,c^2}]$ has only finitely many cluster variables. In this article, for $G = \mathrm{SL}_{r+1}(\mathbb{C})$, we obtain explicit forms of all the cluster variables in $\mathbb{C}[G^{e,c^2}]$ by considering its additive categorification via preprojective algebras, and describe them in terms of monomial realizations of Demazure crystals.

1. Introduction

A cluster algebra is a commutative ring generated by so-called "cluster variables", which has been introduced in order to study certain combinatorial properties of dual (semi) canonical bases by Fomin and Zelevinsky ([7]). Nowadays, it has influenced to remarkably wide areas of mathematics and physics.

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In [2], Berenstein et al. constructed the upper cluster algebra structures on the coordinate algebra $\mathbb{C}[G^{u,v}]$ of double Bruhat cell $G^{u,v}$, where G is a simply-connected simple algebraic group over $\mathbb{C}$ and u, v are elements of the associated Weyl group W. Recently, Goodearl and Yakimov showed that $\mathbb{C}[G^{u,v}]$ also has a cluster algebra structure ([12]). In [9] Geiss et al. initiated categorification of cluster algebras by considering semi-canonical bases.

Cluster algebras which have only finitely many cluster variables are called *finite type*. In [8], cluster algebras of finite type are studied thoroughly, and are classified by the set of Cartan matrices up to coefficients. For a fixed Cartan matrix, all the cluster variables are parametrized by the set of "almost positive roots", which is, a union of all positive roots and negative simple roots corresponding to the Cartan matrix. Here, it is defined that the type of such cluster algebra to be the type of the corresponding Cartan matrix. Let $c \in W$ be a Coxeter element whose length $l(c)$ satisfies $l(c^2) = 2l(c) = 2\mathrm{rank}(G)$. It is known that one can realize a cluster algebra of finite type on the coordinate ring $\mathbb{C}[G^{e,c^2}]$, whose type coincides with the Cartan-Killing type of G [2].

The theory of crystal base has been invented by Kashiwara, whose basic properties admit several kinds of explicit descriptions. Each description provides interesting applications to combinatorics, mathematical physics, and representation theories of finite groups, etc. [17]. A *monomial realization* is one of such descriptions of crystal bases, each element of crystal base is described as a Laurent monomial in double-indexed variables $\{Y_{s,i} | s \in \mathbb{Z}, i \in \{1, 2, \cdots, \mathrm{rank}(G)\}\}$ [15,20], which has been motivated by q-characters and then it matches to express the whole structure of crystals.

In [13,14], we showed that certain cluster variables of $\mathbb{C}[G^{u,e}]$ $(u \in W)$ become Laurent polynomials of $\{Y_{s,i}\}$ with positive coefficients by taking a specific transformation $H \times (\mathbb{C}^\times)^{l(u)} \to G^{u,e}$ (H is a maximal torus of G), and these polynomials coincide with the total sums of monomial realizations of lower Demazure crystals in the case G is type A, B, C or D. For a reduced expression of Weyl group element $w = s_{i_1} \cdots s_{i_n}$ and crystal base $B(\lambda)$ (λ is a dominant weight), Demazure crystal $B(\lambda)_w$ and lower Demazure crystal $B^-(\lambda)_w$ are the following subset of $B(\lambda)$:

$$B(\lambda)_w = \{\tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_n}^{a_n} b_\lambda | a_1, \cdots, a_n \in \mathbb{Z}_{\geq 0}\} \setminus \{0\},$$
$$B^-(\lambda)_w = \{\tilde{e}_{i_1}^{a_1} \cdots \tilde{e}_{i_n}^{a_n} b_\lambda^- | a_1, \cdots, a_n \in \mathbb{Z}_{\geq 0}\} \setminus \{0\},$$

where b_λ (resp. b_λ^-) is the highest (resp. lowest) weight vector in $B(\lambda)$. Then we treated only a part of the cluster variables so-called initial cluster variables. And we did not reveal the meaning of the highest weights of crystal bases appearing in the initial cluster variables. To see more universal relations between the cluster algebras and crystal bases, we need to treat all the cluster variables in the coordinate rings.

From this point of view, in this article, we intended to consider the coordinate ring $\mathbb{C}[G^{e,c^2}]$ for $G = \mathrm{SL}_{r+1}(\mathbb{C})$ $(r \geq 3)$ which has only finitely many cluster variables, where c is the Coxeter element such that a reduced word $\mathbf{i}$ of c^2 can be written as $\mathbf{i} = (2, 4, 6, \cdots, R, 1, 3, 5, \cdots, R', 2, 4, 6, \cdots, R, 1, 3, 5, \cdots, R')$ with $(R, R') = (r, r-1)$ if

r is even, $(R, R') = (r-1, r)$ if r is odd. The aim of the article is to reveal relation between all the cluster variables in $\mathbb{C}[G^{e,c^2}]$ and crystal bases. One of our main results is that each cluster variable in $\mathbb{C}[G^{e,c^2}]$ becomes the total sum of monomial realization of Demazure crystals by applying a coordinate transformation $\overline{x}^G_{\mathbf{i}} : H \times (\mathbb{C}^\times)^{2r} \to G^{e,c^2}$. More precisely, the initial cluster variables coincide with the sums of monomials in the Demazure crystals $B(\Lambda_k)_{w_k}$ with some $k \in \{1, 2, \cdots, r\}$ and $w_k \in W$ (see Proposition 6.2), where Λ_k is the k-th fundamental weight ($1 \leq k \leq r$). The other cluster variables are also obtained as the sums of monomials in the direct sum of Demazure crystals in the form $B(\sum_{s=a}^b \Lambda_s)_w \oplus \bigoplus_{t=1}^p B(\lambda_t)_{w_t}$ with some $w, w_t \in W$, $p, a, b \in \mathbb{Z}_{>0}$ and $\lambda_t \in \sum_{s=a}^b \Lambda_s - \sum_{i\in\{1,2,\cdots,r\}} \mathbb{Z}_{\geq 0}\alpha_i$, where α_i is the i-th simple root. As a corollary of these results, we see that a natural correspondence $-\alpha_k \mapsto B(\Lambda_k)_{w_k}$, $\sum_{s=a}^b \alpha_s \to B(\sum_{s=a}^b \Lambda_s)_w \oplus \bigoplus_{t=1}^p B(\lambda_t)_{w_t}$ gives a parametrization of the cluster variables in $\mathbb{C}[G^{e,c^2}]$ by the set of almost positive roots.

As an example, let us consider the case $G = \mathrm{SL}_4(\mathbb{C})$ (type A_3 algebraic group). For the monomial realization of the crystal $B(\Lambda_2)$ of type A_3, its crystal graph in terms of monomials is as follows:

$$
\begin{tikzcd}
Y_{1,2} \arrow[r, "2"] & \frac{Y_{1,1}Y_{1,3}}{Y_{2,2}} \arrow[r, "1"] \arrow[d, "3"'] & \frac{Y_{1,3}}{Y_{2,1}} \arrow[d, "3"] & \\
& \frac{Y_{1,1}}{Y_{2,3}} \arrow[r, "1"'] & \frac{Y_{2,2}}{Y_{2,1}Y_{2,3}} \arrow[r, "2"'] & \frac{1}{Y_{3,2}}
\end{tikzcd}
\tag{1.1}
$$

On the other hand, taking the Coxeter element $c = s_2s_1s_3 \in W$, specific initial cluster variables in $\mathbb{C}[G^{e,c^2}]$ are given by minors $D_{1,2}$, $D_{12,24}$ and $D_{123,124}$ (see Theorem 3.7), where $D_{\{1,2,\cdots,k\},\{i_1,i_2,\cdots,i_k\}}$ denote the minor of matrices in $\mathrm{SL}_4(\mathbb{C})$, whose rows are labelled by $\{1, 2, \cdots, k\}$, columns are labelled by $\{i_1, i_2, \cdots, i_k\}$. Using the biregularly isomorphism $\overline{x}^G_{\mathbf{i}} : H \times (\mathbb{C}^\times)^6 \to G^{e,c^2}$ ($\mathbf{i} := (2, 1, 3, 2, 1, 3)$, $6 = l(c^2)$) in Proposition 2.4, we have

$$D_{12,24} \circ \overline{x}^G_{\mathbf{i}}(a; \mathbf{Y}) = a_1a_2 \left(Y_{1,2} + \frac{Y_{1,1}Y_{1,3}}{Y_{2,2}} + \frac{Y_{1,3}}{Y_{2,1}} + \frac{Y_{2,2}}{Y_{2,1}Y_{2,3}} + \frac{Y_{1,1}}{Y_{2,3}} \right),$$

where we set $a := \mathrm{diag}(a_1, a_2, a_3, a_4) \in H$ and $\mathbf{Y} := (Y_{1,2}, Y_{1,1}, Y_{1,3}, Y_{2,2}, Y_{2,1}, Y_{2,3}) \in (\mathbb{C}^\times)^6$. Comparing with the above crystal graph (1.1) of $B(\Lambda_2)$, we see that the set of terms $\{Y_{1,2}, \frac{Y_{1,1}Y_{1,3}}{Y_{2,2}}, \frac{Y_{1,3}}{Y_{2,1}}, \frac{Y_{2,2}}{Y_{2,1}Y_{2,3}}, \frac{Y_{1,1}}{Y_{2,3}}\}$ in $D_{12,24} \circ \overline{x}^G_{\mathbf{i}}$ coincides with the monomial realization of the Demazure crystal $B(\Lambda_2)_{s_3s_1s_2}$ (see 5.2). Similarly, we get

$$D_{1,2} \circ \overline{x}^G_{\mathbf{i}}(a; \mathbf{Y}) = a_1 \left(Y_{1,1} + \frac{Y_{2,2}}{Y_{2,1}} \right), \quad D_{123,124} \circ \overline{x}^G_{\mathbf{i}}(a; \mathbf{Y}) = a_1a_2a_3 \left(Y_{1,3} + \frac{Y_{2,2}}{Y_{2,3}} \right),$$

which coincide with the total sums of monomials in Demazure crystals $B(\Lambda_1)_{s_3s_1s_2}$, $B(\Lambda_3)_{s_3s_1s_2}$ respectively up to torus parts. All other cluster variables in $\mathbb{C}[G^{e,c^2}]$ are given as

$$(D_{12,12}D_{13,34}) \circ \overline{x}_{\mathbf{i}}^G = a_1^2 a_2 a_3 Y_{2,2},$$

$$D_{12,14} \circ \overline{x}_{\mathbf{i}}^G = a_1 a_2 \left(Y_{1,2}Y_{2,1} + \frac{Y_{1,1}Y_{1,3}Y_{2,1}}{Y_{2,2}} + \frac{Y_{1,1}Y_{2,1}}{Y_{2,3}} \right),$$

$$D_{12,23} \circ \overline{x}_{\mathbf{i}}^G = a_1 a_2 \left(Y_{1,2}Y_{2,3} + \frac{Y_{1,1}Y_{1,3}Y_{2,3}}{Y_{2,2}} + \frac{Y_{1,3}Y_{2,3}}{Y_{2,1}} \right), \quad D_{1,3} \circ \overline{x}_{\mathbf{i}}^G = a_1 Y_{2,3},$$

$$D_{123,134} \circ \overline{x}_{\mathbf{i}}^G = a_1 a_2 a_3 Y_{2,1}, \quad D_{12,13} \circ \overline{x}_{\mathbf{i}}^G = a_1 a_2 \left(Y_{1,2}Y_{2,1}Y_{2,3} + \frac{Y_{1,1}Y_{1,3}Y_{2,1}Y_{2,3}}{Y_{2,2}} \right),$$

which coincide with the total sums of monomials in Demazure crystals $B(\Lambda_2)_e$, $B(\Lambda_1 + \Lambda_2)_{s_3 s_2}$, $B(\Lambda_2 + \Lambda_3)_{s_1 s_2}$, $B(\Lambda_3)_e$, $B(\Lambda_1)_e$ and $B(\Lambda_1 + \Lambda_2 + \Lambda_3)_{s_2}$ respectively up to torus parts. These results imply the statement of Theorem 6.8 for $r = 3$. Thus, the correspondence $-\alpha_i \mapsto B(\Lambda_i)_{s_3 s_1 s_2}$, $\alpha_i \mapsto B(\Lambda_i)_e$ $(i = 1, 2, 3)$, $\alpha_1 + \alpha_2 \mapsto B(\Lambda_1 + \Lambda_2)_{s_3 s_2}$, $\alpha_2 + \alpha_3 \mapsto B(\Lambda_2 + \Lambda_3)_{s_1 s_2}$, and $\alpha_1 + \alpha_2 + \alpha_3 \mapsto B(\Lambda_1 + \Lambda_2 + \Lambda_3)_{s_2}$ yields an alternative parametrization of all cluster variables in $\mathbb{C}[G^{e,c^2}]$ by the set of almost positive roots. Here, if one takes some specific cluster as an initial cluster, the parametrization in [8] seems to coincide with ours, which will not be discussed in this article.

For the proof of main results, we use the *additive categorification* of the coordinate ring $\mathbb{C}[L^{e,c^2}]$ which has been invented by Geiss et al. [9]. Each cluster in $\mathbb{C}[L^{e,c^2}]$ is associated with a *cluster-tilting module* of the *preprojective algebra* (see Sect. 4), and each cluster variable is associated with a direct summand of the corresponding cluster-tilting module. There exists a remarkable formula to calculate such cluster variables explicitly (Proposition 4.13) [5,9]. With the help of this formula and the additive categorification, we shall obtain the explicit forms of all cluster variables.

The article is organized as follows. In section 2, we recall properties of (reduced) double Bruhat cells $G^{u,v}$ and $L^{u,v}$. In section 3, after a concise reminder on cluster algebras, we review an isomorphism between the coordinate ring of a double Bruhat cell $G^{e,v}$ and a cluster algebra $\mathcal{A}(\mathbf{i})$. In section 4, we recall the cluster algebra structure of $\mathbb{C}[L^{e,v}]$ and basic notions of preprojective algebras. We also review the additive categorifications of the cluster algebras $\mathbb{C}[L^{e,v}]$ following [5,9]. In section 5, we shortly review the definition of monomial realizations of crystal bases. Section 6 is devoted to present our main results, which provide a relation between all cluster variables in $\mathbb{C}[G^{e,c^2}]$ and monomial realizations of Demazure crystals. In Section 7, we complete the proof of the main theorems.

2. Factorization theorem

In this section, we shall introduce (reduced) double Bruhat cells $G^{u,v}$, $L^{u,v}$, and their properties [4,6]. For $l \in \mathbb{Z}_{>0}$, we set $[1, l] := \{1, 2, \cdots, l\}$.

2.1. Double Bruhat cells

Let G be a simple complex algebraic group of classical type, B and B_- be two opposite Borel subgroups in G, $N \subset B$ and $N_- \subset B_-$ be their unipotent radicals, $H := B \cap B_-$ a maximal torus. We set $\mathfrak{g} := \mathrm{Lie}(G)$ with the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}$. Let e_i, f_i ($i \in [1, r]$) be the generators of $\mathfrak{n}$, $\mathfrak{n}_-$ and h_i be the i-th simple coroot ($i \in [1, r]$). For $i \in [1, r]$ and $t \in \mathbb{C}$, we set

$$x_i(t) := \exp(te_i), \quad y_i(t) := \exp(tf_i). \tag{2.1}$$

Let $W := \langle s_i | i = 1, \cdots, r\rangle$ be the Weyl group of $\mathfrak{g}$, where $\{s_i\}$ are the simple reflections. We identify the Weyl group W with $\mathrm{Norm}_G(H)/H$. An element

$$\overline{s_i} := x_i(-1)y_i(1)x_i(-1) \tag{2.2}$$

is in $\mathrm{Norm}_G(H)$, which is a representative of $s_i \in W = \mathrm{Norm}_G(H)/H$ [21]. For $u \in W$, let $u = s_{i_1} \cdots s_{i_n}$ be its reduced expression. Then we write $\overline{u} = \overline{s_{i_1}} \cdots \overline{s_{i_n}}$, call $l(u) := n$ the length of u. We have two kinds of Bruhat decompositions of G as follows:

$$G = \coprod_{u \in W} B\overline{u}B = \coprod_{u \in W} B_-\overline{u}B_-.$$

Then, for u, $v \in W$, we define the *double Bruhat cell* $G^{u,v}$ as follows:

$$G^{u,v} := B\overline{u}B \cap B_-\overline{v}B_-.$$

We also define the *reduced double Bruhat cell* $L^{u,v}$ as follows:

$$L^{u,v} := N\overline{u}N \cap B_-\overline{v}B_- \subset G^{u,v}.$$

Definition 2.1. Let $v = s_{j_n} \cdots s_{j_1}$ be a reduced expression of $v \in W$ ($j_n, \cdots, j_1 \in [1, r]$). Then the finite sequence $\mathbf{i} := (j_n, \cdots, j_1)$ is called a *reduced word* for v.

For example, the sequence $(2, 1, 3, 2, 1, 3)$ is a reduced word of the longest element $s_2s_1s_3s_2s_1s_3$ of the Weyl group of type A_3. In this paper, we mainly treat (reduced) double Bruhat cells of the form $G^{e,v} := B \cap B_-\overline{v}B_-$, $L^{e,v} := N \cap B_-\overline{v}B_-$.

2.2. Factorization theorem

In this subsection, we shall introduce the isomorphisms between double Bruhat cell $G^{e,v}$ and $H \times (\mathbb{C}^\times)^{l(v)}$, and between $L^{e,v}$ and $(\mathbb{C}^\times)^{l(v)}$. For $i \in [1, r]$ and $t \in \mathbb{C}^\times$, we set $\alpha_i^\vee(t) := t^{h_i}$, where if t is written as $t = \exp(k)$ with some $k \in \mathbb{C}$, we set $t^{h_i} := \exp(kh_i)$.

For a reduced word $\mathbf{i} = (i_1, \cdots, i_n)$ ($i_1, \cdots, i_n \in [1, r]$), we define a map $x_{\mathbf{i}}^G : H \times \mathbb{C}^n \to G$ as

$$x_{\mathbf{i}}^G(a;t_1,\cdots,t_n) := a\cdot x_{i_1}(t_1)\cdots x_{i_n}(t_n). \tag{2.3}$$

Theorem 2.2. *[4,6] For $v\in W$ and its reduced word $\mathbf{i}$, the map $x_{\mathbf{i}}^G$ is a biregular isomorphism from $H\times(\mathbb{C}^\times)^{l(v)}$ to a Zariski open subset of $G^{e,v}$. The map $(\mathbb{C}^\times)^{l(v)}\to L^{e,v}$, $(t_1,\cdots,t_n)\mapsto x_{\mathbf{i}}^G(1;t_1,\cdots,t_n)$ is a biregular isomorphism to a Zariski open subset of $L^{e,v}$.*

For $\mathbf{i}=(i_1,\cdots,i_n)$ $(i_1,\cdots,i_n\in[1,r])$, we define a map $\overline{x}_{\mathbf{i}}^G : H\times(\mathbb{C}^\times)^n\to G^{e,v}$ as

$$\overline{x}_{\mathbf{i}}^G(a;t_1,\cdots,t_n) = a x_{i_1}(t_1)\alpha_{i_1}^\vee(t_1)x_{i_2}(t_2)\alpha_{i_2}^\vee(t_2)\cdots x_{i_n}(t_n)\alpha_{i_n}^\vee(t_n),$$

where $a\in H$ and $(t_1,\cdots,t_n)\in(\mathbb{C}^\times)^n$.

Now, let $G=\mathrm{SL}_{r+1}(\mathbb{C})$ and $c\in W$ be a Coxeter element such that a reduced word $\mathbf{i}$ of c^2 can be written as

$$\mathbf{i}=\begin{cases}(2,4,6,\cdots,r,1,3,5,\cdots,r-1,2,4,6,\cdots,r,1,3,5,\cdots,r-1) & \text{if } r \text{ is even},\\ (2,4,6,\cdots,r-1,1,3,5,\cdots,r,2,4,6,\cdots,r-1,1,3,5,\cdots,r) & \text{if } r \text{ is odd}.\end{cases} \tag{2.4}$$

Remark 2.3. In the rest of the paper, we use double indexed variables $Y_{s,j}$ $(s\in\mathbb{Z},\ j\in[1,r])$. If we see the variables $Y_{s,0}$, $Y_{s,j}$ $(r+1\le j)$ then we understand $Y_{s,0}=Y_{s,j}=1$. For example, if $l=1$ then $Y_{s,l-1}=1$.

Proposition 2.4. *In the above setting, the map $\overline{x}_{\mathbf{i}}^G$ is a biregular isomorphism between $H\times(\mathbb{C}^\times)^{2r}$ and a Zariski open subset of G^{e,c^2}.*

Proof. Let j_k be the k-th index of $\mathbf{i}$ in (2.4) from the right, which means that $\mathbf{i}=(j_{2r},\cdots,j_{r+1},j_r,\cdots,j_2,j_1)$. Note that $j_{i+r}=j_i$ $(1\le i\le r)$. In this proof, we use the notation

$$\mathbf{Y} := (Y_{1,j_r},\cdots,Y_{1,j_1},Y_{2,j_r},\cdots,Y_{2,j_2},Y_{2,j_1}),$$

for variables instead of $(t_1,\cdots,t_{2r})\in(\mathbb{C}^\times)^{2r}$.

We define a map $\phi : H\times(\mathbb{C}^\times)^{2r}\to H\times(\mathbb{C}^\times)^{2r}$,

$$\phi(a;\mathbf{Y}) = (\Phi_H(a;\mathbf{Y});\Phi_{1,j_r}(\mathbf{Y}),\cdots,\Phi_{1,j_1}(\mathbf{Y}),\Phi_{2,j_r}(\mathbf{Y}),\cdots,\Phi_{2,j_2}(\mathbf{Y}),\Phi_{2,j_1}(\mathbf{Y})),$$

where

$$\Phi_H(a;\mathbf{Y}) := a\cdot\prod_{i=1}^r\prod_{j=1}^2\alpha_i^\vee(Y_{j,i}), \tag{2.5}$$

and for $l\in\{1,2,\cdots,r\}$,

$$\Phi_{1,l}(\mathbf{Y}) := \begin{cases} \frac{(Y_{1,l-1}Y_{2,l-1})(Y_{1,l+1}Y_{2,l+1})}{Y_{1,l}Y_{2,l}^2} & \text{if } l \text{ is even,} \\ \frac{(Y_{2,l-1})(Y_{2,l+1})}{Y_{1,l}Y_{2,l}^2} & \text{if } l \text{ is odd,} \end{cases} \tag{2.6}$$

$$\Phi_{2,l}(\mathbf{Y}) := \begin{cases} \frac{(Y_{2,l-1})(Y_{2,l+1})}{Y_{2,l}} & \text{if } l \text{ is even,} \\ \frac{1}{Y_{2,l}} & \text{if } l \text{ is odd.} \end{cases} \tag{2.7}$$

Note that ϕ is a biregular isomorphism since we can construct the inverse map $\psi : H \times (\mathbb{C}^\times)^{2r} \to H \times (\mathbb{C}^\times)^{2r}$,

$$\psi(a;\mathbf{Y}) = (\Psi_H(a;\mathbf{Y}); \Psi_{1,j_r}(\mathbf{Y}), \cdots, \Psi_{1,j_1}(\mathbf{Y}), \Psi_{2,j_r}(\mathbf{Y}), \cdots, \Psi_{2,j_1}(\mathbf{Y}))$$

of ϕ as follows:

$$\Psi_{1,l}(\mathbf{Y}) := \begin{cases} (Y_{1,l-1}Y_{1,l}Y_{1,l+1}Y_{2,l-3}Y_{2,l-2}Y_{2,l+2}Y_{2,l+3})^{-1} & \text{if } l \text{ is even,} \\ (Y_{1,l}Y_{2,l-2}Y_{2,l-1}Y_{2,l+1}Y_{2,l+2})^{-1} & \text{if } l \text{ is odd,} \end{cases}$$

$$\Psi_{2,l}(\mathbf{Y}) := \begin{cases} (Y_{2,l-1}Y_{2,l}Y_{2,l+1})^{-1} & \text{if } l \text{ is even,} \\ \frac{1}{Y_{2,l}} & \text{if } l \text{ is odd,} \end{cases}$$

$$\Psi_H(a;\mathbf{Y}) := a \cdot \left(\prod_{i=1}^{r} \prod_{j=1}^{2} \alpha_i^\vee(\Psi_{j,i}(\mathbf{Y})) \right)^{-1}.$$

Then, the map ψ is the inverse map of ϕ.

Let us prove

$$\overline{x}_{\mathbf{i}}^G(a;\mathbf{Y}) = (x_{\mathbf{i}}^G \circ \phi)(a;\mathbf{Y}),$$

which implies that $\overline{x}_{\mathbf{i}}^G : H \times (\mathbb{C}^\times)^{2r} \to G^{e,c^2}$ is a biregular isomorphism by Theorem 2.2. First, it is known that for $1 \leq i,\ j \leq r$ and $s,\ t \in \mathbb{C}^\times$,

$$\alpha_j^\vee(s)x_i(t) = \begin{cases} x_i(s^2t)\alpha_i^\vee(s) & \text{if } i = j, \\ x_i(s^{-1}t)\alpha_j^\vee(s) & \text{if } |i-j| = 1, \\ x_i(t)\alpha_j^\vee(s) & \text{otherwise.} \end{cases} \tag{2.8}$$

On the other hand, it follows from the definition (2.3) of $x_{\mathbf{i}}^G$ and (2.5) that

$$(x_{\mathbf{i}}^G \circ \phi)(a;\mathbf{Y}) = a \cdot \left(\prod_{i=1}^{r} \prod_{s=1}^{2} \alpha_i^\vee(Y_{s,i}) \right) \times x_{j_r}(\Phi_{1,j_r}(\mathbf{Y})) \cdots x_{j_1}(\Phi_{1,j_1}(\mathbf{Y}))$$
$$\times x_{j_r}(\Phi_{2,j_r}(\mathbf{Y})) \cdots x_{j_2}(\Phi_{2,j_2}(\mathbf{Y}))x_{j_1}(\Phi_{2,j_1}(\mathbf{Y})).$$

For each even l $(1 \leq l \leq r)$, we can move

$$\alpha_1^\vee(Y_{1,1})\alpha_3^\vee(Y_{1,3})\cdots\alpha_{l-3}^\vee(Y_{1,l-3})\alpha_{l-1}^\vee(Y_{1,l-1})\prod_{i=l}^{r}\alpha_i^\vee(Y_{1,i})\prod_{i=1}^{r}\alpha_i^\vee(Y_{2,i})$$

to the right of $x_l(\Phi_{1,l}(\mathbf{Y}))$ by using the relations (2.8):

$$\begin{aligned}
&\alpha_1^\vee(Y_{1,1})\alpha_3^\vee(Y_{1,3})\cdots\alpha_{l-3}^\vee(Y_{1,l-3})\alpha_{l-1}^\vee(Y_{1,l-1})\left(\prod_{i=l}^{r}\alpha_i^\vee(Y_{1,i})\prod_{i=1}^{r}\alpha_i^\vee(Y_{2,i})\right)x_l(\Phi_{1,l}(\mathbf{Y}))\\
&= x_l\left(\Phi_{1,l}(\mathbf{Y})\frac{Y_{1,l}^2Y_{2,l}^2}{Y_{1,l-1}Y_{2,l-1}Y_{1,l+1}Y_{2,l+1}}\right)\alpha_1^\vee(Y_{1,1})\cdots\alpha_{l-1}^\vee(Y_{1,l-1})\prod_{i=l}^{r}\alpha_i^\vee(Y_{1,i})\prod_{i=1}^{r}\alpha_i^\vee(Y_{2,i})\\
&= x_l(Y_{1,l})\alpha_1^\vee(Y_{1,1})\alpha_3^\vee(Y_{1,3})\cdots\alpha_{l-3}^\vee(Y_{1,l-3})\alpha_{l-1}^\vee(Y_{1,l-1})\prod_{i=l}^{r}\alpha_i^\vee(Y_{1,i})\prod_{i=1}^{r}\alpha_i^\vee(Y_{2,i}).
\end{aligned}$$

Similarly, we can also move $\alpha_1^\vee(Y_{2,1})\alpha_3^\vee(Y_{2,3})\cdots\alpha_{l-1}^\vee(Y_{2,l-1})\prod_{i=l}^{r}\alpha_i^\vee(Y_{2,i})$ to the right of $x_l(\Phi_{2,l}(\mathbf{Y}))$:

$$\begin{aligned}
&\alpha_1^\vee(Y_{2,1})\alpha_3^\vee(Y_{2,3})\cdots\alpha_{l-3}^\vee(Y_{2,l-3})\alpha_{l-1}^\vee(Y_{2,l-1})\left(\prod_{i=l}^{r}\alpha_i^\vee(Y_{2,i})\right)x_l(\Phi_{2,l}(\mathbf{Y}))\\
&\qquad = x_l(Y_{2,l})\alpha_1^\vee(Y_{2,1})\alpha_3^\vee(Y_{2,3})\cdots\alpha_{l-3}^\vee(Y_{2,l-3})\alpha_{l-1}^\vee(Y_{2,l-1})\prod_{i=l}^{r}\alpha_i^\vee(Y_{2,i}).
\end{aligned}$$

For odd l, we obtain

$$\begin{aligned}
&\alpha_l^\vee(Y_{1,l})\alpha_{l+2}^\vee(Y_{1,l+2})\cdots\alpha_{j_1}^\vee(Y_{1,j_1})\alpha_1^\vee(Y_{2,1})\alpha_2^\vee(Y_{2,2})\cdots\alpha_r^\vee(Y_{2,r})x_l(\Phi_{1,l}(\mathbf{Y}))\\
&\qquad = x_l(Y_{1,l})\alpha_l^\vee(Y_{1,l})\alpha_{l+2}^\vee(Y_{1,l+2})\cdots\alpha_{j_1}^\vee(Y_{1,j_1})\alpha_1^\vee(Y_{2,1})\alpha_2^\vee(Y_{2,2})\cdots\alpha_r^\vee(Y_{2,r}),\\
&\alpha_l^\vee(Y_{2,l})\alpha_{l+2}^\vee(Y_{2,l+2})\cdots\alpha_{j_1}^\vee(Y_{2,j_1})x_l(\Phi_{2,l}(\mathbf{Y})) = x_l(Y_{2,l})\alpha_l^\vee(Y_{2,l})\alpha_{l+2}^\vee(Y_{2,l+2})\cdots\alpha_{j_1}^\vee(Y_{2,j_1}).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
(x_{\mathbf{i}}^G\circ\phi)(a;\mathbf{Y}) &= a\cdot x_{j_r}(Y_{1,j_r})\alpha_{j_r}^\vee(Y_{1,j_r})\cdots x_{j_1}(Y_{1,j_1})\alpha_{j_1}^\vee(Y_{1,j_1})\\
&\quad x_{j_r}(Y_{2,j_r})\alpha_{j_r}^\vee(Y_{2,j_r})\cdots x_{j_2}(Y_{2,j_2})\alpha_{j_2}^\vee(Y_{2,j_2})x_{j_1}(Y_{2,j_1})\alpha_{j_1}^\vee(Y_{2,j_1}) = \overline{x}_{\mathbf{i}}^G(a;\mathbf{Y}). \quad \square
\end{aligned}$$

3. Cluster algebras

Following [2,6,7,11], we review the definitions of cluster algebras and their generators called cluster variables. It is known that any coordinate ring of double Bruhat cells possesses the cluster algebra structure, and some minors play roles of the cluster variables [12]. We will clarify a relation between cluster variables on double Bruhat cells and crystal bases in Sect. 6.

We set $[-1,-l] := \{-1,-2,\cdots,-l\}$ for $l \in \mathbb{Z}_{>0}$. For $n,m \in \mathbb{Z}_{>0}$, let $x_1,\cdots,x_n,x_{n+1},\cdots,x_{n+m}$ be commutative variables and $\mathcal{F} := \mathbb{C}(x_1,\cdots,x_n,x_{n+1},\cdots,x_{n+m})$ be the field of rational functions.

3.1. Cluster algebras of geometric type

In this subsection, we recall the definitions of cluster algebras. Let $\tilde{B} = (b_{ij})_{1\le i\le n+m,\ 1\le j\le n}$ be an $(n+m)\times n$ integer matrix. The *principal part* B of $\tilde{B}$ is obtained from $\tilde{B}$ by deleting the last m rows. For $\tilde{B}$ and $k \in [1,n]$, the new $(n+m)\times n$ integer matrix $\mu_k(\tilde{B}) = (b'_{ij})$ is defined by

$$b'_{ij} := \begin{cases} -b_{ij} & \text{if } i=k \text{ or } j=k, \\ b_{ij} + \frac{|b_{ik}|b_{kj}+b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases}$$

One calls $\mu_k(\tilde{B})$ the *matrix mutation* in direction k of $\tilde{B}$. If there exists a positive integer diagonal matrix D such that DB is skew symmetric, we say B is *skew symmetrizable*. Then we also say $\tilde{B}$ is skew symmetrizable. It is easily verified that if $\tilde{B}$ is skew symmetrizable then $\mu_k(\tilde{B})$ is also skew symmetrizable [11, Proposition 3.6]. We can also verify that $\mu_k\mu_k(\tilde{B}) = \tilde{B}$. If $\mathbf{y} := (y_1,\cdots,y_n,x_{n+1},\cdots,x_{n+m})$ is an algebraically independent subset that generates $\mathcal{F}$, we call the pair $(\mathbf{y},\tilde{B})$ *seed*. For $1 \le k \le n$, a new cluster variable y'_k is defined by the following *exchange relation*.

$$y_k y'_k = \prod_{1\le i\le n+m,\ b_{ik}>0} y_i^{b_{ik}} + \prod_{1\le i\le n+m,\ b_{ik}<0} y_i^{-b_{ik}}. \tag{3.1}$$

Let $\mu_k(\mathbf{x})$ be the set of variables obtained from $\mathbf{y}$ by replacing y_k by y'_k. Ones call the pair $(\mu_k(\mathbf{y}),\mu_k(\tilde{B}))$ the *mutation* in direction k of the seed $(\mathbf{y},\tilde{B})$ and denote by $\mu_k((\mathbf{y},\tilde{B}))$.

Now, we can repeat this process of mutation and obtain a set of seeds inductively. Hence, each seed consists of an $(n+m)$-tuple of variables and a matrix. Ones call this $(n+m)$-tuple and matrix *cluster* and *exchange matrix* respectively. Variables in cluster is called *cluster variables*. In particular, the variables $x_{n+1},\cdots,x_{n+m}$ are called *frozen cluster variables*.

Definition 3.1. [6,11] Let $\tilde{B}$ be an integer matrix whose principal part is skew symmetrizable, $\mathbf{x} = (x_1,\cdots,x_{n+m})$ and $\Sigma = (\mathbf{x},\tilde{B})$ a seed. We set $\mathbb{A} := \mathbb{Z}[x_{n+1}^{\pm 1},\cdots,x_{n+m}^{\pm 1}]$. The *cluster algebra* (of geometric type) $\mathcal{A} = \mathcal{A}(\Sigma)$ over $\mathbb{A}$ associated with seed Σ is defined as the $\mathbb{A}$-subalgebra of $\mathcal{F}$ generated by all cluster variables in all seeds which can be obtained from Σ by sequences of mutations. Then Σ is called an *initial seed* of $\mathcal{A}$.

3.2. Cluster algebra $\mathcal{A}(\mathbf{i})$

In the rest of this section, let $G = \mathrm{SL}_{r+1}(\mathbb{C})$ be the complex simple algebraic group of type A_r and $\mathfrak{g} := \mathrm{Lie}(G)$.

Let $\mathbf{i} = (j_{2r}, \cdots, j_2, j_1)$ be the reduced word for $c^2 \in W$ defined in (2.4). Let us define the cluster algebra $\mathcal{A}(\mathbf{i})$ associated with $\mathbf{i}$. It satisfies that $\mathcal{A}(\mathbf{i}) \otimes \mathbb{C}$ is isomorphic to the coordinate ring $\mathbb{C}[G^{e,c^2}]$ of the double Bruhat cell [2]. Let $\lfloor x \rfloor$ denote the integer part of x.

Following [2], we define a quiver $\Gamma_{\mathbf{i}}$ as follows. The vertices of $\Gamma_{\mathbf{i}}$ are the variables x_k ($k \in [-1, -r] \cup [1, 2r]$). The arrows are as follows:

$$\begin{tikzcd}
x_{-1} \arrow[d] & x_{-2} \arrow[d] & x_{-3} \arrow[d] & x_{-4} \arrow[d] & & x_{-j_1} \arrow[d] & x_{-j_1-1} \arrow[d] \\
x_{2r-\lfloor \frac{r}{2} \rfloor} \arrow[r] \arrow[d] & x_{2r} \arrow[ul] \arrow[ur] \arrow[d] & x_{2r-\lfloor \frac{r}{2} \rfloor-1} \arrow[l] \arrow[r] \arrow[d] & x_{2r-1} \arrow[ul] \arrow[d] & \cdots & x_{r+1} \arrow[r] \arrow[d] & x_{r+\lfloor \frac{r}{2} \rfloor+1} \arrow[ul] \arrow[d] \\
x_{r-\lfloor \frac{r}{2} \rfloor} & x_r \arrow[ul] \arrow[ur] & x_{r-\lfloor \frac{r}{2} \rfloor-1} & x_{r-1} \arrow[ul] & & x_1 & x_{\lfloor \frac{r}{2} \rfloor+1} \arrow[ul]
\end{tikzcd} \tag{3.2}$$

where if r is odd then $j_1 = r$ and the vertices x_{-j_1-1}, $x_{r+\lfloor \frac{r}{2} \rfloor+1}$, $x_{\lfloor \frac{r}{2} \rfloor+1}$ and arrows adjacent to these vertices are removed. For k ($1 \leq k \leq \lfloor \frac{r+1}{2} \rfloor$), vertices and arrows around the vertex x_k in the quiver $\Gamma_{\mathbf{i}}$ are described as

$$\begin{tikzcd}
 & x_{-j_k+2} \arrow[d] & x_{-j_k+1} \arrow[d] & x_{-j_k} \arrow[d] & x_{-j_k-1} \arrow[d] & \\
\cdots & x_{r+k+1} \arrow[r] \arrow[d] & x_{r+\lfloor \frac{r}{2} \rfloor+k+1} \arrow[ul] \arrow[ur] \arrow[d] & x_{r+k} \arrow[l] \arrow[r] \arrow[d] & x_{r+\lfloor \frac{r}{2} \rfloor+k} \arrow[ul] \arrow[d] & \cdots \\
 & x_{k+1} & x_{\lfloor \frac{r}{2} \rfloor+k+1} \arrow[ul] \arrow[ur] & x_k & x_{\lfloor \frac{r}{2} \rfloor+k} \arrow[ul] &
\end{tikzcd} \tag{3.3}$$

For k ($\lfloor \frac{r+1}{2} \rfloor < k \leq r$), it is described as

$$\begin{tikzcd}
 & x_{-j_k+1} \arrow[d] & x_{-j_k} \arrow[d] & x_{-j_k-1} \arrow[d] & x_{-j_k-2} \arrow[d] & \\
\cdots & x_{r+k-\lfloor \frac{r}{2} \rfloor} \arrow[r] \arrow[d] & x_{r+k} \arrow[ul] \arrow[ur] \arrow[d] & x_{r+k-\lfloor \frac{r}{2} \rfloor-1} \arrow[l] \arrow[r] \arrow[d] & x_{r+k-1} \arrow[ul] \arrow[d] & \cdots \\
 & x_{k-\lfloor \frac{r}{2} \rfloor} & x_k \arrow[ul] \arrow[ur] & x_{k-\lfloor \frac{r}{2} \rfloor-1} & x_{k-1} \arrow[ul] &
\end{tikzcd} \tag{3.4}$$

Example 3.2. Let us consider the case $G = \mathrm{SL}_5(\mathbb{C})$ and $\mathbf{i} = (2, 4, 1, 3, 2, 4, 1, 3)$. The quiver $\Gamma_{\mathbf{i}}$ is described as

$$\begin{tikzcd}
x_{-1} \arrow[d] & x_{-2} \arrow[d] & x_{-3} \arrow[d] & x_{-4} \arrow[d] \\
x_6 \arrow[r] \arrow[d] & x_8 \arrow[ul] \arrow[ur] \arrow[d] & x_5 \arrow[l] \arrow[r] \arrow[d] & x_7 \arrow[ul] \arrow[d] \\
x_2 & x_4 \arrow[ul] \arrow[ur] & x_1 & x_3 \arrow[ul]
\end{tikzcd}$$

Next, let us define a matrix $\tilde{B} = \tilde{B}(\mathbf{i})$.

Definition 3.3. Let $\tilde{B}(\mathbf{i})$ be an integer matrix with rows labelled by all the indices in $[-1,-r] \cup [1,2r]$ and columns labelled by all the indices in $[r+1,2r]$. For $k \in [-1,-r] \cup [1,2r]$ and $l \in [r+1,2r]$, an entry b_{kl} of $\tilde{B}(\mathbf{i})$ is determined as follows:

$$b_{kl} := \begin{cases} 1 & \text{if } x_k \to x_l \text{ in } \Gamma_{\mathbf{i}}, \\ -1 & \text{if } x_l \to x_k \text{ in } \Gamma_{\mathbf{i}}, \\ 0 & \text{otherwise.} \end{cases}$$

The principal part $B(\mathbf{i})$ of $\tilde{B}(\mathbf{i})$ is the submatrix $(b_{i,j})_{i,j\in[r+1,2r]}$. We also define $\Sigma_{\mathbf{i}} := (\mathbf{x}, \tilde{B}(\mathbf{i}))$.

Proposition 3.4. *[2] $\tilde{B}(\mathbf{i})$ is skew symmetric.*

In general, for a family of variables $\mathbf{y} = (y_i)_{i\in[-1,-r]\cup[1,2r]}$ and a $(3r)\times r$ integer matrix $\tilde{B} = (b_{i,j})_{i\in[-1,-r]\cup[1,2r],j\in[r+1,2r]}$ whose submatrix $(b_{i,j})_{i,j\in[r+1,2r]}$ is skew symmetric, let $\Gamma((\mathbf{y}, \tilde{B}))$ be a quiver whose vertices are $y_{-r}, \cdots, y_{-1}, y_1 \cdots, y_{2r}$, and whose arrows are determined as follows: For $i \in [-1,-r] \cup [1,2r]$ and $j \in [r+1,2r]$, there exist $|b_{i,j}|$ arrows $y_i \to y_j$ (resp. $y_j \to y_i$) if $b_{i,j} > 0$ (resp. $b_{i,j} < 0$). We can easily check that $\Gamma((\mathbf{x}, \tilde{B}(\mathbf{i}))) = \Gamma_{\mathbf{i}}$. When there exist b arrows $y_i \to y_j$, we write $y_i \overset{b}{\to} y_j$ or $y_j \overset{-b}{\to} y_i$ $(b \geq 0)$.

Lemma 3.5. *[11] Let $(\mathbf{y}, \tilde{B})$ be a seed, where $\mathbf{y} = (y_i)_{i\in[-1,-r]\cup[1,2r]}$ and $\tilde{B} = (b_{i,j})$ is a $(3r) \times r$-skew symmetric matrix. For $k \in [r+1,2r]$, the quiver $\Gamma((\mu_k(\mathbf{y}), \mu_k(\tilde{B})))$ has vertices $y_{-r}, \cdots, y_{-1}, y_1, \cdots, y'_k, \cdots, y_{2r}$ and its arrows are determined as follows:*

1. *If $y_i \overset{b}{\to} y_k$ (resp. $y_k \overset{b}{\to} y_i$) in $\Gamma((\mathbf{y}, \tilde{B}))$ then $y'_k \overset{b}{\to} y_i$ (resp. $y_i \overset{b}{\to} y'_k$) in $\Gamma((\mu_k(\mathbf{y}), \mu_k(\tilde{B})))$.*
2. *We suppose that there exist arrows $y_i \overset{b}{\to} y_k$ and $y_k \overset{b'}{\to} y_j$ in $\Gamma((\mathbf{y}, \tilde{B}))$ with b, $b' \geq 0$ and either $i \in [r+1,2r]$ or $j \in [r+1,2r]$. If $y_j \overset{a}{\to} y_i$ in $\Gamma((\mathbf{y}, \tilde{B}))$, then $y_j \overset{a-bb'}{\to} y_i$ in $\Gamma((\mu_k(\mathbf{y}), \mu_k(\tilde{B})))$.*
3. *The rest of the arrows are the same as the one of $\Gamma((\mathbf{y}, \tilde{B}))$.*

Definition 3.6. [2] By Definition 3.1 and Proposition 3.4, we can construct the cluster algebra. We denote this cluster algebra by $\mathcal{A}(\mathbf{i})$.

3.3. Cluster algebras on double Bruhat cells

For a reduced expression $v = s_{j_n} s_{j_{n-1}} \cdots s_{j_1}$ of $v \in W$, its reduced word $\mathbf{i} = (j_n, \cdots, j_1)$ and $k \in [1,n]$, we set

$$v_{>k} = v_{>k}(\mathbf{i}) := s_{j_1} s_{j_2} \cdots s_{j_{n-k}}. \tag{3.5}$$

For $k \in [1, 2r]$, we define $\Delta(k; \mathbf{i})(x) := D_{[1,j_k], c^2_{>2r-k+1}(\mathbf{i})[1,j_k]}(x)$, and for $k \in [-1, -r]$, $\Delta(k; \mathbf{i})(x) := D_{[1,|k|], c^{-2}[1,|k|]}(x)$.

Finally, we set $F(\mathbf{i}) := \{\Delta(k; \mathbf{i})(x) | k \in [-1, -r] \cup [1, 2r]\}$. It is known that the set $F(\mathbf{i})$ is an algebraically independent generating set for the field of rational functions $\mathbb{C}(G^{e,c^2})$ [6, Theorem 1.12]. Then, we have the following.

Theorem 3.7. *[2,9,12] The isomorphism of fields $\varphi : \mathcal{F} \to \mathbb{C}(G^{e,c^2})$ defined by $\varphi(x_k) = \Delta(k; \mathbf{i})$ $(k \in [-1, -r] \cup [1, 2r])$ restricts to an isomorphism of algebras $\mathcal{A}(\mathbf{i}) \otimes \mathbb{C} \to \mathbb{C}[G^{e,c^2}]$.*

For $k \in [1, r]$, the correspondence of the initial cluster variables under the isomorphism in Theorem 3.7 are as follows:

$$\begin{aligned}
x_{-j_k} &\mapsto D_{[1,j_k], c^{-2}[1,j_k]} = D_{[1,j_k], s_{j_1} s_{j_2} \cdots s_{j_{2r}}[1,j_k]} = D_{[1,j_k], s_{j_1} s_{j_2} \cdots s_{j_{r+k}}[1,j_k]} \\
&= D_{[1,j_k], c^2_{>r-k}[1,j_k]}, \\
x_{r+k} &\mapsto D_{[1,j_k], c^2_{>r-k+1}[1,j_k]} = D_{[1,j_k], s_{j_1} s_{j_2} \cdots s_{j_{r+k-1}}[1,j_k]} \\
&= D_{[1,j_k], s_{j_1} s_{j_2} \cdots s_{j_k}[1,j_k]} = D_{[1,j_k], c^2_{>2r-k}[1,j_k]}, \\
x_k &\mapsto D_{[1,j_k], c^2_{>2r-k+1}[1,j_k]} = D_{[1,j_k], s_{j_1} s_{j_2} \cdots s_{j_{k-1}}[1,j_k]} = D_{[1,j_k],[1,j_k]}.
\end{aligned}$$

3.4. Finite type

Let $\mathcal{S}$ be the set of all seeds of a cluster algebra $\mathcal{A}$. If $\mathcal{S}$ is finite, then $\mathcal{A}$ is said to be of *finite type*. In this subsection, we shall review cluster algebras of finite type [8].

Let $B = (b_{ij})$ be an integer square matrix. The *Cartan counter part* of B is a generalized Cartan matrix $A = A(B) = (a_{i,j})$ defined as follows:

$$a_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ -|b_{i,j}| & \text{if } i \neq j. \end{cases}$$

Theorem 3.8. *[8] The cluster algebra $\mathcal{A}$ is of finite type if and only if there exists a seed $\Sigma = (\mathbf{y}, \tilde{B})$ such that $\mathcal{A} = \mathcal{A}(\Sigma)$ and the Cartan counter part $A(B)$ is a Cartan matrix of finite type, where B is the principal part of $\tilde{B}$.*

By this theorem, we can define the *type* of each cluster algebra of finite type mirroring the Cartan-Killing classification.

Let Φ be the root system associated with a Cartan matrix, with the set of simple roots $\Pi = \{\alpha_i |\ i \in [1, r]\}$ and the set of positive roots $\Phi_{>0}$. The set of *almost positive roots* $\Phi_{\geq -1}$ is defined by $\Phi_{\geq -1} := \Phi_{>0} \cup -\Pi$.

Theorem 3.9. *[8]*

(*i*) *For a cluster algebra $\mathcal{A}$ of finite type, the number of the cluster variables in $\mathcal{A}$ is equal to $|\Phi_{\geq -1}|$, where Φ is the root system associated with the Cartan matrix of the same type as $\mathcal{A}$.*

(*ii*) *Let $c \in W$ be a Coxeter element of G whose length $l(c)$ satisfies $l(c^2) = 2l(c) = 2\mathrm{rank}(G)$. Then the coordinate ring $\mathbb{C}[G^{e,c^2}]$ has a structure of cluster algebra of finite type under the isomorphism in Theorem 3.7, and its type is the Cartan-Killing type of G.*

4. Additive categorifications of cluster algebras

We fix an element $v \in W$ and set $n := l(v)$. In this section, we set $G = \mathrm{SL}_{r+1}(\mathbb{C})$ and review the additive categorifications of the coordinate rings $\mathbb{C}[L^{e,v}]$ according to [1,5,9].

4.1. Preprojective algebras and category $\mathcal{C}_v$

Let $Q = (Q_0, Q_1, s, t)$ be a Dynkin quiver of type A and

$$\Lambda = \mathbb{C}\overline{Q}/(\mathfrak{C})$$

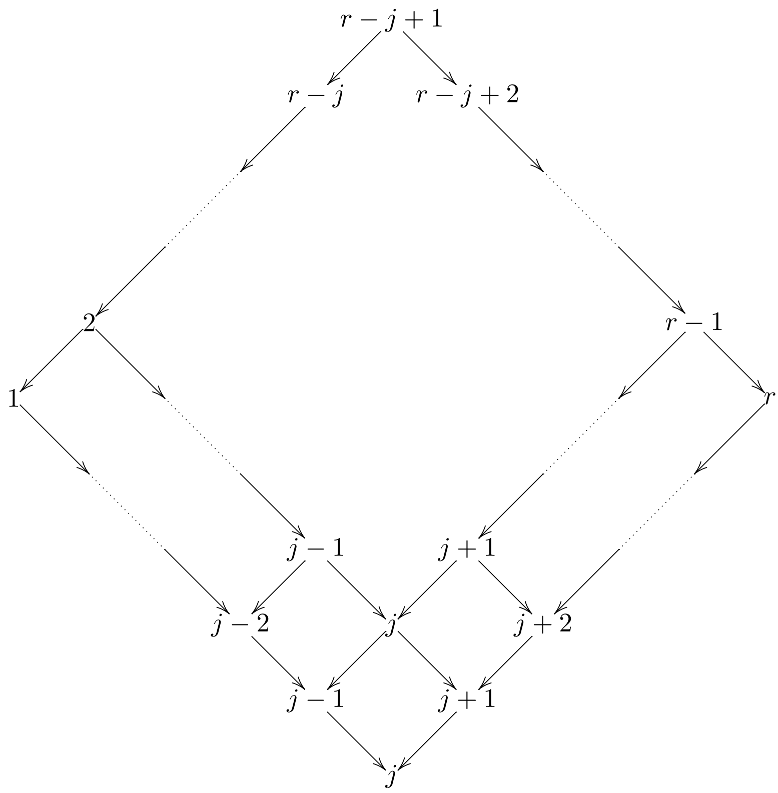
the associated preprojective algebra. Here $\overline{Q}$ is the double quiver of Q:

$$1 \rightleftarrows 2 \rightleftarrows 3 \rightleftarrows \cdots \rightleftarrows r\ ,$$

$\mathbb{C}\overline{Q}$ is its path algebra, and $(\mathfrak{C})$ is the ideal generated by

$$\mathfrak{C} = \sum_{a \in Q_1} (a^* a - a a^*),$$

where if $a \in Q_1$ is the arrow from i to j then a^* is the arrow in $\overline{Q}$ from j to i Let $\widehat{I}_1, \cdots, \widehat{I}_r$ be the indecomposable injective Λ-modules which have the simple socle isomorphic to $S_1, \cdots, S_r$, respectively, where S_i is the 1-dimensional simple Λ-module which corresponds to the vertex i in Q. The module $\widehat{I}_j$ is described as follows:



(4.1)

In (4.1), each vertex k $(1 \leq k \leq r)$ means a basis of $\widehat{I}_j$, and each arrow $k \to k+1$ (resp. $k \to k-1$) means the action of the edge $k \to k+1$ (resp. $k \to k-1$) $\in \Lambda$ on the basis k. The vertex $e_k \in \Lambda$ acts on each basis k' as

$$e_k.k' = \begin{cases} k' & \text{if } k = k', \\ 0 & \text{if } k \neq k'. \end{cases}$$

For example, the vertex $e_j \in \Lambda$ acts on the basis j located at the bottom of (4.1) identically, and all other paths act trivially. Thus, 1-dimensional submodule generated by this basis j is isomorphic to the simple module S_j.

Let $\mathrm{mod}(\Lambda)$ be the category of finite dimensional Λ-modules. Note that though in [9] the category $\mathrm{nil}(\Lambda)$ is treated, we consider the category $\mathrm{mod}(\Lambda)$ instead of $\mathrm{nil}(\Lambda)$ since $\mathrm{mod}(\Lambda) = \mathrm{nil}(\Lambda)$ holds in our setting. For $j \in Q_0$ and Λ-module X in $\mathrm{mod}(\Lambda)$, let $\mathrm{soc}_j(X)$ be the sum of all submodules U of X with $U \cong S_j$. For a sequence $(i_1, \cdots, i_t)$ $(i_1, i_2, \cdots, i_t \in Q_0)$, there exists a unique chain

$$0 = X_0 \subset X_1 \subset \cdots X_t \subset X$$

of submodules such that $X_p/X_{p-1} = \mathrm{soc}_{i_p}(X/X_{p-1})$ $(p = 1, 2, \cdots, t)$. We define $\mathrm{soc}_{(i_1,\cdots,i_t)}(X) := X_t$.

Let $v \in W$ and $\mathbf{i} = (j_n, \cdots, j_1)$ be its reduced word. Without loss of generality, we may assume that for each $j \in [1, r]$, there exist some $k \in [1, n]$ such that $j_k = j$. The Λ-modules $V_k = V_{\mathbf{i},k}$ $(k = 1, 2, \cdots, n) \in \mathrm{mod}(\Lambda)$ are defined as

$$V_k := V_{\mathbf{i},k} = \mathrm{soc}_{(j_k,\cdots,j_1)}(\widehat{I}_{j_k}).$$

Let $V_{\mathbf{i}} := \bigoplus_{k=1}^{n} V_k$ and $\mathcal{C}_{\mathbf{i}}$ be the full subcategory of $\mathrm{mod}(\Lambda)$ whose objects are factor modules of direct sums of finitely many copies of $V_{\mathbf{i}}$. For $j \in [1, r]$, let $m_j := \max\{1 \leq m \leq n | j_m = j\}$ and $I_{\mathbf{i},j} := V_{\mathbf{i},m_j}$. We also set $I_{\mathbf{i}} := I_{\mathbf{i},1} \oplus \cdots \oplus I_{\mathbf{i},r}$. The category $\mathcal{C}_{\mathbf{i}}$ and $I_{\mathbf{i}}$ depend on only v, and do not depend on the choice of reduced word $\mathbf{i}$. Thus, we define

$$\mathcal{C}_v := \mathcal{C}_{\mathbf{i}}, \quad I_v := I_{\mathbf{i}}.$$

A Λ-module C in $\mathcal{C}_v$ is called *$\mathcal{C}_v$-projective* (resp. *$\mathcal{C}_v$-injective*) if $\mathrm{Ext}^1_\Lambda(C, X) = 0$ (resp. $\mathrm{Ext}^1_\Lambda(X, C) = 0$) for all $X \in \mathcal{C}_v$. If C is $\mathcal{C}_v$-projective and $\mathcal{C}_v$-injective, C is said to be *$\mathcal{C}_v$-projective-injective.*

Theorem 4.1. *[3,9] The category $\mathcal{C}_v$ has r indecomposable $\mathcal{C}_v$-projective-injective modules, which are the indecomposable direct summands of I_v.*

Proposition 4.2. *Let $\mathbf{i}$ be the sequence in (2.4), j_k be the k-th index of $\mathbf{i}$ from the right, that is, $\mathbf{i} = (j_{2r}, \cdots, j_{r+1}, j_r, \cdots, j_1)$. Then $V_k = V_{\mathbf{i},k}$ $(1 \leq k \leq 2r)$ is given as follows:*

$$V_k = S_{j_k}, \quad \text{if } 1 \leq k \leq \lfloor \frac{r+1}{2} \rfloor, \tag{4.2}$$

$$V_k = \begin{tikzcd}[column sep=small] j_k-1 \arrow[dr] & & j_k+1 \arrow[dl] \\ & j_k & \end{tikzcd}, \quad \text{if } \lfloor \frac{r+1}{2} \rfloor + 1 \leq k \leq r, \tag{4.3}$$

$$V_k = \begin{tikzcd}[column sep=small] j_k-2 \arrow[dr] & & j_k \arrow[dl] \arrow[dr] & & j_k+2 \arrow[dl] \\ & j_k-1 \arrow[dr] & & j_k+1 \arrow[dl] & \\ & & j_k & & \end{tikzcd}, \quad \text{if } r+1 \leq k \leq \lfloor \frac{r+1}{2} \rfloor + r, \tag{4.4}$$

$$V_k = \begin{tikzcd}[column sep=tiny] j_k-3 \arrow[dr] & & j_k-1 \arrow[dl] \arrow[dr] & & j_k+1 \arrow[dl] \arrow[dr] & & j_k+3 \arrow[dl] \\ & j_k-2 \arrow[dr] & & j_k \arrow[dl] \arrow[dr] & & j_k+2 \arrow[dl] & \\ & & j_k-1 \arrow[dr] & & j_k+1 \arrow[dl] & & \\ & & & j_k & & & \end{tikzcd}, \quad \text{if } \lfloor \frac{r+1}{2} \rfloor + r + 1 \leq k \leq 2r. \tag{4.5}$$

In this case, we have $I_{c^2} = I_{\mathbf{i}} = V_{r+1} \oplus \cdots \oplus V_{2r}$.

Proof. For k with $1 \leq k \leq \lfloor \frac{r+1}{2} \rfloor$, to calculate $V_k = \mathrm{soc}_{(j_k,j_{k-1},\cdots,j_1)}(\hat{I}_{j_k})$, we consider the chain

$$0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_k \subset \hat{I}_{j_k}$$

such that $X_1 = \mathrm{soc}_{j_k}(\hat{I}_{j_k}) = S_{j_k}$, $X_2/X_1 = \mathrm{soc}_{j_{k-1}}(\hat{I}_{j_k}/X_1)$, $X_3/X_2 = \mathrm{soc}_{j_{k-2}}(\hat{I}_{j_k}/X_2)$, $\cdots, X_k/X_{k-1} = \mathrm{soc}_{j_1}(\hat{I}_{j_k}/X_{k-1})$. By (4.1), the module $\hat{I}_{j_k}/S_{j_k}$ has simple submodules isomorphic to S_{j_k-1} and S_{j_k+1}. Since $\hat{I}_{j_k}/S_{j_k}$ has no simple submodules isomorphic to $S_{j_{k-1}}, S_{j_{k-2}}, \cdots, S_{j_1}$, we have $X_1 = X_2 = \cdots = X_k$ and then

$$V_k = X_k = S_{j_k}.$$

Next, for $\lfloor \frac{r+1}{2} \rfloor + 1 \leq k \leq r$, we consider the chain

$$0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_k \subset \hat{I}_{j_k}$$

such that $X_1 = \mathrm{soc}_{j_k}(\hat{I}_{j_k}) = S_{j_k}$, $X_2/X_1 = \mathrm{soc}_{j_{k-1}}(\hat{I}_{j_k}/X_1), \cdots$. In the same way as in (4.2), we get $X_1 = X_2 = \cdots = X_{\lfloor \frac{r}{2} \rfloor} = S_{j_k}$. And $X_{\lfloor \frac{r}{2} \rfloor+1}/X_{\lfloor \frac{r}{2} \rfloor} = \mathrm{soc}_{j_{k-\lfloor \frac{r}{2} \rfloor}}(\hat{I}_{j_k}/X_{\lfloor \frac{r}{2} \rfloor}) = S_{j_k-1}$. So the module $X_{\lfloor \frac{r}{2} \rfloor+1}$ is described as

$$\begin{tikzcd}[column sep=small] j_k-1 \arrow[dr] & \\ & j_k \end{tikzcd}$$

Similarly, we obtain $X_{\lfloor \frac{r}{2} \rfloor+2}/X_{\lfloor \frac{r}{2} \rfloor+1} = \mathrm{soc}_{j_{k-\lfloor \frac{r}{2} \rfloor-1}}(\hat{I}_{j_k}/X_{\lfloor \frac{r}{2} \rfloor+1}) = S_{j_k+1}$. In the same way as in (4.2), we have $V_k = X_k = X_{k-1} = \cdots = X_{\lfloor \frac{r}{2} \rfloor+2}$. Thus, the module V_k is described as

$$\begin{tikzcd}[column sep=small] j_k-1 \arrow[dr] & & j_k+1 \arrow[dl] \\ & j_k & \end{tikzcd}$$

Next, for $r+1 \leq k \leq \lfloor \frac{r+1}{2} \rfloor + r$, we consider the chain

$$0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_k \subset \hat{I}_{j_k}$$

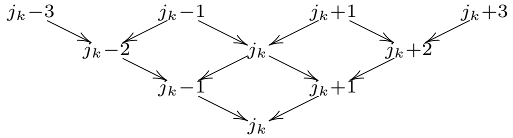
such that $X_1 = \mathrm{soc}_{j_k}(\hat{I}_{j_k}) = S_{j_k}$, $X_2/X_1 = \mathrm{soc}_{j_{k-1}}(\hat{I}_{j_k}/X_1), \cdots$. Note that $j_l = j_{l+r}$ $(1 \leq l \leq r)$. In the same way as in (4.2), we get $X_1 = X_2 = \cdots = X_{\lfloor \frac{r+1}{2} \rfloor-1} = S_{j_k}$. And $X_{\lfloor \frac{r+1}{2} \rfloor}/X_{\lfloor \frac{r+1}{2} \rfloor-1} = \mathrm{soc}_{j_{k-\lfloor \frac{r+1}{2} \rfloor+1}}(\hat{I}_{j_k}/X_{\lfloor \frac{r+1}{2} \rfloor-1}) = S_{j_k-1}$, where we set $S_j := 0$ for $j \leq 0$. We also get $X_{\lfloor \frac{r+1}{2} \rfloor+1}/X_{\lfloor \frac{r+1}{2} \rfloor} = \mathrm{soc}_{j_{k-\lfloor \frac{r+1}{2} \rfloor}}(\hat{I}_{j_k}/X_{\lfloor \frac{r+1}{2} \rfloor}) = S_{j_k+1}$, and

$$X_{\lfloor \frac{r+1}{2} \rfloor+1} = X_{\lfloor \frac{r+1}{2} \rfloor+2} = \cdots = X_{r-1}.$$

We also obtain $X_r/X_{r-1} = \mathrm{soc}_{j_{k-r+1}}(\hat{I}_{j_k}/X_{r-1}) = S_{j_k-2}$, $X_{r+1}/X_r = S_{j_k}$, $X_{r+2}/X_{r+1} = S_{j_k+2}$ and $X_{r+2} = X_{r+3} = \cdots = X_k$. Therefore, the module $X_k = V_k$ is described as

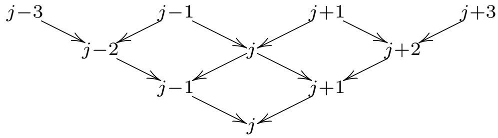
$$\begin{tikzcd}[column sep=small] j_k-2 \arrow[dr] & & j_k \arrow[dl] \arrow[dr] & & j_k+2 \arrow[dl] \\ & j_k-1 \arrow[dr] & & j_k+1 \arrow[dl] & \\ & & j_k & & \end{tikzcd}$$

Finally, for $\lfloor \frac{r+1}{2} \rfloor + r + 1 \leq k \leq 2r$, we can verify that the module V_k is described as:

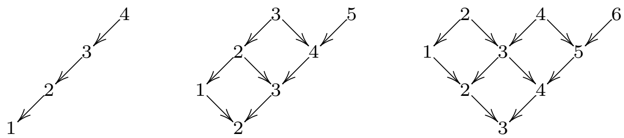


by the same argument as in (4.2), (4.3) and (4.4). $\square$

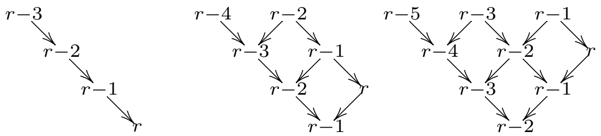
Remark 4.3. When we see the quiver



or its subquiver, if $j = 1$, 2 or 3, we understand it means



respectively. Similarly, if $j = r$, $r - 1$ or $r - 2$, we understand it means



4.2. Mutation

For a Λ-module T in $\mathrm{mod}(\Lambda)$, let $\mathrm{add}(T)$ denote the subcategory of $\mathrm{mod}(\Lambda)$ whose objects are all Λ-modules which are isomorphic to finite direct sums of direct summands of T.

Definition 4.4. [1,5,9]

(i) A Λ-module T is *rigid* if $\mathrm{Ext}^1_\Lambda(T, T) = 0$.
(ii) For a rigid module T in $\mathcal{C}_v$, we say T is a $\mathcal{C}_v$-*cluster-tilting* module if $\mathrm{Ext}^1_\Lambda(T, X) = 0$ with $X \in \mathcal{C}_v$ implies $X \in \mathrm{add}(T)$.
(iii) A Λ-module T is said to be *basic*, if it is decomposed to a direct sum of pairwise non-isomorphic indecomposable modules.

(iv) Let T, X and $Y \in \mathrm{mod}(\Lambda)$. A morphism $f \in \mathrm{Hom}_\Lambda(X, Y)$ (resp. $f \in \mathrm{Hom}_\Lambda(Y, X)$) is said to be a *left* (resp. *right*) add(T)-*approximation* of X if $Y \in \mathrm{add}(T)$ and for an arbitrary $Y' \in \mathrm{add}(T)$ and $f' \in \mathrm{Hom}_\Lambda(X, Y')$ (resp. $f' \in \mathrm{Hom}_\Lambda(Y', X)$), there exists $g \in \mathrm{Hom}_\Lambda(Y, Y')$ (resp. $g \in \mathrm{Hom}_\Lambda(Y', Y)$) and $f' = g \circ f$ (resp. $f' = f \circ g$).

(v) For V, $W \in \mathrm{mod}(\Lambda)$, a morphism $f \in \mathrm{Hom}_\Lambda(V, W)$ is said to be *left* (resp. *right*) *minimal* if every endomorphism $g \in \mathrm{End}_\Lambda(W)$ (resp. $g \in \mathrm{End}_\Lambda(V)$) such that $g \circ f = f$ (resp. $f \circ g = f$) is an isomorphism.

Proposition 4.5. *[5,9,10] Let $T = T_1 \oplus T_2 \oplus \cdots \oplus T_n$ be a basic $\mathcal{C}_v$-cluster-tilting object. We suppose that the $\{T_i\}_{i=1,2,\cdots,n}$ are indecomposable summands of T and $T_{n-r+1}, \cdots, T_n$ are the $\mathcal{C}_v$-projective-injective modules. Then for $k \in \{1, 2, \cdots, n-r\}$, there is a short exact sequence*

$$0 \to T_k \xrightarrow{f} \overline{T_k} \xrightarrow{g} T_k^* \to 0 \tag{4.6}$$

such that

(i) f is a left minimal left $\mathrm{add}(T/T_k)$*-approximation,*
(ii) g is a right minimal right $\mathrm{add}(T/T_k)$*-approximation,*
(iii) T_k^* is an indecomposable Λ-module,
(iv) $T_k^* \notin \mathrm{add}(T)$,
(v) $T/T_k \oplus T_k^*$ is a basic $\mathcal{C}_v$-cluster-tilting object.

Definition 4.6. [5,9] In the setting of the previous proposition, the mutation $\mu_{T_k}(T)$ of T in direction T_k is defined as

$$\mu_{T_k}(T) := T/T_k \oplus T_k^*. \tag{4.7}$$

We call the short exact sequence (4.6) in Proposition 4.5 the *exchange sequence* associated to the direct summand T_k of T.

For a basic module $T = T_1 \oplus \cdots \oplus T_n$ in $\mathcal{C}_v$, let Γ_T be the quiver of $\mathrm{End}_\Lambda(T)^{\mathrm{op}}$, that is, $\mathrm{End}_\Lambda(T)^{\mathrm{op}} \cong \mathbb{C}\Gamma_T/(R)$ with an admissible ideal (R) [1]. Setting

$$\mathrm{Rad}(T_i, T_j) = \begin{cases} \mathrm{Hom}_\Lambda(T_i, T_j) & \text{if } i \neq j, \\ \{\text{nilpotent elements of } \mathrm{End}_\Lambda(T_i)\} & \text{if } i = j, \end{cases}$$

we have the following:

Lemma 4.7. *[1,5] The quiver Γ_T has n vertices indexed by $\{1, 2, \cdots, n\}$, and for $1 \leq i, j \leq n$, the number of arrows $j \to i$ is equal to the dimension of the space*

$$\frac{\mathrm{Rad}(T_i, T_j)}{\sum_{k=1}^n \mathrm{Rad}(T_k, T_j) \circ \mathrm{Rad}(T_i, T_k)}.$$

Definition 4.8. Let $T = T_1 \oplus \cdots \oplus T_n$ be a basic module in $\mathcal{C}_v$. For $i, j \in [1, n]$ and a non-zero homomorphism $f \in \mathrm{Hom}_\Lambda(T_i, T_j)$, it is said that f is *factorizable* in the direct summands of T if it belongs to $\sum_{k=1}^n \mathrm{Rad}(T_k, T_j) \circ \mathrm{Rad}(T_i, T_k)$.

Let $B(\Gamma_T) = (b_{i,j})$ denote $n \times (n-r)$-matrix defined by

$$b_{i,j} = (\text{number of arrows } j \to i \text{ in } \Gamma_T) - (\text{number of arrows } i \to j \text{ in } \Gamma_T).$$

For $\mathbf{i} = (j_n, \cdots, j_1) \in Q_0^n$, we define a quiver $\overline{\Gamma}_{\mathbf{i}}$ as follows: For $k \in [1, n]$, we use the notation

$$k^- := \max\{0, 1 \le s \le k-1 | i_s = i_k\},$$

$$k^+ := \min\{k+1 \le s \le n, n+1 | i_s = i_k\}.$$

The vertices of $\overline{\Gamma}_{\mathbf{i}}$ are $1, 2, \cdots, n$. For two vertices $k, l \in [1, n]$ with $l < k$, there exists an arrow $k \to l$ (resp. $l \to k$) if and only if $l = k^-$ (resp. $k < l^+ \le k^+$ and $a_{i_k, i_l} < 0$).

Theorem 4.9. *[3,9,10] Let $n = l(v)$ and $\mathbf{i} = (j_n, \cdots, j_1)$ be a reduced word of v.*

(i) The module $V_{\mathbf{i}}$ defined in 4.1 is a basic $\mathcal{C}_v$-cluster-tilting object and $\Gamma_{V_{\mathbf{i}}} = \overline{\Gamma}_{\mathbf{i}}$.
(ii) Let $T = T_1 \oplus T_2 \oplus \cdots \oplus T_n$ be a basic $\mathcal{C}_v$-cluster-tilting object. For $1 \le k \le n-r$, we have $B(\Gamma_{\mu_{T_k}(T)}) = \mu_k(B(\Gamma_T))$.
(iii) For a basic $\mathcal{C}_v$-cluster-tilting object $T = T_1 \oplus T_2 \oplus \cdots \oplus T_n$ and $1 \le k \le n-r$, the exchange sequence associated to the direct summand T_k of T is

$$0 \to T_k \to \bigoplus_{i \to k \text{ in } \Gamma_T} T_i \to T_k^* \to 0.$$

Example 4.10. Let $\mathbf{i}$ be the reduced word in (2.4). By Theorem 4.9 (i), for $1 \le k \le \lfloor \frac{r+1}{2} \rfloor$, the quiver $\Gamma_{V_{\mathbf{i}}}$ is described as

$$\begin{tikzcd}
\cdots & r+k+1 \arrow[r] \arrow[d] & r+\lfloor \frac{r}{2} \rfloor+k+1 \arrow[d] & r+k \arrow[l] \arrow[r] \arrow[d] & r+\lfloor \frac{r}{2} \rfloor+k \arrow[d] & \cdots \\
 & k+1 \arrow[r] & \lfloor \frac{r}{2} \rfloor+k+1 \arrow[ul] \arrow[ur] & k \arrow[l] \arrow[r] & \lfloor \frac{r}{2} \rfloor+k \arrow[ul] &
\end{tikzcd} \tag{4.8}$$

Proposition 4.11. *In the setting of Proposition 4.2, let*

$$0 \to V_k \to \overline{V_k} \to V_k^* \to 0$$

be the exchange sequence associated to the direct summand V_k of $V_{\mathbf{i}}$ $(1 \le k \le r)$. Then the indecomposable module V_k^* is given as follows:

(1) *For k with $1 \le k < \lfloor \frac{r+1}{2} \rfloor$, the modules V_k^* and $V^*_{\lfloor \frac{r+1}{2} \rfloor}$ are given as*

$$\begin{tikzcd}[column sep=small]
j_k-2 \arrow[dr] & & j_k \arrow[dl] \arrow[dr] & & j_k+2 \arrow[dl] \\
& j_k-1 & & j_k+1 &
\end{tikzcd} \quad , \quad
\begin{tikzcd}[column sep=small]
& 3 \arrow[dl] \\
2 &
\end{tikzcd} \tag{4.9}$$

respectively.

(2) *For k with $\lfloor \frac{r+1}{2} \rfloor + 1 \le k \le r$, the module V_k^* is given as*

$$\begin{tikzcd}[column sep=small]
j_k-3 \arrow[dr] & & j_k-1 \arrow[dl] \arrow[dr] & & j_k+1 \arrow[dl] \arrow[dr] & & j_k+3 \arrow[dl] \\
& j_k-2 \arrow[dr] & & j_k \arrow[dl] \arrow[dr] & & j_k+2 \arrow[dl] & \\
& & j_k-1 & & j_k+1 & &
\end{tikzcd} \tag{4.10}$$

Proof. (1) For $1 \le k \le \lfloor \frac{r+1}{2} \rfloor$, recall that $V_k = S_{j_k}$. In $\{V_i |\ 1 \le i \le 2r,\ i \ne k\}$, the module V_{r+k} has the simple socle isomorphic to S_{j_k} and the others do not so since their simple socles are S_l $(l \ne j_k)$ by (4.3), (4.4), (4.5). The module V_{r+k} is described as

$$\begin{tikzcd}[column sep=small]
j_k-2 \arrow[dr] & & j_k \arrow[dl] \arrow[dr] & & j_k+2 \arrow[dl] \\
& j_k-1 \arrow[dr] & & j_k+1 \arrow[dl] & \\
& & j_k & &
\end{tikzcd}$$

and bottom j_k means a basis generating the simple socle isomorphic to S_{j_k} ((4.1), (4.4)). Hence, there exists an injective homomorphism $V_k \to V_{r+k}$, and its image is the simple socle. By the above argument, we have $\mathrm{Hom}_\Lambda(V_k, V_{r+k}) \cong \mathbb{C}$ and $\mathrm{Rad}(V_k, V_t) = \{0\}$ for $t \ne r+k$. We get $\overline{V_k} = V_{r+k}$ by Lemma 4.7 and Theorem 4.9 (iii), which yields that V_k^* $(1 \le k < \lfloor \frac{r+1}{2} \rfloor)$ and $V^*_{\lfloor \frac{r+1}{2} \rfloor}$ are described as

$$\begin{tikzcd}[column sep=small]
j_k-2 \arrow[dr] & & j_k \arrow[dl] \arrow[dr] & & j_k+2 \arrow[dl] \\
& j_k-1 & & j_k+1 &
\end{tikzcd} \quad , \quad
\begin{tikzcd}[column sep=small]
& 3 \arrow[dl] \\
2 &
\end{tikzcd}$$

respectively.

(2) Next, for $\lfloor \frac{r+1}{2} \rfloor + 1 \le k \le r$, the module V_k is given as (4.3). The module V_{r+k} is described as (4.5) and it has the submodule isomorphic to V_k, which is generated by the basis $j_k - 1$, j_k and $j_k + 1$ lower one in (4.5). Let c_{j_k-1}, c_{j_k} and c_{j_k+1} denote these three bases. Thus, there exists an injective homomorphism $V_k \to V_{r+k}$. Since V_k has the simple quotients isomorphic to S_{j_k-1}, S_{j_k+1}, there exist surjective homomorphisms $V_k \to V_{k-\lfloor \frac{r}{2} \rfloor} = S_{j_k-1}$ and $V_k \to V_{k-\lfloor \frac{r}{2} \rfloor -1} = S_{j_k+1}$ (note that $j_{k-\lfloor \frac{r}{2} \rfloor} = j_k - 1$ and $j_{k-\lfloor \frac{r}{2} \rfloor - 1} = j_k + 1$). The modules $V_{r+k-\lfloor \frac{r}{2} \rfloor}$ and $V_{r+k-\lfloor \frac{r}{2} \rfloor -1}$ have the simple submodules isomorphic to S_{j_k-1} and S_{j_k+1} respectively. However, homomorphisms $V_k \to V_{r+k-\lfloor \frac{r}{2} \rfloor}$ and $V_k \to V_{r+k-\lfloor \frac{r}{2} \rfloor -1}$ are factorizable in the direct summands of $V_{\mathbf{i}}$ since they are equal

to the composite maps $V_k \to V_{k-\lfloor\frac{r}{2}\rfloor} \to V_{r+k-\lfloor\frac{r}{2}\rfloor}$ and $V_k \to V_{k-\lfloor\frac{r}{2}\rfloor-1} \to V_{r+k-\lfloor\frac{r}{2}\rfloor-1}$ respectively. Moreover, we see that $\mathrm{Rad}(V_k, V_t) = 0$ for $t \neq r+k,\ k-\lfloor\frac{r}{2}\rfloor,\ k-\lfloor\frac{r}{2}\rfloor-1$ since V_t does not have submodule isomorphic to V_k, S_{j_k-1} and S_{j_k+1}. From this, the homomorphisms $V_k \to V_{r+k}$, $V_k \to V_{k-\lfloor\frac{r}{2}\rfloor}$ and $V_k \to V_{k-\lfloor\frac{r}{2}\rfloor-1}$ are not factorizable in the direct summands of $V_{\mathbf{i}}$. Therefore, the exchange sequence associated to the direct summand V_k of $V_{\mathbf{i}}$ is

$$0 \to V_k \to V_{r+k} \oplus S_{j_k-1} \oplus S_{j_k+1} \to V_k^* \to 0$$

by Lemma 4.7 and Theorem 4.9 (iii). The image of the homomorphism $V_k \to V_{r+k} \oplus S_{j_k-1} \oplus S_{j_k+1}$ is 3-dimensional and it can be explicitly written as $\mathbb{C}(c_{j_k-1} + d_{j_k-1}) \oplus \mathbb{C}(c_{j_k}) \oplus \mathbb{C}(c_{j_k+1} + e_{j_k+1})$ with some non-zero elements $d_{j_k-1} \in S_{j_k-1}$, $e_{j_k+1} \in S_{j_k+1}$. By the above argument, the module $V_k^* = (V_{r+k} \oplus S_{j_k-1} \oplus S_{j_k+1})/(\mathbb{C}(c_{j_k-1} \oplus d_{j_k-1} \oplus e_{j_k+1}) \oplus \mathbb{C}(c_{j_k} \oplus d_{j_k-1} \oplus e_{j_k+1}) \oplus \mathbb{C}(c_{j_k+1} \oplus d_{j_k-1} \oplus e_{j_k+1}))$ is described as follows:

$$\begin{tikzcd}[column sep=small]
j_k-3 \arrow[dr] & & j_k-1 \arrow[dl] \arrow[dr] & & j_k+1 \arrow[dl] \arrow[dr] & & j_k+3 \arrow[dl] \\
& j_k-2 \arrow[dr] & & j_k \arrow[dl] \arrow[dr] & & j_k+2 \arrow[dl] & \\
& & j_k-1 & & j_k+1 & &
\end{tikzcd}$$

□

Proposition 4.12. *The modules* $(\mu_{V_r}\mu_{V_{\lfloor\frac{r+1}{2}\rfloor}} V_{\mathbf{i}})_r$ *and* $(\mu_{V_{k-\lfloor\frac{r}{2}\rfloor-1}}\mu_{V_k} V_{\mathbf{i}})_{k-\lfloor\frac{r}{2}\rfloor-1}$ $(\lfloor\frac{r+1}{2}\rfloor + 2 \leq k \leq r)$ *are described as*

$$\begin{tikzcd}[column sep=tiny, row sep=small]
& 3 \arrow[dl] \arrow[dr] & & 5 \arrow[dl] \\
2 \arrow[dr] & & 4 \arrow[dl] & \\
& 3 & &
\end{tikzcd}
,\qquad
\begin{tikzcd}[column sep=tiny, row sep=small]
j_k-3 \arrow[dr] & & j_k-1 \arrow[dl] \arrow[dr] & & j_k+1 \arrow[dl] \arrow[dr] & & j_k+3 \arrow[dl] \\
& j_k-2 \arrow[dr] & & j_k \arrow[dl] & & j_k+2 & \\
& & j_k-1 & & & &
\end{tikzcd}$$

respectively. Note that $j_{\lfloor\frac{r+1}{2}\rfloor} = 1$.

4.3. Cluster algebra structure of $\mathbb{C}[L^{e,v}]$

For a Λ-module X and a sequence $\mathbf{k} = (k_1, \cdots, k_s)$ $(k_t \in [1, r])$, let $\mathcal{F}_{\mathbf{k},X}$ denote the projective variety of composition series of X:

$$0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_s = X,$$

such that each subfactor X_t/X_{t-1} is isomorphic to the simple Λ-module S_{k_t} $(1 \leq t \leq s)$. Recall that we set $x_i(t) := \exp(te_i)$ in (2.1).

Proposition 4.13. *[5,9] For each Λ-module X in* $\mathrm{mod}(\Lambda)$*, there exists a unique function* $\varphi_X \in \mathbb{C}[N]$ *such that for any sequence* $\mathbf{i} = (i_1, \cdots, i_k)$ $(1 \leq i_1, \cdots, i_k \leq r)$*,*

$$\varphi_X(x_{i_1}(t_1)x_{i_2}(t_2)\cdots x_{i_k}(t_k)) = \sum_{\mathbf{a}=(a_1,\cdots,a_k)\in(\mathbb{Z}_{\geq 0})^k} \chi_c(\mathcal{F}_{\mathbf{i}^{\mathbf{a}},X})\frac{t_1^{a_1}\cdots t_k^{a_k}}{a_1!\cdots a_k!},$$

where χ_c is the Euler characteristic, and for $\mathbf{a}=(a_1,a_2,\cdots,a_k)$,

$$\mathbf{i}^{\mathbf{a}} := (\underbrace{i_1,\cdots,i_1}_{a_1},\underbrace{i_2,\cdots,i_2}_{a_2},\cdots,\underbrace{i_k,\cdots,i_k}_{a_k}).$$

Note that we can write $x_{i_1}(t_1)x_{i_2}(t_2)\cdots x_{i_k}(t_k) = x^G_{\mathbf{i}}(1;t_1,\cdots,t_k)$, where 1 is the identity element of H and $x^G_{\mathbf{i}}$ is defined in (2.3).

For a Λ-module X in $\mathrm{mod}(\Lambda)$ and $\mathbf{i}=(i_1,\cdots,i_k)$, $\mathbf{a}=(a_1,a_2,\cdots,a_k)\in(\mathbb{Z}_{\geq 0})^k$, let $\mathcal{F}_{\mathbf{i},\mathbf{a},X}$ be the projective variety of partial composition series of X

$$0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_k = X$$

such that each subfactor X_t/X_{t-1} is isomorphic to $S_{i_t}^{a_t}$ for all $1\leq t\leq k$. Then we have $\chi_c(\mathcal{F}_{\mathbf{i}^{\mathbf{a}},X}) = \chi_c(\mathcal{F}_{\mathbf{i},\mathbf{a},X})a_1!a_2!\cdots a_k!$ [9]. Therefore, in the setting of Proposition 4.13,

$$\varphi_X(x_{i_1}(t_1)x_{i_2}(t_2)\cdots x_{i_k}(t_k)) = \sum_{\mathbf{a}=(a_1,\cdots,a_k)\in(\mathbb{Z}_{\geq 0})^k} \chi_c(\mathcal{F}_{\mathbf{i},\mathbf{a},X})t_1^{a_1}\cdots t_k^{a_k}. \tag{4.11}$$

Example 4.14. In the setting of Proposition 4.2 and 4.11, let us calculate φ_{V_k} $(1\leq k\leq r)$ and $\varphi_{(\mu_k V_{\mathbf{i}})_k}$ $(1\leq k\leq r)$. We set $\mathbf{Y} := (Y_{1,j_r},\cdots,Y_{1,j_1},Y_{2,j_r},\cdots,Y_{2,j_2},Y_{2,j_1})$.

For $\mathbf{i}$ in (2.4), let us consider the variety of flags $\mathcal{F}_{\mathbf{i}^{\mathbf{a}},V_k}$. Let j_k be the k-th index of $\mathbf{i}$ from the right. We write $\mathbf{a}\in(\mathbb{Z}_{\geq 0})^{2r}$ as follows:

$$\mathbf{a} = (a_{1,j_r},\cdots,a_{1,j_2},a_{1,j_1},a_{2,j_r},\cdots,a_{2,j_2},a_{2,j_1}).$$

By Proposition 4.2, for $1\leq k\leq \lfloor\frac{r+1}{2}\rfloor$, since $V_k = S_{j_k}$, if $\mathcal{F}_{\mathbf{i}^{\mathbf{a}},V_k}\neq\phi$ then $\mathbf{i}^{\mathbf{a}} = (j_k)$, which implies $a_{1,j_k}=1$ and other $a_{1,j},a_{2,j}$ are equal to 0, or $a_{2,j_k}=1$ and other $a_{1,j},a_{2,j}$ are equal to 0. In this case, $\mathcal{F}_{\mathbf{i}^{\mathbf{a}},V_k}$ is a point $(=(0\subset S_{j_k}=V_k))$. Thus, Proposition 4.13 means that

$$\varphi_{V_k}(x^G_{\mathbf{i}}(1;\mathbf{Y})) = Y_{1,j_k}+Y_{2,j_k}.$$

Next, for $\lfloor\frac{r+1}{2}\rfloor+1\leq k\leq r$, the module V_k is described as (4.3). If $\mathcal{F}_{\mathbf{i}^{\mathbf{a}},V_k}\neq\phi$ then $\mathbf{i}^{\mathbf{a}} = (j_k,j_k-1,j_k+1)$ or $\mathbf{i}^{\mathbf{a}} = (j_k,j_k+1,j_k-1)$, which implies

$$\begin{gathered} a_{1,j_k}=a_{1,j_k-1}=a_{1,j_k+1}=1, \quad \text{or} \quad a_{1,j_k}=a_{1,j_k-1}=a_{2,j_k+1}=1, \\ \text{or}\quad a_{1,j_k}=a_{2,j_k-1}=a_{2,j_k+1}=1, \quad \text{or} \quad a_{2,j_k}=a_{2,j_k-1}=a_{2,j_k+1}=1, \\ \text{or}\quad a_{1,j_k}=a_{1,j_k+1}=a_{2,j_k-1}=1, \end{gathered}$$

and the all others are equal to 0. Thus, by Proposition 4.13,

$$\varphi_{V_k}(x^G_{\mathbf{i}}(1;\mathbf{Y})) = Y_{1,j_k}Y_{1,j_k-1}Y_{1,j_k+1} + Y_{1,j_k}Y_{1,j_k-1}Y_{2,j_k+1} + Y_{1,j_k}Y_{2,j_k-1}Y_{2,j_k+1}$$
$$+Y_{2,j_k}Y_{2,j_k-1}Y_{2,j_k+1} + Y_{1,j_k}Y_{2,j_k-1}Y_{1,j_k+1}. \tag{4.12}$$

Similarly, it follows from (4.9) that for $1 \le k < \lfloor \frac{r+1}{2} \rfloor$,

$$\varphi_{(\mu_k V)_k}(x^G_{\mathbf{i}}(1;\mathbf{Y})) = \sum_{(*)} Y_{a,j_k-1}Y_{b,j_k+1}Y_{c,j_k-2}Y_{d,j_k}Y_{e,j_k+2}, \tag{4.13}$$

where $(*)$ is condition for a, b, c, d and $e : 1 \le a \le c \le 2$, $1 \le b \le d \le 2$, $1 \le a \le d \le 2$ and $1 \le b \le e \le 2$. And

$$\varphi_{(\mu_{\lfloor \frac{r+1}{2} \rfloor} V)_{\lfloor \frac{r+1}{2} \rfloor}}(x^G_{\mathbf{i}}(1;\mathbf{Y})) = \sum_{1\le a\le b\le 2} Y_{a,2}Y_{b,3}. \tag{4.14}$$

For $\lfloor \frac{r+1}{2} \rfloor + 1 \le k \le r$, it follows from (4.10) that

$$\varphi_{(\mu_k V)_k}(x^G_{\mathbf{i}}(1;\mathbf{Y}))$$
$$= Y_{1,j_k-1}Y_{1,j_k+1}Y_{2,j_k-2}Y_{2,j_k}Y_{2,j_k+2}Y_{2,j_k-3}Y_{2,j_k-1}Y_{2,j_k+1}Y_{2,j_k+3}. \tag{4.15}$$

For two basic $\mathcal{C}_v$-cluster-tilting modules R, R', we denote $R \sim R'$ if R is obtained from R' by a sequence of mutations (4.7).

For $v \in W$, let $L(\mathcal{C}_v) := L(\mathcal{C}_v, V_{\mathbf{i}})$ be the subalgebra of $\mathbb{C}[N]$ generated by $\{\varphi_{R_1}, \varphi_{R_2}, \cdots, \varphi_{R_n} | R_1 \oplus R_2 \oplus \cdots \oplus R_n \in \mathrm{Ob}(\mathcal{C}_v) \quad \sim V_{\mathbf{i}}\}$. Let $\tilde{L}(\mathcal{C}_v)$ be the algebra obtained from $L(\mathcal{C}_v)$ by formally inverting the elements φ_P for all $\mathcal{C}_v$-projective-injective module P. That is, $\tilde{L}(\mathcal{C}_v)$ is the localization of the ring $L(\mathcal{C}_v)$ with respect to φ_P. It follows by Theorem 4.9 that $\tilde{L}(\mathcal{C}_v)$ has a cluster algebra structure.

Theorem 4.15. *[9] For $v \in W$, the coordinate ring $\mathbb{C}[L^{e,v}]$ has a cluster algebra structure. For each reduced word $\mathbf{i} = (j_n, \cdots, j_1)$ of v, the pair $((\varphi_{V_{\mathbf{i},n}}, \cdots, \varphi_{V_{\mathbf{i},1}}), B(\Gamma_{V_{\mathbf{i}}}))$ provides an initial seed of the cluster algebra. Moreover, the restriction to $L^{e,v}$ gives a natural isomorphism of cluster algebras*

$$\tilde{L}(\mathcal{C}_v) \cong \mathbb{C}[L^{e,v}].$$

Furthermore, using the notation as in (3.5), we have $\varphi_{V_{\mathbf{i},k}} = D_{[1,j_k],v_{>n-k}[1,j_k]}|_{L^{e,v}}$.

5. Monomial realizations and Demazure crystals

In Sect. 6, we shall describe cluster variables in a cluster algebra of finite type in terms of the *monomial realizations* of Demazure crystals. Let us recall the notion of crystal base and its monomial realization in this section. Let $\mathfrak{g}$ be a complex simple Lie algebra with an index set $I = \{1, 2, \cdots, r\}$, a Cartan matrix $A = (a_{i,j})$, and the weight lattice P. We take e_i, f_i, h_i as in Sect. 2.1.

5.1. Monomial realizations of crystals

In this subsection, we shall review the monomial realizations of crystals [15,17,20].

Definition 5.1. [16,18] A *crystal* associated with a Cartan matrix A is a set B together with the maps $\mathrm{wt} : B \to P$, $\tilde{e}_i$, $\tilde{f}_i : B \cup \{0\} \to B \cup \{0\}$ and ε_i, $\varphi_i : B \to \mathbb{Z} \cup \{-\infty\}$, $i \in I$, satisfying some properties ((7.1)-(7.5) in [18]).

We call $\tilde{e}_i$ and $\tilde{f}_i$ ($i \in I$) the *Kashiwara operators*. Let $U_q(\mathfrak{g})$ be the quantum enveloping algebra [16] associated with the Cartan matrix A with an indeterminate q. Let $V(\lambda)$ ($\lambda \in P^+ = \oplus_{i\in I}\mathbb{Z}_{\geq 0}\Lambda_i$) be the finite dimensional irreducible representation of $U_q(\mathfrak{g})$ which has the highest weight vector v_λ, and $B(\lambda)$ be the crystal base of $V(\lambda)$. The crystal base $B(\lambda)$ has a crystal structure.

Let us introduce the monomial realization [15,20] which realizes each element of $B(\lambda)$ as a certain Laurent monomial. First, fix a cyclic sequence of the indices $\cdots(i_1, i_2, \cdots, i_r)(i_1, i_2, \cdots, i_r)\cdots$ such that $\{i_1, i_2, \cdots, i_r\} = I$. And we can associate this sequence with a family of integers $p = (p_{j,i})_{j,i\in I,\ j\neq i}$ such that

$$p_{i_a,i_b} = \begin{cases} 1 & \text{if} \quad a < b, \\ 0 & \text{if} \quad a > b. \end{cases}$$

Second, for the doubly-indexed variables $\{Y_{s,i} \,|\, i \in I,\ s \in \mathbb{Z}\}$, we define the set of monomials

$$\mathcal{Y} := \left\{ Y = \prod_{s\in\mathbb{Z},\ i\in I} Y_{s,i}^{\zeta_{s,i}} \,\middle|\, \zeta_{s,i} \in \mathbb{Z},\ \zeta_{s,i} = 0 \text{ except for finitely many } (s,i) \right\}.$$

Finally, we define maps $\mathrm{wt} : \mathcal{Y} \to P$, ε_i, $\varphi_i : \mathcal{Y} \to \mathbb{Z}$, $i \in I$ as follows. For $Y = \prod_{s\in\mathbb{Z},\ i\in I} Y_{s,i}^{\zeta_{s,i}} \in \mathcal{Y}$, set

$$\mathrm{wt}(Y) := \sum_{i,s} \zeta_{s,i}\Lambda_i, \quad \varphi_i(Y) := \max\left\{ \sum_{k\leq s} \zeta_{k,i} \,|\, s \in \mathbb{Z} \right\}, \quad \varepsilon_i(Y) := \varphi_i(Y) - \mathrm{wt}(Y)(h_i). \tag{5.1}$$

We set

$$A_{s,i} := Y_{s,i}Y_{s+1,i} \prod_{j\neq i} Y_{s+p_{j,i},j}^{a_{j,i}} \tag{5.2}$$

and define the Kashiwara operators as follows

$$\tilde{f}_i Y = \begin{cases} A^{-1}_{n_{f_i},i} Y & \text{if} \quad \varphi_i(Y) > 0, \\ 0 & \text{if} \quad \varphi_i(Y) = 0, \end{cases} \qquad \tilde{e}_i Y = \begin{cases} A_{n_{e_i},i} Y & \text{if} \quad \varepsilon_i(Y) > 0, \\ 0 & \text{if} \quad \varepsilon_i(Y) = 0, \end{cases}$$

where

$$n_{f_i} := \min \left\{ n \,\middle|\, \varphi_i(Y) = \sum_{k \le n} \zeta_{k,i} \right\}, \qquad n_{e_i} := \max \left\{ n \,\middle|\, \varphi_i(Y) = \sum_{k \le n} \zeta_{k,i} \right\}.$$

Then the following theorem holds:

Theorem 5.2. *[15,20]*

(i) For the set $p = (p_{j,i})$ as above, $(\mathcal{Y}, \mathrm{wt}, \varphi_i, \varepsilon_i, \tilde{f}_i, \tilde{e}_i)_{i \in I}$ is a crystal. When we emphasize p, we write $\mathcal{Y}$ as $\mathcal{Y}(p)$.
(ii) If a monomial $Y \in \mathcal{Y}(p)$ satisfies $\varepsilon_i(Y) = 0$ for all $i \in I$, then the connected component in the sense of crystal graph containing Y is isomorphic to $B(\mathrm{wt}(Y))$.

5.2. Demazure crystals

The crystal $B(\lambda)$ ($\lambda \in P^+$) has the unique element u_λ which satisfies $\mathrm{wt}(u_\lambda) = \lambda$ and $e_i u_\lambda = 0$ for all $i \in I$. We call u_λ the *highest weight vector* of $B(\lambda)$. For $w \in W$, the *Demazure crystal* $B(\lambda)_w \subset B(\lambda)$ is inductively defined as follows.

Definition 5.3. Let u_λ be the highest weight vector of $B(\lambda)$. For the identity element e of W, we set $B(\lambda)_e := \{u_\lambda\}$. For $w \in W$, if $s_i w < w$,

$$B(\lambda)_w := \{\tilde{f}_i^k b \mid k \ge 0,\ b \in B(\lambda)_{s_i w},\ \tilde{e}_i b = 0\} \setminus \{0\}.$$

Theorem 5.4. *[19] For $w \in W$, let $w = s_{i_1} \cdots s_{i_n}$ be an arbitrary reduced expression. Let u_λ be the highest weight vector of $B(\lambda)$. Then*

$$B(\lambda)_w = \{\tilde{f}_{i_1}^{a(1)} \cdots \tilde{f}_{i_n}^{a(n)} u_\lambda | a(1), \cdots, a(n) \in \mathbb{Z}_{\ge 0}\} \setminus \{0\}.$$

Lemma 5.5. *Let us consider the case of type A_r and the cyclic sequence is*

$$\begin{cases} (2, 4, \cdots, r, 1, 3, 5, \cdots, r-1) & \text{if } r \text{ is even}, \\ (2, 4, \cdots, r-1, 1, 3, 5, \cdots, r) & \text{if } r \text{ is odd}. \end{cases}$$

In this case, (5.2) is written

$$A_{1,i} = \begin{cases} \frac{Y_{1,i} Y_{2,i}}{Y_{1,i-1} Y_{1,i+1}} & \text{if } i \text{ is even}, \\ \frac{Y_{1,i} Y_{2,i}}{Y_{2,i-1} Y_{2,i+1}} & \text{if } i \text{ is odd}. \end{cases} \tag{5.3}$$

In general, if each factor of a monomial $Y \in \mathcal{Y}$ has non-negative degree, then $\varepsilon_i(Y) = 0$ for all $i \in I$. In particular, we have $\varepsilon_i(Y_{1,j}) = 0$ for $j \in I$. Thus, we can consider the monomial realization of crystal base $B(\Lambda_j)$ with the highest weight vector $Y_{1,j}$. The following is its partial crystal graph:

$$\begin{tikzcd}
Y_{1,j} \arrow[r, "\tilde{f}_j"] & Y_{1,j}A_{1,j}^{-1} \arrow[r, "\tilde{f}_{j+1}"] \arrow[d, "\tilde{f}_{j-1}"'] & Y_{1,j}A_{1,j}^{-1}A_{1,j+1}^{-1} \arrow[r] \arrow[d, "\tilde{f}_{j-1}"] & \cdots \\
 & Y_{1,j}A_{1,j}^{-1}A_{1,j-1}^{-1} \arrow[r, "\tilde{f}_{j+1}"'] & Y_{1,j}A_{1,j}^{-1}A_{1,j+1}^{-1}A_{1,j-1}^{-1} \arrow[r] & \cdots
\end{tikzcd}$$

6. Cluster variables and crystals

In the rest of the article, we set $G = \mathrm{SL}_{r+1}(\mathbb{C})$ $(r \geq 3)$ and only treat the Coxeter element $c \in W$ such that a reduced word $\mathbf{i}$ of c^2 can be written as (2.4). Let j_k be the k-th index of $\mathbf{i}$ from the right, and we consider the monomial realization associated with the sequence $(j_r, \cdots, j_2, j_1)$ (Sect. 5.1). Thus, the setting below is the same as in Lemma 5.5. For $a, b \in \mathbb{Z}_{\geq 0}$ with $a \leq b$, we set $[a, b] := \{a, a+1, \cdots, b\}$. In this section, we describe the cluster variables on the double Bruhat cell G^{e,c^2} as the total sum of monomial realizations of Demazure crystals.

Let $\mathbb{V} := ((\varphi_{\mathbb{V}})_{2r}, \cdots, (\varphi_{\mathbb{V}})_{r+1}, (\varphi_{\mathbb{V}})_r, \cdots, (\varphi_{\mathbb{V}})_1, (\varphi_{\mathbb{V}})_{-r}, \cdots, (\varphi_{\mathbb{V}})_{-1})$, where $(\varphi_{\mathbb{V}})_k \in \mathbb{C}[G^{e,c^2}]$ are defined as follows:

$$(\varphi_{\mathbb{V}})_k = \begin{cases} D_{[1,j_k], c^2_{>2r-k}[1,j_k]} & \text{if } 1 \leq k \leq 2r, \\ D_{[1,|k|],[1,|k|]} & \text{if } -r \leq k \leq -1. \end{cases}$$

By Theorem 3.7 and Theorem 3.9, we can regard $\mathbb{C}[G^{e,c^2}]$ as a cluster algebra of finite type and $\mathbb{V}$ as its initial cluster. Moreover, $(\varphi_{\mathbb{V}})_{2r}, \cdots, (\varphi_{\mathbb{V}})_{r+1}$ and $(\varphi_{\mathbb{V}})_{-r}, \cdots, (\varphi_{\mathbb{V}})_{-1}$ are frozen. From Theorem 4.15, for $k \in [1, 2r]$,

$$(\varphi_{\mathbb{V}})_k|_{L^{e,c^2}} = \varphi_{V_k}. \tag{6.1}$$

Thus, we can rewrite (3.3) as

$$\begin{tikzcd}
 & (\varphi_{\mathbb{V}})_{r+k+1} \arrow[d] & (\varphi_{\mathbb{V}})_{r+\lfloor\frac{r}{2}\rfloor+k+1} \arrow[d] & (\varphi_{\mathbb{V}})_{r+k} \arrow[d] & (\varphi_{\mathbb{V}})_{r+\lfloor\frac{r}{2}\rfloor+k} \arrow[d] & \\
\cdots & (\varphi_{\mathbb{V}})_{k+1} \arrow[r] \arrow[d] & (\varphi_{\mathbb{V}})_{\lfloor\frac{r}{2}\rfloor+k+1} \arrow[d] \arrow[ul] \arrow[ur] & (\varphi_{\mathbb{V}})_{k} \arrow[l] \arrow[r] \arrow[d] & (\varphi_{\mathbb{V}})_{\lfloor\frac{r}{2}\rfloor+k} \arrow[d] \arrow[ul] & \cdots \\
 & (\varphi_{\mathbb{V}})_{-j_k+2} & (\varphi_{\mathbb{V}})_{-j_k+1} \arrow[ul] \arrow[ur] & (\varphi_{\mathbb{V}})_{-j_k} & (\varphi_{\mathbb{V}})_{-j_k-1} \arrow[ul] &
\end{tikzcd} \tag{6.2}$$

Comparing with (4.8), we see that the matrix $B(\Gamma_{V_{\mathbf{i}}})$ is a submatrix of $-\tilde{B}(\mathbf{i})$, which is obtained by deleting rows labelled by $(\varphi_{\mathbb{V}})_{-r}, \cdots, (\varphi_{\mathbb{V}})_{-1}$ (note that - sign of $-\tilde{B}(\mathbf{i})$ is needed to match the setting of [2] and [9]). Note also that there are some differences between the quiver $\Gamma_{V_{\mathbf{i}}}$ in (4.8) and the quiver obtained from $\Gamma_{\mathbf{i}}$ by deleting the bottom row, that is, the arrows between the frozen cluster variables such as $r + \lfloor \frac{r}{2} \rfloor + k + 1$, $r + k$, $r + \lfloor \frac{r}{2} \rfloor + k$ in (4.8).

In the rest of the paper, for simplicity, we will drop frozen variables from a cluster in $\mathbb{C}[G^{e,c^2}]$, e.g., $\mathbb{V} = ((\varphi_{\mathbb{V}})_r, \cdots, (\varphi_{\mathbb{V}})_1)$. We will order the cluster variables $(\varphi_{\mathbb{V}})_1, \cdots, (\varphi_{\mathbb{V}})_r$ from the right in $\mathbb{V}$ as above, and let μ_k denote the mutation of the k-th cluster variable from the right. For a cluster $\mathbb{T}$ in $\mathbb{C}[G^{e,c^2}]$, let $(\varphi_{\mathbb{T}})_k$ denote the k-th (non-frozen) cluster variable from the right:

$$\mathbb{T} := ((\varphi_{\mathbb{T}})_r, \cdots, (\varphi_{\mathbb{T}})_1).$$

Each cluster variable is a regular function on G^{e,c^2}, and by Proposition 2.4, it can be seen as a function on $H \times (\mathbb{C}^\times)^{2r}$. Then, let us consider the following change of variables:

Definition 6.1. Along with (2.4), we set the variables $\mathbf{Y} \in (\mathbb{C}^\times)^{2r}$ as

$$\mathbf{Y} := \begin{cases} (Y_{1,2}, Y_{1,4}, \cdots, Y_{1,r}, Y_{1,1}, Y_{1,3}, \cdots, Y_{1,r-1}, Y_{2,2}, \cdots, Y_{2,r}, Y_{2,1}, \cdots, Y_{2,r-1}) \\ \quad r \text{ is even,} \\ (Y_{1,2}, Y_{1,4}, \cdots, Y_{1,r-1}, Y_{1,1}, Y_{1,3}, \cdots, Y_{1,r}, Y_{2,2}, \cdots, Y_{2,r-1}, Y_{2,1}, \cdots, Y_{2,r}) \\ \quad r \text{ is odd.} \end{cases} \tag{6.3}$$

Then for $a \in H$ and cluster $\mathbb{T}$ in $\mathbb{C}[G^{e,c^2}]$, we define

$$(\varphi^G_{\mathbb{T}})_k(a; \mathbf{Y}) := (\varphi_{\mathbb{T}})_k \circ \overline{x}^G_{\mathbf{i}}(a; \mathbf{Y}), \quad (1 \leq k \leq r),$$

where $\overline{x}^G_{\mathbf{i}}$ is as in 2.2.

Due to the property of minors, for $a \in H$, $x \in G$, $w \in W$, $i, j \in I$ and $t \in \mathbb{C}$, we get

$$D_{[1,i],w[1,i]}(ax) = a^{\Lambda_i} D_{[1,i],w[1,i]}(x), \ D_{[1,i],[1,i]}(xx_j(t)) = D_{[1,i],[1,i]}(x), \tag{6.4}$$

where $x_j(t) \in N$ is the one in (2.1) and if $a = T^h$ ($T \in \mathbb{C}^\times$, $h \in \mathrm{Lie}(H)$), then $a^{\Lambda_i} := t^{\Lambda_i(h)}$.

Proposition 6.2. *(1) For k $(1 \leq k \leq \lfloor \frac{r+1}{2} \rfloor)$,*

$$(\varphi^G_{\mathbb{V}})_k(a; \mathbf{Y}) = a^{\Lambda_{j_k}} Y_{1,j_k}(1 + A^{-1}_{1,j_k}),$$

and for k $(\lfloor \frac{r+1}{2} \rfloor + 1 \leq k \leq r)$,

$$(\varphi^G_{\mathbb{V}})_k(a;\boldsymbol{Y}) = a^{\Lambda_{j_k}} Y_{1,j_k}(1 + A^{-1}_{1,j_k} + A^{-1}_{1,j_k}A^{-1}_{1,j_k-1} + A^{-1}_{1,j_k}A^{-1}_{1,j_k+1} + A^{-1}_{1,j_k}A^{-1}_{1,j_k-1}A^{-1}_{1,j_k+1}).$$

For each k $(1 \le k \le r)$, there exists a monomial realization μ of the crystal base $B(\Lambda_{j_k})$ such that $(\varphi^G_{\mathbb{V}})_k(a;\boldsymbol{Y}) = a^{\Lambda_{j_k}} \sum_{b\in B(\Lambda_{j_k})_{c^2_{>2r-k}}} \mu(b)$.

(2) For k $(1 \le k < \lfloor \frac{r+1}{2} \rfloor)$, putting $J := j_k = 2\lfloor \frac{r+1}{2} \rfloor - 2k + 1$,

$$\begin{aligned}(\varphi^G_{(\mu_k\mathbb{V})})_k(a;\boldsymbol{Y}) = a^{\Lambda_{J-1}+\Lambda_{J+1}}&(Y_{1,J-2}Y_{1,J}Y_{1,J+2}(1 + A^{-1}_{1,J-2})(1 + A^{-1}_{1,J+2}) \\ &+ Y_{1,J-1}Y_{2,J}Y_{1,J+1}(1 + A^{-1}_{1,J-1} + A^{-1}_{1,J-1}A^{-1}_{1,J-2})(1 + A^{-1}_{1,J+1} + A^{-1}_{1,J+1}A^{-1}_{1,J+2})),\end{aligned}$$

$$(\varphi^G_{(\mu_{\lfloor \frac{r+1}{2}\rfloor}\mathbb{V})})_{\lfloor \frac{r+1}{2}\rfloor}(a;\boldsymbol{Y}) = a^{\Lambda_2}Y_{1,2}Y_{2,1}(1 + A^{-1}_{1,2} + A^{-1}_{1,2}A^{-1}_{1,3}).$$

There exist monomial realizations of μ and μ' of $B(\Lambda_{J-1}+\Lambda_J+\Lambda_{J+1}) \oplus B(\Lambda_{J-2}+\Lambda_J+\Lambda_{J+2})$ and $B(\Lambda_1+\Lambda_2)$ such that

$$(\varphi^G_{(\mu_k\mathbb{V})})_k(a;\boldsymbol{Y}) = a^{\Lambda_{J-1}+\Lambda_{J+1}} \sum_{b\in B_J} \mu(b),$$

$$(\varphi^G_{(\mu_{\lfloor \frac{r+1}{2}\rfloor}\mathbb{V})})_{\lfloor \frac{r+1}{2}\rfloor}(a;\boldsymbol{Y}) = a^{\Lambda_2} \sum_{b\in B(\Lambda_1+\Lambda_2)_{s_3s_2}} \mu'(b),$$

where $B_J := B(\Lambda_{J-1}+\Lambda_J+\Lambda_{J+1})_{s_{J-2}s_{J-1}s_{J+2}s_{J+1}} \oplus B(\Lambda_{J-2}+\Lambda_J+\Lambda_{J+2})_{s_{J-2}s_{J+2}}$. For $\lfloor \frac{r+1}{2}\rfloor < k \le r$, we have $(\varphi^G_{(\mu_k\mathbb{V})})_k(a;\boldsymbol{Y}) = a^{\Lambda_{j_k-1}+\Lambda_{j_k+1}}Y_{2,j_k} = a^{\Lambda_{2r-2k+1}+\Lambda_{2r-2k+3}} \times Y_{2,2r-2k+2}$. The set $\{Y_{2,j_k}\}$ is a monomial realization of the Demazure crystal $B(\Lambda_{j_k})_e = B(\Lambda_{2r-2k+2})_e$.

Proof. In the above setting,

$$(\varphi^G_{\mathbb{V}})_k(a;\mathbf{Y}) = (\varphi_{\mathbb{V}})_k \circ x^G_{\mathbf{i}} \circ \phi(a;\mathbf{Y}),$$

where $\phi : H \times (\mathbb{C}^\times)^{2r} \to H \times (\mathbb{C}^\times)^{2r}$,

$$\phi(a;\mathbf{Y}) = (\Phi_H(a;\mathbf{Y}); \Phi_{1,j_r}(\mathbf{Y}), \cdots, \Phi_{1,j_1}(\mathbf{Y}), \Phi_{2,j_r}(\mathbf{Y}), \cdots, \Phi_{2,j_2}(\mathbf{Y}), \Phi_{2,j_1}(\mathbf{Y}))$$

is the map in the proof of Proposition 2.4. Since $(\varphi_{\mathbb{V}})_k$ is the minor $D_{[1,j_k],c^2_{>2r-k}[1,j_k]}$ (Theorem 3.7), we have

$$(\varphi_{\mathbb{V}})_k \circ x^G_{\mathbf{i}}(a;\mathbf{Y}) = a^{\Lambda_{j_k}}(\varphi_{\mathbb{V}})_k \circ x^G_{\mathbf{i}}(1;\mathbf{Y}), \tag{6.5}$$

where $\mathbf{Y} := (Y_{1,j_{2r}}, \cdots, Y_{1,j_{r+1}}, Y_{2,j_r}, \cdots, Y_{2,j_1})$ and 1 is the identity element of H. By (6.1), we obtain $(\varphi_{\mathbb{V}})_k \circ x^G_{\mathbf{i}}(1;\mathbf{Y}) = \varphi_{V_k} \circ x^G_{\mathbf{i}}(1;\mathbf{Y})$. In Example 4.14, we have calculated $\varphi_{V_k} \circ x^G_{\mathbf{i}}(1;\mathbf{Y})$.

If $1 \le k \le \lfloor \frac{r+1}{2}\rfloor$, j_k is odd. By the fact $\varphi_{V_k} \circ x^G_{\mathbf{i}}(1;\mathbf{Y}) = Y_{1,j_k} + Y_{2,j_k}$, (2.5), (2.6) and (2.7), we get

$$(\varphi_{\mathbb{V}}^G)_k(a;\mathbf{Y}) = (\Phi_H(a;\mathbf{Y}))^{\Lambda_{j_k}}(\Phi_{1,j_k}(a;\mathbf{Y}) + \Phi_{2,j_k}(a;\mathbf{Y}))$$
$$= a^{\Lambda_{j_k}}(Y_{1,j_k}Y_{2,j_k})\left(\frac{Y_{2,j_k-1}Y_{2,j_k+1}}{Y_{1,j_k}Y_{2,j_k}^2} + \frac{1}{Y_{2,j_k}}\right) = a^{\Lambda_{j_k}}Y_{1,j_k}(1 + A_{1,j_k}^{-1}),$$

where $A_{1,j}$ is given in (5.3). By Theorem 5.4 and Lemma 5.5, the set of monomials $\{Y_{1,j_k}, Y_{1,j_k}A_{1,j_k}^{-1}\}$ coincides with the monomial realization of Demazure crystal $B(\Lambda_{j_k})_{s_{j_k}}$, where the monomial corresponding to the highest weight vector is Y_{1,j_k}.

For $\lfloor \frac{r+1}{2} \rfloor + 1 \le k \le r$, j_k is even. In this case, we have calculated $\varphi_{V_k} \circ x_{\mathbf{i}}^G(1;\mathbf{Y})$ in (4.12). Thus, using (2.5), (2.6) and (2.7), one has

$$\begin{aligned}
(\varphi_{\mathbb{V}}^G)_k(a;\mathbf{Y}) &= a^{\Lambda_{j_k}}(Y_{1,j_k}Y_{2,j_k}) \\
&\times\left(\frac{Y_{2,j_k-2}Y_{2,j_k+2}}{Y_{1,j_k}Y_{2,j_k-1}Y_{2,j_k+1}} + \frac{Y_{1,j_k+1}Y_{2,j_k-2}}{Y_{1,j_k}Y_{2,j_k}Y_{2,j_k-1}} + \frac{Y_{1,j_k-1}Y_{1,j_k+1}}{Y_{1,j_k}Y_{2,j_k}^2}\right. \\
&\left.+\frac{1}{Y_{2,j_k}} + \frac{Y_{1,j_k-1}Y_{2,j_k+2}}{Y_{1,j_k}Y_{2,j_k}Y_{2,j_k+1}}\right) \\
&= a^{\Lambda_{j_k}}\left(Y_{1,j_k} + \frac{Y_{1,j_k-1}Y_{1,j_k+1}}{Y_{2,j_k}} + \frac{Y_{1,j_k+1}Y_{2,j_k-2}}{Y_{2,j_k-1}} + \frac{Y_{1,j_k-1}Y_{2,j_k+2}}{Y_{2,j_k+1}}\right. \\
&\left.+\frac{Y_{2,j_k-2}Y_{2,j_k}Y_{2,j_k+2}}{Y_{2,j_k-1}Y_{2,j_k+1}}\right) \\
&= a^{\Lambda_{j_k}}Y_{1,j_k}(1 + A_{1,j_k}^{-1} + A_{1,j_k}^{-1}A_{1,j_k-1}^{-1} + A_{1,j_k}^{-1}A_{1,j_k+1}^{-1} + A_{1,j_k}^{-1}A_{1,j_k-1}^{-1}A_{1,j_k+1}^{-1}).
\end{aligned}$$

By Theorem 5.4 and Lemma 5.5, the set of monomials $\{Y_{1,j_k}, \frac{Y_{1,j_k-1}Y_{1,j_k+1}}{Y_{2,j_k}}, \frac{Y_{1,j_k+1}Y_{2,j_k-2}}{Y_{2,j_k-1}}, \frac{Y_{1,j_k-1}Y_{2,j_k+2}}{Y_{2,j_k+1}}, \frac{Y_{2,j_k-2}Y_{2,j_k}Y_{2,j_k+2}}{Y_{2,j_k-1}Y_{2,j_k+1}}\}$ coincides with the monomial realization of Demazure crystal $B(\Lambda_{j_k})_{s_{j_k+1}s_{j_k-1}s_{j_k}}$, where the corresponding highest weight vector is Y_{1,j_k}.

Next, let us consider the mutation in direction k of $\mathbb{V}$ by calculating $(\varphi_{(\mu_k\mathbb{V})}^G)_k(a;\mathbf{Y})$ $(1 \le k \le r)$. If $1 \le k \le \lfloor \frac{r+1}{2} \rfloor$, by (4.8) and (6.2),

$$\begin{aligned}
&(\varphi_{(\mu_k\mathbb{V})})_k \circ x_{\mathbf{i}}^G(a;\mathbf{Y}) \\
&= \left(\frac{(\varphi_{\mathbb{V}})_{r+k}(\varphi_{\mathbb{V}})_{-j_k+1}(\varphi_{\mathbb{V}})_{-j_k-1} + (\varphi_{\mathbb{V}})_{-j_k}(\varphi_{\mathbb{V}})_{\lfloor\frac{r}{2}\rfloor+k}(\varphi_{\mathbb{V}})_{\lfloor\frac{r}{2}\rfloor+k+1}}{(\varphi_{\mathbb{V}})_k}\right) \circ x_{\mathbf{i}}^G(a;\mathbf{Y}) \\
&= \frac{a^{\Lambda_{j_k-1}}a^{\Lambda_{j_k}}a^{\Lambda_{j_k+1}}}{a^{\Lambda_{j_k}}} \cdot \left(\frac{\varphi_{V_{r+k}} + \varphi_{V_{\lfloor\frac{r}{2}\rfloor+k}}\varphi_{V_{\lfloor\frac{r}{2}\rfloor+k+1}}}{\varphi_{V_k}}\right) \circ x_{\mathbf{i}}^G(1;\mathbf{Y}) \\
&= a^{\Lambda_{j_k-1}}a^{\Lambda_{j_k+1}} \cdot (\varphi_{(\mu_k V)_k}) \circ x_{\mathbf{i}}^G(1;\mathbf{Y}),
\end{aligned} \tag{6.6}$$

where we use (6.4). From (4.13), for $1 \le k < \lfloor \frac{r+1}{2} \rfloor$ we get

$$(\varphi_{(\mu_k\mathbb{V})}^G)_k(a;\mathbf{Y}) = (\varphi_{(\mu_k\mathbb{V})})_k \circ x_{\mathbf{i}}^G \circ \phi(a;\mathbf{Y})$$
$$= {\Phi_H}^{\Lambda_{j_k-1}}{\Phi_H}^{\Lambda_{j_k+1}} \cdot \sum_{(*)} \Phi_{b_1,j_k-1}\Phi_{b_2,j_k+1}\Phi_{b_3,j_k-2}\Phi_{b_4,j_k}\Phi_{b_5,j_k+2},$$

where $(*)$ is the condition for b_1, b_2, b_3, b_4 and b_5 : $1 \le b_1 \le b_3 \le 2$, $1 \le b_2 \le b_4 \le 2$, $1 \le b_1 \le b_4 \le 2$ and $1 \le b_2 \le b_5 \le 2$. We can easily verify that if $b_4 = 2$, then $1 \le b_1 \le b_3 \le 2$ and $1 \le b_2 \le b_5 \le 2$. If $b_4 = 1$, then $b_1 = b_2 = 1$ and $1 \le b_3,\ b_5 \le 2$. By (2.6) and (2.7), we see that

$$\Phi_{1,j} = A_{1,j}^{-1}\Phi_{2,j} \tag{6.7}$$

for $j \in [1, r]$. Thus,

$$\begin{aligned}
&\sum_{(*),\ b_4=2} \Phi_{b_1,j_k-1}\Phi_{b_2,j_k+1}\Phi_{b_3,j_k-2}\Phi_{b_4,j_k}\Phi_{b_5,j_k+2} = \\
&\qquad \Phi_{2,j_k-1}\Phi_{2,j_k+1}\Phi_{2,j_k-2}\Phi_{2,j_k}\Phi_{2,j_k+2}(1 + A_{1,j_k-1}^{-1} + A_{1,j_k-1}^{-1}A_{1,j_k-2}^{-1}) \\
&\qquad\qquad \times (1 + A_{1,j_k+1}^{-1} + A_{1,j_k+1}^{-1}A_{1,j_k+2}^{-1}) \\
&\quad = \frac{Y_{2,j_k}}{Y_{2,j_k-1}Y_{2,j_k+1}}(1 + A_{1,j_k-1}^{-1} + A_{1,j_k-1}^{-1}A_{1,j_k-2}^{-1})(1 + A_{1,j_k+1}^{-1} + A_{1,j_k+1}^{-1}A_{1,j_k+2}^{-1}) \quad , \\
&\qquad \sum_{(*),\ b_4=1} \Phi_{b_1,j_k-1}\Phi_{b_2,j_k+1}\Phi_{b_3,j_k-2}\Phi_{b_4,j_k}\Phi_{b_5,j_k+2} \\
&\qquad = \Phi_{1,j_k-1}\Phi_{1,j_k+1}\Phi_{2,j_k-2}\Phi_{1,j_k}\Phi_{2,j_k+2}(1 + A_{1,j_k-2}^{-1})(1 + A_{1,j_k+2}^{-1}) \\
&\qquad = \frac{Y_{1,j_k-2}Y_{1,j_k}Y_{1,j_k+2}}{Y_{1,j_k-1}Y_{2,j_k-1}Y_{1j_k+1}Y_{2,j_k+1}}(1 + A_{1,j_k-2}^{-1})(1 + A_{1,j_k+2}^{-1}).
\end{aligned}$$

By the above argument, we get

$$\begin{aligned}
(\varphi^G_{(\mu_k\mathbb{V})})_k(a;\mathbf{Y}) &= a^{\Lambda_{j_k-1}}a^{\Lambda_{j_k+1}}(Y_{1,j_k-2}Y_{1,j_k}Y_{1,j_k+2}(1 + A_{1,j_k-2}^{-1})(1 + A_{1,j_k+2}^{-1}) \\
&\quad + Y_{1,j_k-1}Y_{2,j_k}Y_{1,j_k+1}(1 + A_{1,j_k-1}^{-1} + A_{1,j_k-1}^{-1}A_{1,j_k-2}^{-1})(1 + A_{1,j_k+1}^{-1} + A_{1,j_k+1}^{-1}A_{1,j_k+2}^{-1})).
\end{aligned}$$

By the definition of Kashiwara operators in 5.1, we see that $Y_{1,j_k-2}Y_{1,j_k}Y_{1,j_k+2}(1 + A_{1,j_k-2}^{-1})(1 + A_{1,j_k+2}^{-1})$ are the total sum of $\mu(B(\Lambda_{j_k-2} + \Lambda_{j_k} + \Lambda_{j_k+2})_{s_{j_k-2}s_{j_k+2}})$, and $Y_{1,j_k-1}Y_{2,j_k}Y_{1,j_k+1}(1+A_{1,j_k-1}^{-1}+A_{1,j_k-1}^{-1}A_{1,j_k-2}^{-1})(1+A_{1,j_k+1}^{-1}+A_{1,j_k+1}^{-1}A_{1,j_k+2}^{-1})$ is the total sum of $\mu(B(\Lambda_{j_k-1} + \Lambda_{j_k} + \Lambda_{j_k+1})_{s_{j_k-2}s_{j_k-1}s_{j_k+2}s_{j_k+1}})$. Note that $\Lambda_{j_k-2} + \Lambda_{j_k} + \Lambda_{j_k+2} = (\Lambda_{j_k-1} + \Lambda_{j_k} + \Lambda_{j_k+1}) - \alpha_{j_k-1} - \alpha_{j_k} - \alpha_{j_k+1}$.

Arguing similarly, we obtain

$$(\varphi^G_{(\mu_{\lfloor \frac{r+1}{2}\rfloor}\mathbb{V})})_{\lfloor \frac{r+1}{2}\rfloor}(a;\mathbf{Y}) = a^{\Lambda_2}Y_{1,2}Y_{2,1}(1 + A_{1,2}^{-1} + A_{1,2}^{-1}A_{1,3}^{-1}).$$

The polynomial $Y_{1,2}Y_{2,1}(1+A_{1,2}^{-1}+A_{1,2}^{-1}A_{1,3}^{-1})$ is the total sum of the monomial realization of the Demazure crystal $\mu'(B(\Lambda_1 + \Lambda_2)_{s_3s_2})$.

Similarly, for $\lfloor \frac{r+1}{2}\rfloor < k \le r$, it follows from (3.4) and (4.15) that

$$\begin{aligned}
(\varphi^G_{(\mu_k\mathbb{V})})_k(a;\mathbf{Y}) &= \Phi_H{}^{\Lambda_{j_k-1}}\Phi_H{}^{\Lambda_{j_k+1}} \times \Phi_{1,j_k-1}\Phi_{1,j_k+1}\Phi_{2,j_k-2}\Phi_{2,j_k}\Phi_{2,j_k+2} \\
&\quad \times\Phi_{2,j_k-3}\Phi_{2,j_k-1}\Phi_{2,j_k+1}\Phi_{2,j_k+3} = a^{\Lambda_{j_k-1}}a^{\Lambda_{j_k+1}}Y_{2,j_k}.
\end{aligned}$$

The monomial Y_{2,j_k} is the monomial realization of the Demazure crystal $B(\Lambda_{j_k})_e$. $\square$

We obtain the following Proposition 6.3 by Proposition 4.12 and the same argument in the above example.

Proposition 6.3. *We have the cluster variable*

$$(\varphi^G_{(\mu_r\mu_{\lfloor \frac{r+1}{2}\rfloor}\mathbb{V})})_r(a;\mathbf{Y}) = a^{\Lambda_3}Y_{2,1},$$

and the set $\{Y_{2,1}\}$ *is a monomial realization of the Demazure crystal* $B(\Lambda_1)_e$. *For* $\lfloor \frac{r}{2}\rfloor + 2 \le k \le r$, *we obtain*

$$\begin{aligned}&(\varphi^G_{(\mu_{k-\lfloor \frac{r}{2}\rfloor-1}\mu_k\mathbb{V})})_{k-\lfloor \frac{r}{2}\rfloor-1}(a;\mathbf{Y})\\ &\quad = a^{\Lambda_{2r-2k+1}+\Lambda_{2r-2k+4}}Y_{2,2r-2k+3}Y_{1,2r-2k+4}(1+A^{-1}_{1,2r-2k+4}+A^{-1}_{1,2r-2k+4}A^{-1}_{1,2r-2k+5}),\end{aligned}$$

and the set $\{Y_{2,2r-2k+3}Y_{1,2r-2k+4},\ Y_{2,2r-2k+3}Y_{1,2r-2k+4}A^{-1}_{1,2r-2k+4},\ Y_{2,2r-2k+3}\times Y_{1,2r-2k+4}A^{-1}_{1,2r-2k+4}A^{-1}_{1,2r-2k+5}\}$ *coincides with the monomial realization of the Demazure crystal* $B(\Lambda_{2r-2k+3}+\Lambda_{2r-2k+4})_{s_{2r-2k+5}s_{2r-2k+4}}$, *where* $Y_{2,2r-2k+3}Y_{1,2r-2k+4}$ *is the corresponding highest weight vector in* $B(\Lambda_{2r-2k+3}+\Lambda_{2r-2k+4})$.

In the following Proposition 6.4, 6.6 and 6.7, we shall give the explicit expressions of all the other cluster variables in $\mathbb{C}[G^{e,c^2}]$. We use the notation as in (2.5), (2.6), (2.7) and (5.3), and set $\phi(\mathbf{Y}) := (\Phi_{1,j_r}(\mathbf{Y}),\cdots,\Phi_{1,j_1}(\mathbf{Y}),\Phi_{2,j_r}(\mathbf{Y}),\cdots,\Phi_{2,j_1}(\mathbf{Y}))$. We abbreviate $\Phi_H(a;\mathbf{Y})$ to Φ_H. For the integers b, c ($b<c$) and x, we set

$$A[b,c;x] := \left(\prod_{s=b}^{c-1} A_{1,x-2s-2}A_{1,x-2s-3}\right)^{-1} A^{-1}_{1,x-2c-2} = \prod_{s=x-2c-2}^{x-2b-2} A^{-1}_{1,s}.$$

For $p\in\mathbb{Z}_{>0}$ and $\mathbf{b}=(b_i)_{i=1}^p\in(\mathbb{Z}_{\ge 0})^p$, $\mathbf{c}=(c_i)_{i=1}^p\in(\mathbb{Z}_{\ge 0})^p$ such that $b_i<c_i$ ($1\le i\le p$), we also set

$$\alpha[\mathbf{b},\mathbf{c};x] := \sum_{t=x-2c_1-2}^{x-2b_1-2}\alpha_t+\cdots+\sum_{t=x-2c_p-2}^{x-2b_p-2}\alpha_t,$$

where when $s\le 0$, we understand $A_{1,s}=1$, $\alpha_s=0$. For $l\in\mathbb{Z}_{\ge 0}$, we define

$$\begin{aligned}R^p_l := \{(\mathbf{b},\mathbf{c})\in(\mathbb{Z}_{\ge 0})^p\times(\mathbb{Z}_{\ge 0})^p \mid \mathbf{b}=(b_i)_{i=1}^p,\ \mathbf{c}=(c_i)_{i=1}^p,\\ 0\le b_1<c_1<\cdots<b_p<c_p\le l\}.\end{aligned}$$

For $(\mathbf{b},\mathbf{c})\in R^p_l$, we define $[\mathbf{b},\mathbf{c}] := [b_1,c_1]\cup\cdots\cup[b_p,c_p]$.

Proposition 6.4. *For $k \in [\lfloor \frac{r+1}{2} \rfloor + 1, r-1]$ and $l \in [0, r-k-1]$, let $\mu[l]$ be the following iteration of mutations*

$$\mu[l] := (\mu_{k-\lfloor \frac{r}{2} \rfloor + l}\mu_{k+l+1}\mu_{k+l}) \cdots (\mu_{k-\lfloor \frac{r}{2} \rfloor + 1}\mu_{k+2}\mu_{k+1})(\mu_{k-\lfloor \frac{r}{2} \rfloor}\mu_{k+1}\mu_k).$$

(a) *We have the cluster variable*

$$\begin{aligned}
&(\varphi^G_{(\mu[l]\mathbb{V})})_{k-\lfloor \frac{r}{2} \rfloor + l}(a; \mathbf{Y}) \\
&= \Phi_H^{(\sum_{s=0}^{l+2} \Lambda_{j_k-2s+1}) + (\sum_{s=0}^{l-1} \Lambda_{j_k-2s-2})} \varphi_{(\mu[l]V)_{k-\lfloor \frac{r}{2} \rfloor + l}} \circ x_{\mathbf{i}}^G(1; \phi(\mathbf{Y})) \\
&= a^{(\sum_{s=0}^{l+2} \Lambda_{j_k-2s+1}) + (\sum_{s=0}^{l-1} \Lambda_{j_k-2s-2})} H_1 \left(\prod_{q \in [0,l-1]} (1 + A^{-1}_{1,j_k-2q-2}) \right. \\
&\left. + \sum_{p>0} \sum_{(\mathbf{b},\mathbf{c}) \in R^p_{l-1}} \prod_{i=1}^{p} A[b_i, c_i; j_k] \prod_{q \in [0,l-1] \setminus [\mathbf{b},\mathbf{c}]} (1 + A^{-1}_{1,j_k-2q-2}) \right).
\end{aligned}$$

(b) *We also obtain the cluster variable*

$$\begin{aligned}
&(\varphi^G_{(\mu_{k+l+1}\mu[l]\mathbb{V})})_{k+l+1}(a; \mathbf{Y}) \\
&= \Phi_H^{\sum_{s=0}^{l} (\Lambda_{j_k-2s+1} + \Lambda_{j_k-2s-2})} \varphi_{(\mu_{k+l+1}\mu[l]V)_{k+l+1}} \circ x_{\mathbf{i}}^G(1; \phi(\mathbf{Y})) \\
&= a^{\sum_{s=0}^{l} (\Lambda_{j_k-2s+1} + \Lambda_{j_k-2s-2})} H_2 \\
&\times \left((1 + A^{-1}_{1,j_k-2l-2} + A^{-1}_{1,j_k-2l-2} A^{-1}_{1,j_k-2l-3}) \prod_{q \in [0,l-1]} (1 + A^{-1}_{1,j_k-2q-2}) \right. \\
&+ \sum_{p>0} \sum_{(\mathbf{b},\mathbf{c}) \in R^p_l} (1 - \delta_{c_p,l} + A^{-1+\delta_{c_p,l}}_{1,j_k-2l-2}(1 + A^{-1}_{1,j_k-2l-3})) \\
&\left. \times \prod_{i=1}^{p} A[b_i, c_i; j_k] \prod_{q \in [0,l-1] \setminus [\mathbf{b},\mathbf{c}]} (1 + A^{-1}_{1,j_k-2q-2}) \right),
\end{aligned}$$

where

$$H_1 := \left(\prod_{t=0}^{l-1} Y_{1,j_k-2t-2} \right) \left(\prod_{t=0}^{l} Y_{2,j_k-2t-1} \right), \quad H_2 := \left(\prod_{t=0}^{l} Y_{1,j_k-2t-2} Y_{2,j_k-2t-1} \right).$$

Example 6.5. If $r = 10$, $k = 6$ and $l = 2$, then $\mu[2] = \mu_3\mu_9\mu_8\mu_2\mu_8\mu_7\mu_1\mu_7\mu_6$, $j_6 = 10$ and $H_1 = Y_{2,5}Y_{1,6}Y_{2,7}Y_{1,8}Y_{2,9}$ in the notation of Proposition 6.4. Note that

$$R^p_1 = \begin{cases} \{(0,1)\} & \text{if } p = 1, \\ \phi & \text{otherwise.} \end{cases}$$

It follows from Proposition 6.4 (a) that

$$\begin{aligned}(\varphi^G_{(\mu[2]\mathbb{V})})_3(a;\mathbf{Y}) &= a^{\Lambda_3+\Lambda_5+\Lambda_6+\Lambda_7+\Lambda_8+\Lambda_9}Y_{2,5}Y_{1,6}Y_{2,7}Y_{1,8}Y_{2,9}(1+A^{-1}_{1,8})(1+A^{-1}_{1,6})\\ &\quad +a^{\Lambda_3+\Lambda_5+\Lambda_6+\Lambda_7+\Lambda_8+\Lambda_9}Y_{2,5}Y_{1,6}Y_{2,7}Y_{1,8}Y_{2,9}A[0,1;10]\\ &= a^{\Lambda_3+\Lambda_5+\Lambda_6+\Lambda_7+\Lambda_8+\Lambda_9}Y_{2,5}Y_{1,6}Y_{2,7}Y_{1,8}Y_{2,9}(1+A^{-1}_{1,8}+A^{-1}_{1,6}+A^{-1}_{1,6}A^{-1}_{1,8})\\ &\quad +a^{\Lambda_3+\Lambda_5+\Lambda_6+\Lambda_7+\Lambda_8+\Lambda_9}Y_{1,5}Y_{2,5}Y_{1,7}Y_{1,9}Y_{2,9}. \end{aligned}\tag{6.8}$$

In the same setting, let us calculate $(\varphi^G_{(\mu_9\mu[2]\mathbb{V})})_9(a;\mathbf{Y})$. Note that

$$R_2^p = \begin{cases}\{(0,1),(0,2),(1,2)\} & \text{if } p=1,\\ \phi & \text{otherwise,}\end{cases}$$

and $H_2 = Y_{1,4}Y_{2,5}Y_{1,6}Y_{2,7}Y_{1,8}Y_{2,9}$. Thus, by Proposition 6.4 (b),

$$\begin{aligned}&a^{-(\Lambda_4+\Lambda_6+\Lambda_7+\Lambda_8+\Lambda_9)}(\varphi^G_{(\mu_9\mu[2]\mathbb{V})})_9(a;\mathbf{Y})\\ &= Y_{1,4}Y_{2,5}Y_{1,6}Y_{2,7}Y_{1,8}Y_{2,9}(1+A^{-1}_{1,8})(1+A^{-1}_{1,6})(1+A^{-1}_{1,4}+A^{-1}_{1,4}A^{-1}_{1,3})\\ &\quad +Y_{1,4}Y_{2,5}Y_{1,6}Y_{2,7}Y_{1,8}Y_{2,9}A[0,1;10](1+A^{-1}_{1,4}+A^{-1}_{1,4}A^{-1}_{1,3})\\ &\quad +Y_{1,4}Y_{2,5}Y_{1,6}Y_{2,7}Y_{1,8}Y_{2,9}A[0,2;10](1+A^{-1}_{1,3})\\ &\quad +Y_{1,4}Y_{2,5}Y_{1,6}Y_{2,7}Y_{1,8}Y_{2,9}A[1,2;10](1+A^{-1}_{1,3})\\ &= Y_{1,4}Y_{2,5}Y_{1,6}Y_{2,7}Y_{1,8}Y_{2,9}(1+A^{-1}_{1,6}+A^{-1}_{1,8}+A^{-1}_{1,6}A^{-1}_{1,8})(1+A^{-1}_{1,4}+A^{-1}_{1,4}A^{-1}_{1,3})\\ &\quad +Y_{1,4}Y_{1,5}Y_{2,5}Y_{1,7}Y_{1,9}Y_{2,9}(1+A^{-1}_{1,4}+A^{-1}_{1,4}A^{-1}_{1,3})\\ &\quad +Y_{1,3}Y_{1,5}Y_{2,6}Y_{1,7}Y_{1,9}Y_{2,9}(1+A^{-1}_{1,3})\\ &\quad +Y_{1,3}Y_{1,5}Y_{1,7}Y_{2,7}Y_{1,8}Y_{2,9}(1+A^{-1}_{1,3}).\end{aligned}\tag{6.9}$$

Proposition 6.6. *For $k \in [1, \lfloor \frac{r+1}{2}\rfloor - 2]$ and $l \in [0, \lfloor \frac{r+1}{2}\rfloor - k - 2]$, let $\mu'[l]$ be the following iteration of mutations*

$$\begin{aligned}\mu'[l] := {}&(\mu_{k+l+1}\mu_{\lfloor\frac{r}{2}\rfloor+k+l+2}\mu_{\lfloor\frac{r}{2}\rfloor+k+l+1})\cdots(\mu_{k+2}\mu_{\lfloor\frac{r}{2}\rfloor+k+3}\mu_{\lfloor\frac{r}{2}\rfloor+k+2})\\ &\times(\mu_{k+1}\mu_{\lfloor\frac{r}{2}\rfloor+k+2}\mu_{\lfloor\frac{r}{2}\rfloor+k+1})\mu_k.\end{aligned}$$

(a) If $j_k < r$, we have the cluster variable

$$\begin{aligned}&(\varphi^G_{(\mu'[l]\mathbb{V})})_{k+l+1}(a;\boldsymbol{Y}) = a^{\sum_{s=0}^{l+1}\Lambda_{j_k-2s-2}+\Lambda_{j_k-2s+1}}H_3\\ &\times\Bigg((1+A^{-1}_{1,j_k+1}+A^{-1}_{1,j_k+1}A^{-1}_{1,j_k+2})\prod_{q\in[1,l+1]}(1+A^{-1}_{1,j_k-2q+1})\end{aligned}$$

$$+\sum_{p>0}\sum_{(\mathbf{b},\mathbf{c})\in R^p_{l+1}}(1-\delta_{b_1,0}+A^{-1+\delta_{b_1,0}}_{1,j_k+1}(1+A^{-1}_{1,j_k+2}))$$

$$\left.\times\prod_{i=1}^{p}A[b_i,c_i;j_k+3]\prod_{q\in[1,l+1]\setminus([\mathbf{b},\mathbf{c}])}(1+A^{-1}_{1,j_k-2q+1})\right),$$

and if $j_k=r$*, we have*

$$(\varphi^G_{(\mu'[l]\mathbb{V})})_{k+l+1}(a;\boldsymbol{Y})=a^{\sum_{s=0}^{l+1}\Lambda_{r-2s-2}+\Lambda_{r-2s+1}}H_3\left(\prod_{q\in[1,l+1]}(1+A^{-1}_{1,r-2q+1})\right.$$

$$\left.+\sum_{p>0}\sum_{(\mathbf{b},\mathbf{c})\in R^p_{l+1},b_1>0}\prod_{i=1}^{p}A[b_i,c_i;r+3]\prod_{q\in[1,l+1]\setminus([\mathbf{b},\mathbf{c}])}(1+A^{-1}_{1,r-2q+1})\right).$$

(b) *If* $j_k<r$*, we also obtain the cluster variable*

$$(\varphi^G_{(\mu_{\lfloor\frac{r}{2}\rfloor+k+l+2}\mu'[l]\mathbb{V})})_{\lfloor\frac{r}{2}\rfloor+k+l+2}(a;\boldsymbol{Y})=a^{\sum_{s=0}^{l-1}\Lambda_{j_k-2s-2}+\sum_{s=0}^{l+2}\Lambda_{j_k-2s+1}}H_4$$

$$\times\left(\prod_{q\in[1,l+1]}(1+A^{-1}_{1,j_k-2q+1})(1+A^{-1}_{1,j_k-2l-3}+A^{-1}_{1,j_k-2l-3}A^{-1}_{1,j_k-2l-4})\right.$$

$$\times(1+A^{-1}_{1,j_k+1}+A^{-1}_{1,j_k+1}A^{-1}_{1,j_k+2})+\sum_{p>0}\sum_{(\mathbf{b},\mathbf{c})\in R^p_{l+2}}(1-\delta_{b_1,0}+A^{-1+\delta_{b_1,0}}_{1,j_k+1}(1+A^{-1}_{1,j_k+2}))$$

$$\times(1-\delta_{c_p,l+2}+A^{-1+\delta_{c_p,l+2}}_{1,j_k-2l-3}(1+A^{-1}_{1,j_k-2l-4}))$$

$$\left.\times\prod_{i=1}^{p}A[b_i,c_i;j_k+3]\prod_{q\in[1,l+1]\setminus([\mathbf{b},\mathbf{c}])}(1+A^{-1}_{1,j_k-2q+1})\right),$$

and if $j_k=r$*, we have*

$$(\varphi^G_{(\mu_{\lfloor\frac{r}{2}\rfloor+k+l+2}\mu'[l]\mathbb{V})})_{\lfloor\frac{r}{2}\rfloor+k+l+2}(a;\boldsymbol{Y})=a^{\sum_{s=0}^{l-1}\Lambda_{r-2s-2}+\sum_{s=0}^{l+2}\Lambda_{r-2s+1}}H_4$$

$$\times\left((1+A^{-1}_{1,r-2l-3}+A^{-1}_{1,r-2l-3}A^{-1}_{1,r-2l-4})\prod_{q\in[1,l+1]}(1+A^{-1}_{1,r-2q+1})\right.$$

$$+\sum_{p>0}\sum_{(\mathbf{b},\mathbf{c})\in R^p_{l+2},\ b_1>0}\prod_{i=1}^{p}A[b_i,c_i;r+3]$$

$$\left.\times(1-\delta_{c_p,l+2}+A^{-1+\delta_{c_p,l+2}}_{1,r-2l-3}(1+A^{-1}_{1,r-2l-4}))\prod_{q\in[1,l+1]\setminus([\mathbf{b},\mathbf{c}])}(1+A^{-1}_{1,r-2q+1})\right),$$

where $H_3 := \left(\prod_{t=0}^{l+1} Y_{1,j_k-2t+1}Y_{2,j_k-2t}\right)$, $H_4 := \left(\prod_{t=0}^{l+2} Y_{1,j_k-2t+1}\right)\left(\prod_{t=0}^{l+1} Y_{2,j_k-2t}\right) = H_3 \times Y_{1,j_k-2l-3}$. *If* $j_k = r$, *then we understand* $Y_{1,j_k+1} = 1$ *and* $\Lambda_{j_k+1} = 0$.

Proposition 6.7. *For* $l \in [0, \lfloor \frac{r}{2} \rfloor - 2]$, *let* $\mu''[l]$ *be the following iteration of mutations*

$$\mu''[l] := (\mu_{\lfloor \frac{r+1}{2} \rfloor -l-1}\mu_{r-l-1}\mu_{r-l}) \cdots (\mu_{\lfloor \frac{r+1}{2} \rfloor -2}\mu_{r-2}\mu_{r-1})(\mu_{\lfloor \frac{r+1}{2} \rfloor -1}\mu_{r-1}\mu_{r})\mu_{\lfloor \frac{r+1}{2} \rfloor}.$$

(*a*) *We have the cluster variable*

$$(\varphi^G_{(\mu''[l]\mathbb{V})})_{\lfloor \frac{r+1}{2} \rfloor -l-1}(a;\boldsymbol{Y}) = a^{\sum_{s=0}^{l+1}\Lambda_{2s+3}+\sum_{s=0}^{l}\Lambda_{2s+2}} \cdot H_5 \left(\prod_{q\in[1,l+1]}(1+A^{-1}_{1,2q}) + \sum_{p>0}\sum_{(\mathbf{b},\mathbf{c})\in R^p_{l+1}, b_1>0}\prod_{i=1}^{p} A[-c_i,-b_i;2]\prod_{q\in[1,l+1]\setminus[\mathbf{b},\mathbf{c}]}(1+A^{-1}_{1,2q})\right).$$

(*b*) *We also obtain the cluster variable*

$$\begin{aligned}(\varphi^G_{(\mu_{r-l-1}\mu''[l]\mathbb{V})})_{r-l-1}(a;\boldsymbol{Y}) &= a^{\sum_{s=0}^{l-1}\Lambda_{2s+3}+\sum_{s=0}^{l+1}\Lambda_{2s+2}} H_6 \\ &\times \left((1+A^{-1}_{1,2l+4}+A^{-1}_{1,2l+4}A^{-1}_{2l+5}))\prod_{q\in[1,l+1]}(1+A^{-1}_{1,2q})\right. \\ &+\sum_{p>0}\sum_{(\mathbf{b},\mathbf{c})\in R^p_{l+2}, b_1>0}(1-\delta_{c_p,l+2}+A_{1,2l+4}^{-1+\delta_{c_p,l+2}}(1+A^{-1}_{2l+5})) \\ &\left.\times\prod_{i=1}^{p}A[-c_i,-b_i;2]\prod_{q\in[1,l+1]\setminus[\mathbf{b},\mathbf{c}]}(1+A^{-1}_{1,2q})\right),\end{aligned}$$

where $H_5 := \left(\prod_{t=0}^{l} Y_{1,2t+2}\right)\left(\prod_{t=0}^{l+1} Y_{2,2t+1}\right)$, $H_6 := \left(\prod_{t=0}^{l+1} Y_{1,2t+2}Y_{2,2t+1}\right) = H_5 \times Y_{1,2t+4}$.

Furthermore, if r *is odd, then we get the cluster variable*

$$\begin{aligned}&(\varphi^G_{(\mu_1\mu_{\frac{r+3}{2}}\mu''[\frac{r-1}{2}-2]\mathbb{V})})_1(a;\boldsymbol{Y}) \\ &= a^{\sum_{s=0}^{\frac{r-5}{2}}\Lambda_{2s+3}+\sum_{s=0}^{\frac{r-3}{2}}\Lambda_{2s+2}}\left(\prod_{t=0}^{\frac{r-3}{2}}Y_{1,2t+2}\right)\left(\prod_{t=0}^{\frac{r-1}{2}}Y_{2,2t+1}\right)\left(\prod_{q\in[1,\frac{r-1}{2}]}(1+A^{-1}_{1,2q})\right. \\ &\left.+\sum_{p>0}\sum_{(\mathbf{b},\mathbf{c})\in R^p_{\frac{r-1}{2}},\ b_1>0}\prod_{i=1}^{p}A[-c_i,-b_i;2]\prod_{q\in[1,\frac{r-1}{2}]\setminus[\mathbf{b},\mathbf{c}]}(1+A^{-1}_{1,2q})\right),\end{aligned}$$

where we set $\mu''[-1] := \mu_2$ *when* $r = 3$.

The following theorem is the main result, which describes all the cluster variables in $\mathbb{C}[G^{e,c^2}]$ in terms of the monomials in Demazure crystals. We use the notation as in Proposition 6.4, 6.6 and 6.7.

Theorem 6.8. *There exist certain Demazure crystals such that each cluster variable in $\mathbb{C}[G^{e,c^2}]$ is the total sum of their monomial realizations. More precisely, it is given by Proposition 6.2, 6.3 and the following:*

(1) *Let $k \in [\lfloor \frac{r+1}{2} \rfloor + 1, r-1]$, $l \in [0, r-k-1]$ and $J := j_k = 2r-2k+2$.*
   (a) The cluster variable $(\varphi^G_{(\mu[l]\mathbb{V})})_{k-\lfloor \frac{r}{2} \rfloor + l}(a; \mathbf{Y})$ is the total sum of monomials in

$$B\left(\sum_{s=2r-2k+1-2l}^{2r-2k+1} \Lambda_s\right)_{w_1} \oplus \bigoplus_{\substack{p>0 \\ (\mathbf{b},\mathbf{c}) \in R^p_{l-1}}} B\left(\left(\sum_{s=2r-2k+1-2l}^{2r-2k+1} \Lambda_s\right) - \alpha[\mathbf{b},\mathbf{c};J]\right)_{w_1(\mathbf{b},\mathbf{c})}.$$

   (b) The cluster variable $(\varphi^G_{(\mu_{k+l+1}\mu[l]\mathbb{V})})_{k+l+1}(a; \mathbf{Y})$ is the total sum of monomials in

$$B\left(\sum_{s=2r-2k-2l}^{2r-2k+1} \Lambda_s\right)_{w_2} \oplus \bigoplus_{\substack{p>0 \\ (\mathbf{b},\mathbf{c}) \in R^p_{l}}} B\left(\left(\sum_{s=2r-2k-2l}^{2r-2k+1} \Lambda_s\right) - \alpha[\mathbf{b},\mathbf{c};J]\right)_{w_2(\mathbf{b},\mathbf{c})}.$$

(2) *Let $k \in [1, \lfloor \frac{r+1}{2} \rfloor - 2]$, $l \in [0, \lfloor \frac{r+1}{2} \rfloor - k - 2]$ and $J := j_k = 2\lfloor r+1/2 \rfloor - 2k + 1$.*
   (a) The cluster variable $(\varphi^G_{(\mu'[l]\mathbb{V})})_{k+l+1}(a; \mathbf{Y})$ is the total sum of monomials in

$$B\left(\sum_{s=2\lfloor \frac{r+1}{2} \rfloor -2k-2l-1}^{2\lfloor \frac{r+1}{2} \rfloor -2k+2} \Lambda_s\right)_{w_3} \oplus \bigoplus_{\substack{p>0 \\ (\mathbf{b},\mathbf{c}) \in R^p_{l+1} \\ \text{if } j_k=r \Rightarrow b_1>0}} B\left(\left(\sum_{s=2\lfloor \frac{r+1}{2} \rfloor -2k-2l-1}^{2\lfloor \frac{r+1}{2} \rfloor -2k+2} \Lambda_s\right) - \alpha[\mathbf{b},\mathbf{c};J+3]\right)_{w_3(\mathbf{b},\mathbf{c})},$$

   (b) $(\varphi^G_{(\mu_{\lfloor \frac{r}{2} \rfloor+k+l+2}\mu'[l]\mathbb{V})})_{\lfloor \frac{r}{2} \rfloor+k+l+2}(a; \mathbf{Y})$ is the total sum of monomials in

$$B\left(\sum_{s=2\lfloor \frac{r+1}{2} \rfloor -2k-2l-2}^{2\lfloor \frac{r+1}{2} \rfloor -2k+2} \Lambda_s\right)_{w_4} \oplus \bigoplus_{\substack{p>0 \\ (\mathbf{b},\mathbf{c}) \in R^p_{l+2} \\ \text{if } j_k=r \Rightarrow b_1>0}} B\left(\left(\sum_{s=2\lfloor \frac{r+1}{2} \rfloor -2k-2l-2}^{2\lfloor \frac{r+1}{2} \rfloor -2k+2} \Lambda_s\right) - \alpha[\mathbf{b},\mathbf{c};J+3]\right)_{w_4(\mathbf{b},\mathbf{c})}.$$

(3) *Let* $l \in [0, \lfloor \frac{r}{2} \rfloor - 2]$.
(a) The cluster variable $(\varphi^G_{(\mu''[l]\mathbb{V})})_{\lfloor \frac{r+1}{2} \rfloor - l - 1}(a; \boldsymbol{Y})$ *is the total sum of monomials in*

$$B\left(\sum_{s=1}^{2l+3} \Lambda_s\right)_{w_5} \oplus \bigoplus_{\substack{p>0 \\ (\mathbf{b},\mathbf{c}) \in R^p_{l+1}, b_1>0}} B\left(\left(\sum_{s=1}^{2l+3} \Lambda_s\right) - \alpha[-\mathbf{c}, -\mathbf{b}; 2]\right)_{w_5(\mathbf{b},\mathbf{c})}.$$

(b) The variable $(\varphi^G_{(\mu_{r-l-1}\mu''[l]\mathbb{V})})_{r-l-1}(a; \boldsymbol{Y})$ *is the total sum of monomials in*

$$B\left(\sum_{s=1}^{2l+4} \Lambda_s\right)_{w_6} \oplus \bigoplus_{\substack{p>0 \\ (\mathbf{b},\mathbf{c}) \in R^p_{l+2}, b_1>0}} B\left(\left(\sum_{s=1}^{2l+4} \Lambda_s\right) - \alpha[-\mathbf{c}, -\mathbf{b}; 2]\right)_{w_6(\mathbf{b},\mathbf{c})}.$$

If r is odd, then $(\varphi^G_{(\mu_1 \mu_{\frac{r+3}{2}} \mu''[\frac{r-1}{2}-2]\mathbb{V})})_1(a; \boldsymbol{Y})$ *is the total sum of monomials in*

$$B\left(\sum_{s=1}^{r} \Lambda_s\right)_{\prod_{q \in [1, \frac{r-1}{2}]} s_{2q}} \oplus \bigoplus_{\substack{p>0 \\ (\mathbf{b},\mathbf{c}) \in R^p_{\frac{r-1}{2}}, b_1>0}} B\left(\left(\sum_{s=1}^{r} \Lambda_s\right) - \alpha[-\mathbf{c}, -\mathbf{b}; 2]\right)_{\prod_{q \in [1, \frac{r-1}{2}] \setminus [\mathbf{b},\mathbf{c}]} s_{2q}}.$$

The explicit forms of the Weyl group elements in the above formula will be given in the next section.

We obtain the following corollary from Proposition 6.2, Proposition 6.3 and Theorem 6.8. Let Ξ be the set of the non-frozen cluster variables in $\mathbb{C}[G^{e,c^2}]$.

Corollary 6.9. *(1) Each initial cluster variable $\varphi_{\mathbb{V}_k}$ in $\mathbb{C}[G^{e,c^2}]$ is the total sum of monomials in the Demazure crystal $B(\Lambda_{j_k})_{c^2_{>2r-k}}$, where we use the notation as in (3.5).*
(2) For b, $b' \in I$ with $b \le b'$, there uniquely exists a non-initial cluster variable $\varphi[b, b']$ which is given as the total sum of monomials in

$$B(\sum_{j=b}^{b'} \Lambda_j)_w \oplus \bigoplus_{i=1}^{p} B(\lambda_i)_{w_i},$$

with some $p \in \mathbb{Z}_{\ge 0}$, $w, w_i \in W$ and $\lambda_i \in P^+$ such that $\left(\sum_{j=b}^{b'} \Lambda_j\right) - \lambda_i \in \bigoplus_i \mathbb{Z}_{\ge 0}\alpha_i$. Thus, the map $\Phi_{\ge -1} \to \Xi$ defined by

$$-\alpha_{j_k} \mapsto \varphi_{\mathbb{V}_k}, \quad \sum_{j=b}^{b'} \alpha_j \mapsto \varphi[b, b']$$

is a bijection between the almost positive roots $\Phi_{\ge -1}$ and Ξ.

Proof. Let us show (2) since the case (1) is immediate from Proposition 6.2 (1). We consider the following 6 cases.

(i-1) We suppose that b' and b are odd, and $3 \le b \le b' < r$. In this case, putting $k := \frac{2r-b'+1}{2}$ and $l := \frac{b'-b}{2}$, it follows $\lfloor \frac{r+1}{2} \rfloor + 1 \le k \le r-1$ and $0 \le l \le r-k-1$. Using Theorem 6.8 (1)(a), we obtain $\varphi[b,b'] = (\varphi_{(\mu[l]\mathbb{V})})_{k-\lfloor \frac{r}{2} \rfloor + l}$.

(i-2) In the case b and r are odd, and $3 \le b \le r-2$, putting $k := 1$ and $l := \frac{r-b-2}{2}$, we obtain $0 \le l \le \frac{r+1}{2} - 3$ and $j_1 = r$. It follows from Theorem 6.8 (2)(a) that $\varphi[b,r] = (\varphi_{(\mu'[l]\mathbb{V})})_{l+2}$.

(ii-1) We suppose that b' is odd and b is even, and $2 \le b < b' < r$. In this case, putting $k := \frac{2r-b'+1}{2}$ and $l := \frac{b'-b-1}{2}$, it follows $\lfloor \frac{r+1}{2} \rfloor + 1 \le k \le r-1$ and $0 \le l \le r-k-1$. Hence, we get $\varphi[b,b'] = (\varphi_{(\mu_{k+l+1}\mu[l]\mathbb{V})})_{k+l+1}$ by Theorem 6.8 (1)(b).

(ii-2) In the case r is odd, b is even and $2 \le b \le r-3$, putting $k := 1$ and $l := \frac{r-b-3}{2}$, we obtain $0 \le l \le \frac{r+1}{2} - 3$ and $j_1 = r$. So Theorem 6.8 (2)(b) says $\varphi[b,r] = (\varphi_{(\mu_{\lfloor \frac{r}{2} \rfloor + l+3}\mu'[l]\mathbb{V})})_{\lfloor \frac{r}{2} \rfloor + l+3}$.

(iii) We suppose that b' is even and b is odd, and $6 \le b' \le r$, $3 \le b \le b'-3$. In this setting, putting $k := \lfloor \frac{r+1}{2} \rfloor + 1 - \frac{b'}{2}$ and $l := \frac{b'-b-3}{2}$, we have $1 \le k \le \lfloor \frac{r+1}{2} \rfloor - 2$ and $0 \le l \le \lfloor \frac{r+1}{2} \rfloor - k - 2$. Therefore, we can verify $\varphi[b,b'] = (\varphi_{(\mu'[l]\mathbb{V})})_{k+l+1}$ by using Theorem 6.8 (2)(a).

(iv) We suppose that b' and b are even, and $6 \le b' \le r$, $2 \le b \le b'-4$. We set $k := \lfloor \frac{r+1}{2} \rfloor + 1 - \frac{b'}{2}$ and $l := \frac{b'-b-4}{2}$. Then k and l satisfy $1 \le k \le \lfloor \frac{r+1}{2} \rfloor - 2$ and $0 \le l \le \lfloor \frac{r+1}{2} \rfloor - k - 2$. Thus, the conclusion $\varphi[b,b'] = (\varphi_{(\mu_{\lfloor \frac{r}{2} \rfloor + k+l+2}\mu'[l]\mathbb{V})})_{\lfloor \frac{r}{2} \rfloor + k+l+2}$ follows from Theorem 6.8 (2)(b).

(v) Let b' be an odd number ($3 \le b' < r$). Setting $l := \frac{b'-3}{2}$, we see that $0 \le l \le \lfloor \frac{r}{2} \rfloor - 2$. Using Theorem 6.8 (3)(a), we have $\varphi[1,b'] = (\varphi_{(\mu''[l]\mathbb{V})})_{\lfloor \frac{r+1}{2} \rfloor - l-1}$. Similarly, in the case b' is an even number ($4 \le b' \le r$), putting $l := \frac{b'-4}{2}$, we get $\varphi[1,b'] = (\varphi_{(\mu_{r-l-1}\mu''[l]\mathbb{V})})_{r-l-1}$. In particular, if r is odd, then Theorem 6.8 (3) implies $\varphi[1,r] = (\varphi_{(\mu_1\mu_{\frac{r+3}{2}}\mu''[\frac{r-1}{2}-2]\mathbb{V})})_1$.

(vi) The remaining cluster variables are $\varphi[b'-2,b']$ (b' is even, $4 \le b' \le r$), $\varphi[b'-1,b']$, $\varphi[b',b']$ (b' is even, $2 \le b' \le r$) and $\varphi[1,1]$, and if r is odd, $\varphi[r-1,r]$ and $\varphi[r,r]$.

Putting $k := \lfloor \frac{r+1}{2} \rfloor + 1 - \frac{b'}{2}$, we have $1 \le k \le \lfloor \frac{r+1}{2} \rfloor$. By Proposition 6.2 (2), it follows $\varphi[b'-2,b'] = (\varphi_{(\mu_k\mathbb{V})})_k$. In the case r is odd, we have $\varphi[r-1,r] = (\varphi_{(\mu_1\mathbb{V})})_1$. Similarly, we have $\varphi[b',b'] = (\varphi_{(\mu_{r+1-\frac{b'}{2}}\mathbb{V})})_{r+1-\frac{b'}{2}}$ and $\varphi[1,2] = (\varphi_{(\mu_{\lfloor \frac{r+1}{2} \rfloor}\mathbb{V})})_{\lfloor \frac{r+1}{2} \rfloor}$.

From Proposition 6.3, setting $K := r+2-\frac{b'}{2}$ for $4 \le b'$, we obtain $\varphi[b'-1,b'] = (\varphi_{(\mu_{K-\lfloor \frac{r}{2} \rfloor -1}\mu_K\mathbb{V})})_{K-\lfloor \frac{r}{2} \rfloor -1}$. In the case r is odd then we have $\varphi[r,r] = (\varphi_{(\mu_1\mu_{\frac{r-1}{2}+2}\mathbb{V})})_1$.

The same proposition means $\varphi[1,1] = (\varphi_{(\mu_r\mu_{\lfloor \frac{r+1}{2} \rfloor}\mathbb{V})})_r$. $\square$

Example 6.10. We consider the same setting as in Example 6.5. Let μ (resp. μ') denote the monomial realization of crystal $B(\Lambda_5+\Lambda_6+\Lambda_7+\Lambda_8+\Lambda_9)$ (resp. $B(2\Lambda_5+$

$\Lambda_7 + 2\Lambda_9)$) such that the highest weight vector is realized as $Y_{2,5}Y_{1,6}Y_{2,7}Y_{1,8}Y_{2,9}$ (resp. $Y_{1,5}Y_{2,5}Y_{1,7}Y_{1,9}Y_{2,9}$). It follows from Theorem 5.2, 5.4 and (6.8) that

$$\begin{aligned}
&(\varphi^G_{(\mu[2]\mathbb{V})})_3(a;\mathbf{Y}) \\
&= a^{\Lambda_3+\Lambda_5+\Lambda_6+\Lambda_7+\Lambda_8+\Lambda_9}\left(\sum_{b\in B(\Lambda_5+\Lambda_6+\Lambda_7+\Lambda_8+\Lambda_9)_{s_6 s_8}} \mu(b) + \sum_{b\in B(2\Lambda_5+\Lambda_7+2\Lambda_9)_e} \mu'(b)\right).
\end{aligned}$$

Similarly, using (6.9),

$$\begin{aligned}
&a^{-(\Lambda_4+\Lambda_6+\Lambda_7+\Lambda_8+\Lambda_9)}(\varphi^G_{(\mu_9\mu[2]\mathbb{V})})_9(a;\mathbf{Y}) \\
&= \sum_{b\in B(\Lambda_4+\Lambda_5+\Lambda_6+\Lambda_7+\Lambda_8+\Lambda_9)_{s_3 s_4 s_6 s_8}} \mu(b) + \sum_{b\in B(\Lambda_4+2\Lambda_5+\Lambda_7+2\Lambda_9)_{s_3 s_4}} \mu'(b) \\
&\quad + \sum_{b\in B(\Lambda_3+\Lambda_5+\Lambda_6+\Lambda_7+2\Lambda_9)_{s_3}} \mu''(b) + \sum_{b\in B(\Lambda_3+\Lambda_5+2\Lambda_7+\Lambda_8+\Lambda_9)_{s_3}} \mu'''(b),
\end{aligned}$$

where μ, μ', μ'' and μ''' are the monomial realizations such that the highest weight vectors are realized by $Y_{1,4}Y_{2,5}Y_{1,6}Y_{2,7}Y_{1,8}Y_{2,9}$, $Y_{1,4}Y_{1,5}Y_{2,5}Y_{1,7}Y_{1,9}Y_{2,9}$, $Y_{1,3}Y_{1,5}Y_{2,6}Y_{1,7}Y_{1,9}Y_{2,9}$ and $Y_{1,3}Y_{1,5}Y_{1,7}Y_{2,7}Y_{1,8}Y_{2,9}$, respectively.

Example 6.11. Let $G = \mathrm{SL}_5(\mathbb{C})$ and $c = s_2s_4s_1s_3$. The all cluster variables in $\mathbb{C}[G^{e,c^2}]$ are

$$\begin{gathered}
(\varphi^G_{\mathbb{V}})_k,\ (\varphi^G_{(\mu_k\mathbb{V})})_k\ \ (1\le k\le 4),\quad (\varphi^G_{(\mu_1\mu_4\mathbb{V})})_1,\ (\varphi^G_{(\mu_4\mu_2\mathbb{V})})_4, \\
(\varphi^G_{(\mu[0]\mathbb{V})})_1,\ (\varphi^G_{(\mu_4\mu[0]\mathbb{V})})_4,\ (\varphi^G_{(\mu''[0]\mathbb{V})})_1,\ (\varphi^G_{(\mu_3\mu''[0]\mathbb{V})})_3,
\end{gathered}$$

where $\mu[0] = \mu_1\mu_4\mu_3$, $\mu''[0] = \mu_1\mu_3\mu_4\mu_2$, and these are described as the total sums of monomials in the following Demazure crystals up to torus parts and parametrized by the almost positive roots as follows:

$(\varphi^G_{\mathbb{V}})_k$	$B(\Lambda_{j_k})_{c^2_{>8-k}}$	$-\alpha_{j_k}$
$(\varphi^G_{(\mu_1\mathbb{V})})_1$	$B(\Lambda_2+\Lambda_3+\Lambda_4)_{s_1s_2s_4} \oplus B(\Lambda_1+\Lambda_3)_{s_1}$	$\alpha_2+\alpha_3+\alpha_4$
$(\varphi^G_{(\mu_2\mathbb{V})})_2$	$B(\Lambda_1+\Lambda_2)_{s_3s_2}$	$\alpha_1+\alpha_2$
$(\varphi^G_{(\mu_3\mathbb{V})})_3$	$B(\Lambda_4)_e$	α_4
$(\varphi^G_{(\mu_4\mathbb{V})})_4$	$B(\Lambda_2)_e$	α_2
$(\varphi^G_{(\mu_1\mu_4\mathbb{V})})_1$	$B(\Lambda_3+\Lambda_4)_{s_4}$	$\alpha_3+\alpha_4$
$(\varphi^G_{(\mu_4\mu_2\mathbb{V})})_4$	$B(\Lambda_1)_e$	α_1
$(\varphi^G_{(\mu[0]\mathbb{V})})_1$	$B(\Lambda_3)_e$	α_3
$(\varphi^G_{(\mu_4\mu[0]\mathbb{V})})_4$	$B(\Lambda_2+\Lambda_3)_{s_1s_2}$	$\alpha_2+\alpha_3$
$(\varphi^G_{(\mu''[0]\mathbb{V})})_1$	$B(\Lambda_1+\Lambda_2+\Lambda_3)_{s_2}$	$\alpha_1+\alpha_2+\alpha_3$
$(\varphi^G_{(\mu_3\mu''[0]\mathbb{V})})_3$	$B(\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4)_{s_2s_4} \oplus B(2\Lambda_1+\Lambda_3)_e$	$\alpha_1+\alpha_2+\alpha_3+\alpha_4$

7. Proof of the main theorem

In this section, we prove Proposition 6.4, 6.6, 6.7 and then finally Theorem 6.8. For $k_1, \cdots, k_s \in [1, r]$, let $\mu_{k_1} \cdots \mu_{k_s} \Gamma_{\mathbf{i}}$ be the quiver of the seed $(\mu_{k_1} \cdots \mu_{k_s}(\mathbb{V}), \mu_{k_1} \cdots \mu_{k_s}(\tilde{B}_{\mathbf{i}}))$ (Sect. 3).

Lemma 7.1. *In the setting of Proposition 6.4, we have*

$$(\varphi^G_{(\mu_{k+l+2}\mu_{k+l+1}\mu[l]\mathbb{V})})_{k+l+2}(a; \boldsymbol{Y}) = (\varphi^G_{(\mu_{k+l+2}\mathbb{V})})_{k+l+2}(a; \boldsymbol{Y}).$$

Proof. In the quiver $\Gamma_{\mathbf{i}}$, by (6.2), the below is around $(\varphi_{\mathbb{V}})_{k+l+2}$:

$$\begin{tikzcd}
 & (\varphi_{\mathbb{V}})_{r+k-\lfloor\frac{r}{2}\rfloor+l+2} \arrow[d] & (\varphi_{\mathbb{V}})_{r+k+l+2} \arrow[d] & (\varphi_{\mathbb{V}})_{r+k-\lfloor\frac{r}{2}\rfloor+l+1} \arrow[d] & (\varphi_{\mathbb{V}})_{r+k+l+1} \arrow[d] & \\
\cdots & (\varphi_{\mathbb{V}})_{k-\lfloor\frac{r}{2}\rfloor+l+2} \arrow[r] \arrow[d] & (\varphi_{\mathbb{V}})_{k+l+2} \arrow[ul] \arrow[ur] \arrow[d] & (\varphi_{\mathbb{V}})_{k-\lfloor\frac{r}{2}\rfloor+l+1} \arrow[l] \arrow[r] \arrow[d] & (\varphi_{\mathbb{V}})_{k+l+1} \arrow[ul] \arrow[d] & \cdots \\
 & (\varphi_{\mathbb{V}})_{-j_k-2l-5} & (\varphi_{\mathbb{V}})_{-j_k-2l-4} \arrow[ul] \arrow[ur] & (\varphi_{\mathbb{V}})_{-j_k-2l-3} & (\varphi_{\mathbb{V}})_{-j_k-2l-2} \arrow[ul] &
\end{tikzcd}$$

The initial cluster variables changed by $\mu_{k+l+1}\mu[l]$ are $(\varphi_{\mathbb{V}})_k, (\varphi_{\mathbb{V}})_{k+1}, \cdots, (\varphi_{\mathbb{V}})_{k+l+1}$ and $(\varphi_{\mathbb{V}})_{k-\lfloor\frac{r}{2}\rfloor}, (\varphi_{\mathbb{V}})_{k-\lfloor\frac{r}{2}\rfloor+1}, \cdots, (\varphi_{\mathbb{V}})_{k-\lfloor\frac{r}{2}\rfloor+l}$, which are not connected with $(\varphi_{\mathbb{V}})_{k+l+2}$ directly in the above quiver. Hence, Lemma 3.5 says that the arrows incident to $(\varphi_{\mathbb{V}})_{k+l+2}$ in $\Gamma_{\mathbf{i}}$ coincide with the ones in $\mu_{k+l+1}\mu[l]\Gamma_{\mathbf{i}}$. Thus, we get

$$\begin{aligned}
&(\varphi^G_{(\mu_{k+l+2}\mu_{k+l+1}\mu[l]\mathbb{V})})_{k+l+2} \\
&= \frac{1}{(\varphi^G_{\mathbb{V}})_{k+l+2}} \Big((\varphi^G_{\mathbb{V}})_{r+k-\lfloor\frac{r}{2}\rfloor+l+1} (\varphi^G_{\mathbb{V}})_{r+k-\lfloor\frac{r}{2}\rfloor+l+2} (\varphi^G_{\mathbb{V}})_{-j_k-2l-4} \\
&\quad + (\varphi^G_{\mathbb{V}})_{k-\lfloor\frac{r}{2}\rfloor+l+1} (\varphi^G_{\mathbb{V}})_{k-\lfloor\frac{r}{2}\rfloor+l+2} (\varphi^G_{\mathbb{V}})_{r+k+l+2} \Big) = (\varphi^G_{(\mu_{k+l+2}\mathbb{V})})_{k+l+2}. \quad \square
\end{aligned}$$

Next, we will order the indecomposable direct summands $V_1, \cdots, V_{2r}$ of $V_{\mathbf{i}}$ from the right:

$$V_{\mathbf{i}} = V_{2r} \oplus \cdots \oplus V_1.$$

For a basic $\mathcal{C}_{c^2}$-cluster-tilting Λ-module $T = T_{2r} \oplus \cdots \oplus T_1$, we write

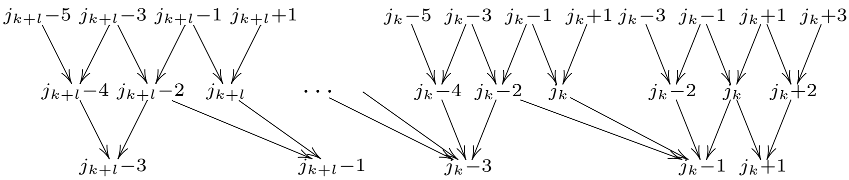
$$\mu_k(T) := \mu_{T_k}(T) = T_{2r} \oplus \cdots \oplus T_{k+1} \oplus T_k^* \oplus T_{k-1} \oplus \cdots \oplus T_1,$$

for $k \in [1, r]$. Let $(\mu_k(T))_l$ denote the l-th indecomposable direct summand of $\mu_k(T)$ from the right.

In the following Lemma 7.2-7.4, the notations in Remark 4.3 are applied.

Lemma 7.2. *We use the notation as in Proposition 6.4 and let j_k be the k-th index of* $\mathbf{i}$ *in* (2.4) *from the right.*

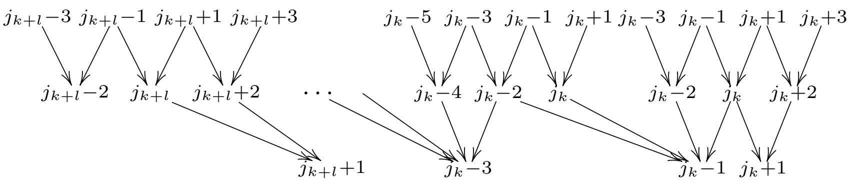
(a) *The module $(\mu[l](V_{\mathbf{i}}))_{k-\lfloor \frac{r}{2} \rfloor+l}$ is described as follows:*



(7.1)

Note that we have $j_{k+l} = j_k - 2l$ from (2.4).

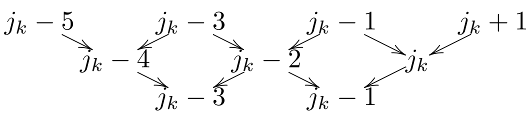
(b) *The module $(\mu_{k+l+1}\mu[l](V_{\mathbf{i}}))_{k+l+1}$ is described as follows:*



(7.2)

Proof. Using the induction on l, we shall prove (a) and (b) simultaneously.

First, let us prove (a) and (b) for $l = 0$. As have seen in Proposition 4.2 (4.2), $(\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k-\lfloor \frac{r}{2} \rfloor} = (V_{\mathbf{i}})_{k-\lfloor \frac{r}{2} \rfloor} = S_{j_k-1}$. We have already obtained $(\mu_k(V_{\mathbf{i}}))_k = V_k^*$ in Example 4.11 (4.10). Similarly, $(\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1}$ is

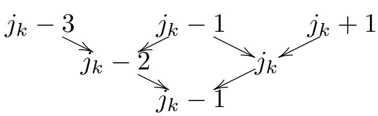


(7.3)

Hence, the modules $(\mu_k(V_{\mathbf{i}}))_k$ and $(\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1}$ have the simple submodule isomorphic to S_{j_k-1}. So there exist injective homomorphisms

$$(\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k-\lfloor \frac{r}{2} \rfloor} = S_{j_k-1} \to (\mu_k(V_{\mathbf{i}}))_k, \quad (\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k-\lfloor \frac{r}{2} \rfloor} \to (\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1}.$$

Let e_{j_k-1} denote a basis vector in $(\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k-\lfloor \frac{r}{2} \rfloor} = S_{j_k-1}$, and let $e'_{j_k-1} \in (\mu_k(V_{\mathbf{i}}))_k$ and $e''_{j_k-1} \in (\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1}$ be the images of e_{j_k-1} respectively. Note that since $j_{r+k-\lfloor \frac{r}{2} \rfloor} = j_{k-\lfloor \frac{r}{2} \rfloor} = j_k - 1$, the module $V_{r+k-\lfloor \frac{r}{2} \rfloor}$ is described as



(7.4)

and has the simple socle isomorphic to S_{j_k-1} (4.4). So there exists an injective homomorphism $S_{j_k-1} \to V_{r+k-\lfloor \frac{r}{2} \rfloor}$. However, this map is factorizable in the direct summands of $\mu_{k+1}\mu_k(V_{\mathbf{i}})$ since it is the same composite map as $S_{j_k-1} \to (\mu_{k+1}\mu_k(V_{\mathbf{i}}))_k = (\mu_k(V_{\mathbf{i}}))_k \to V_{r+k-\lfloor \frac{r}{2} \rfloor}$. Moreover, we can verify that $\mathrm{Hom}(S_{j_k-1}, V_t) = \{0\}$ for $t \neq r+k-\lfloor \frac{r}{2} \rfloor$, $k-\lfloor \frac{r}{2} \rfloor$, k, $k+1$. From Lemma 4.7 and Theorem 4.9 (iii), the exchange sequence associated to the direct summand S_{j_k-1} of $\mu_{k+1}\mu_k(V_{\mathbf{i}})$ is as follows:

$$0 \to S_{j_k-1} \to (\mu_k(V_{\mathbf{i}}))_k \oplus (\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1} \to (\mu_{k-\lfloor \frac{r}{2} \rfloor}\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k-\lfloor \frac{r}{2} \rfloor} \to 0,$$

where the image of the injective homomorphism $S_{j_k-1} \to (\mu_k(V_{\mathbf{i}}))_k \oplus (\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1}$ is $\mathbb{C}(e'_{j_k-1} + e''_{j_k-1})$. Therefore, the module

$$(\mu_{k-\lfloor \frac{r}{2} \rfloor}\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k-\lfloor \frac{r}{2} \rfloor} = ((\mu_k(V_{\mathbf{i}}))_k \oplus (\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1})/\mathbb{C}(e'_{j_k-1} + e''_{j_k-1}) \tag{7.5}$$

is described as follows:

$$\begin{tikzcd}[column sep=tiny] j_k-5 \arrow[dr] & & j_k-3 \arrow[dl] \arrow[dr] & & j_k-1 \arrow[dl] \arrow[dr] & & j_k+1 \arrow[dl] & & j_k-3 \arrow[dr] & & j_k-1 \arrow[dl] \arrow[dr] & & j_k+1 \arrow[dl] \arrow[dr] & & j_k+3 \arrow[dl] \\ & j_k-4 \arrow[dr] & & j_k-2 \arrow[dl] \arrow[drrrrrrr] & & j_k \arrow[drrrrr] & & & & j_k-2 & & j_k \arrow[dl] \arrow[dr] & & j_k+2 \arrow[dl] & \\ & & j_k-3 & & & & & & & & j_k-1 & & j_k+1 & & \end{tikzcd} \tag{7.6}$$

Since $j_{k+1} = j_k - 2$, we have the claim (a) for $l = 0$.

Next, let us prove the claim (b) for $l = 0$. We have seen that $(\mu_{k-\lfloor \frac{r}{2} \rfloor}\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1} = (\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1}$ is described as (7.3). It follows from (7.6) that $(\mu_{k-\lfloor \frac{r}{2} \rfloor}\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k-\lfloor \frac{r}{2} \rfloor}$ has the submodule isomorphic to $(\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1}$. Hence, we can find an injective homomorphism

$$(\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1} \to (\mu_{k-\lfloor \frac{r}{2} \rfloor}\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k-\lfloor \frac{r}{2} \rfloor}. \tag{7.7}$$

Note that the homomorphism (7.7) is not factorizable in the direct summands of $(\mu_{k-\lfloor \frac{r}{2} \rfloor}\mu_{k+1}\mu_k(V_{\mathbf{i}}))$ since no direct summand in $(\mu_{k-\lfloor \frac{r}{2} \rfloor}\mu_{k+1}\mu_k(V_{\mathbf{i}}))$ has submodules isomorphic to $(\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1}$. By (7.3) and (7.4), we see that the module $(\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1}$ has the quotient isomorphic to $V_{r+k-\lfloor \frac{r}{2} \rfloor}$. Then, we have a surjective homomorphism $(\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1} \to V_{r+k-\lfloor \frac{r}{2} \rfloor}$, which is, indeed, factorizable in the direct summands of $(\mu_{k-\lfloor \frac{r}{2} \rfloor}\mu_{k+1}\mu_k(V_{\mathbf{i}}))$ since it can be written as the composite map as follows: We label each basis of $V_{r+k-\lfloor \frac{r}{2} \rfloor}$ (7.4) as

$$\begin{tikzcd}[column sep=small] c^{(3)}_{j_k-3} \arrow[dr] & & c^{(3)}_{j_k-1} \arrow[dl] \arrow[dr] & & c^{(3)}_{j_k+1} \arrow[dl] \\ & c^{(2)}_{j_k-2} \arrow[dr] & & c^{(2)}_{j_k} \arrow[dl] & \\ & & c^{(1)}_{j_k-1} & & \end{tikzcd}$$

and each basis of $(\mu_{k-\lfloor \frac{r}{2} \rfloor}\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k-\lfloor \frac{r}{2} \rfloor}$ (7.6) as

$$\begin{tikzcd}[column sep=tiny]
d^{(3)}_{j_k-5} \arrow[dr] & & d^{(3)}_{j_k-3} \arrow[dl] \arrow[dr] & & d^{(3)}_{j_k-1} \arrow[dl] \arrow[dr] & & d^{(3)}_{j_k+1} \arrow[dl] & & e^{(3)}_{j_k-3} \arrow[dr] & & e^{(3)}_{j_k-1} \arrow[dl] \arrow[dr] & & e^{(3)}_{j_k+1} \arrow[dl] \arrow[dr] & & e^{(3)}_{j_k+3} \arrow[dl] \\
& d^{(2)}_{j_k-4} \arrow[dr] & & d^{(2)}_{j_k-2} \arrow[dl] \arrow[drrrrrrr] & & d^{(2)}_{j_k} \arrow[drrrrr] & & & & e^{(2)}_{j_k-2} \arrow[dr] & & e^{(2)}_{j_k} \arrow[dl] \arrow[dr] & & e^{(2)}_{j_k+2} \arrow[dl] & \\
& & d^{(1)}_{j_k-3} & & & & & & & & e^{(1)}_{j_k-1} & & e^{(1)}_{j_k+1} & &
\end{tikzcd}$$

Then we can define the surjective homomorphism $(\mu_{k-\lfloor\frac{r}{2}\rfloor}\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k-\lfloor\frac{r}{2}\rfloor} \to V_{r+k-\lfloor\frac{r}{2}\rfloor}$ by $e^{(1)}_{j_k-1} \mapsto c^{(1)}_{j_k-1}$, $d^{(2)}_j$ and $e^{(2)}_j \mapsto c^{(2)}_j$ $(j = j_k,\ j_k-2)$, $d^{(3)}_j$ and $e^{(3)}_j \mapsto c^{(3)}_j$ $(j = j_k+1,\ j_k-1,\ j_k-3)$ and mapping all others to 0. Then the homomorphism $(\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1} \to V_{r+k-\lfloor\frac{r}{2}\rfloor}$ coincides with the composite map

$$(\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1} \to (\mu_{k-\lfloor\frac{r}{2}\rfloor}\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k-\lfloor\frac{r}{2}\rfloor} \to V_{r+k-\lfloor\frac{r}{2}\rfloor},$$

where the first map is the one in (7.7).

The other non-zero homomorphisms from $(\mu_{k-\lfloor\frac{r}{2}\rfloor}\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1} = (\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1}$ to the direct summands of $(\mu_{k-\lfloor\frac{r}{2}\rfloor}\mu_{k+1}\mu_k(V_{\mathbf{i}}))$ are factored through $(\mu_{k-\lfloor\frac{r}{2}\rfloor}\mu_{k+1} \times \mu_k(V_{\mathbf{i}}))_{k-\lfloor\frac{r}{2}\rfloor}$. Thus, the exchange sequence associated to the direct summand $(\mu_{k-\lfloor\frac{r}{2}\rfloor} \times \mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1} = (\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1}$ of $\mu_{k-\lfloor\frac{r}{2}\rfloor}\mu_{k+1}\mu_k(V_{\mathbf{i}})$ is as follows:

$$0 \to (\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1} \to (\mu_{k-\lfloor\frac{r}{2}\rfloor}\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k-\lfloor\frac{r}{2}\rfloor} \to (\mu_{k+1}\mu_{k-\lfloor\frac{r}{2}\rfloor}\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1} \to 0.$$

By the above argument, we see that the module $(\mu_{k+1}\mu_{k-\lfloor\frac{r}{2}\rfloor}\mu_{k+1}\mu_k(V_{\mathbf{i}}))_{k+1}$ is described as

$$\begin{tikzcd}[column sep=tiny]
j_k-3 \arrow[dr] & & j_k-1 \arrow[dl] \arrow[dr] & & j_k+1 \arrow[dl] \arrow[dr] & & j_k+3 \arrow[dl] \\
& j_k-2 & & j_k \arrow[dr] & & j_k+2 \arrow[dl] & \\
& & & & j_k+1 & &
\end{tikzcd} \tag{7.8}$$

which means the claim (b) for $l = 0$.

Next, we assume that the claims (a) and (b) are shown for $0, 1, \cdots, l$. Let us consider the claim (a) for $l+1$, and then construct the exchange sequence associated to the direct summand $(\mu_{k+l+2}\mu_{k+l+1}\mu[l](V_{\mathbf{i}}))_{k-\lfloor\frac{r}{2}\rfloor+l+1}$ of $\mu_{k+l+2}\mu_{k+l+1}\mu[l](V_{\mathbf{i}})$ as in (7.10) below. Since the mutation $\mu_{k-\lfloor\frac{r}{2}\rfloor+l+1}$ does not appear in $\mu_{k+l+2}\mu_{k+l+1}\mu[l]$, we have $(\mu_{k+l+2}\mu_{k+l+1}\mu[l](V_{\mathbf{i}}))_{k-\lfloor\frac{r}{2}\rfloor+l+1} = (V_{\mathbf{i}})_{k-\lfloor\frac{r}{2}\rfloor+l+1} = S_{j_k-2l-3}$ (see (4.2)). By the induction hypothesis, the module $(\mu_{k+l+2}\mu_{k+l+1}\mu[l](V_{\mathbf{i}}))_{k-\lfloor\frac{r}{2}\rfloor+l} = (\mu[l](V_{\mathbf{i}}))_{k-\lfloor\frac{r}{2}\rfloor+l}$ is described as (7.1), and it has the simple submodule isomorphic to S_{j_k-2l-3}. It follows from Theorem 4.9 and a similar argument to the proof of Lemma 7.1 that the module $(\mu_{k+l+2}\mu_{k+l+1}\mu[l](V_{\mathbf{i}}))_{k+l+2}$ is the same as $(\mu_{k+l+2}(V_{\mathbf{i}}))_{k+l+2}$, and is described as follows:

$$
\begin{tikzcd}[column sep=tiny]
j_k-2l-7 \arrow[dr] & & j_k-2l-5 \arrow[dl] \arrow[dr] & & j_k-2l-3 \arrow[dl] \arrow[dr] & & j_k-2l-1 \arrow[dl] \\
& j_k-2l-6 \arrow[dr] & & j_k-2l-4 \arrow[dl] \arrow[dr] & & j_k-2l-2 \arrow[dl] & \\
& & j_k-2l-5 & & j_k-2l-3 & &
\end{tikzcd}
\tag{7.9}
$$

Hence, this module $(\mu_{k+l+2}(V_{\mathbf{i}}))_{k+l+2}$ has the simple submodule isomorphic to S_{j_k-2l-3}. It follows from (4.4) and $j_{r+k-\lfloor\frac{r}{2}\rfloor+l+1} = j_{k-\lfloor\frac{r}{2}\rfloor+l+1} = j_k-2l-3$ that the module $(\mu_{k+l+2}\mu_{k+l+1}\mu[l](V_{\mathbf{i}}))_{r+k-\lfloor\frac{r}{2}\rfloor+l+1} = (V_{\mathbf{i}})_{r+k-\lfloor\frac{r}{2}\rfloor+l+1}$ is described as

$$
\begin{tikzcd}[column sep=tiny, row sep=small]
j_k-2l-5 \arrow[dr] & & j_k-2l-3 \arrow[dl] \arrow[dr] & & j_k-2l-1 \arrow[dl] \\
& j_k-2l-4 \arrow[dr] & & j_k-2l-2 \arrow[dl] & \\
& & j_k-2l-3 & &
\end{tikzcd}
$$

and there exists an injective homomorphism $S_{j_k-2l-3} \to (V_{\mathbf{i}})_{r+k-\lfloor\frac{r}{2}\rfloor+l+1}$. But, this map is factorizable since it can be written as the composite map $S_{j_k-2l-3} \to (\mu_{k+l+2}(V_{\mathbf{i}}))_{k+l+2} \to (V_{\mathbf{i}})_{r+k-\lfloor\frac{r}{2}\rfloor+l+1}$. By the induction hypothesis, the other direct summands of $\mu_{k+l+2}\mu_{k+l+1}\mu[l](V_{\mathbf{i}})$ do not have the simple submodule isomorphic to S_{j_k-2l-3}. Thus, the exchange sequence associated to the direct summand $(\mu_{k+l+2}\mu_{k+l+1}\mu[l](V_{\mathbf{i}}))_{k-\lfloor\frac{r}{2}\rfloor+l+1} = S_{j_k-2l-3}$ of $\mu_{k+l+2}\mu_{k+l+1}\mu[l](V_{\mathbf{i}})$ is

$$
\begin{aligned}
0 \to S_{j_k-2l-3} \to (\mu_{k+l+2}(V_{\mathbf{i}}))_{k+l+2} \oplus (\mu[l](V_{\mathbf{i}}))_{k-\lfloor\frac{r}{2}\rfloor+l} \to \\
(\mu_{k-\lfloor\frac{r}{2}\rfloor+l+1}\mu_{k+l+2}\mu_{k+l+1}\mu[l](V_{\mathbf{i}}))_{k-\lfloor\frac{r}{2}\rfloor+l+1} \to 0.
\end{aligned}
\tag{7.10}
$$

The module $(\mu[l+1](V_{\mathbf{i}}))_{k-\lfloor\frac{r}{2}\rfloor+l+1} = (\mu_{k-\lfloor\frac{r}{2}\rfloor+l+1}\mu_{k+l+2}\mu_{k+l+1}\mu[l](V_{\mathbf{i}}))_{k-\lfloor\frac{r}{2}\rfloor+l+1}$ is described as

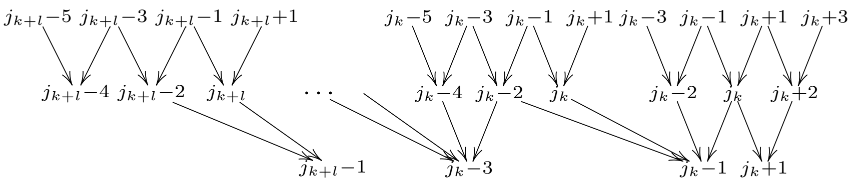
$$
\begin{tikzcd}[column sep=tiny]
j_{k+l}-7 \arrow[dr] & & j_{k+l}-5 \arrow[dl] \arrow[dr] & & j_{k+l}-3 \arrow[dl] \arrow[dr] & & j_{k+l}-1 \arrow[dl] & & & & j_k-5 \arrow[dr] & & j_k-3 \arrow[dl] \arrow[dr] & & j_k-1 \arrow[dl] \arrow[dr] & & j_k+1 \arrow[dl] & j_k-3 \arrow[dr] & & j_k-1 \arrow[dl] \arrow[dr] & & j_k+1 \arrow[dl] \arrow[dr] & & j_k+3 \arrow[dl] \\
& j_{k+l}-6 \arrow[dr] & & j_{k+l}-4 \arrow[dl] \arrow[drrrr] & & j_{k+l}-2 \arrow[drr] & & & \cdots \arrow[drrrr] & \arrow[drrr] & & j_k-4 \arrow[dr] & & j_k-2 \arrow[dl] \arrow[drrrrrr] & & j_k \arrow[drrrr] & & & j_k-2 \arrow[dr] & & j_k \arrow[dl] \arrow[dr] & & j_k+2 \arrow[dl] & \\
& & j_{k+l}-5 & & & & & j_{k+l}-3 & & & & & j_k-3 & & & & & & & j_k-1 & & j_k+1 & &
\end{tikzcd}
\tag{7.11}
$$

Taking $j_{k+l} = j_k - 2l$ into account, we get (a) for $l+1$.

Next, we consider the claim (b) for $l+1$. The module $(\mu[l+1](V_{\mathbf{i}}))_{k+l+2} = (\mu_{k+l+2}\mu_{k+l+1}\mu[l](V_{\mathbf{i}}))_{k+l+2}$ is described as (7.9). By the description (7.11) of the module $(\mu[l+1](V_{\mathbf{i}}))_{k-\lfloor\frac{r}{2}\rfloor+l+1}$, it has the submodule isomorphic to $(\mu_{k+l+2}\mu_{k+l+1}\mu[l](V_{\mathbf{i}}))_{k+l+2}$. Using the same argument in the proof of claim (b) for $l=0$, there exists an injective homomorphism $(\mu_{k+l+2}\mu_{k+l+1}\mu[l](V_{\mathbf{i}}))_{k+l+2} \to (\mu[l+1](V_{\mathbf{i}}))_{k-\lfloor\frac{r}{2}\rfloor+l+1}$, which is not factorizable in the direct summands of $(\mu[l+1](V_{\mathbf{i}}))$, and the other non-zero homomorphisms from $(\mu_{k+l+2}\mu_{k+l+1}\mu[l](V_{\mathbf{i}}))_{k+l+2}$ to the direct summands of $(\mu[l+1](V_{\mathbf{i}}))$ are factored through $(\mu[l+1](V_{\mathbf{i}}))_{k-\lfloor \frac{r}{2} \rfloor+l+1}$. Thus, the exchange sequence associated to the direct summand $(\mu[l+1](V_{\mathbf{i}}))_{k+l+2}$ of $(\mu[l+1](V_{\mathbf{i}}))$ is

$$0 \to (\mu[l+1](V_{\mathbf{i}}))_{k+l+2} \to (\mu[l+1](V_{\mathbf{i}}))_{k-\lfloor \frac{r}{2} \rfloor+l+1} \to (\mu_{k+l+2}\mu[l+1](V_{\mathbf{i}}))_{k+l+2} \to 0, \tag{7.12}$$

which yields the following description of the module $(\mu_{k+l+2}\mu[l+1](V_{\mathbf{i}}))_{k+l+2}$:



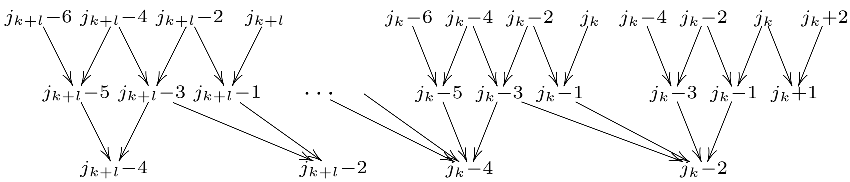
(7.13)

Because of $j_{k+l+1} = j_{k+l} - 2$, we get (b) for $l+1$. $\square$

We can similarly verify the following two lemmas.

Lemma 7.3. *We use the notation as in Proposition 6.6 and let j_k be the k-th index of* $\mathbf{i}$ (2.4) *from the right.*

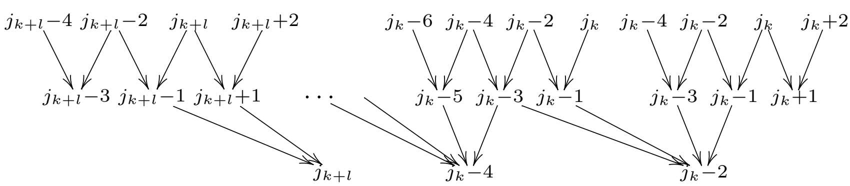
(*a*) *The module $(\mu'[l](V_{\mathbf{i}}))_{k+l+1}$ is described as follows:*



(7.14)

Note that $j_{k+l} = j_k - 2l$ from (2.4).

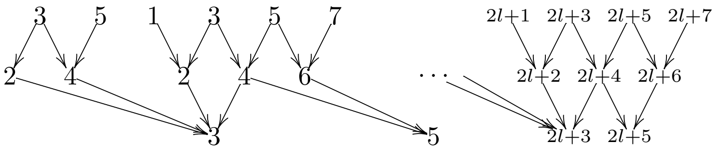
(*b*) *The module $(\mu_{\lfloor \frac{r}{2} \rfloor+k+l+2}\mu'[l](V_{\mathbf{i}}))_{\lfloor \frac{r}{2} \rfloor+k+l+2}$ is described as follows:*



(7.15)

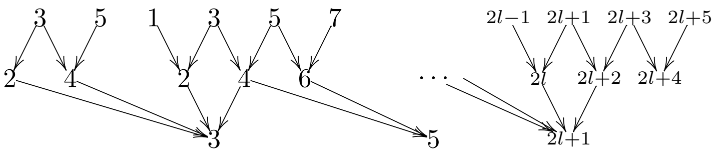
Lemma 7.4. *We use the notation as in Proposition 6.7.*

(*a*) *The module* $(\mu''[l](V_{\mathbf{i}}))_{\lfloor\frac{r+1}{2}\rfloor-l-1}$ *is described as follows:*



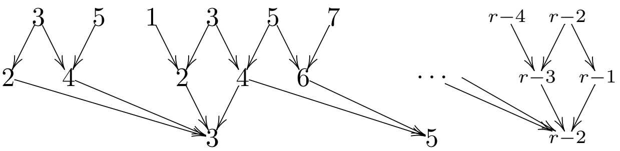
(7.16)

(*b*) *The module* $(\mu_{r-l-1}\mu''[l](V_{\mathbf{i}}))_{r-l-1}$ *is described as follows:*



(7.17)

Furthermore, if r *is odd, then the module* $(\mu_1\mu_{\frac{r+3}{2}}\mu''[\frac{r-1}{2}-2](V_{\mathbf{i}}))_1$ *is described as*



(7.18)

Next, let us prove Proposition 6.4. The following is an overview of the proof: We will use the induction on l. Using the exchange relation (3.1), Theorem 4.9(ii), (6.4), (6.5) and the induction hypothesis, calculations of $(\varphi^G_{(\mu[l+1]\mathbb{V})})_{k-\lfloor\frac{r}{2}\rfloor+l+1}(a;\mathbf{Y})$, $(\varphi^G_{(\mu_{k+l+2}\mu[l]\mathbb{V})})_{k+l+2}(a;\mathbf{Y})$ are reduced to those of $\varphi_{(\mu[l+1]V)_{k-\lfloor\frac{r}{2}\rfloor+l+1}}\circ x^G_{\mathbf{i}}(1;\phi(\mathbf{Y}))$, $\varphi_{(\mu_{k+l+2}\mu[l+1]V)_{k+l+2}}\circ x^G_{\mathbf{i}}(1;\phi(\mathbf{Y}))$, respectively. Then we can calculate $\varphi_{(\mu[l+1]V)_{k-\lfloor\frac{r}{2}\rfloor+l+1}}\circ x^G_{\mathbf{i}}(1;\phi(\mathbf{Y}))$ and $\varphi_{(\mu_{k+l+2}\mu[l+1]V)_{k+l+2}}\circ x^G_{\mathbf{i}}(1;\phi(\mathbf{Y}))$ by the explicit forms of the modules $(\mu[l+1]V)_{k-\lfloor\frac{r}{2}\rfloor+l+1}$, $(\mu_{k+l+2}\mu[l+1]V)_{k+l+2}$ in Lemma 7.2, Proposition 4.13, the formulas (4.11) and (6.7).

Proof of Proposition 6.4. For any cluster $\mathbb{T}=((\varphi_{\mathbb{T}})_j)_{j\in[1,2r]\cup[-r,-1]}$, $(\varphi_{\mathbb{T}})_s$ $(s\in[r+1,2r]\cup[-r,-1])$ is frozen, then we have $(\varphi_{\mathbb{T}})_s=(\varphi_{\mathbb{V}})_s$. Using the induction on l, let us prove Proposition 6.4 (a) and (b) simultaneously. For $l=0$, let us calculate $(\varphi^G_{(\mu[0]\mathbb{V})})_{k-\lfloor\frac{r}{2}\rfloor}$. In Sect. 6, we see that the vertices and the arrows around the vertex $(\varphi_{\mathbb{V}})_k$ in the quiver $\Gamma_{\mathbf{i}}$ are described as follows:

$$\begin{tikzcd}
 & (\varphi_{\mathbb{V}})_{r+k+1} \arrow[d] & (\varphi_{\mathbb{V}})_{r+k-\lfloor\frac{r}{2}\rfloor} \arrow[d] & (\varphi_{\mathbb{V}})_{r+k} \arrow[d] & (\varphi_{\mathbb{V}})_{r+k-\lfloor\frac{r}{2}\rfloor-1} \arrow[d] & \\
\cdots & (\varphi_{\mathbb{V}})_{k+1} \arrow[d] \arrow[ur] & (\varphi_{\mathbb{V}})_{k-\lfloor\frac{r}{2}\rfloor} \arrow[d] \arrow[l] \arrow[r] & (\varphi_{\mathbb{V}})_{k} \arrow[d] \arrow[ul] \arrow[ur] & (\varphi_{\mathbb{V}})_{k-\lfloor\frac{r}{2}\rfloor-1} \arrow[d] \arrow[l] & \cdots \\
 & (\varphi_{\mathbb{V}})_{-j_k+2} \arrow[ur] & (\varphi_{\mathbb{V}})_{-j_k+1} & (\varphi_{\mathbb{V}})_{-j_k} \arrow[ul] \arrow[ur] & (\varphi_{\mathbb{V}})_{-j_k-1} &
\end{tikzcd}$$

Applying the mutation $\mu_{k+1}\mu_k$ to this quiver, the arrows between $(\varphi_{\mathbb{V}})_{k-\lfloor\frac{r}{2}\rfloor}$ and $(\varphi_{\mathbb{V}})_{-s}$ $(1 \le s \le r)$

$$\begin{tikzcd}
{} \arrow[r] & (\varphi_{\mathbb{V}})_{k+1} \arrow[d] & (\varphi_{\mathbb{V}})_{k-\lfloor\frac{r}{2}\rfloor} \arrow[l] \arrow[r] \arrow[d] & (\varphi_{\mathbb{V}})_{k} \arrow[d] & {} \arrow[l] \\
& (\varphi_{\mathbb{V}})_{-j_k+2} \arrow[ul] \arrow[ur] & (\varphi_{\mathbb{V}})_{-j_k+1} & (\varphi_{\mathbb{V}})_{-j_k} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

are transformed to

$$\begin{tikzcd}
{} & (\varphi_{\mu_{k+1}\mu_k(\mathbb{V})})_{k+1} \arrow[l] \arrow[r] & (\varphi_{\mathbb{V}})_{k-\lfloor\frac{r}{2}\rfloor} \arrow[d] & (\varphi_{\mu_k(\mathbb{V})})_{k} \arrow[l] \arrow[r] & {} \\
& (\varphi_{\mathbb{V}})_{-j_k+2} \arrow[u] & (\varphi_{\mathbb{V}})_{-j_k+1} & (\varphi_{\mathbb{V}})_{-j_k} \arrow[u] &
\end{tikzcd}$$

by Lemma 3.5. Similarly, the arrows between $(\varphi_{(\mu_{k+1}\mu_k\mathbb{V})})_{k-\lfloor\frac{r}{2}\rfloor} = (\varphi_{\mathbb{V}})_{k-\lfloor\frac{r}{2}\rfloor}$ and $(\varphi_{\mu_{k+1}\mu_k\mathbb{V}})_s$ $(1 \le s \le 2r)$ in $\mu_{k+1}\mu_k(\Gamma_{\mathbf{i}})$ are

$$\begin{tikzcd}
(\varphi_{\mathbb{V}})_{r+k-\lfloor\frac{r}{2}\rfloor+1} & & (\varphi_{\mathbb{V}})_{r+k-\lfloor\frac{r}{2}\rfloor} & & (\varphi_{\mathbb{V}})_{r+k-\lfloor\frac{r}{2}\rfloor-1} \\
& (\varphi_{(\mu_{k+1}\mu_k\mathbb{V})})_{k+1} \arrow[r] & (\varphi_{\mathbb{V}})_{k-\lfloor\frac{r}{2}\rfloor} \arrow[ull] \arrow[u] \arrow[urr] & (\varphi_{(\mu_k\mathbb{V})})_{k} \arrow[l] &
\end{tikzcd}$$

Thus, by the exchange relation (3.1),

$$\begin{aligned}
&(\varphi^G_{(\mu[0]\mathbb{V})})_{k-\lfloor\frac{r}{2}\rfloor} \\
&= \frac{(\varphi^G_{(\mu_k\mathbb{V})})_k(\varphi^G_{(\mu_{k+1}\mu_k\mathbb{V})})_{k+1} + (\varphi^G_{\mathbb{V}})_{-j_k+1}(\varphi^G_{\mathbb{V}})_{r+k-\lfloor\frac{r}{2}\rfloor-1}(\varphi^G_{\mathbb{V}})_{r+k-\lfloor\frac{r}{2}\rfloor}(\varphi^G_{\mathbb{V}})_{r+k-\lfloor\frac{r}{2}\rfloor+1}}{(\varphi^G_{\mathbb{V}})_{k-\lfloor\frac{r}{2}\rfloor}}.
\end{aligned}$$

By (6.6) in the proof of Proposition 6.2, we can write

$$\begin{aligned}
(\varphi^G_{(\mu_k\mathbb{V})})_k &= (\varphi_{(\mu_k V)_k}) \circ x^G_{\mathbf{i}}(\Phi_H;\phi(\mathbf{Y})) \\
&= (\Phi_H(\mathbf{Y}))^{\Lambda_{j_k-1}}(\Phi_H(\mathbf{Y}))^{\Lambda_{j_k+1}} \cdot (\varphi_{(\mu_k V)_k}) \circ x^G_{\mathbf{i}}(1;\phi(\mathbf{Y})). \qquad (7.19)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(\varphi^G_{(\mu_{k+1}\mu_k\mathbb{V})})_{k+1} &= (\varphi^G_{(\mu_{k+1}\mathbb{V})})_{k+1} \\
&= (\Phi_H(\mathbf{Y}))^{\Lambda_{j_k-3}}(\Phi_H(\mathbf{Y}))^{\Lambda_{j_k-1}} \cdot (\varphi_{(\mu_{k+1}V)_{k+1}}) \circ x^G_{\mathbf{i}}(1;\phi(\mathbf{Y})). \qquad (7.20)
\end{aligned}$$

Therefore, using (6.4), (6.5), (7.19), (7.20) and Theorem 4.9 (ii), we obtain

$$\begin{aligned}
(\varphi^G_{(\mu[0]\mathbb{V})})_{k-\lfloor\frac{r}{2}\rfloor}(a;\mathbf{Y}) &= \frac{(\Phi_H(\mathbf{Y}))^{2\Lambda_{j_k-1}+\Lambda_{j_k-3}+\Lambda_{j_k+1}}}{(\Phi_H(\mathbf{Y}))^{\Lambda_{j_k-1}}} \\
&\times \frac{(\varphi_{(\mu_k V)_k})(\varphi_{(\mu_{k+1}\mu_k V)_{k+1}}) + (\varphi_{V_{r+k-\lfloor\frac{r}{2}\rfloor-1}})(\varphi_{V_{r+k-\lfloor\frac{r}{2}\rfloor}})(\varphi_{V_{r+k-\lfloor\frac{r}{2}\rfloor+1}})}{(\varphi_{V_{k-\lfloor\frac{r}{2}\rfloor}})} \circ x^G_{\mathbf{i}}(1;\phi(\mathbf{Y}))
\end{aligned}$$

$$
\begin{aligned}
&= (\Phi_H(\mathbf{Y}))^{\Lambda_{j_k-1}+\Lambda_{j_k-3}+\Lambda_{j_k+1}} \times (\varphi_{(\mu[0]V)_{k-\lfloor \frac{r}{2} \rfloor}}) \circ x_{\mathbf{i}}^G(1;\phi(\mathbf{Y})) \qquad (7.21)\\
&= a^{\Lambda_{j_k-1}+\Lambda_{j_k-3}+\Lambda_{j_k+1}} Y_{1,j_k-1}Y_{2,j_k-1}Y_{1,j_k-3}Y_{2,j_k-3}Y_{1,j_k+1}Y_{2,j_k+1} \times \\
&\quad (\varphi_{(\mu[0]V)_{k-\lfloor \frac{r}{2} \rfloor}}) \circ x_{\mathbf{i}}^G(1;\phi(\mathbf{Y})).
\end{aligned}
$$

The module $(\mu[0]V)_{k-\lfloor \frac{r}{2} \rfloor}$ is described as (7.6). Using Proposition 4.13 and (4.11), let us calculate $(\varphi_{(\mu[0]V)_{k-\lfloor \frac{r}{2} \rfloor}}) \circ x_{\mathbf{i}}^G(1;\phi(\mathbf{Y}))$. To do that we will find

$$
\mathbf{a} = (a_{1,j_r}, \cdots, a_{1,j_2}, a_{1,j_1}, a_{2,j_r}, \cdots, a_{2,j_2}, a_{2,j_1}) \in (\mathbb{Z}_{\geq 0})^{2r}
$$

satisfying $\mathcal{F}_{\mathbf{i}^{\mathbf{a}},(\mu[0]V)_{k-\lfloor \frac{r}{2} \rfloor}} \neq \phi$ (or equivalently, $\mathcal{F}_{\mathbf{i},\mathbf{a},(\mu[0]V)_{k-\lfloor \frac{r}{2} \rfloor}} \neq \phi$). If $\mathcal{F}_{\mathbf{i}^{\mathbf{a}},(\mu[0]V)_{k-\lfloor \frac{r}{2} \rfloor}} \neq \phi$, by counting the number of the bases in (7.6) and the fact that the dimension at j_k+1 is 3, we have $a_{1,j_k+1}+a_{2,j_k+1}=3$. Considering similarly,

$$
\begin{aligned}
&a_{1,j_k+1}+a_{2,j_k+1} = a_{1,j_k-1}+a_{2,j_k-1} = a_{1,j_k-3}+a_{2,j_k-3} = 3,\\
&a_{1,j_k}+a_{2,j_k} = a_{1,j_k-2}+a_{2,j_k-2} = 2, \qquad (7.22)\\
&a_{1,j_k+3}+a_{2,j_k+3} = a_{1,j_k+2}+a_{2,j_k+2} = a_{1,j_k-4}+a_{2,j_k-4} = a_{1,j_k-5}+a_{2,j_k-5} = 1.
\end{aligned}
$$

Since the module $(\mu[0]V)_{k-\lfloor \frac{r}{2} \rfloor}$ does not have the simple submodules isomorphic to $S_{j_r}, S_{j_{r-1}}, \cdots, S_{j_{\lfloor \frac{r+1}{2} \rfloor+1}}$, we have $a_{1,j_r} = a_{1,j_{r-1}} = \cdots = a_{1,j_{\lfloor \frac{r+1}{2} \rfloor+1}} = 0$, which yields $a_{2,j_k-4}=1$, $a_{2,j_k-2}=2$, $a_{2,j_k}=2$ and $a_{2,j_k+2}=1$. We can also check that $a_{1,j_k-3} = a_{1,j_k-1} = a_{1,j_k+1} = 1$. Thus, $\mathcal{F}_{\mathbf{i}^{\mathbf{a}},(\mu[0]V)_{k-\lfloor \frac{r}{2} \rfloor}} \neq \phi$ if and only if

$$
\begin{aligned}
\mathbf{i}^{\mathbf{a}} = (&j_k-3, j_k-1, j_k+1, j_k-4, j_k-2, j_k-2, j_k, j_k, j_k+2,\\
&j_k-5, j_k-3, j_k-3, j_k-1, j_k-1, j_k+1, j_k+1, j_k+3). \qquad (7.23)
\end{aligned}
$$

Then we show that $\mathcal{F}_{\mathbf{i},\mathbf{a},(\mu[0]V)_{k-\lfloor \frac{r}{2} \rfloor}}$ is a point. Here, we use the notation as in (4.11). By the above argument and (2.5), (2.6), (2.7), we have

$$
\begin{aligned}
(\varphi^G_{(\mu[0]\mathbb{V})})_{k-\lfloor \frac{r}{2} \rfloor}(a;\mathbf{Y}) &= a^{\Lambda_{j_k-1}+\Lambda_{j_k-3}+\Lambda_{j_k+1}} Y_{1,j_k-1}Y_{2,j_k-1}Y_{1,j_k-3}Y_{2,j_k-3}Y_{1,j_k+1}Y_{2,j_k+1}\\
&\times \Phi_{1,j_k-3}(\mathbf{Y})\Phi_{1,j_k-1}(\mathbf{Y})\Phi_{1,j_k+1}(\mathbf{Y})\Phi_{2,j_k-4}(\mathbf{Y})\Phi^2_{2,j_k-2}(\mathbf{Y})\Phi^2_{2,j_k}(\mathbf{Y})\Phi^2_{2,j_k}(\mathbf{Y})\\
&\times \Phi_{2,j_k+2}(\mathbf{Y})\Phi_{2,j_k-5}(\mathbf{Y})\Phi^2_{2,j_k-3}(\mathbf{Y})\Phi^2_{2,j_k-1}(\mathbf{Y})\Phi^2_{2,j_k+1}(\mathbf{Y})\Phi_{2,j_k+3}(\mathbf{Y})\\
&= a^{\Lambda_{j_k-1}+\Lambda_{j_k-3}+\Lambda_{j_k+1}} Y_{2,j_k-1},
\end{aligned}
$$

which implies the claim (a) for $l=0$.

Next, let us consider the claim (b) for $l=0$. By Lemma 3.5, the arrows between $(\varphi_{(\mu[0]\mathbb{V})})_{k+1} = (\varphi_{(\mu_{k+1}\mathbb{V})})_{k+1}$ and $(\varphi_{(\mu[0]\mathbb{V})})_s$ ($s \in [-r,-1] \cup [1,2r]$) in $\mu[0](\Gamma_{\mathbf{i}})$, are

$$\begin{tikzcd}
 & (\varphi\mathbb{V})_{r+k+1} & & (\varphi\mathbb{V})_{r+k-\lfloor\frac{r}{2}\rfloor-1} \\
(\varphi\mathbb{V})_{k-\lfloor\frac{r}{2}\rfloor+1} & (\varphi_{(\mu_{k+1}\mathbb{V})})_{k+1} \arrow[l] \arrow[u] \arrow[urr] \arrow[dr] \arrow[d] & (\varphi_{(\mu[0]\mathbb{V})})_{k-\lfloor\frac{r}{2}\rfloor} \arrow[l] & \\
 & (\varphi\mathbb{V})_{-j_k+2} & (\varphi\mathbb{V})_{-j_k+1} &
\end{tikzcd}$$

Thus,

$$\begin{aligned}
&(\varphi^G_{(\mu_{k+1}\mu[0]\mathbb{V})})_{k+1}(a;\mathbf{Y}) \\
&= \frac{(\varphi^G_{(\mu[0]\mathbb{V})})_{k-\lfloor\frac{r}{2}\rfloor}(\varphi^G_{\mathbb{V}})_{-j_k+2} + (\varphi^G_{\mathbb{V}})_{r+k-\lfloor\frac{r}{2}\rfloor-1}(\varphi^G_{\mathbb{V}})_{r+k+1}(\varphi^G_{\mathbb{V}})_{k-\lfloor\frac{r}{2}\rfloor+1}(\varphi^G_{\mathbb{V}})_{-j_k+1}}{(\varphi^G_{(\mu_{k+1}\mathbb{V})})_{k+1}}.
\end{aligned}$$

Using (6.4), (6.5), (7.20), (7.21) and Theorem 4.9 (ii), we obtain

$$\begin{aligned}
&(\varphi^G_{(\mu_{k+1}\mu[0]\mathbb{V})})_{k+1}(a;\mathbf{Y}) = (\Phi_H(a;\mathbf{Y}))^{\Lambda_{j_k-2}+\Lambda_{j_k+1}} \\
&\times \frac{(\varphi_{(\mu[0]V)_{k-\lfloor\frac{r}{2}\rfloor}}) + (\varphi_{V_{r+k-\lfloor\frac{r}{2}\rfloor-1}})(\varphi_{V_{r+k+1}})(\varphi_{V_{k-\lfloor\frac{r}{2}\rfloor+1}})}{(\varphi_{(\mu_{k+1}V)_{k+1}})} \circ x^G_{\mathbf{i}}(1;\phi(\mathbf{Y})) \\
&= a^{\Lambda_{j_k-2}+\Lambda_{j_k+1}} Y_{1,j_k-2}Y_{2,j_k-2}Y_{1,j_k+1}Y_{2,j_k+1} \times (\varphi_{(\mu_{k+1}\mu[0]V)_{k+1}}) \circ x^G_{\mathbf{i}}(1;\phi(\mathbf{Y})).
\end{aligned} \tag{7.24}$$

Applying a similar argument as in (7.23) to the module $(\mu_{k+1}\mu[0]V)_{k+1}$ in (7.8), for $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^{2r}$, we find that $\mathcal{F}_{\mathbf{i}^{\mathbf{a}},(\mu_{k+1}\mu[0]V)_{k+1}} \neq \phi$ if and only if

$$\begin{aligned}
\mathbf{i}^{\mathbf{a}} = &(j_k+1, j_k-2, j_k, j_k+2, j_k-3, j_k-1, j_k+1, j_k+3), \\
&(j_k-2, j_k+1, j_k, j_k+2, j_k-3, j_k-1, j_k+1, j_k+3), \\
&(j_k-2, j_k-3, j_k+1, j_k, j_k+2, j_k-1, j_k+1, j_k+3).
\end{aligned}$$

Therefore, it follows from (6.7) and Proposition 4.13 that

$$\begin{aligned}
&(\varphi_{(\mu_{k+1}\mu[0]V)_{k+1}}) \circ x^G_{\mathbf{i}}(1;\phi(\mathbf{Y})) \\
&= \Phi_{1,j_k+1}\Phi_{2,j_k-2}\Phi_{2,j_k}\Phi_{2,j_k+2}\Phi_{2,j_k-3}\Phi_{2,j_k-1}\Phi_{2,j_k+1}\Phi_{2,j_k+3} \\
&+\Phi_{1,j_k-2}\Phi_{1,j_k+1}\Phi_{2,j_k}\Phi_{2,j_k+2}\Phi_{2,j_k-3}\Phi_{2,j_k-1}\Phi_{2,j_k+1}\Phi_{2,j_k+3} \\
&+\Phi_{1,j_k-2}\Phi_{1,j_k-3}\Phi_{1,j_k+1}\Phi_{2,j_k}\Phi_{2,j_k+2}\Phi_{2,j_k-1}\Phi_{2,j_k+1}\Phi_{2,j_k+3} \\
&= \Phi_{1,j_k+1}\Phi_{2,j_k-2}\Phi_{2,j_k}\Phi_{2,j_k+2}\Phi_{2,j_k-3}\Phi_{2,j_k-1}\Phi_{2,j_k+1} \\
&\times\Phi_{2,j_k+3}(1 + A^{-1}_{1,j_k-2} + A^{-1}_{1,j_k-2}A^{-1}_{1,j_k-3}) \\
&= \frac{Y_{2,j_k-1}}{Y_{2,j_k-2}Y_{1,j_k+1}Y_{2,j_k+1}}(1 + A^{-1}_{1,j_k-2} + A^{-1}_{1,j_k-2}A^{-1}_{1,j_k-3}).
\end{aligned} \tag{7.25}$$

Substituting (7.25) for (7.24), we obtain

$$(\varphi^G_{(\mu_{k+1}\mu[0]\mathbb{V})})_{k+1}(a;\mathbf{Y}) = a^{\Lambda_{j_k-2}+\Lambda_{j_k+1}} Y_{1,j_k-2}Y_{2,j_k-1}(1 + A^{-1}_{1,j_k-2} + A^{-1}_{1,j_k-2}A^{-1}_{1,j_k-3}),$$

which means the claim (b) for $l = 0$.

Next, assuming that the claims (a) and (b) for $0, 1, \cdots, l$, let us prove the claims for $l + 1$. Using Lemma 7.1, we have $(\varphi_{(\mu_{k+l+2}\mu_{k+l+1}\mu[l]\mathbb{V})})_{k+l+2} = (\varphi_{(\mu_{k+l+2}\mathbb{V})})_{k+l+2}$. By Lemma 3.5, we see that the arrows between $(\varphi\mathbb{V})_{k-\lfloor\frac{r}{2}\rfloor+l+1}$ and $(\varphi\mathbb{V})_{-s}$ ($s \in [1, r]$) in $\mu_{k+l+2}\mu_{k+l+1}\mu[l]\Gamma_{\mathbf{i}}$ are as follows:

$$\begin{tikzcd}[column sep=small] (\varphi\mathbb{V})_{k-\lfloor\frac{r}{2}\rfloor+l+1} \arrow[d] & \\ (\varphi\mathbb{V})_{-(j_k-2l-3)} & (\varphi\mathbb{V})_{-(j_k-2l-2)} \arrow[ul] \end{tikzcd}$$

It follows from the exchange sequence (7.10), Lemma 4.7 and Theorem 4.9 that the arrows from $(\varphi_{(\mu_{k+l+2}\mu_{k+l+1}\mu[l]\mathbb{V})})_s$ $(1 \le s \le 2r)$ to $(\varphi\mathbb{V})_{k-\lfloor\frac{r}{2}\rfloor+l+1}$ are

$$(\varphi_{(\mu_{k+l+2}\mathbb{V})})_{k+l+2} \to (\varphi\mathbb{V})_{k-\lfloor\frac{r}{2}\rfloor+l+1}, \quad (\varphi_{(\mu[l]\mathbb{V})})_{k-\lfloor\frac{r}{2}\rfloor+l} \to (\varphi\mathbb{V})_{k-\lfloor\frac{r}{2}\rfloor+l+1}.$$

Similarly, since there exist non-factorizable homomorphisms in the direct summands of $(\mu_{k+l+2}\mu_{k+l+1}\mu[l]V)$ from $(\mu_{k+l+1}\mu[l]V)_{k+l+1}$, $V_{r+k-\lfloor\frac{r}{2}\rfloor+l+2}$, $V_{r+k-\lfloor\frac{r}{2}\rfloor+l+1}$ and $V_{r+k-\lfloor\frac{r}{2}\rfloor+l}$ to $V_{k-\lfloor\frac{r}{2}\rfloor+l+1} = S_{j_k-2l-3}$, we see that the arrows in $(\mu_{k+l+2}\mu_{k+l+1}\mu[l]\Gamma_{\mathbf{i}})$ from $(\varphi\mathbb{V})_{k-\lfloor\frac{r}{2}\rfloor+l+1}$ to $(\varphi_{(\mu_{k+l+2}\mu_{k+l+1}\mu[l]\mathbb{V})})_s$ $(1 \le s \le 2r)$ are

$$\begin{gathered}(\varphi\mathbb{V})_{k-\lfloor\frac{r}{2}\rfloor+l+1} \to (\varphi_{(\mu_{k+l+1}\mu[l]\mathbb{V})})_{k+l+1}, \quad (\varphi\mathbb{V})_{k-\lfloor\frac{r}{2}\rfloor+l+1} \to (\varphi\mathbb{V})_{r+k-\lfloor\frac{r}{2}\rfloor+l+2}, \\ (\varphi\mathbb{V})_{k-\lfloor\frac{r}{2}\rfloor+l+1} \to (\varphi\mathbb{V})_{r+k-\lfloor\frac{r}{2}\rfloor+l+1}, \quad (\varphi\mathbb{V})_{k-\lfloor\frac{r}{2}\rfloor+l+1} \to (\varphi\mathbb{V})_{r+k-\lfloor\frac{r}{2}\rfloor+l}.\end{gathered}$$

Hence, by the induction hypothesis of the claim (b) and the same way as in (7.21), we obtain the following:

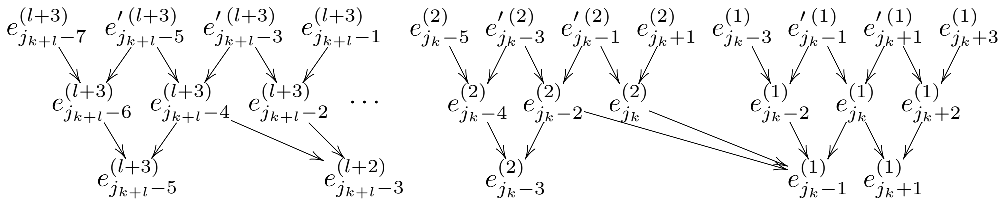
$$\begin{aligned}(\varphi^G_{(\mu[l+1]\mathbb{V})})_{k-\lfloor\frac{r}{2}\rfloor+l+1} &= (\Phi_H(a;\mathbf{Y}))^{(\sum_{s=0}^{l+3}\Lambda_{j_k-2s+1})+(\sum_{s=0}^{l}\Lambda_{j_k-2s-2})} \\ &\quad \times \varphi_{(\mu[l+1]V)_{k-\lfloor\frac{r}{2}\rfloor+l+1}} \circ x^G_{\mathbf{i}}(1;\phi(\mathbf{Y})). \end{aligned} \tag{7.26}$$

The module $(\mu[l+1]V)_{k-\lfloor\frac{r}{2}\rfloor+l+1}$ is described as (7.11). Using Proposition 4.13, let us calculate $\varphi_{(\mu[l+1]V)_{k-\lfloor\frac{r}{2}\rfloor+l+1}} \circ x^G_{\mathbf{i}}(1;\phi(\mathbf{Y}))$:

For $\mathbf{a} = (a_{1,j_r}, \cdots, a_{1,j_1}, a_{2,j_r}, \cdots, a_{1,j_1}) \in (\mathbb{Z}_{\ge 0})^{2r}$, if the variety $\mathcal{F}_{\mathbf{i},\mathbf{a},(\mu[l+1]V)_{k-\lfloor\frac{r}{2}\rfloor+l+1}}$ is non-empty, we have

$$\begin{aligned} &a_{1,j_k+1} = a_{2,j_k+2} = a_{2,j_k+3} = 1, \quad a_{1,j_k-1} = 1, \ a_{2,j_k} = 2, \ a_{2,j_k+1} = 2, \ a_{2,j_k-1} = 3, \\ &a_{1,j_{k+l}-5} = a_{2,j_{k+l}-6} = a_{2,j_{k+l}-7} = 1, \\ &a_{1,j_{k+l}-3} = 1, \ a_{2,j_{k+l}-4} = 2, \ a_{2,j_{k+l}-5} = 2, \ a_{2,j_{k+l}-3} = 3, \\ &a_{1,j_k-2t-3} + a_{2,j_k-2t-3} = 5 \quad (0 \le t \le l-1), \\ &a_{1,j_k-2t-2} + a_{2,j_k-2t-2} = 3 \quad (0 \le t \le l), \end{aligned} \tag{7.27}$$

by the same argument in the proof of the claim (a) for $l = 0$. Denoting the bases in (7.11) by



we see that all 1-dimensional simple submodules of $(\mu[l+1]V)_{k-\lfloor \frac{r}{2} \rfloor+l+1}$ are $\mathbb{C}e^{(t)}_{j_k-2t+1}$ $(1 \le t \le l+3)$, $\mathbb{C}e^{(1)}_{j_k+1}$ and $\mathbb{C}(e^{(t+1)}_{j_k-2t-2} - e^{(t+2)}_{j_k-2t-2} + e^{(t+3)}_{j_k-2t-2})$ $(0 \le t \le l)$, which are isomorphic to S_{j_k-2t+1}, S_{j_k+1} and S_{j_k-2t-2}, respectively. We can also see that all 1-dimensional simple submodules isomorphic to one of $\{S_{j_k-2t-3} | 0 \le t \le l-1\}$ of the quotient module

$$\frac{(\mu[l+1]V)_{k-\lfloor \frac{r}{2} \rfloor+l+1}}{(\mathbb{C}(e^{(t+1)}_{j_k-2t-2} - e^{(t+2)}_{j_k-2t-2} + e^{(t+3)}_{j_k-2t-2}) \oplus \mathbb{C}(e^{(t+2)}_{j_k-2t-4} - e^{(t+3)}_{j_k-2t-4} + e^{(t+4)}_{j_k-2t-4}))}$$

are $\mathbb{C}e^{(t+2)}_{j_k-2t-3}$ and $\mathbb{C}(e^{(t+1)}_{j_k-2t-3} - e'^{(t+2)}_{j_k-2t-3} + e'^{(t+3)}_{j_k-2t-3} - e^{(t+4)}_{j_k-2t-3})$ $(0 \le t \le l-1)$. Thus, $\mathcal{F}_{\mathbf{i},\mathbf{a},(\mu[l+1]V)_{k-\lfloor \frac{r}{2} \rfloor+l+1}} \neq \phi$ if and only if $\mathbf{a}$ satisfies the following in addition to (7.27): For each $t \in [0, l-1]$,

$$(a_{1,j_k-2t-2}, a_{1,j_k-2t-3}, a_{1,j_k-2t-4}) = (1,2,1), (0,1,0), (0,1,1), (1,1,0) \text{ or } (1,1,1)$$
$$\text{and all other } a_{1,i},\ a_{2,i} = 0.$$

Let us calculate the monomial M corresponding to $(a_{1,j_k-2t-2}, a_{1,j_k-2t-3}, a_{1,j_k-2t-4}) = (0,1,0)$ for all $t \in [0, l-1]$, which means that $a_{1,j_k-2s+1} = 1$ $(0 \le s \le l+3)$ and $a_{1,j_k-2s+2} = 0$ $(0 \le s \le l+4)$. Thus, it is calculated as

$$\begin{aligned} M &= \Phi_{1,j_k+1} \times \prod_{s=0}^{l+2} (\Phi_{1,j_k-2s-1} \Phi_{2,j_k-2s-2} \Phi_{2,j_k-2s-3} \\ &\qquad \Phi_{2,j_k-2s} \Phi_{2,j_k-2s-1} \Phi_{2,j_k-2s+2} \Phi_{2,j_k-2s+1} \Phi_{2,j_k-2s+3}) \\ &= \frac{Y_{2,j_k} Y_{2,j_k+2}}{Y_{1,j_k+1} Y^2_{2,j_k+1}} \prod_{s=0}^{l+2} \frac{Y_{2,j_k-2s+1}}{Y_{1,j_k-2s-1} Y_{2,j_k-2s-1} Y_{2,j_k-2s+2}}. \end{aligned} \tag{7.28}$$

For $p \ge 1$ and $(\mathbf{b}, \mathbf{c}) \in R^p_l$ $(\mathbf{b} = \{b_i\}_{i=1}^p,\ \mathbf{c} = \{c_i\}_{i=1}^p)$, the monomial corresponding to $a_{1,j_k-2t-3} = 2,\ a_{1,j_k-2t-2} = a_{1,j_k-2t-4} = 1$ for $t \in [b_1, c_1-1] \cup \cdots \cup [b_p, c_p-1]$, and $a_{1,j_k-2t-3} = 1$ for $t \in [1, l-1] \setminus ([b_1, c_1-1] \cup \cdots \cup [b_p, c_p-1])$, and $a_{1,j_k-2t-2} = 0$ for $t \in [0, l] \setminus [\mathbf{b}, \mathbf{c}]$ is $M \times A[b_1, c_1; j_k] \cdots A[b_p, c_p; j_k]$ by (6.7). Using (6.7) again, we see that the partial sum of $\varphi_{(\mu[l+1]V)_{k-\lfloor \frac{r}{2} \rfloor+l+1}} \circ x^G_{\mathbf{i}}(1; \phi(\mathbf{Y}))$ corresponding to $a_{1,j_k-2t-3} =$

2, $a_{1,j_k-2t-2}=a_{1,j_k-2t-4}=1$ for $t\in[b_1,c_1-1]\cup\cdots\cup[b_p,c_p-1]$ and $a_{1,j_k-2t-2}=0$ or 1 for $t\in[0,l]\setminus[\mathbf{b},\mathbf{c}]$ is

$$M\times A[b_1,c_1;j_k]\cdots A[b_p,c_p;j_k]\prod_{t\in[0,l]\setminus([b_1,c_1]\cup\cdots\cup[b_p,c_p])}(1+A^{-1}_{1,j_k-2t-2}).$$

On the other hand, by (2.5) and (7.28),

$$(\Phi_H(a;\mathbf{Y}))^{(\sum_{s=0}^{l+3}\Lambda_{j_k-2s+1})+(\sum_{s=0}^{l}\Lambda_{j_k-2s-2})}\times M$$
$$=a^{(\sum_{s=0}^{l+3}\Lambda_{j_k-2s+1})+(\sum_{s=0}^{l}\Lambda_{j_k-2s-2})}\prod_{s=0}^{l}Y_{1,j_k-2s-2}\prod_{s=0}^{l+1}Y_{2,j_k-2s-1}.$$

Hence, by (7.26), we have

$$(\varphi^G_{(\mu[l+1]\mathbb{V})})_{k-\lfloor\frac{r}{2}\rfloor+l+1}=a^{(\sum_{s=0}^{l+3}\Lambda_{j_k-2s+1})+(\sum_{s=0}^{l}\Lambda_{j_k-2s-2})}\prod_{s=0}^{l}Y_{1,j_k-2s-2}\prod_{s=0}^{l+1}Y_{2,j_k-2s-1}$$
$$\times\sum_{p\geq 0,\ (\mathbf{b},\mathbf{c})\in R_l^p}A[b_1,c_1;j_k]\cdots A[b_p,c_p;j_k]\prod_{t\in[0,l]\setminus([b_1,c_1]\cup\cdots\cup[b_p,c_p])}(1+A^{-1}_{1,j_k-2t-2}),$$

which implies the claim (a) for $l+1$.

Finally, let us prove the claim (b) for $l+1$. By the direct calculation, the arrows between $(\varphi_{(\mu_{k+l+2}\mu_{k+l+1}\mu[l]\mathbb{V})})_{k+l+2}=(\varphi_{(\mu_{k+l+2}\mathbb{V})})_{k+l+2}$ and $(\varphi_{\mathbb{V}})_{-s}$ $(s\in[1,r])$ in $\mu[l+1]\Gamma_{\mathbf{i}}$ are as follows:

$$\begin{tikzcd}
(\varphi_{(\mu_{k+l+2}\mathbb{V})})_{k+l+2} \arrow[dr] & \\
(\varphi_{\mathbb{V}})_{-(j_k-2l-4)} \arrow[u] & (\varphi_{\mathbb{V}})_{-(j_k-2l-3)}
\end{tikzcd}$$

The exchange sequence (7.12), Lemma 4.7 and Theorem 4.9 imply that the arrows from $(\varphi_{(\mu[l+1]\mathbb{V})})_s$ $(1\leq s\leq 2r)$ to $(\varphi_{(\mu_{k+l+2}\mathbb{V})})_{k+l+2}$ are

$$(\varphi_{(\mu[l+1]\mathbb{V})})_{k-\lfloor\frac{r}{2}\rfloor+l+1}\to(\varphi_{(\mu_{k+l+2}\mathbb{V})})_{k+l+2}.$$

The arrows from $(\varphi_{(\mu_{k+l+2}\mathbb{V})})_{k+l+2}$ to $(\varphi_{(\mu[l+1]\mathbb{V})})_s$ $(1\leq s\leq 2r)$ are

$$(\varphi_{(\mu_{k+l+2}\mathbb{V})})_{k+l+2}\to(\varphi_{(\mu_{k+l+1}\mu[l]\mathbb{V})})_{k+l+1},\quad(\varphi_{(\mu_{k+l+2}\mathbb{V})})_{k+l+2}\to(\varphi_{\mathbb{V}})_{r+k+l-\lfloor\frac{r}{2}\rfloor},$$
$$(\varphi_{(\mu_{k+l+2}\mathbb{V})})_{k+l+2}\to(\varphi_{\mathbb{V}})_{r+k+l+2},\quad(\varphi_{(\mu_{k+l+2}\mathbb{V})})_{k+l+2}\to(\varphi_{\mathbb{V}})_{k+l-\lfloor\frac{r}{2}\rfloor+2}.$$

By the same way as in (7.26), we have

$$(\varphi^G_{(\mu_{k+l+2}\mu[l+1]\mathbb{V})})_{k+l+2}=(\Phi_H(a;\mathbf{Y}))^{(\sum_{s=0}^{l+1}\Lambda_{j_k-2s+1}+\Lambda_{j_k-2s-2})}$$
$$\times\varphi_{(\mu_{k+l+2}\mu[l+1]V)_{k+l+2}}\circ x^G_{\mathbf{i}}(1;\phi(\mathbf{Y})).\tag{7.29}$$

The module $(\mu_{k+l+2}\mu[l+1]V)_{k+l+2}$ is described as (7.13), and it has the simple submodules S'_{j_k-2t} isomorphic to S_{j_k-2t} $(1 \leq t \leq l+2)$. The quotient modules $(\mu_{k+l+2}\mu[l+1]V)_{k+l+2}/(S'_{j_k-2t} \oplus S'_{j_k-2t-2})$ and $(\mu_{k+l+2}\mu[l+1]V)_{k+l+2}/(S'_{j_k-2l-4})$ have the simple submodules isomorphic S_{j_k-2t-1} and S_{j_k-2l-5} respectively. Therefore, for $\mathbf{a} = (a_{1,j_r}, \cdots, a_{1,j_1}, a_{2,j_r}, \cdots, a_{1,j_1}) \in (\mathbb{Z}_{\geq 0})^{2r}$, the variety $\mathcal{F}_{\mathbf{i},\mathbf{a},(\mu_{k+l+2}\mu[l+1]V)_{k+l+2}}$ is non-empty if and only if

$$\begin{aligned}
&a_{1,j_k+1} = a_{2,j_k+2} = a_{2,j_k+3} = 1,\ a_{1,j_k-1} = 1,\ a_{2,j_k} = a_{2,j_k+1} = 2,\ a_{2,j_k-1} = 3,\\
&a_{1,j_k-2t-3} + a_{2,j_k-2t-3} = 5 \quad (0 \leq t \leq l-2),\\
&a_{1,j_k-2t-2} + a_{2,j_k-2t-2} = 3 \quad (0 \leq t \leq l-1),\\
&a_{1,j_k-2l-1} + a_{2,j_k-2l-1} = 4, \quad a_{1,j_k-2l-3} + a_{2,j_k-2l-3} = 2,\\
&a_{1,j_k-2l-5} + a_{2,j_k-2l-5} = 1, \quad a_{1,j_k-2l-2} + a_{2,j_k-2l-2} = 2,\\
&a_{1,j_k-2l-4} + a_{2,j_k-2l-4} = 1,
\end{aligned}$$

and for each $t \in [0, l-1]$,

$$\begin{aligned}
&(a_{1,j_k-2t-2}, a_{1,j_k-2t-3}, a_{1,j_k-2t-4}) = (1,2,1), (0,1,0), (0,1,1), (1,1,0) \text{ or } (1,1,1)\\
&(a_{1,j_k-2l-2}, a_{1,j_k-2l-3}, a_{1,j_k-2l-4}) = (1,1,1), (0,0,0), (0,0,1), (1,0,0) \text{ or } (1,0,1)\\
&0 \leq a_{1,j_k-2l-5} \leq a_{1,j_k-2l-4} \leq 1, \quad \text{all other } a_{1,i},\ a_{2,i} = 0.
\end{aligned}$$

Let us calculate the monomial M' corresponding to $(a_{1,j_k-2t-2}, a_{1,j_k-2t-3}, a_{1,j_k-2t-4}) = (0,1,0)$ for all $t \in [0, l-1]$ and $a_{1,j_k-2l-3} = a_{1,j_k-2l-4} = a_{1,j_k-2l-5} = 0$. It is calculated as

$$\begin{aligned}
M' &= \prod_{s=0}^{l+1} (\Phi_{1,j_k-2s+1}\Phi_{2,j_k-2s-2}\Phi_{2,j_k-2s-3}\\
&\quad \Phi_{2,j_k-2s}\Phi_{2,j_k-2s-1}\Phi_{2,j_k-2s+2}\Phi_{2,j_k-2s+1}\Phi_{2,j_k-2s+3})\\
&= \prod_{s=0}^{l+1} \frac{Y_{2,j_k-2s-1}}{Y_{1,j_k-2s+1}Y_{2,j_k-2s+1}Y_{2,j_k-2s-2}}. \qquad (7.30)
\end{aligned}$$

For $p \geq 1$ and $(\mathbf{b}, \mathbf{c}) \in R^p_{l+1}$ ($\mathbf{b} = \{b_i\}_{i=1}^p$, $\mathbf{c} = \{c_i\}_{i=1}^p$) such that $c_p < l+1$, the monomial corresponding to $a_{1,j_k-2t-3} = 2$, $a_{1,j_k-2t-2} = a_{1,j_k-2t-4} = 1$ for $t \in [b_1, c_1-1] \cup \cdots \cup [b_p, c_p-1]$, $a_{1,j_k-2t-3} = 1$ for $t \in [1, l-1] \setminus ([b_1, c_1-1] \cup \cdots \cup [b_p, c_p-1])$, $a_{1,j_k-2l-3} = a_{1,j_k-2l-5} = 0$ and $a_{1,j_k-2t-2} = 0$ for $t \in [0, l+1] \setminus [\mathbf{b}, \mathbf{c}]$ is $M' \times A[b_1, c_1; j_k] \cdots A[b_p, c_p; j_k]$ by (6.7). Using (6.7) again, we see that the partial sum of $\varphi_{(\mu_{k+l+2}\mu[l+1]V)_{k+l+2}} \circ x^G_{\mathbf{i}}(1; \phi(\mathbf{Y}))$ corresponding to $a_{1,j_k-2t-3} = 2$, $a_{1,j_k-2t-2} = a_{1,j_k-2t-4} = 1$ $(t \in [b_1, c_1-1] \cup \cdots \cup [b_p, c_p-1])$ and $a_{1,j_k-2t-2} = 1$ or 0 for $t \in [0, l+1] \setminus [\mathbf{b}, \mathbf{c}]$ and $0 \leq a_{1,j_k-2l-5} \leq a_{1,j_k-2l-4} \leq 1$ is

$$M' \times A[b_1, c_1; j_k] \cdots A[b_p, c_p; j_k] \prod_{t \in [0,l+1] \setminus ([b_1,c_1] \cup \cdots \cup [b_p,c_p])} (1 + A^{-1}_{1,j_k-2t-2})$$
$$\times (1 + A^{-1}_{1,j_k-2l-4} + A^{-1}_{1,j_k-2l-4} A^{-1}_{1,j_k-2l-5}).$$

Similarly, for $p \geq 1$ and $(\mathbf{b}, \mathbf{c}) \in R^p_{l+1}$ ($\mathbf{b} = \{b_i\}^p_{i=1}$, $\mathbf{c} = \{c_i\}^p_{i=1}$) such that $c_p = l+1$, the monomial corresponding to $a_{1,j_k-2t-3} = 2$, $a_{1,j_k-2t-2} = a_{1,j_k-2t-4} = 1$ for $t \in [b_1, c_1 - 1] \cup \cdots \cup [b_p, l]$, $a_{1,j_k-2t-3} = 1$ for $t \in [1, l] \setminus ([b_1, c_1 - 1] \cup \cdots \cup [b_p, l-1])$, $a_{1,j_k-2l-3} = a_{1,j_k-2l-4} = 1$, $a_{1,j_k-2l-5} = 0$ and $a_{1,j_k-2t-2} = 0$ for $t \in [0, l+1] \setminus ([b_1, c_1] \cup \cdots \cup [b_p, l+1])$ is $M' \times A[b_1, c_1; j_k] \cdots A[b_p, c_p; j_k]$. We see that the partial sum of $\varphi_{(\mu_{k+l+2}\mu[l+1]V)_{k+l+2}} \circ x^G_{\mathbf{i}}(1; \phi(\mathbf{Y}))$ corresponding to $a_{1,j_k-2t-3} = 2$, $a_{1,j_k-2t-2} = a_{1,j_k-2t-4} = 1$ ($t \in [b_1, c_1 - 1] \cup \cdots \cup [b_p, l-1]$) and $a_{1,j_k-2l-2} = a_{1,j_k-2l-3} = a_{1,j_k-2l-4} = 1$ is $M' \times (1 + A^{-1}_{1,j_k-2l-5}) \times A[b_1, c_1; j_k] \cdots A[b_p, c_p; j_k] \prod_{t \in [0,l+1] \setminus ([b_1,c_1] \cup \cdots \cup [b_p,l+1])} (1 + A^{-1}_{1,j_k-2t-2})$. On the other hand, by (2.5) and (7.30),

$$(\Phi_H(a; \mathbf{Y}))^{(\sum_{s=0}^{l+1} \Lambda_{j_k-2s+1} + \Lambda_{j_k-2s-2})} \times M'$$
$$= a^{(\sum_{s=0}^{l+1} \Lambda_{j_k-2s+1} + \Lambda_{j_k-2s-2})} \prod_{s=0}^{l+1} Y_{1,j_k-2s-2} \prod_{s=0}^{l+1} Y_{2,j_k-2s-1}.$$

Hence, by (7.29), we have

$$(\varphi^G_{(\mu_{k+l+2}\mu[l+1]\mathbb{V})})_{k+l+2} = a^{(\sum_{s=0}^{l+1} \Lambda_{j_k-2s+1} + \Lambda_{j_k-2s-2})} \prod_{s=0}^{l+1} Y_{1,j_k-2s-2} \prod_{s=0}^{l+1} Y_{2,j_k-2s-1}$$
$$\times \sum_{p \geq 0,\ (\mathbf{b},\mathbf{c}) \in R^p_{l+1}} A[b_1, c_1; j_k] \cdots A[b_p, c_p; j_k] (1 - \delta_{c_p,l+1} + A^{-1+\delta_{c_p,l+1}}_{1,j_k-2l-4} (1 + A^{-1}_{1,j_k-2l-5}))$$
$$\times \prod_{t \in [0,l+1] \setminus ([\mathbf{b},\mathbf{c}])} (1 + A^{-1}_{1,j_k-2t-2}),$$

which implies the claim (b) for $l+1$. □

The proofs for Proposition 6.6 and 6.7 are similar to the one for Proposition 6.4.

Proof of Theorem 6.8. Let us set w_i, $w_i(\mathbf{b}, \mathbf{c})$ in our claim as

$$w_1 := \prod_{q \in [0,l-1]} s_{j_k-2q-2}, \quad w_1(\mathbf{b}, \mathbf{c}) := \prod_{q \in [0,l-1] \setminus [\mathbf{b},\mathbf{c}]} s_{j_k-2q-2},$$
$$w_2 := s_{j_k-2l-3} s_{j_k-2l-2} \prod_{q \in [0,l-1]} s_{j_k-2q-2},$$
$$w_2(\mathbf{b}, \mathbf{c}) := s_{j_k-2l-3} s^{1-\delta_{c_p,l}}_{j_k-2l-2} \prod_{q \in [0,l-1] \setminus [\mathbf{b},\mathbf{c}]} s_{j_k-2q-2},$$

$$w_3 := s_{j_k+2}s_{j_k+1}\prod_{q\in[1,l+1]} s_{j_k-2q+1},$$

$$w_3(\mathbf{b},\mathbf{c}) := s_{j_k+2}s_{j_k+1}^{1-\delta_{b_1,0}}\prod_{q\in[1,l+1]\setminus[\mathbf{b},\mathbf{c}]} s_{j_k-2q+1},$$

$$w_4 := s_{j_k+2}s_{j_k+1}s_{j_k-2l-4}s_{j_k-2l-3}\prod_{q\in[1,l+1]} s_{j_k-2q+1},$$

$$w_4(\mathbf{b},\mathbf{c}) := s_{j_k+2}s_{j_k+1}^{1-\delta_{b_1,0}}s_{j_k-2l-4}s_{j_k-2l-3}^{1-\delta_{c_p,l+2}}\prod_{q\in[1,l+1]\setminus[\mathbf{b},\mathbf{c}]} s_{j_k-2q+1},$$

$$w_5 := \prod_{q\in[1,l+1]} s_{2q},\quad w_5(\mathbf{b},\mathbf{c}) := \prod_{q\in[1,l+1]\setminus[\mathbf{b},\mathbf{c}]} s_{2q},$$

$$w_6 := s_{2l+5}s_{2l+4}\prod_{q\in[1,l+1]} s_{2q},\quad w_6(\mathbf{b},\mathbf{c}) := s_{2l+5}s_{2l+4}^{1-\delta_{c_p,l+2}}\prod_{q\in[1,l+1]\setminus[\mathbf{b},\mathbf{c}]} s_{2q}$$

and prove only (1) since (2) and (3) are proven in the same way as (1).
(a) For $p\geq 0$ and $(\mathbf{b},\mathbf{c})\in R^p_{l-1}$ ($\mathbf{b}=\{b_i\}_{i=1}^p$, $\mathbf{c}=\{c_i\}_{i=1}^p$), let $\mu_1 : B\Big(\Big(\sum_{s=j_k-2l-1}^{j_k-1}\Lambda_s\Big)-\alpha[\mathbf{b},\mathbf{c};j_k]\Big)\to\mathcal{Y}$ be the monomial realization which maps the highest weight vector in $B\left(\left(\sum_{s=j_k-2l-1}^{j_k-1}\Lambda_s\right)-\alpha[\mathbf{b},\mathbf{c};j_k]\right)$ to the monomial $H_1[\mathbf{b},\mathbf{c}] := H_1\cdot A[b_1,c_1;j_k]\cdots A[b_p,c_p;j_k]$, where $\mathcal{Y}$ is defined in 5.1. By Proposition 6.4, we need show that $H_1[\mathbf{b},\mathbf{c}]\prod_{q\in[0,l-1]\setminus[\mathbf{b},\mathbf{c}]}(1+A^{-1}_{1,j_k-2q-2})$ coincides with

$$\sum_{b\in B\left(\left(\sum_{s=j_k-2l-1}^{j_k-1}\Lambda_s\right)-\alpha[\mathbf{b},\mathbf{c};j_k]\right)_{\prod_{q\in[0,l-1]\setminus[\mathbf{b},\mathbf{c}]} s_{j_k-2q-2}}} \mu_1(b). \tag{7.31}$$

First, let us show that each factor in the monomial $H_1[\mathbf{b},\mathbf{c}]$ has non-negative degree. For $1\leq i\leq p$, we can easily see that

$$\begin{aligned}
A[b_i,c_i;j_k] &= \left(\prod_{s=b_i}^{c_i-1} A^{-1}_{1,j_k-2s-2}A^{-1}_{1,j_k-2s-3}\right)A^{-1}_{1,j_k-2c_i-2}\\
&= \left(\prod_{s=b_i}^{c_i-1}\frac{Y_{1,j_k-2s-1}Y_{2,j_k-2s-4}}{Y_{1,j_k-2s-2}Y_{2,j_k-2s-3}}\right)\frac{Y_{1,j_k-2c_i-3}Y_{1,j_k-2c_i-1}}{Y_{1,j_k-2c_i-2}Y_{2,j_k-2c_i-2}}\\
&= \frac{\prod_{s=b_i}^{c_i+1}Y_{1,j_k-2s-1}\prod_{s=b_i+1}^{c_i-1}Y_{2,j_k-2s-2}}{\prod_{s=b_i}^{c_i}Y_{1,j_k-2s-2}\prod_{s=b_i+1}^{c_i}Y_{2,j_k-2s-1}}.
\end{aligned}\tag{7.32}$$

Hence, each factor in the monomial $H_1[\mathbf{b},\mathbf{c}]$ has non-negative degree. Since the monomial $A_{1,i}$ has the weight α_i (see (5.1), (5.3)), the monomial $H_1[\mathbf{b},\mathbf{c}]$ has the weight $\left(\sum_{s=j_k-2l-1}^{j_k-1}\Lambda_s\right)-\alpha[\mathbf{b},\mathbf{c};j_k]$. Furthermore, by (7.32), we can verify that for $q\in[1,l-1]\setminus[\mathbf{b},\mathbf{c}]$, in the monomial $H_1[\mathbf{b},\mathbf{c}]$, the factor Y_{1,j_k-2q-2} has the degree 1, and the factor Y_{2,j_k-2q-2} does not appear. Thus, the definition of Kashiwara operators in 5.1 implies that

$$\tilde{f}_{j_k-2q-2}H_1[\mathbf{b},\mathbf{c}] = H_1[\mathbf{b},\mathbf{c}] \cdot A^{-1}_{1,j_k-2q-2},$$

and $\tilde{f}^2_{j_k-2q-2}H_1[\mathbf{b},\mathbf{c}] = 0$. More generally, by the definition of the monomials $A_{1,i}$ ($i \in [1,r]$) in (5.3), for $q_1,\cdots,q_m \in [1,l-1] \setminus [\mathbf{b},\mathbf{c}]$ ($m \in \mathbb{Z}_{\geq 0}$), if $q \in [1,l-1] \setminus [\mathbf{b},\mathbf{c}]$ and $q \neq q_1,\cdots,q_m$, then in the monomial $H_1[\mathbf{b},\mathbf{c}]\prod_{s=1}^m A^{-1}_{1,j_k-2q_s-2}$, the factor Y_{1,j_k-2q-2} has the degree 1, and factors $Y^{\pm 1}_{2,j_k-2q-2}$ do not appear. Hence,

$$\tilde{f}_{j_k-2q-2}(H_1[\mathbf{b},\mathbf{c}]\prod_{s=1}^m A^{-1}_{1,j_k-2q_s-2}) = (H_1[\mathbf{b},\mathbf{c}]\prod_{s=1}^m A^{-1}_{1,j_k-2q_s-2}) \cdot A^{-1}_{1,j_k-2q-2},$$

and $\tilde{f}^2_{j_k-2q-2}(H_1[\mathbf{b},\mathbf{c}]\prod_{s=1}^m A^{-1}_{1,j_k-2q_s-2}) = 0$.

Let $\mathrm{id}_{\mathcal{Y}}$ be the identity map on the set $\mathcal{Y}$. By the above argument, we obtain

$$H_1[\mathbf{b},\mathbf{c}] \prod_{q\in[0,l-1]\setminus[\mathbf{b},\mathbf{c}]} (1 + A^{-1}_{1,j_k-2q-2}) = \left(\prod_{q\in[0,l-1]\setminus[\mathbf{b},\mathbf{c}]} (\mathrm{id}_{\mathcal{Y}} + \tilde{f}_{1,j_k-2q-2})\right) H_1[\mathbf{b},\mathbf{c}],$$

and from Theorem 5.4, it coincides with (7.31). Now, the proof of Theorem 6.8 (1)(a) has been completed.

Finally, let us show (b). For $p \geq 0$ and $(\mathbf{b},\mathbf{c}) \in R^p_l$ ($\mathbf{b} = \{b_i\}^p_{i=1}, \mathbf{c} = \{c_i\}^p_{i=1}$), let

$$\mu_2 : B\left(\left(\sum_{s=j_k-2l-2}^{j_k-1} \Lambda_s\right) - \alpha[\mathbf{b},\mathbf{c};j_k]\right) \to \mathcal{Y}$$

be the monomial realization which maps the highest weight vector to the monomial $H_2[\mathbf{b},\mathbf{c}] := H_2 \cdot A[b_1,c_1;j_k]\cdots A[b_p,c_p;j_k]$. By Proposition 6.4, we need show that

$$H_2[\mathbf{b},\mathbf{c}] \cdot (1 - \delta_{c_p,l} + A^{-1+\delta_{c_p,l}}_{1,j_k-2l-2}(1 + A^{-1}_{1,j_k-2l-3})) \prod_{q\in[0,l-1]\setminus[\mathbf{b},\mathbf{c}]} (1 + A^{-1}_{1,j_k-2q-2}) \tag{7.33}$$

coincides with

$$\sum_{b\in B(\mathbf{b},\mathbf{c})} \mu_2(b), \tag{7.34}$$

where $B(\mathbf{b},\mathbf{c}) = B\left(\left(\sum_{s=j_k-2l-2}^{j_k-1} \Lambda_s\right) - \alpha[\mathbf{b},\mathbf{c};j_k]\right)_{s_{j_k-2l-3}s^{1-\delta_{c_p,l}}_{j_k-2l-2}\prod_{q\in[0,l-1]\setminus[\mathbf{b},\mathbf{c}]} s_{j_k-2q-2}}$.

In the same way as (a), we see that each factor in the monomial $H_2[\mathbf{b},\mathbf{c}]$ has non-negative power, and it has the weight $\left(\sum_{s=j_k-2l-2}^{j_k-1} \Lambda_s\right) - \alpha[\mathbf{b},\mathbf{c};j_k]$. For $q_1,\cdots,q_m \in [0,l-1] \setminus [\mathbf{b},\mathbf{c}]$ ($m \in \mathbb{Z}_{\geq 0}$), if $q \in [0,l-1] \setminus [\mathbf{b},\mathbf{c}]$ and $q \neq q_1,\cdots,q_m$, then in the monomial

$H_2[\mathbf{b},\mathbf{c}]\prod_{s=1}^{m} A^{-1}_{1,j_k-2q_s-2}$, the factor Y_{1,j_k-2q-2} has the degree 1, and factors $Y^{\pm 1}_{2,j_k-2q-2}$ do not appear. Hence,

$$\tilde{f}_{j_k-2q-2}(H_2[\mathbf{b},\mathbf{c}]\prod_{s=1}^{m} A^{-1}_{1,j_k-2q_s-2}) = (H_2[\mathbf{b},\mathbf{c}]\prod_{s=1}^{m} A^{-1}_{1,j_k-2q_s-2}) \cdot A^{-1}_{1,j_k-2q-2},$$

and $\tilde{f}^2_{j_k-2q-2}(H_2[\mathbf{b},\mathbf{c}]\prod_{s=1}^{m} A^{-1}_{1,j_k-2q_s-2}) = 0$. Moreover, if $c_p < l$, we have

$$\tilde{f}_{j_k-2l-2}(H_2[\mathbf{b},\mathbf{c}]\prod_{s=1}^{m} A^{-1}_{1,j_k-2q_s-2}) = (H_2[\mathbf{b},\mathbf{c}]\prod_{s=1}^{m} A^{-1}_{1,j_k-2q_s-2}) \cdot A^{-1}_{1,j_k-2l-2} \tag{7.35}$$

and $\tilde{f}^2_{j_k-2l-2}(H_2[\mathbf{b},\mathbf{c}]\prod_{s=1}^{m} A^{-1}_{1,j_k-2q_s-2}) = 0$. It follows from the explicit forms of $H_2[\mathbf{b},\mathbf{c}]$ and A^{-1}_{1,j_k-2q_s-2} that in the monomial (7.35), the factor Y_{1,j_k-2l-3} has the degree 1, and factors $Y^{\pm 1}_{2,j_k-2l-3}$ do not appear. Hence, the definition of Kashiwara operators in 5.1 implies that

$$\begin{aligned}&\tilde{f}_{j_k-2l-3}((H_2[\mathbf{b},\mathbf{c}]\prod_{s=1}^{m} A^{-1}_{1,j_k-2q_s-2})A^{-1}_{1,j_k-2l-2})\\ &= ((H_2[\mathbf{b},\mathbf{c}]\prod_{s=1}^{m} A^{-1}_{1,j_k-2q_s-2})A^{-1}_{1,j_k-2l-2}) \cdot A^{-1}_{1,j_k-2l-3},\end{aligned}$$

and $\tilde{f}^2_{j_k-2l-3}((H_2[\mathbf{b},\mathbf{c}]\prod_{s=1}^{m} A^{-1}_{1,j_k-2q_s-2})A^{-1}_{1,j_k-2l-2}) = 0$. Similarly, if $c_p = l$, we obtain $\tilde{f}_{j_k-2l-3}(H_2[\mathbf{b},\mathbf{c}]\prod_{s=1}^{m} A^{-1}_{1,j_k-2q_s-2}) = (H_2[\mathbf{b},\mathbf{c}]\prod_{s=1}^{m} A^{-1}_{1,j_k-2q_s-2})A^{-1}_{1,j_k-2l-3}$, and $\tilde{f}^2_{j_k-2l-3}(H_2[\mathbf{b},\mathbf{c}]\prod_{s=1}^{m} A^{-1}_{1,j_k-2q_s-2}) = 0$. By the above argument, the sum in (7.33) is the same as

$$((1-\delta_{c_p,l})\mathrm{id}_{\mathcal{Y}} + \tilde{f}^{-1+\delta_{c_p,l}}_{j_k-2l-2}(\mathrm{id}_{\mathcal{Y}} + \tilde{f}_{j_k-2l-3}))\left(\prod_{q\in[0,l-1]\setminus[\mathbf{b},\mathbf{c}]}(\mathrm{id}_{\mathcal{Y}} + \tilde{f}_{1,j_k-2q-2})\right)H_2[\mathbf{b},\mathbf{c}],$$

and from Theorem 5.4, it coincides with (7.34). □

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