

Hurwitz spaces of Galois coverings of \mathbb{P}^1 , whose Galois groups are Weyl groups

Vassil Kanev

Università di Palermo, Dipartimento di Matematica ed Applicazioni, Via Archirafi n. 34, 90123 Palermo, Italy

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Abstract

We prove the irreducibility of the Hurwitz spaces which parametrize equivalence classes of Galois coverings of \mathbb{P}^1 , whose Galois group is an arbitrary Weyl group, and the local monodromies are reflections. This generalizes a classical theorem due to Lüroth, Clebsch and Hurwitz.

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0. Introduction

A classical theorem due to Lüroth, Clebsch and Hurwitz states that the Hurwitz space $\mathcal{H}_{d,n}(\mathbb{P}^1)$, which parametrizes equivalence classes of irreducible coverings of \mathbb{P}^1 of degree d simply ramified in n points, is irreducible [17]. Coverings of \mathbb{P}^1 of degree d with a fixed monodromy group $G \subset S_d$, the related Galois coverings with Galois group G , and the corresponding Hurwitz spaces were studied in connection with the inverse Galois problem (see [25] and the references therein). The irreducibility of such spaces is a relevant problem for this theory and was verified in few cases [2,11,12].

E-mail address: kanev@math.unipa.it.

Replacing S_d by an arbitrary Weyl group one may ask whether the theorem of Clebsch and Hurwitz could be generalized to Galois coverings of \mathbb{P}^1 , whose Galois groups are Weyl groups. Coverings of this type are interesting on their own. They appear in the study of spectral curves, integrable systems, generalized Prym varieties, Prym–Tyurin varieties [8,18,19,23]. The generalized Prym maps yield morphisms from the Hurwitz spaces of coverings, with monodromy groups contained in a Weyl group, to Siegel modular varieties which parametrize Abelian varieties with a fixed polarization type. If one proves the irreducibility and the unirationality of some Hurwitz spaces, and the dominance of the Prym maps, this would imply the unirationality of the corresponding Siegel modular varieties. This idea was successfully realized in proving the unirationality of $\mathcal{A}_3(1, 1, d)$ and $\mathcal{A}_3(1, d, d)$ for $d \leq 4$ [20,21] (the case $d = 5$ is a work in progress). We hope the method may be extended considering coverings with monodromy groups an arbitrary irreducible Weyl group. The irreducibility of the Hurwitz spaces is the first issue to address here. For coverings of \mathbb{P}^1 Hurwitz showed in [17] that the problem of irreducibility is reduced to a purely algebraic problem about transitivity of certain actions of the braid groups on tuples of elements of the monodromy group (see Section 1 for details). We mention that analogous reduction, involving braid groups of Riemann surfaces of positive genus, was found in [22] for Hurwitz spaces of coverings of a fixed smooth, projective curve of positive genus.

In the present paper we generalize the result of Lüroth, Clebsch and Hurwitz and prove in Theorem 2.7 the irreducibility of the Hurwitz spaces, which parametrize equivalence classes of Galois coverings of \mathbb{P}^1 , whose Galois group is an arbitrary Weyl group W , and which have simple branching, in the sense that every local monodromy is a reflection. We notice that when W is the Weyl group of an irreducible root system of type D_r , B_r or C_r the result is already known. The case $W(D_r)$ is treated in [2]. The case $W(B_r)$, which is the same as $W(C_r)$, is easily reduced to the theorem of Lüroth, Clebsch and Hurwitz. One consequence of our result is the topological classification of the coverings we consider. Namely, Clebsch gave a normal form for the local monodromies of a simple covering $\pi : X \rightarrow \mathbb{P}^1$ ([5], cf. [13, proof of Proposition 1.5]). Our Corollary 2.8 gives a normal form for the local monodromies when the monodromy group is a Weyl group and the branching is simple.

We mention two other recent papers where the problem of irreducibility of Hurwitz spaces of coverings of \mathbb{P}^1 was studied. Let W be a finite irreducible Coxeter group of rank r . S. Humphries studied in [16] Hurwitz actions of the r -strand braid group on r -tuples of reflections of W and one of his results has the following corollary. Let W be of type A_r , B_r , D_r , E_6 , E_7 , F_4 , I_3 or I_4 . Theorem 1.2 [16] implies the irreducibility of the Hurwitz space parametrizing the irreducible Galois covers of \mathbb{P}^1 with Galois group W , branched in $r + 1$ points, in r of which the local monodromies are reflections. F. Vetro studied in [24] coverings of \mathbb{P}^1 of degree $2r$ whose monodromy group is contained in $W(B_r) \subset S_{2r}$. She proved the irreducibility of the corresponding Hurwitz space when the local monodromies at all branch points, except possibly one, are reflections. We do not know whether our Theorems 2.5 and 2.7 remain valid if one replaces Weyl groups by finite Coxeter groups. While only Weyl groups are relevant for the Siegel modular varieties, a possible generalization to finite Coxeter groups might be of interest for the inverse Galois theory (see, e.g., [25]).

1. Hurwitz spaces and Weyl groups

In Sections 1.1–1.5 we recall some facts about Hurwitz spaces. The references for this material are [10,13,25].

1.1. Let $\pi : X \rightarrow \mathbb{P}^1$ be a Galois covering with Galois group G . We assume G acts on X on the left. We call π a G -covering for short. Let $D \subset \mathbb{P}^1$ be the discriminant locus of π and let $b_0 \in \mathbb{P}^1 - D$. We consider the fundamental groups with multiplication defined by means of composition of arcs $\gamma_1 * \gamma_2$, where one first travels along γ_1 , and then along γ_2 . Let $x_0 \in \pi^{-1}(b_0)$. The monodromy homomorphism $m_{x_0} : \pi_1(\mathbb{P}^1 - D, b_0) \rightarrow G$ is defined as follows. If γ is a closed arc in $\mathbb{P}^1 - D$ based at b_0 , let $\hat{\gamma} : [0, 1] \rightarrow X - \pi^{-1}(D)$ be its lifting, which starts at x_0 and ends at gx_0 . Then $m_{x_0}([\gamma]) = g$. We notice that if instead, one considers Galois coverings, where G acts on the right, then $m_{x_0}([\gamma]) = g$ if $\hat{\gamma}(0) = x_0$, $\hat{\gamma}(1) = x_0g^{-1}$.

Let ℓ be an arc which connects b_0 with a point $b \in D$ and contains none of the other points of D . Let γ be a closed arc which begins at b_0 , travels along ℓ to a point near b , makes a small counterclockwise loop around b , and returns to b_0 along ℓ . The element $t = m_{x_0}([\gamma])$ is the *local monodromy* around b along γ . The conjugacy class $\{gtg^{-1} \mid g \in G\}$ depends neither on the choice of γ , nor on the choice of $x_0 \in \pi^{-1}(b_0)$, nor on the choice of $b_0 \in \mathbb{P}^1 - D$. It characterizes the ramification type at the discriminant point $b \in D$.

Let $n = |D|$. An *arc system* is a collection of n simple arcs (= embedded intervals), which join b_0 with the points of D , and do not meet outside b_0 . One defines an ordering of an arc system by choosing arbitrarily the first one and numbering the arcs by the directions of departure in counterclockwise order. One associates to an arc system n closed arcs as above. We call the obtained $\gamma_1, \dots, \gamma_n$ a *standard system* of closed arcs. Their homotopy classes generate $\pi_1(\mathbb{P}^1 - D, b_0)$ with the only relation $\gamma_1 * \dots * \gamma_n \simeq 1$. Let $t_i = m_{x_0}([\gamma_i])$. One has $t_1 \dots t_n = 1$. Conversely, let the n -tuple (t_1, \dots, t_n) of elements of G satisfies $t_i \neq 1$ for $\forall i$, and $t_1 \dots t_n = 1$. Let $D \subset \mathbb{P}^1 - b_0$ be an arbitrary set of n points, let $\gamma_1, \dots, \gamma_n$ be a simple system of closed arcs constructed as above, and let $m : \pi_1(\mathbb{P}^1 - D, b_0) \rightarrow G$ be the homomorphism defined by $m_{x_0}([\gamma_i]) = t_i$. Then, by Riemann's existence theorem, there is a G -covering $\pi : X \rightarrow \mathbb{P}^1$ branched in D , and a point $x_0 \in \pi^{-1}(b_0)$, such that $m_{x_0} = m$. The topological cover $X - \pi^{-1}(D)$ is connected, equivalently X is irreducible, if and only if t_1, \dots, t_n generate G .

1.2. Definition. An n -tuple (t_1, \dots, t_n) of elements of a group G which satisfy $t_i \neq 1$ for $\forall i$ and $t_1 \dots t_n = 1$ is called a *Hurwitz system*.

1.3. Two G -coverings $\pi : X \rightarrow \mathbb{P}^1$ and $\pi' : X' \rightarrow \mathbb{P}^1$ are called equivalent if there is a G -equivariant isomorphism $f : X \rightarrow X'$ such that $\pi' = \pi \circ f$. Suppose furthermore π and π' are not branched in $b_0 \in \mathbb{P}^1$. Let $x_0 \in \pi^{-1}(b_0)$, $x'_0 \in \pi'^{-1}(b_0)$. The pairs $(\pi : X \rightarrow \mathbb{P}^1, x_0)$ and $(\pi' : X' \rightarrow \mathbb{P}^1, x'_0)$ are called equivalent if there is an isomorphism f as above, which satisfies furthermore $f(x_0) = x'_0$.

Let C_1, \dots, C_k be conjugacy classes of G , $C_i \neq C_j$ if $i \neq j$. Let $\underline{n} = n_1C_1 + \dots + n_kC_k$ be a formal sum where $n_i \in \mathbb{N}$. Let $|\underline{n}| = n_1 + \dots + n_k$. In this paper we study two types of Hurwitz spaces, which we first define as sets. The points of $\mathcal{H}_{G;\underline{n}}(\mathbb{P}^1)$ are

the equivalence classes of G -coverings $[\pi : X \rightarrow \mathbb{P}^1]$ with $n = |\underline{n}|$ discriminant points, such that n_i of these points have local monodromies belonging to C_i , $i = 1, \dots, k$, and, moreover, X is irreducible. The points of $\mathcal{H}_{G;\underline{n}}(\mathbb{P}^1, b_0)$ are the equivalence classes of pairs $[\pi : X \rightarrow \mathbb{P}^1, x_0]$, where π is as above, and furthermore it is unramified at $b_0 \in \mathbb{P}^1$ and $\pi(x_0) = b_0$. The following properties are known:

(i) $\mathcal{H}_{G;\underline{n}}(\mathbb{P}^1) \neq \emptyset \Leftrightarrow \mathcal{H}_{G;\underline{n}}(\mathbb{P}^1, b_0) \neq \emptyset \Leftrightarrow$ there exists a Hurwitz system $(t_1, \dots, t_n) \in G^n$, $t_1 \cdots t_n = 1$, such that t_1, \dots, t_n generate G (cf. Section 1.1). We assume for the next properties that the Hurwitz spaces are non-empty.

(ii) $\mathcal{H}_{G;\underline{n}}(\mathbb{P}^1)$ has a canonical complex analytic structure such that the map given by $[X \rightarrow \mathbb{P}^1] \mapsto D$ is a finite, étale holomorphic map $\mathcal{H}_{G;\underline{n}}(\mathbb{P}^1) \rightarrow (\mathbb{P}^1)^{(n)} - \Delta$. Here Δ is the codimension one subvariety consisting of effective non-simple divisors of degree n . Similarly $\mathcal{H}_{G;\underline{n}}(\mathbb{P}^1, b_0)$ has a canonical complex analytic structure and a finite étale covering $\mathcal{H}_{G;\underline{n}}(\mathbb{P}^1, b_0) \rightarrow (\mathbb{P}^1 - b_0)^{(n)} - \Delta$.

(iii) The complex analytic spaces and the coverings of (ii) are algebraic.

(iv) Let us fix a $D \in (\mathbb{P}^1 - b_0)^{(n)}$ and let us choose a standard system of closed arcs $\gamma_1, \dots, \gamma_n$. Varying $m : \pi_1(\mathbb{P}^1 - D, b_0) \rightarrow G$ let us identify the fiber of $\mathcal{H}_{G;\underline{n}}(\mathbb{P}^1, b_0) \rightarrow (\mathbb{P}^1 - b_0)^{(n)} - \Delta$ over D with the set of Hurwitz systems $(t_1, \dots, t_n) \in G^n$, $t_1 \cdots t_n = 1$, such that n_i of its elements belong to C_i , $i = 1, \dots, k$, and t_1, \dots, t_n generate G . Consider the monodromy action of the braid group $\pi_1((\mathbb{P}^1 - b_0)^{(n)} - \Delta, D)$ on the fiber over $D = \{b_1, \dots, b_n\}$. Then the action of the elementary braids σ_i, σ_i^{-1} , which fix b_j for $j \neq i, i+1$ and rotate b_i, b_{i+1} at angles $\pm\pi$, is given by the following formulae:

$$\begin{aligned} (t_1, \dots, t_{i-1}, t_i, t_{i+1}, t_{i+2}, \dots, t_n) &\mapsto (t_1, \dots, t_{i-1}, t_i t_{i+1} t_i^{-1}, t_i, t_{i+2}, \dots, t_n), \\ (t_1, \dots, t_{i-1}, t_i, t_{i+1}, t_{i+2}, \dots, t_n) &\mapsto (t_1, \dots, t_{i-1}, t_{i+1}, t_i^{-1} t_i t_{i+1}, t_{i+2}, \dots, t_n). \end{aligned} \quad (1)$$

We call such transformations of n -tuples *elementary transformations* or *braid moves*. They determine uniquely the monodromy action of the braid group since the elementary braids generate it.

(v) The image of the forgetful map $[X \rightarrow \mathbb{P}^1, x_0] \mapsto [X \rightarrow \mathbb{P}]$ is a Zariski open, dense subset $\mathcal{U}(b_0) \subset \mathcal{H}_{G;\underline{n}}(\mathbb{P}^1)$ consisting of $[X \rightarrow \mathbb{P}]$ which are unramified at b_0 . Let us denote by $[t_1, \dots, t_n]$ the orbit of an n -tuple of elements of G with respect to the conjugacy action $(t_1, \dots, t_n) \mapsto (s t_1 s^{-1}, \dots, s t_n s^{-1})$, $s \in G$. Then the fiber of $\mathcal{U}(b_0) \rightarrow (\mathbb{P}^1 - b_0)^{(n)} - \Delta$ over D may be identified with the set $\{[t_1, \dots, t_n]\}$ where (t_1, \dots, t_n) runs over all Hurwitz systems satisfying the conditions of (iv). This set is called *Nielsen class* and denoted by $Ni(\underline{n}, G)$. The monodromy action of the braid group $\pi_1((\mathbb{P} - b_0)^{(n)} - \Delta, D)$ on the Nielsen class is determined by formulae (1).

1.4. Definition. We call two n -tuples of elements of G *braid-equivalent*, if one can be obtained from the other by a finite sequence of elementary transformations (1). We denote the braid equivalence by \sim .

1.5. Using (ii), (iv) and (v) of Section 1.3 one obtains the following conclusion:

The Hurwitz space $\mathcal{H}_{G;\underline{n}}(\mathbb{P}^1, b_0)$ is irreducible if and only if every two Hurwitz systems satisfying the conditions of (iv) are braid-equivalent. The Hurwitz space $\mathcal{H}_{G;\underline{n}}(\mathbb{P}^1)$ is ir-

reducible if and only if every two G -orbits of Hurwitz systems satisfying the conditions of (v) are braid-equivalent

1.6. Lemma. ([11, p. 102], [25, Lemma 9.4]) *Let $(t_1, \dots, t_n) \in G^n$ be a Hurwitz system. Then*

- (i) $(t_1, t_2, \dots, t_n) \sim (t_2, \dots, t_n, t_1)$;
- (ii) if $s \in \langle t_1, \dots, t_n \rangle \subset G$ then $(t_1, \dots, t_n) \sim (st_1s^{-1}, \dots, st_ns^{-1})$.

1.7. Corollary. *The forgetful map $\mathcal{H}_{G;\underline{n}}(\mathbb{P}^1, b_0) \rightarrow \mathcal{H}_{G;\underline{n}}(\mathbb{P}^1)$ establishes a bijective correspondence between the connected (i.e., irreducible) components of the two Hurwitz spaces.*

1.8. Lemma. *Suppose the n -tuple $(t_1, \dots, t_n) \in G^n$ contains the adjacent pair $t_k = t$, $t_{k+1} = t^{-1}$. Then performing a sequence of elementary transformations (1), one may move the pair (t, t^{-1}) at any two consecutive places, without changing the other elements of the n -tuple.*

Proof. This follows from the braid equivalences $(u, t, t^{-1}) \sim (t, t^{-1}ut, t^{-1}) \sim (t, t^{-1}, u)$ and $(t, t^{-1}, u) \sim (t, t^{-1}ut, t^{-1}) \sim (u, t, t^{-1})$. \square

1.9. Lemma. *Let (t_1, \dots, t_n) be an n -tuple of elements of G such that $t_{i+1} = t_i^{-1}$. Let H be the subgroup generated by $t_1, \dots, t_{i-1}, t_{i+2}, \dots, t_n$. Then for every $h \in H$ one has*

$$(t_1, \dots, t_{i-1}, ht_ih^{-1}, ht_{i+1}h^{-1}, t_{i+2}, \dots, t_n) \sim (t_1, \dots, t_n).$$

Proof. Let H_1 be the subset consisting of elements h such that

$$(t_1, \dots, t_{i-1}, hth^{-1}, ht^{-1}h^{-1}, t_{i+2}, \dots, t_n) \sim (t_1, \dots, t_{i-1}, t, t^{-1}, t_{i+2}, \dots, t_n)$$

holds for every $t \in G$. By reflexivity and transitivity of \sim it follows that H_1 is a subgroup of G . The statement of the lemma would be proved if we show that $t_j \in H_1$ for every $j \neq i, i+1$. Using Lemma 1.8 we move (t, t^{-1}) to the right of t_j . Then we have $(t_j, t, t^{-1}) \sim (t_j t t_j^{-1}, t_j, t^{-1}) \sim (t_j t t_j^{-1}, t_j t^{-1} t_j^{-1}, t_j)$. We then move the pair $(t_j t t_j^{-1}, t_j t^{-1} t_j^{-1})$ back to the initial position. \square

1.10. Let R be a root system in a real vector space V (see [3, Chapter VI] or [14, Chapter III]). Let $r = \dim V$ be the rank of R . We assume R is reduced, i.e., for each $\alpha \in R$ one has $R\alpha \cap R = \{\alpha, -\alpha\}$. Let $W = W(R)$ be the Weyl group generated by the reflections $\{s_\alpha \mid \alpha \in R\}$. Let (\mid) be a W -invariant inner product of V . Following [3] we denote $\alpha^\vee = \frac{2\alpha}{(\alpha|\alpha)}$ and $n(x, \alpha) = \frac{2(x|\alpha)}{(\alpha|\alpha)}$. Then $s_\alpha = x - n(x, \alpha)\alpha$. The values of $n(\alpha, \beta)$ for $\alpha, \beta \in R$ are given in [3, Chapter VI, §1.3]. Let C be a chamber, $B(C) = \{\alpha_1, \dots, \alpha_r\}$ be the corresponding base of R , $R = R^+ \cup R^-$ be the decomposition into positive and negative roots. Every total ordering of V determines uniquely a base of R , composed of the set of positive roots indecomposable into sums of other positive roots [15, §1.3] and [3, Chapter VI, §7]. Viceversa, given a base of R , and choosing its linear ordering, one

may consider the corresponding lexicographic ordering of V . The simple system associated with this total ordering is the given base of R . Suppose R is an irreducible root system. If R is simply laced, i.e., of type A_r, D_r, E_6, E_7 or E_8 , then it consists of one W -orbit. If R is non-simply laced, i.e., of type B_r, C_r, F_4 or G_2 , then it consists of two W -orbits $R = R_s \cup R_\ell$, called short and long roots, respectively.

1.11. Definition. Let R be a root system and let W be its Weyl group. A Galois covering $\pi : X \rightarrow \mathbb{P}^1$ with Galois group W is called *simply ramified* if every local monodromy is a reflection.

1.12. Simply ramified W -coverings of \mathbb{P}^1 yield Hurwitz systems (t_1, \dots, t_n) , where t_i are reflections in W . Applying the canonical homomorphism $\epsilon : W \rightarrow \{1, -1\}$ we see that n is even for these Hurwitz systems.

Let $R = R^{(1)} \sqcup \dots \sqcup R^{(k)}$ be the decomposition into disjoint union of irreducible root systems. Let $W^{(i)} = W(R^{(i)})$ be the corresponding Weyl groups. Every reflection in W belongs to some $W^{(i)}$ and every conjugacy class of reflections in W is a conjugacy class of reflections in some $W^{(i)}$ with respect to $W^{(i)}$. Simplifying the notation of Section 1.3 we may specify the branching data of a simply ramified W -covering $\pi : X \rightarrow \mathbb{P}^1$ by $\underline{n} = (\underline{n}^{(1)}, \dots, \underline{n}^{(k)})$ where: if $R^{(i)}$ is simply laced $\underline{n}^{(i)} = n^{(i)}$ and denotes the number of discriminant points whose local monodromies are reflections in $W^{(i)}$; if $R^{(i)}$ is non-simply laced then $\underline{n}^{(i)} = (n_s^{(i)}, n_\ell^{(i)})$ where $n_s^{(i)}$, respectively $n_\ell^{(i)}$, denotes the number of discriminant points whose local monodromies are reflections with respect to short roots, respectively long roots, in $W^{(i)}$. Our aim is to prove that the Hurwitz spaces $\mathcal{H}_{W; \underline{n}}(\mathbb{P}^1, b_0)$ and $\mathcal{H}_{W; \underline{n}}(\mathbb{P}^1)$ are irreducible by studying braid equivalences between Hurwitz systems of reflections in W .

1.13. Definition. Let $\{t_1, \dots, t_m\}$ be a subset of a group G . Replacing a pair t_i, t_j by $t_i, t_i t_j t_i^{-1}$ one obtains a new set $\{t'_1, \dots, t'_m\}$ where $t'_k = t_k$ if $k \neq j$ and $t_j = t_i t_j t_i^{-1}$. This transformation of subsets of a group is called *Nielsen transformation*.

The following theorem is a particular case of a well-known result [6,7,9]. The part concerning Nielsen transformations, which we need, is treated only in [1] in a special case, so we include a simple proof of it.

1.14. Theorem. Let $R \subset V$ be a root system, let $W = W(R)$ be its Weyl group, let \geq be a total ordering of V , and let $\{\alpha_1, \dots, \alpha_r\}$ be the corresponding base of R . Let $T = \{t_1, \dots, t_m\} \subset W$ be a set of reflections, $t_i = t_{\beta_i}$ with $\beta_i \in R^+$. Let W' be the subgroup generated by T , let $R' = W' \cdot \{\beta_1, \dots, \beta_m\}$ and let V' be the span of R' . Then R' is a root system in V' and W' is its Weyl group. Furthermore, if $\{\alpha'_1, \dots, \alpha'_\ell\}$ is the base of R' associated with the total ordering induced by \geq on V' , then the set $\{s_{\alpha'_1}, \dots, s_{\alpha'_\ell}\}$ may be obtained from T by a finite sequence of Nielsen transformations.

Proof. First we notice that if s_α, s_β are reflections in W then $s_\alpha s_\beta s_\alpha = s_{s_\alpha(\beta)}$. Let (\mid) be a W -invariant inner product of V . For each $\alpha \in R$, $\alpha = a_1 \alpha_1 + \dots + a_r \alpha_r$, let $ht(\alpha) =$

$a_1 + \dots + a_r$. If $s = s_\alpha$ is a reflection with $\alpha \in R^+$, let $h(s) = ht(\alpha)$. We extend this definition to sets of reflections $A = \{\sigma_i\}$ letting $h(A) = \sum h(\sigma_i)$.

Claim. Suppose $\alpha, \beta \in R^+$ and $(\alpha | \beta) > 0$. Then there is a Nielsen transformation $\{s_\alpha, s_\beta\} \mapsto \{s_{\alpha'}, s_{\beta'}\}$ such that $h(\{s_\alpha, s_\beta\}) > h(\{s_{\alpha'}, s_{\beta'}\})$.

For the proof of the claim we may obviously suppose that R is irreducible. We may furthermore suppose $\|\beta\| \geq \|\alpha\|$. First let $\|\alpha\| = \|\beta\|$. We have $s_\alpha(\beta) = \beta - \alpha \in R$. If $\beta - \alpha \in R^+$ then $h(s_{s_\alpha(\beta)}) = ht(\beta - \alpha) < ht(\beta)$ and we may decrease h considering the Nielsen transformation $\{s_\alpha, s_\beta\} \mapsto \{s_\alpha, s_\alpha s_\beta s_\alpha\}$. If $\beta - \alpha \in R^-$, then $s_\beta(\alpha) = \alpha - \beta \in R^+$ and we consider $\{s_\beta s_\alpha s_\beta, s_\beta\}$. If $\|\beta\| > \|\alpha\|$ then $s_\beta(\alpha) = \alpha - \beta$ and $s_\alpha(\beta) = \beta - c\alpha$ where $c = 2$ if R is of type B_n, C_n or F_4 and $c = 3$ if R is of type G_2 . If $\alpha - \beta \in R^+$ we consider as above $\{s_\beta s_\alpha s_\beta, s_\beta\}$. If $s_\alpha(\beta) = \beta - c\alpha \in R^+$ then we consider $\{s_\alpha, s_\alpha s_\beta s_\alpha\}$. It remains to deal with the cases when $\|\beta\| > \|\alpha\|$, $\alpha - \beta \in R^-$, $\beta - c\alpha \in R^-$. Suppose first R is of type B_n, C_n or F_4 , so $c = 2$. Then $\alpha - \beta \in R^-$ implies $ht(\alpha) < \beta$, so the positive root $2\alpha - \beta$ satisfies $ht(2\alpha - \beta) < ht(\beta)$. Therefore $h(s_\alpha s_\beta s_\alpha) = h(s_{2\alpha - \beta}) < h(s_\beta)$ and we may decrease h by the Nielsen transformation $\{s_\alpha, s_\beta\} \mapsto \{s_\alpha, s_\alpha s_\beta s_\alpha\}$. Finally, let R be of type G_2 . With the assumptions above we have

$$h(s_\alpha) + h(s_\alpha s_\beta s_\alpha) = ht(\alpha) + ht(3\alpha - \beta) = 4ht(\alpha) - ht(\beta),$$

$$h(s_\beta s_\alpha s_\beta) + h(s_\beta) = ht(\beta - \alpha) + ht(\beta) = 2ht(\beta) - ht(\alpha).$$

It is impossible that both numbers are greater than or equal to $ht(\alpha) + ht(\beta)$. This proves the claim.

The claim shows that if among the reflections of T there are two $t_i = s_{\beta_i}, t_j = s_{\beta_j}$ with $\beta_i, \beta_j \in R^+$ and $(\beta_i | \beta_j) > 0$, then performing a Nielsen transformation of T we may decrease $h(T)$. Since h assumes a finite number of positive values on sets of $\leq m$ elements we conclude that after a finite number of Nielsen transformations we obtain a set of reflections $T' = \{s_{\alpha'_1}, \dots, s_{\alpha'_\ell}\}$, where $\alpha'_i \in R^+$ and $(\alpha'_i | \alpha'_j) \leq 0$ for $i \neq j$. We have furthermore that $n(\alpha'_i, \alpha'_j) \cdot n(\alpha'_j, \alpha'_i) \in \{0, 1, 2, 3\}$ if $i \neq j$ since $\alpha'_i, \alpha'_j \in R$. By the classification of Coxeter graphs and Dynkin diagrams (see, e.g., [14]) it follows that $\alpha'_1, \dots, \alpha'_\ell$ is a base of a root system. Replacing T by a Nielsen transformation changes neither $W' = \langle T \rangle$ nor $R' = W' \cdot \{\beta_1, \dots, \beta_m\}$. We conclude that R' is a root system with base $\alpha'_1, \dots, \alpha'_\ell$ and, moreover, since all α'_i are positive roots in R this is the unique base of R' associated with the total ordering \geq . \square

1.15. Corollary. Let $R \subset V$ be a root system, let W be its Weyl group, and let $S = \{s_{\alpha_1}, \dots, s_{\alpha_r}\}$ be a set of simple reflections corresponding to a base of R . Let $T = \{t_1, \dots, t_m\}$ be a set of reflections which generate W . Then one can obtain S from T by a finite sequence of Nielsen transformations.

Proof. Let us choose a linear ordering of the set $\{\alpha_1, \dots, \alpha_r\}$ and let \geq be the corresponding lexicographic ordering of V . Using the notation of the theorem we have $W' = W$,

$R' = R$ (see [15, Example 1.14]), $V' = V$ and $\{\alpha'_1, \dots, \alpha'_\ell\} = \{\alpha_1, \dots, \alpha_r\}$ since the total ordering \geq determines uniquely a base of $R = R'$. \square

2. Irreducibility of Hurwitz spaces

2.1. Suppose $R \subset V$ is an irreducible root system. Let us normalize the W -invariant inner product (\mid) so that $(\alpha \mid \alpha) = 2$ for every root α if R is simply laced and $(\alpha \mid \alpha) = 2$ for every short root if R is non-simply laced. In the latter case for every long root β one has $(\beta \mid \beta) = 4$ if R is of type B_r, C_r or F_4 and $(\beta \mid \beta) = 6$ if R is of type G_2 . Let us choose a chamber $C \subset V$. Let λ be the dominant root if R is simply laced, and let λ be the dominant short root if R is non-simply laced. It is known that:

$$\text{if } \alpha \in R^+ \text{ and } \alpha \neq \lambda \quad \text{then } (\lambda \mid \alpha^\vee) \text{ equals } 0 \text{ or } 1$$

(cf. [3, Chapter VI, §1.3]). With the fixed normalization of (\mid) we have that $\alpha^\vee = \alpha$ if R is of type A_r, D_r, E_6, E_7, E_8 , or if R is of type B_r, C_r, F_4, G_2 and α is a short root. So in these cases we have that $\alpha \in R^+$ and $\alpha \neq \lambda$ implies that $(\lambda \mid \alpha)$ equals 0 or 1. If R is of type B_r, C_r, F_4 or G_2 and β is a positive long root then $(\lambda \mid \beta)$ equals 0 or $(\beta \mid \beta)/2$.

2.2. Proposition. *Let R be a root system and let W be its Weyl group. Let (t_1, \dots, t_n) be a Hurwitz system where t_i are reflections in W . Then (t_1, \dots, t_n) is braid-equivalent to a Hurwitz system (t'_1, \dots, t'_n) such that $t'_1 = t'_2$.*

Proof. By Theorem 1.14 we may suppose without loss of generality that the set $\{t_1, \dots, t_n\}$ generates W . Let $R = R^{(1)} \sqcup \dots \sqcup R^{(k)}$ be the decomposition into disjoint union of irreducible root systems. Let $W^{(i)} = W(R^{(i)})$. If $\alpha \in R^{(i)}, \beta \in R^{(j)}$ and $i \neq j$ then $(\alpha \mid \beta) = 0$ and s_α commutes with s_β . Thus one may perform several braid moves to the Hurwitz system (t_1, \dots, t_n) to the effect of obtaining a concatenation $(T^{(1)}, T^{(2)}, \dots, T^{(k)})$, where $T^{(i)}$ contains all reflections in T which belong to $W^{(i)}$ in the order they appear in T . Since W is a direct product of $W^{(i)}$, the product of the reflections in $T^{(i)}$ equals 1 for each i . It suffices to prove the proposition for $T^{(1)}$, so we may suppose without loss of generality that R is an irreducible root system. The case $rk(R) = 1, W = S_2$ is trivial, so we may furthermore suppose $r = rk(R) \geq 2$. If $T = (t_1, \dots, t_n)$ is braid-equivalent to a sequence with two equal reflections, then we may move them by elementary transformations to the first two places and obtain the required Hurwitz system. Let us suppose by way of contradiction that

(*) *No sequence braid-equivalent to (t_1, \dots, t_n) contains two equal reflections.*

Step 1. Let $t_i = s_{\gamma_i}$ with $\gamma_i \in R^+$. Let λ be the dominant root defined in Section 2.1. If among t_1, \dots, t_n the reflection s_λ is present, we move it to the first place by a sequence of elementary transformations of the type $(\sigma, \tau) \mapsto (\tau, \tau\sigma\tau)$. Similarly we may move to the front all reflections s_{γ_i} with $(\lambda \mid \gamma_i) > 0$. We obtain a Hurwitz system, braid-equivalent to (t_1, \dots, t_n) , of the form

$$(s_{\beta_1}, s_{\beta_2}, \dots, s_{\beta_k}, s_{\beta_{k+1}}, \dots, s_{\beta_n}) \quad (2)$$

where:

- (i) $\beta_i \in R^+$ and $\beta_i \neq \beta_j$ for $i \neq j$ (assumption (*));
- (ii) $(\lambda | \beta_i) > 0$ for $i = 1, \dots, k$ and $(\lambda | \beta_j) = 0$ for $j \geq k + 1$;
- (iii) if $\lambda \in \{\beta_1, \dots, \beta_k\}$ then $\lambda = \beta_1$.

Notice that $k \geq 1$ since W cannot be generated by reflections s_α with $(\lambda | \alpha) = 0$. Among the Hurwitz systems, braid-equivalent to (t_1, \dots, t_n) and satisfying conditions (i)–(iii), we consider those for which

- (iv) k is minimal possible.

Step 2. We claim there is a Hurwitz system braid-equivalent to (t_1, \dots, t_n) , which satisfies conditions (i)–(iv) of Step 1, and furthermore

- (v) $(\beta_i | \beta_j) \geq 0$ for every i, j with $1 \leq i, j \leq k$

holds. First suppose $\beta_1 = \lambda$. Then $(\beta_1, \beta_i) \geq 0$ for $i \geq 2$ since λ is a dominant root. Suppose there is a pair i, j with $2 \leq i < j \leq k$ such that $(\beta_i | \beta_j) < 0$. If $\|\beta_i\| = \|\beta_j\|$ then $s_{\beta_i}(\beta_j) = \beta_j + \beta_i$ and $\|s_{\beta_i}(\beta_j)\| = \|\beta_j\|$. We have $(\lambda | s_{\beta_i}(\beta_j)) = (\lambda | \beta_i) + (\lambda | \beta_j) > \max\{(\lambda | \beta_i), (\lambda | \beta_j)\}$. From Section 2.1 this is possible only if $\|\beta_i\| = \|\beta_j\| = 2$, and $\lambda = \beta_i + \beta_j$. Performing several braid moves among $s_{\beta_2}, \dots, s_{\beta_k}$ we place s_{β_i} adjacent to s_{β_j} . The braid move $(s_{\beta_i}, s_{\beta_j}) \mapsto (s_{\beta_i}s_{\beta_j}s_{\beta_i}, s_{\beta_i}) = (s_\lambda, s_{\beta_i})$ yields a sequence with two reflections equal to s_λ . This contradicts assumption (*). If $\|\beta_i\| > \|\beta_j\|$ then $s_{\beta_i}(\beta_j) = \beta_j + \beta_i$. This is a root with $(\lambda | \beta_j + \beta_i) = 1 + \frac{1}{2}\|\beta_i\| \geq 3$. This is impossible by the choice of λ in Section 2.1. One reasons similarly in the case $\|\beta_i\| < \|\beta_j\|$ considering $s_{\beta_j}(\beta_i) = \beta_i + \beta_j$. The claim of Step 2 is proved if $\beta_1 = \lambda$. Let $\beta_1 \neq \lambda$. Suppose $(\beta_i | \beta_j) < 0$ for some pair i, j with $1 \leq i < j \leq k$. The same arguments as above show that the only possibility is $\|\beta_i\| = \|\beta_j\| = 2$, $\beta_i + \beta_j = \lambda$, in which case performing braid moves among the first k reflections one obtains a Hurwitz system containing s_λ . One moves s_λ to the first place by elementary transformations. The obtained Hurwitz system is of the type of Step 1, since none of $s_{\beta_{k+1}}, \dots, s_{\beta_n}$ has been changed by these transformations, and by the minimality of k (condition (iv) of Step 1). We already treated such cases in this step, so there is a braid-equivalent Hurwitz system for which condition (v) holds.

Step 3. We claim that for a Hurwitz system which satisfies conditions (i)–(v) of Step 1 and Step 2 one of the following alternatives holds:

- (vi) if $\beta_1 \neq \lambda$ then $(\beta_i | \beta_j) = 0$ for $\forall i \neq j$ with $1 \leq i, j \leq k$;
- (vi') if $\beta_1 = \lambda$ then $(\beta_i | \beta_j) = 0$ for $\forall i \neq j$ with $2 \leq i, j \leq k$.

Suppose $(\beta_i | \beta_j) > 0$ for some pair $i \neq j$ with $1 \leq i, j \leq k$ and $\beta_i \neq \lambda \neq \beta_j$. We may assume $\|\beta_i\| \geq \|\beta_j\|$. If $\|\beta_i\| = \|\beta_j\|$ then $s_{\beta_j}(\beta_i) = \beta_i - \beta_j$ and $(\lambda | s_{\beta_j}(\beta_i)) = \frac{1}{2}\|\beta_i\| - \frac{1}{2}\|\beta_j\| = 0$. If $\|\beta_i\| > \|\beta_j\|$ then $\|\beta_j\| = 2$ and $(\lambda | \beta_j) = 1$ since $\lambda \neq \beta_j$. Furthermore $s_{\beta_j}(\beta_i) = \beta_i - n(\beta_i, \beta_j)\beta_j$ and $(\lambda | s_{\beta_j}(\beta_i)) = \frac{1}{2}\|\beta_i\| - n(\beta_i, \beta_j) = 0$. In both

cases $(\lambda \mid s_{\beta_j}(\beta_i)) = 0$. Let β be the positive root belonging to $\{s_{\beta_j}(\beta_i), -s_{\beta_j}(\beta_i)\}$. Performing several braid moves among the first k reflections we place s_{β_i} adjacent to s_{β_j} . We obtain either $(\dots, s_{\beta_i}, s_{\beta_j}, \dots) \sim (\dots, s_{\beta_j}, s_{\beta_i}, \dots)$ or $(\dots, s_{\beta_j}, s_{\beta_i}, \dots) \sim (\dots, s_{\beta_i}, s_{\beta_j}, \dots)$ with $(\lambda \mid s_{\beta}) = 0$. In either case we move s_{β} to the k th place by successive braid moves. The obtained Hurwitz system contradicts the minimality of k required in condition (iv) of Step 1.

Step 4. We claim alternative (vi) of Step 3 yields a contradiction. We have β_1, \dots, β_k are k mutually orthogonal roots. The involution $\sigma = s_{\beta_1} \cdots s_{\beta_k}$ has eigenvalues 1 and -1 . Let V_+ and V_- be the corresponding eigenspaces, $V = V_+ \oplus V_-$. By [4, Section 2] one has $V_+ = H_{\beta_1} \cap \cdots \cap H_{\beta_k}$ and $V_- = V_+^\perp$. The identity $s_{\beta_1} \cdots s_{\beta_k} s_{\beta_{k+1}} \cdots s_{\beta_n} = 1$ and $(\lambda \mid \beta_i) = 0$ for $i \geq k+1$ implies $\sigma(\lambda) = \lambda$. Hence $\lambda \in V_+$, i.e., $(\lambda \mid \beta_i) = 0$ for $1 \leq i \leq k$. This is a contradiction with condition (ii) of Step 1 (we recall $k \geq 1$ since $s_{\beta_1}, \dots, s_{\beta_n}$ generate W).

Step 5. If R is not of type G_2 then alternative (vi') of Step 3 yields a contradiction. First we prove $k \leq 2$ in this case. Suppose among β_2, \dots, β_k there are two roots with length 2. We may assume without loss of generality that $\|\beta_2\| = \|\beta_3\| = 2$. We have $s_{\beta_2}(\lambda) = \lambda - \beta_2$ and $s_{\beta_3}(s_{\beta_2}(\lambda)) = \lambda - \beta_2 - \beta_3 = \beta$. The latter root satisfies $(\lambda \mid \beta) = 0$. Performing the braid moves $(s_\lambda, s_{\beta_1}, s_{\beta_2}, \dots) \sim (s_{\beta_1, s_\lambda - \beta_1}, s_{\beta_2}, \dots) \sim (s_{\beta_1}, s_{\beta_2}, s_\beta, \dots)$, and moving s_β to the k th place by elementary transformations, we obtain a Hurwitz system with $n - k + 1$ reflections which fix λ . This contradicts condition (iv) of Step 1. Therefore among β_2, \dots, β_k there is at most one root of length 2. If R is of type B_r, C_r or F_4 suppose among β_2, \dots, β_k there is a long root. We may assume without loss of generality this is β_2 . Then $s_{\beta_2}(\lambda) = \lambda - \beta_2 = \beta$ and $(\lambda \mid \beta) = (\lambda \mid \lambda) - (\lambda \mid \beta_2) = 2 - \frac{1}{2}\|\beta_2\| = 0$. Performing the braid move $(s_\lambda, s_{\beta_2}, \dots) \sim (s_{\beta_2}, s_\beta, \dots)$, and moving s_β to the k th place by successive elementary transformations, we obtain again a Hurwitz system which contradicts the minimality of k . Our claim that $k \leq 2$ is proved. We have $s_\lambda = s_{\beta_2} s_{\beta_3} \cdots s_{\beta_n}$. If $k = 1$ then $(\lambda \mid \beta_i) = 0$ for $i \geq 2$. Applying both sides of the above equality to λ we obtain the absurdity $-\lambda = \lambda$. If $k = 2$ then $(\lambda \mid \beta_i) = 0$ for $i \geq 3$. Applying both sides of the above equality to λ we obtain $-\lambda = s_\lambda(\lambda) = s_{\beta_2}(\lambda)$. Therefore $\beta_2 = \lambda$ which contradicts assumption (*).

Step 6. It remains to consider alternative (vi') of Step 3 when R is of type G_2 . The same argument as in Step 5 shows that among β_2, \dots, β_k there is at most one root of length 2. We claim there is also at most one long root among β_2, \dots, β_k . Indeed, if β_i, β_j are long roots, then $s_{\beta_i}(\lambda) = \lambda - \beta_i$, $s_{\beta_j}(s_{\beta_i}(\lambda)) = \lambda - \beta_i - \beta_j$. The root $\lambda - \beta_i - \beta_j$ satisfies $(\lambda \mid \lambda - \beta_i - \beta_j) = 2 - 3 - 3 = -4$. This is an absurdity since for $\forall \gamma \in R$ one has $(\lambda \mid \gamma) \in \{0, \pm 1, \pm 3\}$. The case $k \leq 2$ is impossible by the argument of Step 5. It remains to consider the case where $k = 3$ and $\{\beta_2, \beta_3\}$ consists of a short and a long root which are orthogonal. Without loss of generality we may assume $\|\beta_2\| = 2$, $\|\beta_3\| = 6$. The number n is even, so $n \geq 4$. If among β_4, \dots, β_n there were no long roots, then s_{β_3} would be contained in the subgroup generated by reflections with respect to short roots. This is impossible since the only reflections in the latter subgroup are s_γ with γ a short root in R . Without loss of generality we may assume β_4 is a long root. We obtain a Hurwitz system $(s_\lambda, s_{\beta_2}, s_{\beta_3}, s_{\beta_4}, \dots)$ braid-equivalent to (t_1, \dots, t_n) where $\|\beta_2\| = 2$, $\|\beta_3\| = \|\beta_4\| = 6$, $(\lambda \mid \beta_2) = 1$, $(\lambda \mid \beta_3) = 3$, $(\lambda \mid \beta_4) = 0$, $(\beta_2 \mid \beta_3) = 0$. Table X of [3, Chapter VI] lists the positive roots in R of type G_2 : $R^+ = R_s^+ \cup R_\ell^-$, $R_s^+ = \{\alpha_1, \alpha'_1, \omega_1\}$, $R_\ell^- = \{\alpha_2, \alpha'_2, \omega_2\}$

where α_1, α_2 is a base of R and ω_1, ω_2 are the fundamental weights. We have $\lambda = \omega_1$. The following two possibilities for the quadruple $(s_\lambda, s_{\beta_2}, s_{\beta_3}, s_{\beta_4})$ may occur:

$$(s_{\omega_1}, s_{\alpha_1}, s_{\omega_2}, s_{\alpha_2}) \quad \text{or} \quad (s_{\omega_1}, s_{\alpha'_1}, s_{\alpha'_2}, s_{\alpha_2}). \quad (3)$$

In the first case we perform the following braid moves:

$$\begin{aligned} (s_{\omega_1}, s_{\alpha_1}, s_{\omega_2}, s_{\alpha_2}) &\sim (s_{\omega_1}, s_{\omega_2}, s_{\alpha_1}, s_{\alpha_2}) \sim (s_{\omega_1}, s_{\omega_2}, s_{\alpha_2}, s_{\alpha'_1}) \sim (s_{\omega_1}, s_{\omega_2}, s_{\alpha'_1}, s_{-\omega_2}) \\ &\sim (s_{\omega_1}, s_{-\omega_1}, s_{\omega_2}, s_{-\omega_2}) = (s_{\omega_1}, s_{\omega_1}, s_{\omega_2}, s_{\omega_2}). \end{aligned}$$

This contradicts assumption (*). The product of the first quadruple of (3) equals 1 and conjugating it by s_{ω_1} one obtains the second quadruple. By Lemma 1.6 the second quadruple is braid-equivalent to the first one, hence braid-equivalent to $(s_{\omega_1}, s_{\omega_1}, s_{\omega_2}, s_{\omega_2})$. This contradicts assumption (*).

We proved that assumption (*) leads to a contradiction. Therefore the Hurwitz system (t_1, \dots, t_n) is braid-equivalent to some (t'_1, \dots, t'_n) with $t'_1 = t'_2$. The proposition is proved. \square

2.3. Proposition. *Let R be a root system and let W be its Weyl group. Let (t_1, \dots, t_n) be a Hurwitz system of reflections in W . Then (t_1, \dots, t_n) is braid-equivalent to a Hurwitz system (t'_1, \dots, t'_n) , where $t'_{2i-1} = t'_{2i}$ for $i = 1, \dots, n/2$.*

Proof. Use Proposition 2.2 and induction on the even number n . \square

2.4. Let R be an irreducible root system of rank r with Weyl group W . In Section 1.3 we defined the Hurwitz spaces $\mathcal{H}_{W;\underline{n}}(\mathbb{P}^1, b_0)$ and $\mathcal{H}_{W;\underline{n}}(\mathbb{P}^1)$. If R is simply laced, i.e., of type A_r, D_r, E_6, E_7 or E_8 , one has $\underline{n} = n$, and the spaces parametrize irreducible Galois covers of \mathbb{P}^1 branched in n points, whose local monodromies are reflections in W . If R is non-simply laced, i.e., of type B_r, C_r, F_4 or G_2 , one has $\underline{n} = (n_s, n_\ell)$, and the spaces parametrize irreducible Galois covers of \mathbb{P}^1 branched in $n = n_s + n_\ell$ points, with n_s discriminant points whose local monodromies are reflections with respect to short roots, and n_ℓ discriminant points whose local monodromies are reflections with respect to long roots. In the non-simply laced case let r_s , respectively r_ℓ , denote the number of short roots, respectively long roots, in the Dynkin diagram of R . One has $r = r_s + r_\ell$, and for types B_r, C_r, F_4, G_2 the pair (r_s, r_ℓ) equals respectively $(1, r-1), (r-1, 1), (2, 2), (1, 1)$ (cf. [3, Chapter VI]).

2.5. Theorem. *Let R be an irreducible root system of rank r with Weyl group W :*

- (i) *The Hurwitz spaces $\mathcal{H}_{W;\underline{n}}(\mathbb{P}^1, b_0)$ and $\mathcal{H}_{W;\underline{n}}(\mathbb{P}^1)$ are irreducible when non-empty.*
- (ii) *If R is of type A_r, D_r, E_6, E_7 or E_8 then $\mathcal{H}_{W;\underline{n}}(\mathbb{P}^1, b_0) \neq \emptyset$, equivalently $\mathcal{H}_{W;\underline{n}}(\mathbb{P}^1) \neq \emptyset$, if and only if $n \geq 2r$.*
- (iii) *If R is of type B_r, C_r, F_4 or G_2 then $\mathcal{H}_{W;\underline{n}}(\mathbb{P}^1, b_0) \neq \emptyset$, equivalently $\mathcal{H}_{W;\underline{n}}(\mathbb{P}^1) \neq \emptyset$, if and only if $n_s \equiv 0 \pmod{2}$, $n_\ell \equiv 0 \pmod{2}$, $n_s \geq 2r_s$ and $n_\ell \geq 2r_\ell$.*

Proof. By Section 1.3(i) and Corollary 1.7 it suffices to prove the statements for $\mathcal{H}_{W;\underline{n}}(\mathbb{P}^1, b_0)$. Let $\alpha_1, \dots, \alpha_r$ be a base of R . If R is simply laced let us choose an arbitrary root α . If R is non-simply laced let us choose an arbitrary short root α and an arbitrary long root β . According to Section 1.5 the following claim proves part (i) and one of the directions of parts (ii) and (iii).

Claim. Let (t_1, \dots, t_n) be a Hurwitz system of reflections in W such that t_1, \dots, t_n generate W . If R is simply laced then $n \geq 2r$ and the Hurwitz system is braid-equivalent to

$$(s_{\alpha_1}, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_2}, \dots, s_{\alpha_r}, s_{\alpha_r}, s_{\alpha}, \dots, s_{\alpha}), \quad (4)$$

where s_{α} appears $n - 2r$ times. If R is non-simply laced then n_s and n_{ℓ} are even, $n_s \geq 2r_s$, $n_{\ell} \geq 2r_{\ell}$ and the Hurwitz system is braid-equivalent to

$$(s_{\alpha_1}, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_2}, \dots, s_{\alpha_r}, s_{\alpha_r}, s_{\alpha}, \dots, s_{\alpha}, s_{\beta}, \dots, s_{\beta}), \quad (5)$$

where s_{α} appears $n_s - 2r_s$ times and s_{β} appears $n_{\ell} - 2r_{\ell}$ times.

Let $n = 2m$. According to Proposition 2.3 the Hurwitz system (t_1, \dots, t_n) is braid-equivalent to (t'_1, \dots, t'_n) where $t'_{2i-1} = t'_{2i} = s_{\beta_i}$ for $i = 1, \dots, m$. One has $\langle s_{\beta_1}, \dots, s_{\beta_m} \rangle = \langle t_1, \dots, t_n \rangle = W$. According to Corollary 1.15 there is a finite sequence of Nielsen transformations by which one can obtain $\{s_{\alpha_1}, \dots, s_{\alpha_r}\}$ from $\{s_{\beta_1}, \dots, s_{\beta_m}\}$. Lemma 1.9 shows that if a Hurwitz system contains two adjacent pairs of involutions (s, s) and (t, t) then replacing (t, t) by (sts, sts) one obtains a braid-equivalent Hurwitz system. This implies that, extending the above sequence of Nielsen transformations to Nielsen transformations of Hurwitz systems composed of pairs of elements of the corresponding sets, one obtains a sequence of braid-equivalences. Eventually we obtain a Hurwitz system composed of pairs $(s_{\alpha_i}, s_{\alpha_i})$, and every such pair with $1 \leq i \leq r$ does appear. Using Lemma 1.8 we may replace the obtained Hurwitz system by a braid-equivalent one, in which the first $2r$ elements are $(s_{\alpha_1}, s_{\alpha_1}, \dots, s_{\alpha_r}, s_{\alpha_r})$. These $2r$ reflections generate W , so by Lemma 1.9 we may replace any of the remaining pairs (s_{γ}, s_{γ}) by (s_{α}, s_{α}) if $\|\gamma\| = \|\alpha\|$, and by (s_{β}, s_{β}) if $\|\gamma\| = \|\beta\|$. This proves the claim.

For the proof of the other direction of parts (ii) and (iii) notice that, if the specified inequalities for the number of discriminant points are valid, one may define Hurwitz systems by (4) and (5) and apply Section 1.3(i). \square

Performing a sequence of braid moves to a given Hurwitz system may be viewed in two ways. Either one fixes a simple system of closed arcs in $\mathbb{P}^1 - D$ based at b_0 and varies the homomorphism $m: \pi_1(\mathbb{P}^1 - D, b_0) \rightarrow W$, thus obtaining information about the connected components of the Hurwitz space $\mathcal{H}_{W;\underline{n}}(\mathbb{P}^1, b_0)$, or one fixes the monodromy map $m: \pi_1(\mathbb{P}^1 - D, b_0) \rightarrow W$ and varies the simple arc system, thus obtaining a normal form for the local monodromies of a given covering, and eventually determining the topological type of the covering. So the proof of the theorem yields the following result, in which we use the notation introduced in Section 2.4.

2.6. Corollary. Let $\pi : X \rightarrow \mathbb{P}^1$ be an irreducible Galois cover, whose Galois group is the Weyl group of an irreducible root system R of rank r . Suppose π is simply ramified, i.e., every local monodromy is a reflection. Let D be the discriminant locus of π , let $|D| = n$, let $b_0 \in \mathbb{P}^1 - D$, let $x_0 \in \pi^{-1}(b_0)$, and let $m = m_{x_0} : \pi_1(\mathbb{P}^1 - D, b_0) \rightarrow W$ be the monodromy map. Then there is a simple arc system with initial point b_0 and end points in D , such that the local monodromies along the corresponding simple system of closed arcs $\gamma_1, \dots, \gamma_n$ have the form given by the Hurwitz system (4), respectively (5). Namely, fixing a base $\alpha_1, \dots, \alpha_r$ of R , an arbitrary root α if R is simply laced, an arbitrary short root α and an arbitrary long root β if R is non-simply laced, one has:

- (i) if R is of type A_r, D_r, E_6, E_7 or E_8 then

$$(m(\gamma_1), \dots, m(\gamma_n)) = (s_{\alpha_1}, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_2}, \dots, s_{\alpha_r}, s_{\alpha_r}, s_{\alpha}, \dots, s_{\alpha}),$$

where s_{α} appears $n - 2r$ times;

- (ii) if R is of type B_r, C_r, F_4 or G_2 then

$$(m(\gamma_1), \dots, m(\gamma_n)) = (s_{\alpha_1}, s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_2}, \dots, s_{\alpha_r}, s_{\alpha_r}, s_{\alpha}, \dots, s_{\alpha}, s_{\beta}, \dots, s_{\beta}),$$

where s_{α} appears $n_s - 2r_s$ times and s_{β} appears $n_{\ell} - 2r_{\ell}$ times.

We now extend Theorem 2.5 and Corollary 2.6 to simply ramified W -coverings where W is an arbitrary Weyl group. We refer to Sections 1.12 and 2.4 for the notation used. The superscript (i) refers to the irreducible root system $R^{(i)}$.

2.7. Theorem. Let R be a root system with Weyl group W . Let $R = R^{(1)} \sqcup \dots \sqcup R^{(k)}$ be its decomposition into irreducible components:

- (i) The Hurwitz spaces $\mathcal{H}_{W; \underline{n}}(\mathbb{P}^1, b_0)$ and $\mathcal{H}_{W; \underline{n}}(\mathbb{P}^1)$ are irreducible when non-empty.
(ii) $\mathcal{H}_{W; \underline{n}}(\mathbb{P}^1, b_0) \neq \emptyset$, equivalently $\mathcal{H}_{W; \underline{n}}(\mathbb{P}^1) \neq \emptyset$, if and only if for every simply laced component $R^{(i)}$ one has $n^{(i)} \equiv 0 \pmod{2}$, $n^{(i)} \geq 2r^{(i)}$ and for every non-simply laced component $R^{(j)}$ one has $n_s^{(j)} \equiv 0 \pmod{2}$, $n_{\ell}^{(j)} \equiv 0 \pmod{2}$, $n_s^{(j)} \geq 2r_s^{(j)}$, $n_{\ell}^{(j)} \geq 2r_{\ell}^{(j)}$.

Proof. Let (t_1, \dots, t_n) be a Hurwitz system of reflections in W such that t_1, \dots, t_n generate W . It is braid-equivalent to a concatenation $(T^{(1)}, T^{(2)}, \dots, T^{(k)})$ where each $T^{(i)}$ is a Hurwitz system of reflections in $W^{(i)}$ which generate $W^{(i)}$ (cf. the proof of Proposition 2.2). The theorem is proved applying the argument of Theorem 2.5 to each $T^{(i)}$. \square

2.8. Corollary. Let $\pi : X \rightarrow \mathbb{P}^1$ be an irreducible Galois cover, whose Galois group is the Weyl group of an arbitrary root system R . Suppose every local monodromy is a reflection. Let D be the discriminant locus of π , let $b_0 \in \mathbb{P}^1 - D$, and let $D = D^{(1)} \sqcup \dots \sqcup D^{(k)}$ be the disjoint union, corresponding to the decomposition into irreducible components $R = R^{(1)} \sqcup \dots \sqcup R^{(k)}$. Let $n^{(i)} = |D^{(i)}|$. Then there is a simple arc system with initial point b_0 and end points in D , ordered so that the first $n^{(1)}$ arcs end in $D^{(1)}$, the arcs with numbers

$n^{(1)} + 1, \dots, n^{(1)} + n^{(2)}$ end in $D^{(2)}$ etc., such that for every i the local monodromies corresponding to the collection of arcs ending in $D^{(i)}$ are given by the formulae in Corollary 2.6 with R replaced by $R^{(i)}$.

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References

- [1] C. Bennett, Signed Dynkin diagrams and associated groups, in: Group Theory, Granville, OH, 1992, World Scientific, River Edge, NJ, 1993, pp. 30–61.
- [2] R. Biggers, M. Fried, Irreducibility of moduli spaces of cyclic unramified covers of genus g curves, Trans. Amer. Math. Soc. 295 (1) (1986) 59–70.
- [3] N. Bourbaki, Lie Groups and Lie Algebras, Chapters 4–6, Elements of Mathematics, Springer-Verlag, Berlin, 2002.
- [4] R.W. Carter, Conjugacy classes in the Weyl group, Compos. Math. 25 (1972) 1–59.
- [5] A. Clebsch, Zür Theorie der Riemann’schen Flächen, Math. Ann. 6 (1872) 216–230.
- [6] H.S.M. Coxeter, Finite groups generated by reflections and their subgroups generated by reflections, Proc. Cambridge Philos. Soc. 30 (1934) 466–482.
- [7] V.V. Deodhar, A note on subgroups generated by reflections in Coxeter groups, Arch. Math. (Basel) 53 (6) (1989) 543–546.
- [8] R. Donagi, Decomposition of spectral covers, Astérisque 218 (1993) 145–175.
- [9] M. Dyer, Reflection subgroups of Coxeter systems, J. Algebra 135 (1) (1990) 57–73.
- [10] M. Fried, Fields of definition of function fields and Hurwitz families—groups as Galois groups, Comm. Algebra 5 (1) (1977) 17–82.
- [11] M. Fried, R. Biggers, Moduli spaces of covers and the Hurwitz monodromy group, J. Reine Angew. Math. 335 (1982) 87–121.
- [12] M. Fried, H. Völklein, The inverse Galois problem and rational points on moduli spaces, Math. Ann. 290 (4) (1991) 771–800.
- [13] W. Fulton, Hurwitz schemes and irreducibility of moduli of algebraic curves, Ann. of Math. (2) 90 (1969) 542–575.
- [14] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, Grad. Texts in Math., vol. 9, Springer-Verlag, New York, 1978.
- [15] J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge Stud. Adv. Math., vol. 29, Cambridge Univ. Press, Cambridge, 1990.
- [16] S. Humphries, Finite Hurwitz braid actions on sequences of Euclidean reflections, J. Algebra 269 (2003) 556–588.
- [17] A. Hurwitz, Über Riemann’sche Flächen mit gegebenen Verzweigungspunkten, Math. Ann. 39 (1891) 1–61.
- [18] V. Kanev, Spectral curves, simple Lie algebras, and Prym–Tjurin varieties, in: Theta Functions, Bowdoin, 1987, Part 1, Brunswick, ME, 1987, in: Proc. Sympos. Pure Math., vol. 49, Amer. Math. Soc., Providence, RI, 1989, pp. 627–645.
- [19] V. Kanev, Spectral Curves and Prym–Tjurin Varieties. I, in: Abelian Varieties, Egloffstein, 1993, de Gruyter, Berlin, 1995, pp. 151–198.
- [20] V. Kanev, Hurwitz spaces of triple coverings of elliptic curves and moduli spaces of Abelian threefolds, Ann. Mat. Pura Appl. 183 (2004) 333–374.

- [21] V. Kanev, Hurwitz spaces of quadruple coverings of elliptic curves and the moduli space of Abelian threefolds $\mathcal{A}_3(1, 1, 4)$, *Math. Nachr.* 278 (2005) 154–172.
- [22] V. Kanev, Irreducibility of Hurwitz spaces, preprint No. 241, February, 2004, Dipartimento di Matematica e Applicazioni, Università di Palermo, math.AG/0509154.
- [23] R. Scognamillo, Prym–Tjurin [Tyurin] varieties and the Hitchin map, *Math. Ann.* 303 (1) (1995) 47–62.
- [24] F. Vetro, Irreducibility of Hurwitz spaces of coverings with monodromy groups Weyl groups of type $W(B_d)$, *Boll. Unione Mat. Ital.*, in press, preprint No. 279, April, 2005, Dipartimento di Matematica e Applicazioni, Università di Palermo.
- [25] H. Völklein, Groups as Galois Groups, *Cambridge Stud. Adv. Math.*, vol. 53, Cambridge Univ. Press, Cambridge, 1996.