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Structure of internal modules and a formula for the spherical vector of minimal representations

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Abstract

We develop a formula for the spherical vector of minimal representations of simply laced Chevalley groups.

Keywords: Chevalley groups; Minimal representations

1. Introduction

Let G be a simply connected Chevalley group, and $P = MN$ a maximal parabolic subgroup of G. Let $\mathfrak{n}$ be the Lie algebra of N. A choice of Chevalley basis defines a $\mathbb{Z}$-structure on $\mathfrak{n}$. The structure of M orbits over $\mathbb{Z}$ on irreducible subquotients of $\mathfrak{n}$ could be highly non-trivial, and very interesting as Bhargava [B] shows. In the first part of this paper we deal with this question in the case when G is simply laced and N is abelian. In a sense, this is the most banal case. Our results can be described as follows. Let M_{ss} be the "semi-simple" part of M. It is more natural to work with M_{ss}. Starting with the highest root β one can, in a canonical fashion, define a maximal sequence of orthogonal roots $\beta, \beta_1, \dots, \beta_{r-1}$ in the Lie algebra $\mathfrak{n}$. Let $e_\beta, \dots, e_{\beta_{r-1}}$ be the corresponding Chevalley basis elements in $\mathfrak{n}$. Then every $M^{\mathrm{ss}}(\mathbb{Z})$-orbit in $\mathfrak{n}$ contains an

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element

$$de_\beta + d_1 e_{\beta_1} + \cdots + d_{r-1} e_{\beta_{r-1}}$$

such that $d \mid d_1 \mid \ldots \mid d_{r-1}$. Moreover, all d_k can be picked to be non-negative except perhaps d_{r-1}. This result is a generalization of a result of Richardson, Röhrle and Steinberg [RRS], who considered the same question for groups over a field k. Then

$$\mathfrak{n} = \Omega_0 \cup \cdots \cup \Omega_r$$

where $\Omega_0 = \{0\}$ and Ω_j is the $M_{\mathrm{ss}}(k)$-orbit of $e_\beta + e_{\beta_1} + \cdots + e_{\beta_{j-1}}$ except, perhaps, Ω_r which could be a union of orbits parameterized by classes of squares in $k^\times$. Also, the case when $\mathfrak{n}$ is a 27-dimensional representation of $E_6(\mathbb{Z})$ was recently obtained by Krutelevich [K] in his Yale PhD thesis.

Our next result is an application to minimal representations of p-adic groups. Let G be a simple split group of *adjoint* type. Let $P = MN$ be a maximal parabolic subgroup with abelian nil radical. Let Ω_1 be the set of rank $= 1$ elements in the opposite nil-radical $\bar{N}$. The minimal representation of G can be realized as a space of functions f on Ω_1 (see [MS]) such that the action of P is given by

$$\begin{cases} (\pi(n)f)(y) = f(y)\psi(-\langle n, y\rangle) \quad \text{and} \\ (\pi(m)f)(y) = \chi^{s_0}(m)\Delta^{-1/2}(m)f(m^{-1}ym) \end{cases}$$

where ψ is an additive character of $\mathbb{Q}_p$ of conductor 0, $\langle n, y\rangle$ the natural pairing between N and $\bar{N}$, and $\chi^{s_0}(m)$ an unramified character of M, described in Section 3. The main disadvantage of this model is that we do not have any explicit formula for the action of the maximal compact subgroup $K = G(\mathbb{Z}_p)$. In particular, it is not clear a priori how to determine the spherical vector of the minimal representation. We accomplish this as follows. First of all, under the action of $M(\mathbb{Z}_p)$ the orbit Ω_1 decomposes as a union of orbits each containing $p^m e_{-\beta}$ for some integer m. Thus a spherical vector f, since it is fixed by $M(\mathbb{Z}_p)$, is determined by its value on $p^m e_{-\beta}$ for all integers m. Furthermore, since f is fixed by $N(\mathbb{Z}_p)$ as well, it must vanish on these elements if $m < 0$. To determine f exactly we shall use the fact that it is an eigenvector for the Hecke algebra. More precisely, we have $T_i * f = c_i \cdot f$ where T_i is a Hecke operator corresponding to a miniscule coweight ω_i. Such a coweight exists since we assume that G has a maximal parabolic subgroup with abelian nilpotent radical. The support of the Hecke operator is $K\omega_i K$. The Cartan decomposition implies that $K\omega_i K$ can be written as a union $K\omega_i K = \bigcup_j p_j K$ for some p_j in P. Then

$$T_i * f = \sum_j \pi(p_j) f.$$

Thus the action of T_i can be explicitly calculated since we know how P acts! This gives us a recursive relation

$$c_i \cdot f\left(p^n e_{-\beta}\right) = a_1 f\left(p^{n+1} e_{-\beta}\right) + a_0 f\left(p^n e_{-\beta}\right) + a_{-1} f\left(p^{n-1} e_{-\beta}\right)$$

from which it is not too difficult to determine f completely. In fact, the answer is a geometric series

$$f\left(p^n e_{-\beta}\right) = 1 + p^d + \cdots + p^{nd}$$

where d depends on the pair (G, M). In particular, this formula is a generalization of the well-known formula for GL_2. Indeed, if f is a spherical vector of the representation (parabolically) induced from two unramified characters χ_1 and χ_2, then

$$f\left(p^n e_{-\beta}\right) = \chi_1(p)^n + \chi_1(p)^{n-1}\chi_2(p) + \cdots + \chi_1(p)\chi_2(p)^{n-1} + \chi_2(p)^n.$$

The question of spherical vector was addressed in several papers. For p-adic groups, but working with a different model of the minimal representation, a formula for the spherical vector was found by Kazhdan and Polishchuk in [KP]. For real groups, in a situation similar to ours, the spherical vector was determined in a beautiful paper of Dvorsky and Sahi [DS]. Their result is a bit more restricted, for they assume that $\bar{N}$ is conjugated to N, which is not always the case.

2. Maximal parabolic subalgebras

Let $\mathfrak{g}$ be a simple split Lie algebra over $\mathbb{Z}$ and $\mathfrak{t} \subseteq \mathfrak{g}$ a maximal split Cartan subalgebra. Let Φ be the corresponding root system. We assume that Φ is a simply laced root system, meaning that all roots are of equal length. In particular, the type of Φ is A, D or E. Fix $\Delta = \{\alpha_1, \ldots, \alpha_l\}$, a set of simple roots. Every root can be written as a sum $\alpha = \sum_{i=0}^{l} m_i(\alpha)\alpha_i$ for some integers $m_i(\alpha)$. To every simple root α_i we can attach a subalgebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ such that

$$\begin{cases} \mathfrak{m} = \mathfrak{t} \oplus (\bigoplus_{m_i(\alpha)=0} \mathfrak{g}_\alpha), \\ \mathfrak{n} = \bigoplus_{m_i(\alpha)>0} \mathfrak{g}_\alpha. \end{cases}$$

Note that $\mathfrak{m}^{\mathrm{ss}} = [\mathfrak{m}, \mathfrak{m}]$ is a semi-simple Lie algebra which corresponds to the Dynkin diagram of $\Delta \setminus \{\alpha_i\}$. Let β be the highest root, and $b = n_i(\beta)$. For every j between 1 and b, define

$$\mathfrak{n}_j = \bigoplus_{m_i(\alpha)=j} \mathfrak{g}_\alpha.$$

Then $[\mathfrak{n}_j, \mathfrak{n}_k] \subseteq \mathfrak{n}_{j+k}$. In particular, if $b = 1$ then $\mathfrak{n}$ is commutative. Here is the list of all possible pairs $(\mathfrak{g}, \mathfrak{m})$ with $\mathfrak{n}$ commutative. (The simple root defining $\mathfrak{m}$ will be henceforth denoted by τ.)

$\mathfrak{g}$	A_{n-1}	D_n	D_{n+1}	E_6	E_7
$\mathfrak{m}^{\mathrm{ss}}$	$A_{k-1} \times A_{n-k-1}$	A_{n-1}	D_n	D_5	E_6
$\dim(\mathfrak{n})$	$k(n-k)$	$n(n-1)/2$	$2n$	16	27

Explanation: in the first case, $\mathfrak{n}$ is equal to the set of $k \times (n-k)$ matrices. In the second case it is equal to the set of all skew-symmetric $n \times n$ matrices, and in the third case $\mathfrak{n}$ is the standard representation of $\mathfrak{so}(2n)$. In the fourth case $\mathfrak{n}$ is a 16-dimensional spin representation and, in the fifth and last case, it a 27-dimensional representation of E_6.

We would like to determine $M^{\mathrm{ss}}(\mathbb{Z})$-orbits on $\mathfrak{n}$. Consider the case when $\mathfrak{n}$ is the set of $n \times m$ matrices. As is well known, using row-column operations, every matrix A can be transformed (reduced) into a matrix with integers $d_1 \mid d_2 \mid \ldots$ on the diagonal. The column operations correspond to multiplying A by certain *elementary* matrices. For example, if $m = 2$, then multiplying A from the right by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

corresponds, respectively, to:

(i) Adding the first column of A to the second.
(ii) Permuting the two columns of A.
(iii) Changing signs in the first column of A.

Similarly, row operations correspond to multiplying A by the elementary matrices from the left. An inconvenience here is that the last two matrices are not in $SL_2(\mathbb{Z})$ since they have determinant -1. In order to remedy this, we shall replace them by the following matrices of determinant 1:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Multiplying A by these three matrices corresponds to so-called *strict* column operations:

(i) Adding the first column of A to the second.
(ii) Permuting two columns of A, and changing the signs in one.
(iii) Changing the signs in both columns of A.

Since elementary matrices (of determinant one) generate $SL_n(\mathbb{Z})$, the *strict* row column reduction can be formulated as the following:

Every $SL_n(\mathbb{Z}) \times SL_m(\mathbb{Z})$-orbit in the set of $n \times m$ matrices contains a diagonal matrix $d_1 \mid d_2 \mid \ldots$ where all entries, save perhaps one, are non-negative.

The proof of this result is inductive in nature. The first number d_1 is the GCD of all matrix entries. Using row-column operations we can arrange to have d_1 on the left upper corner, with 0 in all other positions in the first row and column. In this way we reduce to $(n-1) \times (m-1)$.

We claim that this inductive procedure can be done in general. To explain, we need another parabolic subgroup $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{h}$, so-called Heisenberg parabolic subgroup. Here $\mathfrak{l}^{ss} = [\mathfrak{l}, \mathfrak{l}]$ corresponds to the subset of Δ given by $\{\alpha_i \mid \langle \beta, \alpha_i \rangle = 0\}$. The possible cases are following.

$\mathfrak{g}$	A_{n+1}	D_{n+1}	E_6	E_7
$\mathfrak{l}^{ss}$	A_{n-1}	$A_1 \times D_{n-1}$	A_5	D_6

2.1. Fourier–Jacobi towers

(As described in the work of Weissman [W].) Fix a pair (G, M). Let $\mathfrak{g}_1$ be the unique summand of $\mathfrak{l}^{ss}$ which is not contained in $\mathfrak{m}$. Put

$$\begin{cases} \mathfrak{m}_1 = \mathfrak{m} \cap \mathfrak{g}_1, \\ \mathfrak{n}_1 = \mathfrak{n} \cap \mathfrak{g}_1. \end{cases}$$

Thus, starting from a pair $(\mathfrak{g}, \mathfrak{m})$ we have constructed another pair $(\mathfrak{g}_1, \mathfrak{m}_1)$. Note, as a simple observation, that this process can be continued as long as the pair is not equal to (A_n, A_{n-1}), which we will call a terminal pair. The length of the tower

$$\begin{gathered}(\mathfrak{g}, \mathfrak{m}),\\ (\mathfrak{g}_1, \mathfrak{m}_1),\\ \vdots\end{gathered}$$

finishing with a terminal pair, is the rank of $\mathfrak{n}$. In particular, the rank of $\mathfrak{n}_1$ is one less then the rank of $\mathfrak{n}$.

Some examples (of rank 3):

$(\mathfrak{g}, \mathfrak{m})$	(E_7, E_6)	(D_6, A_5)	$(A_5, A_2 \times A_2)$
$(\mathfrak{g}_1, \mathfrak{m}_1)$	(D_6, D_5)	(D_4, A_3)	$(A_3, A_1 \times A_1)$
$(\mathfrak{g}_2, \mathfrak{m}_2)$	$(A_1, -)$	$(A_1, -)$	$(A_1, -)$

In the last tower, the corresponding sequence $\mathfrak{n}$, $\mathfrak{n}_1$ and $\mathfrak{n}_2$ can be identified with 3×3, 2×2 and 1×1 matrices, respectively.

Theorem 2.1. *Fix a pair $(\mathfrak{g}, \mathfrak{m})$ such that the rank of $\mathfrak{n}$ is r. Let β, $\beta_1, \dots, \beta_{r-1}$ be the highest roots for $\mathfrak{g}$, $\mathfrak{g}_1, \dots, \mathfrak{g}_{r-1}$, respectively. Then every $M^{\mathrm{ss}}(\mathbb{Z})$-orbit in $\mathfrak{n}$ contains an element*

$$de_\beta + d_1 e_{\beta_1} + \cdots + d_{r-1} e_{\beta_{r-1}}$$

such that $d \mid d_1 \mid \dots \mid d_{r-1}$. Moreover, all d_k can be picked to be non-negative except perhaps d_{r-1} which can happen only if the terminal pair is $(A_1, -)$.

Proof. The proof is the induction on r. If $r = 1$, then the pair is terminal and we have two cases. If the pair is $(A_1, -)$ then M^{ss} is trivial and orbits are parameterized by integers. If the pair is (A_n, A_{n-1}) then $M^{\mathrm{ss}} = SL_n(\mathbb{Z})$, and $\mathfrak{n} = \mathbb{Z}^n$. Here orbits are parameterized by non-negative integers.

Let Φ_M be the roots of $\mathfrak{m}$ and $\Sigma \subseteq \Phi$ be the set of all roots in $\mathfrak{n}$. Then any element of $\mathfrak{n}$ can be written as

$$n = \sum_{\alpha \in \Sigma} t_\alpha e_\alpha$$

for some integers t_α. If γ is in Φ_M then the adjoint action of the one-parameter group $e_\gamma(u)$ on e_α is given by

$$e_\gamma(t)(e_\alpha) = e_\alpha + t[e_\gamma, e_\alpha].$$

Indeed, $[e_\gamma[e_\gamma, e_\alpha]] = 0$ since $\gamma \neq -\alpha$, so the exponential series defining the action of $e_\gamma(u)$ reduces down to the first two terms.

Now assume that $r > 1$. Let n be in $\mathfrak{n}$. If $n = 0$, then there is nothing to prove. Otherwise, let $\Sigma_1 \subseteq \Sigma$ the set of all roots in $\mathfrak{n}_1$. Then

$$\Sigma = \{\beta\} \cup \Sigma_\beta \cup \Sigma_1$$

where Σ_β is the set of all roots α in Σ such that $\langle \alpha, \beta \rangle = 1$. In order to use induction, we have to show that n contains in its $M_{\mathrm{ss}}(\mathbb{Z})$-orbit an element such that

(i) $t_\beta > 0$ and $t_\alpha = 0$ for all α in Σ_β.
(ii) t_β divides t_α for all α in Σ_1.

We deal first with (i). Recall that the Weyl group W_M of M acts transitively on the set of roots in Σ. After conjugating n by an element in W_M, if necessary, we can assume that

$$0 < |t_\beta| \leqslant |t_\alpha|$$

for all α in Σ such that $t_\alpha \neq 0$. If $t_\alpha \neq 0$ for a root α in Σ_β, then we can write $t_\alpha = q t_\beta + r$ where $|r| < |t_\beta|$. Notice that $\gamma = \alpha - \beta$ is a root. Furthermore, since $n_\tau(\alpha - \beta) = 0$ it is a root in Φ_M. (Recall that τ is the simple root defining $\mathfrak{m}$.) It follows that

$$e_\gamma(q)(t_\beta e_\beta + \cdots + t_\alpha e_\alpha + \cdots) = t_\beta + \cdots + r e_\alpha + \cdots.$$

(This formula is correct if $[e_\gamma, e_\beta] = -e_\alpha$. If $[e_\gamma, e_\beta] = e_\alpha$ then q has to be replaced by $-q$.) In any case, if $t_\alpha \neq 0$ for some α in Σ_β then we can decrease the smallest non-zero coordinate of n. Proceeding in this fashion we can accomplish (i) in finitely many steps.

Next, we deal with (ii). Let α be in Σ_1 such that t_β does not divide t_α. After conjugating by an element of W_{M_1}, if necessary, we can assume that $\alpha = \beta_1$. Let δ be a simple root such that $\langle \beta, \delta \rangle = 1$. Then $\alpha = \beta_1 + \delta$ is a root in Σ_β and

$$e_\delta(1)(t_\beta e_\beta + t_{\beta_1} e_{\beta_1} + \cdots) = t_\beta + \cdots \pm t_{\beta_1} e_\alpha + \cdots.$$

Thus we are back in the situation of the proof of (i) and, in the same fashion, we can decrease the smallest coordinate of n. This process has to stop in finitely many steps. This proves part (ii) and, therefore, the theorem. □

Corollary 2.2. *(See [RRS].) Let k be any field. If $(A_1, -)$ is not the terminal pair, then $\mathfrak{n} = \Omega_0 \cup \cdots \cup \Omega_r$ where Ω_i is the $M^{\mathrm{ss}}(k)$-orbit of $e_\beta + \cdots + e_{\beta_{i-1}}$. If $(A_1, -)$ is the terminal pair then Ω_r is a union of $M(k)$-orbits parameterized by classes of squares in $k^\times$. In any case, elements in Ω_i are said to have rank i.*

3. Degenerate principal series

In this section we shall assume that $G = G_{ad}$ is of adjoint type. We give an explicit model of the minimal representation of G. The discussion here is based on [S] and [W]. Since G is assumed to be of adjoint type, it acts faithfully on the Lie algebra $\mathfrak{g}$ and the torus T of G is isomorphic to $\Lambda_c \otimes k^\times$ where Λ_c is the lattice of integral coweights. It is the lattice dual to the root lattice with respect to the usual form $\langle\cdot,\cdot\rangle$. Let $\lambda(t)$ denote the element $\lambda\otimes t$ in T. It acts on e_α by the formula

$$\lambda(t)e_\alpha\lambda(t)^{-1}=t^{\langle\lambda,\alpha\rangle}e_\alpha.$$

Let τ be the simple root defining P, and ρ and $\bar\rho$ the half-sum of all roots in N and $\bar N$, respectively. Let $\Delta: M\to\mathbb{R}^+$ be the modular character with respect to $\bar N$, which means that

$$\int\limits_{\bar N} f\bigl(mxm^{-1}\bigr)\,dx=\Delta(m)\int\limits_{\bar N} f(x)\,dx.$$

Let ρ_N and $\rho_{\bar N}$ be the half-sum of all roots in $\mathfrak{n}$ and $\bar{\mathfrak{n}}$, respectively. Then the composition of Δ with the embedding of T into M is given by

$$\Delta^{1/2}\bigl(\lambda(p)\bigr)=|p|^{\langle\lambda,\rho_{\bar N}\rangle}.$$

Furthermore, let $\chi: M\to\mathbb{R}^+$ be a character such that $\chi^{2\langle\tau,\rho_N\rangle}=\Delta$. Define the principal series $I(s)=\operatorname{Ind}_{\bar P}^G(\chi^s)$, the space of all locally constant functions on G such that

$$f(\bar n m g)=\chi^s(m)\Delta^{1/2}(m)f(g).$$

There is a non-degenerate G-invariant hermitian pairing $(\cdot,\cdot)_s: I(-s)\times I(s)\to\mathbb{C}$ defined by

$$(f_{-s},f_s)_s=\int\limits_{\bar P\backslash G} f_{-s}(x)\bar f_s(x)\,dx=\int\limits_N f_{-s}(x)\bar f_s(x)\,dx.$$

Here the last equality follows since $\bar PN$ is an open subset of G. Inside $I(s)$ there is a P-submodule of all functions in $I(s)$ supported in the open subset $\bar PN$. This can be identified with $S(N)$, the space of locally constant, compactly supported functions on N. The action of the maximal parabolic $P=MN$ on $S(N)$ is given by

$$\begin{cases}\pi(n)f(x)=f(x+n),\\ \pi(m)f(x)=\chi^s(m)\Delta^{1/2}(m)f(m^{-1}xm).\end{cases}$$

Next, we shall analyze the structure of $S(N)$, as a P-module, using the Fourier transform. To that end, notice that we have a natural pairing $\langle\cdot,\cdot\rangle$ between N and $\bar N$ induced by the Killing form. Thus $\bar N$ can be identified with the dual of N. The Fourier transform is an isomorphism of (vector spaces) $S(N)$ and $S(\bar N)$ defined by

$$\hat f(y)=\int\limits_N f(x)\psi\bigl(\langle x,y\rangle\bigr)\,dx.$$

Using the Fourier transform we shall transfer the action of P from $S(N)$ to $S(\bar N)$. Let $f\in S(\bar N)$, and $m\in M$. Then the Fourier transform of $\pi(m)f$ is

$$\bigl(\widehat{\pi(m)f}\bigr)(y)=\chi^s(m)\Delta^{1/2}(m)\int\limits_N f\bigl(m^{-1}xm\bigr)\psi\bigl(\langle x,y\rangle\bigr)\,dx.$$

We introduce a new variable z by $z = m^{-1}xm$. Then $dx = \Delta(m)^{-1}\,dz$, and the formula simplifies to

$$\widehat{\big(\pi(m)f\big)}(y) = \chi^s(m)\Delta^{-1/2}(m)\hat{f}\big(m^{-1}ym\big).$$

This gives a formula for the action of M on $S(\bar{N})$. Similarly—but much easier—we can derive the action of N on $S(\bar{N})$. The two formulas are summarized below:

$$\begin{cases} (\pi(n)f)(y) = f(y)\psi(-\langle n, y\rangle) \quad \text{and} \\ (\pi(m)f)(y) = \chi^s(m)\Delta^{-1/2}(m)f(m^{-1}ym), \end{cases}$$

where $m \in M$, $n \in N$ and $f \in S(\bar{N})$.

Let Ω_i be the set of elements of rank i in $\bar{N}$. Let S_i be the subset of $S(N)$ of all functions f such that the Fourier transform $\hat{f}$ vanishes on $\bigcup_{j<i}\Omega_j$. Then S_i is a P-submodule, and the quotient S_i/S_{i+1} is isomorphic to $S(\Omega_i)$—the space of locally constant and compactly supported functions on Ω_i—with the action given by the previous formulas. Every subquotient is irreducible by Mackey's lemma.

Let us look now at the special case $s = s_0$ when the minimal $V_{\min}$ representation is the unique submodule of $I(-s_0)$. Notice that the pairing $(\cdot,\cdot)_{s_0}$ restricted to $V_{\min} \times S(N)$ is left non-degenerate. Indeed, any $f \neq 0$ in $V_{\min}$ will give you a non-trivial function when restricted to N (since N is dense in $\bar{P}\backslash G$) and, therefore, a non-trivial distribution of $S(N)$. In fact, we have a bit more. The pairing is left non-degenerate even when restricted to $V_{\min} \times S_1$. To see this recall that $V_{\min}$ is unitarizable. In particular, by a theorem of Howe and Moore, if an element v in $V_{\min}$ is fixed by N then $v = 0$. Since any vector in $V_{\min}$ perpendicular to S_1 is N-fixed it must be zero. This shows that the pairing, restricted to $V_{\min} \times S_1$, is left non-degenerate. Since the N-rank of $V_{\min}$ is one the pairing is trivial on $S_2 \subseteq S_1$. (This is basically a definition of the N-rank.) Thus the pairing descends to a non-degenerate pairing in both variables of $V_{\min}$ and $S_1/S_2 = S(\Omega_1)$, where the action of P on $S(\Omega_1)$ is given by

$$\begin{cases} (\pi(n)f)(y) = f(y)\psi(-\langle n, y\rangle) \quad \text{and} \\ (\pi(m)f)(y) = \chi^{s_0}(m)\Delta^{-1/2}(m)f(m^{-1}ym). \end{cases}$$

Here $m \in M$, $n \in N$ and $f \in S(\Omega_1)$. It follows that $V_{\min}$, as a P-module, embeds into the P-smooth dual of $S(\Omega_1)$. This dual can be described in the following way. While there is no M-invariant measure on Ω_1, there exists a (modular) character δ_1 of M and a measure dy on Ω_1 such that

$$\int_{\Omega_1} f\big(mym^{-1}\big)\,dy = \delta_1(m)\int_{\Omega_1} f(y)\,dy$$

for every locally constant and compactly supported function f on Ω_1. The P-smooth dual of $S(\Omega_1)$ is isomorphic to the space of locally constant, but not necessarily compactly supported, functions on Ω_1 with the action of P given by

$$\begin{cases} (\pi(n)f)(y) = f(y)\psi(-\langle n, y\rangle) \quad \text{and} \\ (\pi(m)f)(y) = \chi_1(m)f(m^{-1}ym), \end{cases}$$

where the character χ_1 is defined by $\chi_1 \cdot (\chi^{s_0} \cdot \Delta^{-1/2}) = \delta_1$. It appears that we have an annoying issue of figuring out what δ_1 is. It turns out that is not necessary. To this end, note that $V_{\min}$ is a quotient of $I(s_0)$ and the pairing of $V_{\min}$ and $I(s_0)$ descends down to a pairing between $V_{\min}$ and $V_{\min}$. It follows that S_1/S_2 is a submodule of $V_{\min}$ (the second factor) which shows that $\chi_1 = \chi^{s_0} \cdot \Delta^{-1/2}$.

The possible cases for s_0 (see [W]) and $\langle \tau, \rho_N \rangle$ are following.

$\mathfrak{g}$	A_{n+1}	A_{2n+1}	D_{n+1}	D_{n+1}	E_6	E_7
$\mathfrak{m}^{ss}$	A_n	$A_n \times A_n$	A_n	D_n	D_5	E_6
s_0	0	n	$n-2$	1	3	5
$\langle \tau, \rho_N \rangle$	$n/2+1$	$n+1$	n	n	6	9

4. Eigenvalues of Hecke operators

Consider the root system of type A_n, D_n or E_n, and let ω_j be the fundamental coweights as in Bourbaki tables. Let $\hat{\omega}_b$ be the fundamental weight corresponding to the unique branching vertex of the Dynkin diagram for D_n and E_n. This is ω_4 for all three exceptional groups. For the root system of type A_n there is no branching point, but we define $\hat{\omega}_b$ to be the fundamental coweight of the middle vertex if n is odd, or the arithmetic mean of the two middle vertices if n is even. Let ρ be the half sum of all positive roots. The Satake parameter of the minimal representation is $\lambda_{\min}(p) \in \hat{G}$, the dual group of G, where

$$\lambda_{\min} = \rho - \hat{\omega}_b.$$

If ω_i is a miniscule fundamental coweight, then the eigenvalue of the Hecke operator $p^{-\langle \rho, \omega_i \rangle} T_i$ on the spherical vector of the minimal representation is

$$\mathrm{Tr}_{V(\omega_i)}\big(\lambda_{\min}(p)\big) = \sum_{\mu \sim \omega_i} p^{\langle \lambda_{\min}, \mu \rangle},$$

the trace of $\lambda_{\min}(p)$ on the representation $V(\omega_i)$ of $\hat{G}$ with the highest weight ω_i. Here the sum is taken over all weights μ of $V(\omega_i)$. (Weight spaces of the miniscule representation are one-dimensional and are Weyl group conjugate to ω_i.) We now give explicit formulas in the following cases:

Case A_{2n-1}, and $\omega_i = \omega_1$, the highest weight of the standard $2n$-dimensional representation. Then the eigenvalue of the Hecke operator $p^{-\langle \rho, \omega_1 \rangle} T_1$ is

$$p^{n-1} + p^{n-2} + \cdots + p + 2 + p^{-1} + \cdots + p^{2-n} + p^{1-n}.$$

Case A_{2n}, and $\omega_i = \omega_1$, the highest weight of the standard $2n$-dimensional representation. Then the eigenvalue of the Hecke operator $p^{-\langle \rho, \omega_1 \rangle} T_1$ is

$$p^{n-1/2} + p^{n-3/2} + \cdots + p^{1/2} + 1 + p^{-1/2} + \cdots + p^{3/2-n} + p^{1/2-n}.$$

Case D_{n+1}, and $\omega_i = \omega_1$, the highest weight of the standard $(2n+2)$-dimensional representation. Then the eigenvalue of the Hecke operator $p^{-\langle \rho, \omega_1 \rangle} T_1$ is

$$p^{n-1} + \cdots + p^2 + 2p + 2 + 2p^{-1} + p^{-2} + \cdots + p^{1-n}.$$

Case E_6, and $\omega_i = \omega_1$, the highest weight of the standard 27-dimensional representation of E_6. In the terminology of Bourbaki, the Satake parameter is

$$\lambda_{\min} = (0, 1, 1, 2, 3, -3, -3, 3).$$

It will be convenient to realize $V(\omega_1)$ as an internal module in E_7. More precisely, consider the root system of type E_7 as in Bourbaki tables. If we remove the last simple root α_7 then we get a root system E_6. As usual, write every positive root of E_7 as $\alpha = \sum m_i(\alpha)\alpha_i$. The subspace

$$\bigoplus_{m_7(\alpha)=1} \mathfrak{g}_\alpha$$

is the 27-dimensional representations of E_6 with the highest weight ω_1, i.e. the first fundamental weight. Thus to tabulate the weights of this representation, we have to write down all roots α of E_7 such that $m_7(\alpha) = 1$ which is the same as $\langle \alpha, \omega_7 \rangle = 1$, where $\omega_7 = e_6 + \frac{1}{2}(e_8 - e_7)$. These are $\pm e_i + e_6$ $(1 \leqslant i \leqslant 5)$, $e_8 - e_7$ (total of 11 roots here) and

$$\frac{1}{2}\left(e_8 - e_7 + e_6 + \sum_{i=1}^{5}(-1)^{\nu(i)} e_i\right)$$

where $\sum \nu(i)$ is odd. This, second, group has 16 roots.

(Warning: ω_7 is the fundamental weight for E_7. While simple roots for E_6 are also simple roots for E_7 this is not true for fundamental weights. First 6 fundamental weights for E_7 are not the fundamental weights for E_6.)

The eigenvalue of the Hecke operator $p^{-\langle \rho, \omega_1 \rangle} T_1$ is

$$\left(\sum_{m_7(\alpha)=1} p^{\langle \lambda_{\min}, \alpha \rangle}\right) = p^6 + p^5 + 2p^4 + 2p^3 + 3p^2 + 3p + 3$$
$$+ 3p^{-1} + 3p^{-2} + 2p^{-3} + 2p^{-4} + p^{-5} + p^{-6}.$$

Case E_7, and $\omega_i = \omega_7$, the highest weight of the 56-dimensional representation of E_7. Here the Satake parameter is

$$\lambda_{\min} = (0, 1, 1, 2, 3, 4, -13/2, 13/2).$$

Again, the representation V_{ω_7} can be written down as an internal module in E_8. Let α_8 be the root for E_8 such that other simple roots belong to E_7. Then the 56-dimensional representation is equal to

$$\bigoplus_{m_8(\alpha)=1} \mathfrak{g}_\alpha.$$

So again we have to tabulate all roots for E_8 such that $\langle \alpha, \omega_8 \rangle = 1$. Since $\omega_8 = e_7 + e_8$, these are $\pm e_i + e_7$ $(1 \leqslant i \leqslant 6)$, $\pm e_i + e_8$ $(1 \leqslant i \leqslant 6)$ and

$$\frac{1}{2}\left(e_8 + e_7 + \sum_{i=1}^{6}(-1)^{\nu(i)} e_i\right)$$

where $\sum \nu(i)$ is even. There are 32 of this last type. Now it is not too difficult to see that the eigenvalue of the Hecke operator $p^{-\langle\rho,\omega_7\rangle}T_7$ for E_7 is

$$\left(\sum_{m_8(\alpha)=1} p^{\langle\lambda_{\min},\alpha\rangle}\right) = p^{\frac{21}{2}} + p^{\frac{19}{2}} + p^{\frac{17}{2}} + 2p^{\frac{15}{2}} + 2p^{\frac{13}{2}} + 3p^{\frac{11}{2}} + 3p^{\frac{9}{2}} + 3p^{\frac{7}{2}} + 4p^{\frac{5}{2}}$$
$$+ 4p^{\frac{3}{2}} + 4p^{\frac{1}{2}} + 4p^{-\frac{1}{2}} + 4p^{-\frac{3}{2}} + 4p^{-\frac{5}{5}} + 3p^{-\frac{7}{2}} + 3p^{-\frac{9}{2}}$$
$$+ 3p^{-\frac{11}{2}} + 2p^{-\frac{13}{2}} + 2p^{-\frac{15}{2}} + p^{-\frac{17}{2}} + p^{-\frac{19}{2}} + p^{-\frac{21}{2}}.$$

5. Satake transform

Let U be the maximal nilpotent subgroup corresponding to our choice of simple roots. Let ω_i be a miniscule fundamental coweight. The purpose of this section is to decompose the double coset $K\omega_i(p)K$ as a union of single cosets $u\mu(p)K$, where $u \in U$. This will be accomplished by means of the Satake transform.

The modular character δ is given by $\delta(\lambda(p))^{1/2} = p^{\langle\rho,\lambda\rangle}$. The Satake transform $S: H_G \to H_T$ is given by

$$S(f)(t) = \delta(t)^{-1/2}\int_N f(tu)\,du.$$

It is known that $S(T_i) = p^{\langle\rho,\omega_i\rangle}V(\omega_i)$ where $V(\omega_i)$ is the fundamental representation of $\hat{G} = G_{sc}$ with the highest weight ω_i. Here we use the identification of H_T with $\mathbb{C}[\Lambda_c]$, the group algebra of the coweight lattice Λ_c. Under this identification $V(\omega_i)$ is a sum of delta functions for all weights μ of $V(\omega_i)$. It follows that $S(T_i)(\mu(p)) = 0$ unless μ is a weight of $V(\omega_i)$ in which case it is equal to $p^{\langle\rho,\omega_i\rangle}$. Proposition 13.1 in [GGS] implies that, for every weight μ of $V(\omega_i)$, the number of single cosets of type $u\mu(p)K$ contained in $K\omega_i(p)K$ is equal to $p^{\langle\rho,\mu+\omega_i\rangle}$.

Proposition 5.1. *Let ω_i be a miniscule fundamental coweight, and μ a Weyl group conjugate of ω_i. If $u\mu(p)K$ is contained in $K\omega_i(p)K$ then it is equal to*

$$\left(\prod_{\alpha>0,\,\langle\alpha,\mu\rangle=1} e_\alpha(t_\alpha)\right)\mu(p)K$$

for some (unique) $t_\alpha \in \mathbb{Z}_p/p\mathbb{Z}_p$.

Proof. Notice that $e_\alpha(t_\alpha)$ commute since the scalar product of μ and any root can be only -1, 0 or 1. In particular, the product in the proposition is well defined. Furthermore, since $e_\alpha(t_\alpha)$ with $t_\alpha \in \mathbb{Z}_p$ are contained in K the single cosets (as defined in the statement) are contained in our double coset. We shall first show uniqueness. If

$$\prod_{\alpha>0,\,\langle\alpha,\mu\rangle=1} e_\alpha(t_\alpha)\mu(p)K = \prod_{\alpha>0,\,\langle\alpha,\mu\rangle=1} e_\alpha\big(t'_\alpha\big)\mu(p)K$$

then

$$\prod_{\alpha>0,\,\langle\alpha,\mu\rangle=1} e_\alpha\big((t_\alpha - t'_\alpha)/p\big) \in K.$$

This is possible if and only if $t_\alpha \equiv t'_\alpha \pmod K$, as claimed. Finally, since we know that the number of single cosets of the form $u\mu(p)K$ is equal to $p^{\langle\rho,\omega_i+\mu\rangle}$, in order to prove the proposition it remains to verify the following lemma.

Lemma 5.2. *Let μ be a Weyl group conjugate of the miniscule coweight ω_i. Then the number of positive roots α such that $\langle\alpha,\mu\rangle = 1$ is equal to $\langle\rho,\omega_i+\mu\rangle$.*

Proof. Let w be a Weyl group element such that $\mu = w(\omega_i)$. Let α be a positive root such that $\langle\alpha,\mu\rangle = 1$. Then

$$1 = \langle\alpha,\mu\rangle = \langle w^{-1}(\alpha), \omega_i\rangle.$$

This implies that $w^{-1}(\alpha) = \beta$ is positive, so we are counting the number of positive roots β such that $w(\beta)$ is positive and $\langle\beta,\omega_i\rangle = 1$. Since $\langle\beta,\omega_i\rangle = 1$ or 0 for every positive root, the number of positive roots α such that $\langle\alpha,\mu\rangle = 1$ is equal to

$$\sum_{\beta>0,\,w(\beta)>0} \langle\beta,\omega_i\rangle.$$

Since (this is well known) $\sum_{\beta>0,\,w(\beta)>0}\beta = \rho + w^{-1}(\rho)$ the lemma follows. □

6. Spherical vector

We would like to determine the spherical vector of the minimal representation. Under the action of $M(\mathbb{Z}_p)$ the orbit Ω_1 decomposes as a union of orbits each containing $p^m e_{-\tau}$ for some integer m. Thus a spherical vector f, since it is fixed by $M(\mathbb{Z}_p)$, is determined by its value on $p^m e_{-\tau}$ for all integers m. In order to simplify notation, let us write

$$\boxed{f(m) = f\big(p^m e_{-\tau}\big)}.$$

Next, since f is fixed by $N(\mathbb{Z}_p)$ as well, $f(m) = 0$ if $m < 0$. To determine f exactly we shall use the fact that it is an eigenvector for the Hecke operator $T_i = \mathrm{Char}(K\omega_i K)$ where ω_i is a miniscule fundamental coweight. As we know from the previous section, the double coset $K\omega_i K$ can be written as a union of single cosets $u\mu(p)K$ where μ is a Weyl group conjugate of ω_i and u is in $U \cap K$. Also, for a fixed μ there are $p^{\langle\rho,\mu+\omega_i\rangle}$ different single cosets. It follows that $e_{-\tau}$ is a highest weight vector for $M \cap U$. Thus, it follows that

$$(T_i * f)(m) = \sum_\mu p^{\langle\rho,\mu+\omega_i\rangle}\chi^{s_0}(\mu)\Delta^{-1/2}(\mu) f\big(m + \langle\mu,\tau\rangle\big).$$

Since $\langle \mu, \tau \rangle$ is equal to -1, 0 or 1, the formula gives a recursion relation as indicated in the introduction. It remains to calculate this formula in every case. But first we state the final result.

Theorem 6.1. *Let Ω_1 be the set of rank one elements in $\bar{N}$. Recall that the Chevalley basis gives a natural coordinate system of $\bar{N}$. If $x \in \Omega_1$, let p^m be the greatest common divisor of all coordinates of x. Then $f(x) = 0$ unless $m \geqslant 0$. If $m \geqslant 0$ then, after normalizing $f(1) = 1$,*

$$f(x) = 1 + p^d + \cdots + p^{md}$$

where d is given by the following table:

$\mathfrak{g}$	A_{n+1}	A_{2n+1}	D_{n+1}	D_{n+1}	E_6	E_7
$\mathfrak{m}^{ss}$	A_n	$A_n \times A_n$	A_n	D_n	D_5	E_6
d	$n/2$	0	1	$n-2$	2	3

Proof. We calculate the recursive relation on a case by case basis using the data from the following tables. The first table includes the half sum of all the positive roots and the simple root τ not in M. The second table gives the characterization of $\chi^{s_0}(\cdot)\Delta^{-1/2}(\cdot)$ in terms of ρ_N, the half sum of the roots in M.

(G, M)	ρ	τ
$(A_{2n-1}, A_{n-1} \times A_{n-1})$	$(n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2} - n)$	$(0, \ldots, 0, 1, -1, 0, \ldots, 0)$
(D_{n+1}, D_n)	$(n, n-1, \ldots, 1, 0)$	$(1, -1, 0, \ldots, 0)$
(D_{n+1}, A_n)	$(n, n-1, \ldots, 1, 0)$	$(0, \ldots, 0, 1, 1)$
(E_6, D_5)	$(0, 1, 2, 3, 4, -4, -4, 4)$	$\frac{1}{2}(1, -1, -1, -1, -1, -1, -1, 1)$
(E_7, E_6)	$(0, 1, 2, 3, 4, 5, -\frac{17}{2}, \frac{17}{2})$	$(0, 0, 0, 0, -1, 1, 0, 0)$

	ρ_N	$\chi^{s_0}\Delta^{-1/2}$
$(A_{2n-1}, A_{n-1} \times A_{n-1})$	$(\frac{n}{2}, \ldots, \frac{n}{2}, -\frac{n}{2}, \ldots, -\frac{n}{2})$	$p^{-\frac{1}{n}\langle \cdot, \rho_N \rangle}$
(D_{n+1}, D_n)	$(n, 0, \ldots, 0)$	$p^{(\frac{1}{n}-1)\langle \cdot, \rho_N \rangle}$
(D_{n+1}, A_n)	$(\frac{n}{2}, \ldots, \frac{n}{2})$	$p^{-\frac{2}{n}\langle \cdot, \rho_N \rangle}$
(E_6, D_5)	$(0, 0, 0, 0, 0, -4, -4, 4)$	$p^{-\frac{1}{2}\langle \cdot, \rho_N \rangle}$
(E_7, E_6)	$(0, 0, 0, 0, 0, 9, -\frac{9}{2}, \frac{9}{2})$	$p^{-\frac{4}{9}\langle \cdot, \rho_N \rangle}$

We start with the case $G = D_{n+1}$ and $M = D_n$. The Weyl group orbit of the highest weight $\omega_1 = e_1$ consists of $\pm e_i$ for $1 \leqslant i \leqslant n+1$. The eigenvalue of T_1 is

$$p^n\left(p^{n-1} + \cdots + p^2 + 2p + 2 + 2p^{-1} + p^{-2} + \cdots + p^{1-n}\right).$$

Next, we shall work out $T_1 * f(m)$ using the action of single cosets. The total number of single cosets is

$$p^{2n} + p^{2n-1} + \cdots + p^{n+1} + 2p^n + p^{n-1} + \cdots + p + 1.$$

In order to calculate the coefficients a_1 and a_{-1} in the recursive relation we are interested in conjugates μ of ω_1 such that $\langle \tau, \mu \rangle = 1$ or -1. They are, followed by the number of cosets of the type $u\mu(p)K$, and the value $\chi^{s_0}(\mu)\Delta^{-1/2}(\mu)$:

μ	$\langle \tau, \mu \rangle$	$p^{\langle \rho, \mu+\omega_1 \rangle}$	$\chi^{s_0}\Delta^{-1/2}$
e_1	1	p^{2n}	p^{1-n}
e_2	-1	p^{2n-1}	1
$-e_1$	-1	1	p^{n-1}
$-e_2$	1	p	1

In particular, it is not difficult to check that the right-hand side of the recursion can be written as

$$\begin{aligned}&\left(p^{n+1} + p\right) f(m+1) + \left(p^{2n-2} + \cdots + p^{n+1} + 2p^n + p^{n-1} + \cdots + p^2\right) f(m)\\ &\quad + \left(p^{2n-1} + p^{n-1}\right) f(m-1).\end{aligned}$$

This gives plenty of reductions with the left-hand side of the recursion, which is the product of the eigenvalue of T_1 and $f(m)$, and the recursion can be rewritten as

$$\left(p^{2n-1} + p^{n+1} + p^{n-1} + p\right) f(m) = \left(p^{n+1} + p\right) f(m+1) + \left(p^{2n-1} + p^{n-1}\right) f(m-1),$$

which is equivalent to

$$p^{n-2}\left[f(m) - f(m-1)\right] = \left[f(m+1) - f(m)\right].$$

This, of course, implies that $f(m) = 1 + p^{n-2} + \cdots + p^{m(n-2)}$ or, in words, it is a geometric series in p^{n-2}.

We now address the case $G = A_{2n-1}$ and $M = A_{n-1} \times A_{n-1}$. The Weyl group of the miniscule weight $\omega_1 = e_1$ consists of the elements e_i ($1 \leqslant i \leqslant 2n$). As before, we need the eigenvalue of T_1, which is

$$p^{\frac{2n-1}{2}}\left(p^{n-1} + \cdots + p + 2 + p^{-1} + \cdots + p^{1-n}\right),$$

because this (times $f(m)$) gives the left-hand side of the recursion formula. Also,

$$\chi^{s_0}(e_i)\Delta^{-1/2}(e_i) = p^{-\frac{1}{n}\langle e_i, \rho_N \rangle} = \begin{cases} p^{-1/2}, & 1 \leqslant i \leqslant n,\\ p^{1/2}, & n < i \leqslant 2n.\end{cases}$$

Notice that only the elements e_n and e_{n+1} have non-zero dot product with τ (1 and -1, respectively), and $p^{\langle\rho,e_i+e_1\rangle} = p^{2n-i}$. Thus, the right-hand side of the equation is

$$p^{-\frac{1}{2}}\bigl[\bigl(p^{2n-1}+\cdots+p^{n+1}\bigr)f(m)+p^n f(m+1)\bigr]$$
$$+\,p^{\frac{1}{2}}\bigl[p^{n-1}f(m-1)+\bigl(p^{n-2}+\cdots+1\bigr)f(m)\bigr].$$

After combining both sides of the equation and simplifying, this becomes

$$f(m)-f(m-1)=f(m+1)-f(m).$$

Hence, $f(m)=m$.

The next case is $G=D_{n+1}$ and $M=D_n$. As is the case when $G=D_{n+1}$ and $M=A_n$, we consider the Weyl group orbit of $\omega_1=(1,0,\ldots,0)$. As noted above, this orbit consists of all elements $\pm e_i$ ($1\leqslant i\leqslant n+1$). First, we tabulate those elements μ such that $\langle\mu,\tau\rangle\neq 0$.

μ	$\langle\mu,\tau\rangle$	$p^{\langle\rho,\mu+\omega_1\rangle}$	$\chi^{s_0}\Delta^{-1/2}$
e_n	1	p^{n+1}	p^{-1}
e_{n+1}	1	p^n	p^{-1}
$-e_n$	-1	p^{n-1}	p
$-e_{n+1}$	-1	p^n	p

The left-hand side of the recursion is identical to the other case with $G=D_{n+1}$, but the right-hand side is

$$f(m+1)\bigl(p^n+p^{n-1}\bigr)+f(m-1)\bigl(p^{n+1}+p^n\bigr)$$
$$+\,f(m)\bigl(\bigl(p^{2n}+\cdots+p^{n+2}\bigr)p^{-1}+\bigl(p^{n-2}+\cdots+1\bigr)p\bigr).$$

After cancellation and simplification the recursion becomes

$$p\bigl[f(m)-f(m-1)\bigr]=\bigl[f(m+1)-f(m)\bigr].$$

Hence, $f(m)=1+p+\cdots+p^m$.

Next we consider $G=E_6$ and $M=D_5$. Recall that the eigenvalue for the Hecke operator T_1 is

$$p^8\bigl(p^6+p^5+2p^4+2p^3+3p^2+3p+3+3p^{-1}+3p^{-2}+2p^{-3}+2p^{-4}+p^{-5}+p^{-6}\bigr)$$
$$=p^{14}+p^{13}+2p^{12}+2p^{11}+3p^{10}+3p^9+3p^8+3p^7+3p^6+2p^5+2p^4+p^3+p^2.$$

As we have seen, there are 27 elements in the orbit of ω_1. We list below those which have the property that $\langle\mu,\tau\rangle \neq 0$ along with the number of cosets of type $u\mu(p)K$ and $\chi^{s_0}(\mu)\Delta^{-1/2}(\mu)$.

μ	$\langle\mu,\tau\rangle$	$p^{\langle\rho,\mu+\omega_1\rangle}$	$p^{-\frac{1}{2}\langle\mu,\rho_N\rangle}$
e_6-e_1	-1	p^4	p^2
e_6+e_2	-1	p^5	p^2
e_6+e_3	-1	p^6	p^2
e_6+e_4	-1	p^7	p^2
e_6+e_5	-1	p^8	p^2
$\frac{1}{2}(-e_1+e_2+e_3+e_4+e_5+e_6-e_7+e_8)$	-1	p^{15}	p^{-1}
$\frac{1}{2}(-e_1-e_2-e_3-e_4-e_5-e_6-e_7+e_8)$	1	p^5	p^{-1}
$\frac{1}{2}(e_1+e_2-e_3-e_4-e_5+e_6-e_7+e_8)$	1	p^6	p^{-1}
$\frac{1}{2}(e_1-e_2+e_3-e_4-e_5+e_6-e_7+e_8)$	1	p^7	p^{-1}
$\frac{1}{2}(e_1-e_2-e_3+e_4-e_5+e_6-e_7+e_8)$	1	p^8	p^{-1}
$\frac{1}{2}(e_1-e_2-e_3-e_4+e_5+e_6-e_7+e_8)$	1	p^9	p^{-1}
e_8-e_7	1	p^{16}	p^{-4}

From the table above we can read off the coefficients of $f(m+1)$ and $f(m-1)$ on the right-hand side. These are

$$f(m-1)\left[p^6+p^7+p^8+p^9+p^{10}+p^{14}\right]$$

and

$$f(m+1)\left[p^4+p^5+p^6+p^7+p^8+p^{12}\right].$$

Similarly, we can tabulate the values of $p^{\langle\rho,\mu+\omega_1\rangle}$ and $p^{-\frac{1}{2}\langle\mu,\rho_N\rangle}$ when $\langle\mu,\tau\rangle = 0$. This will show that the final term on the right-hand side of the equation is

$$f(m)\left[p^2+p^3+p^4+p^5+p^6+p^7+p^8+2p^9+2p^{10}+2p^{11}+p^{12}+p^{13}\right].$$

After subtracting this term from both sides and dividing by $p^4+p^5+p^6+p^7+p^8+p^{12}$ this becomes

$$f(m)\left[p^2+1\right] = f(m-1)p^2 + f(m+1).$$

This is obviously equivalent to

$$p^2\left[f(m)-f(m-1)\right] = \left[f(m+1)-f(m)\right],$$

which implies that $f(m) = 1+p^2+\cdots+p^{2m}$.

We now address the final case: $G = E_7$ and $M = E_6$. As we have already computed the eigenvalue for the Hecke operator $p^{-\langle\omega_7,\rho\rangle}T_7$ we see that the left-hand side of our equation is

$$f(m)\big[p^{24}+p^{23}+p^{22}+2p^{21}+2p^{20}+3p^{19}+3p^{18}+3p^{17}+4p^{16}+4p^{15}+4p^{14}$$
$$+4p^{13}+4p^{12}+4p^{11}+3p^{10}+3p^{9}+3p^{8}+2p^{7}+2p^{6}+p^{5}+p^{4}+p^{3}\big].$$

As in the case of $G=E_6$, one must tabulate each of the 56 elements μ in the orbit of ω_7 along with number of cosets of type $u\mu(p)K$ (which is $p^{\langle\rho,\mu+\omega_7\rangle}$), and the value $\chi_{s_0}(\mu)\Delta^{-1/2}(\mu)$ (which is $p^{-\frac{4}{9}\langle\mu,\rho_N\rangle}$). As before, we do this for those elements μ such that $\langle\mu,\tau\rangle\neq 0$.

μ	$\langle\mu,\tau\rangle$	$p^{\langle\rho,\mu+\omega_7\rangle}$	$p^{-\frac{1}{2}\langle\mu,\rho_N\rangle}$
e_6-e_7	1	p^{27}	p^{-6}
$-e_5-e_7$	1	p^{18}	p^{-2}
$\frac{1}{2}(-e_1+e_2+e_3+e_4-e_5+e_6-e_7-e_8)$	1	p^{17}	p^{-2}
$\frac{1}{2}(e_1-e_2+e_3+e_4-e_5+e_6-e_7-e_8)$	1	p^{16}	p^{-2}
$\frac{1}{2}(e_1+e_2-e_3+e_4-e_5+e_6-e_7-e_8)$	1	p^{15}	p^{-2}
$\frac{1}{2}(e_1+e_2+e_3-e_4-e_5+e_6-e_7-e_8)$	1	p^{14}	p^{-2}
$\frac{1}{2}(-e_1-e_2-e_3+e_4-e_5+e_6-e_7-e_8)$	1	p^{14}	p^{-2}
$\frac{1}{2}(-e_1-e_2+e_3-e_4-e_5+e_6-e_7-e_8)$	1	p^{13}	p^{-2}
$\frac{1}{2}(-e_1+e_2-e_3-e_4-e_5+e_6-e_7-e_8)$	1	p^{12}	p^{-2}
$\frac{1}{2}(e_1-e_2-e_3-e_4-e_5+e_6-e_7-e_8)$	1	p^{11}	p^{-2}
$-e_5-e_8$	1	p	p^{2}
e_6-e_8	1	p^{10}	p^{-2}
e_5-e_7	-1	p^{26}	p^{-2}
$-e_6-e_7$	-1	p^{17}	p^{2}
$\frac{1}{2}(-e_1+e_2+e_3+e_4+e_5-e_6-e_7-e_8)$	-1	p^{16}	p^{2}
$\frac{1}{2}(e_1-e_2+e_3+e_4+e_5-e_6-e_7-e_8)$	-1	p^{15}	p^{2}
$\frac{1}{2}(e_1+e_2-e_3+e_4+e_5-e_6-e_7-e_8)$	-1	p^{14}	p^{2}
$\frac{1}{2}(e_1+e_2+e_3-e_4+e_5-e_6-e_7-e_8)$	-1	p^{13}	p^{2}
$\frac{1}{2}(-e_1-e_2-e_3+e_4+e_5-e_6-e_7-e_8)$	-1	p^{12}	p^{2}
$\frac{1}{2}(-e_1-e_2+e_3-e_4+e_5-e_6-e_7-e_8)$	-1	p^{11}	p^{2}
$\frac{1}{2}(-e_1+e_2-e_3-e_4+e_5-e_6-e_7-e_8)$	-1	p^{11}	p^{2}
$\frac{1}{2}(e_1-e_2-e_3-e_4+e_5-e_6-e_7-e_8)$	-1	p^{10}	p^{2}
$-e_6-e_8$	-1	1	p^{6}
e_5-e_8	-1	p^{11}	p^{2}

So, the right-hand side consists of

$$f(m+1)\big[p^{21}+p^{16}+p^{15}+p^{14}+p^{13}+2p^{12}+p^{11}+p^{10}+p^{9}+p^{8}+p^{3}\big]$$
$$+f(m-1)\big[p^{24}+p^{19}+p^{18}+p^{17}+p^{16}+2p^{15}+p^{14}+p^{13}+p^{12}+p^{11}+p^{6}\big]$$
$$+f(m)\big[p^{23}+p^{22}+p^{21}+2p^{20}+2p^{19}+2p^{18}+2p^{17}+2p^{16}+p^{15}+2p^{14}$$
$$+2p^{13}+p^{12}+2p^{11}+2p^{10}+2p^{9}+2p^{8}+2p^{7}+p^{6}+p^{5}+p^{4}\big].$$

We simplify (just as before) and this yields:

$$p^3\big[f(m) - f(m-1)\big] = \big[f(m+1) - f(m)\big]$$

which implies that $f(m) = 1 + p^3 + \cdots + p^{3m}$. The theorem is proved. $\square$

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