

# The prime spectrum of algebras of quadratic growth

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## Abstract

We study prime algebras of quadratic growth. Our first result is that if  $A$  is a prime monomial algebra of quadratic growth then  $A$  has finitely many prime ideals  $P$  such that  $A/P$  has GK dimension one. This shows that prime monomial algebras of quadratic growth have bounded matrix images. We next show that a prime graded algebra of quadratic growth has the property that the intersection of the non-zero prime ideals  $P$  such that  $A/P$  has GK dimension 2 is non-zero, provided there is at least one such ideal. From this we conclude that a prime monomial algebra of quadratic growth is either primitive or has non-zero locally nilpotent Jacobson radical. Finally, we show that there exists a prime monomial algebra  $A$  of GK dimension two with unbounded matrix images and thus the quadratic growth hypothesis is necessary to conclude that there are only finitely many prime ideals such that  $A/P$  has GK dimension 1.

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## 1. Introduction

We study finitely generated algebras of quadratic growth. Given a field  $k$  and a finitely generated  $k$ -algebra  $A$ , a  $k$ -subspace  $V$  of  $A$  is called a *frame* of  $A$  if  $V$  is finite-dimensional,  $1 \in V$ ,

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and  $V$  generates  $A$  as a  $k$ -algebra. We say that  $A$  has *quadratic growth* if there exist a frame  $V$  of  $A$  and constants  $C_1, C_2 > 0$  such that

$$C_1 n^2 \leq \dim_k(V^n) \leq C_2 n^2 \quad \text{for all } n \geq 1.$$

We note that an algebra of quadratic growth has GK dimension 2. More generally, the GK dimension of a finitely generated  $k$ -algebra  $A$  is defined to be

$$\text{GKdim}(A) = \limsup_{n \rightarrow \infty} \log(\dim(V^n)) / \log n,$$

where  $V$  is a frame of  $A$ . While algebras of quadratic growth have GK dimension 2, it is not the case that an algebra of GK dimension 2 necessarily has quadratic growth. We note that in the case that  $A$  is a finitely generated commutative algebra, GK dimension is equal to Krull dimension. For this reason, GK dimension has seen great use over the years as a useful tool for obtaining non-commutative analogues of results from classical algebraic geometry. For more information about GK dimension we refer the reader to Krause and Lenagan [12].

We consider prime monomial algebras of quadratic growth. A  $k$ -algebra  $A$  is a *monomial algebra* if

$$A \cong k\{x_1, \dots, x_d\}/I,$$

for some ideal  $I$  generated by monomials in  $x_1, \dots, x_d$ . Monomial algebras are useful for many reasons. First, Gröbner basis is the associated monomial algebra to a finitely generated algebra, and thus monomial algebras can be used to answer questions about ideal membership and Hilbert series for general algebras. Second, many questions for algebras reduce to combinatorial problems for monomial algebras and can be studied in terms of forbidden subwords. For these reasons, monomial algebras are a rich area of study. The paper of Belov, Borisenko, and Latyshev [8] gives many useful and interesting results about monomial algebras.

Our first result is the following theorem.

**Theorem 1.1.** *Let  $k$  be a field and let  $A$  be a prime monomial  $k$ -algebra of quadratic growth. Then the set of primes  $P$  such that  $\text{GKdim}(A/P) = 1$  is finite; moreover, all such primes are monomial ideals. In particular,  $A$  has bounded matrix images.*

We also show that if there is a constant  $C > 0$  such that the number of words of length at most  $n$  with non-zero image in  $A$  is at most  $Cn^2$  for  $n$  sufficiently large, then  $A$  has at most  $2C$  primes such that  $A/P$  has GK dimension one. In general, if a finitely generated  $k$ -algebra  $A$  has quadratic growth, we define the *growth constant* of  $A$  to be

$$\text{GC}(A) := \inf_V \limsup_{n \rightarrow \infty} \dim(V^n)/n^2,$$

where the infimum is taken over all frames  $V$  of  $A$ . Theorem 1.1 leads us to make the following conjecture.

**Conjecture 1.2.** *There exists a function  $F : (0, \infty) \rightarrow (0, \infty)$  such that for every field  $k$  and every finitely generated non-PI prime Noetherian  $k$ -algebra  $A$  of quadratic growth,  $A$  has at most  $F(\text{GC}(A))$  prime ideals  $P$  such that  $\text{GKdim}(A/P) = 1$ .*

We know no example of a finitely generated prime Noetherian  $k$ -algebra of quadratic growth with more than  $2\text{GC}(A) + 1$  primes of co-GK dimension 1.

We next turn our attention to graded algebras of quadratic growth. In this case, we give an analogue of Bergman's gap theorem—which states that there are no algebras of GK dimension strictly between 1 and 2—for ideals.

**Theorem 1.3.** *Let  $K$  be a field and let  $A = \bigoplus_{i=0}^{\infty} A_i$  be a finitely generated graded  $K$ -algebra with quadratic growth, generated in degree one. Suppose  $u$  is a non-zero homogeneous element of  $A$  and let  $(u)$  denote the two-sided ideal of  $A$  generated by  $u$ . Then either:*

1. *there is a natural number  $m$  such that*

$$\dim_K \left( (u) \cap \bigoplus_{i=0}^n A_i \right) \geq \frac{(n-m)(n-m-1)}{2}$$

*for all  $n$  sufficiently large; or*

2. *there is a positive constant  $C$  such that*

$$\dim_K \left( A_c(u)A_c \cap \bigoplus_{i=1}^n A_i \right) < Cn$$

*for all  $n$  sufficiently large.*

Moreover, if  $A$  is prime then the former holds.

Using this theorem we are able to obtain the following result about prime ideals in graded algebras of quadratic growth.

**Theorem 1.4.** *Let  $K$  be a field, and let  $A = \bigoplus_{i=0}^{\infty} A_i$  be a prime affine graded non-PI  $K$ -algebra, generated in degree one. If  $A$  has quadratic growth then the intersection of all non-zero prime ideals  $P$  such that  $A/P$  has GK dimension 2 is non-zero, where we take an empty intersection to be all of  $A$ .*

As a result of this theorem and Theorem 1.1 we obtain the following corollary.

**Corollary 1.5.** *Let  $A$  be a finitely generated prime monomial algebra of quadratic growth. Then  $A$  has bounded matrix images and either  $A$  is primitive or has non-zero locally nilpotent Jacobson radical.*

We note that a finitely generated prime monomial algebra that has GK dimension greater than 1 cannot be PI [8]. A conjecture of Small [4, Question 3.2] is that a finitely generated prime Noetherian algebra of quadratic growth is either primitive or PI. By the Jacobson density theorem, primitive algebras are dense subrings of endomorphism rings over a division algebra. For this reason, primitive ideals are an important object of study and classifying primitive ideals is often an important intermediate step in classifying finite-dimensional representations of an algebra.

Finally, we turn to the more general setting of algebras of GK dimension 2. In this case, we are able to show that prime algebras can have unbounded matrix images.

**Theorem 1.6.** *There exists a finitely generated prime monomial algebra  $A$  of GK dimension 2 with unbounded matrix images.*

This answers a question of Small [4, Question 3.1]; Small's question is still open when the additional hypothesis that the algebra be Noetherian is added. Irving [11] obtained a similar example of an algebra of GK dimension 2 with unbounded matrix images; his example is not prime, however.

## 2. Prime ideals of co-GK 1

We begin our investigation of monomial algebras by giving a description of monomial algebras of GK dimension 1. We first introduce the following notation. Given a finite alphabet  $\{x_1, \dots, x_d\}$ , we call an infinite product of the form

$$x_{i_1} x_{i_2} x_{i_3} \cdots$$

a *right infinite word* on  $\{x_1, \dots, x_d\}$ . Left infinite words are defined analogously and doubly infinite words are both left and right infinite. Given a finite alphabet  $\{x_1, \dots, x_d\}$  and a right infinite word  $W$ , we let  $A_W$  denote the algebra  $k\{x_1, \dots, x_d\}/I$ , where  $I$  is the monomial ideal generated by the collection of words which do not occur as a subword of  $W$ . In particular, the images of the distinct subwords of  $W$  form a basis for  $A_W$ .

**Remark 2.1.** Let  $A$  be a finitely generate prime monomial algebra. Then there is a right infinite word  $W$  on a finite alphabet such that  $A \cong A_W$ .

The reason for this is that if  $A = k\{x_1, \dots, x_d\}/I$ , then we can pick an enumeration  $W_1, W_2, \dots$  of the words on  $\{x_1, \dots, x_d\}$  with non-zero image in  $A$ . Then since  $A$  is prime, there exist words  $V_1, V_2, \dots$  such that  $W_1 V_1 W_2 V_2 \cdots V_m W_{m+1}$  has non-zero image in  $A$  for every  $m$ . Let

$$W = W_1 V_1 W_2 V_2 \cdots.$$

Then  $W$  is a right infinite word. Moreover every subword of  $W$  has non-zero image in  $A$  since every subword of  $W$  is a subword of  $W_1 V_1 W_2 V_2 \cdots V_m W_{m+1}$  for some  $m$ . Moreover, every word with non-zero image in  $A$  is a subword of the form  $W_i$  for some  $i$  and hence is a subword of  $W$ . It follows that  $A \cong A_W$ .

**Lemma 2.2.** *Let  $k$  be a field and let  $A$  be a finitely generated prime monomial  $k$ -algebra of GK dimension 1. Then there is a word  $W$  on a finite alphabet such that  $A \cong A_{W^\omega}$ , where  $W^\omega$  is the right infinite word  $WWW \cdots$ . Conversely, if  $W$  is a word on a finite alphabet, then the algebra  $A_{W^\omega}$  is a finitely generated prime algebra of GK dimension 1.*

**Proof.** Since  $A$  is prime, there is a right infinite word  $U$  such that  $A \cong A_U$  by Remark 2.1. Since  $A$  has GK dimension 1 and the images of the subwords of  $U$  form a basis for  $A_U$ , we see that

$U = W_1 W^\omega$  for some words  $W_1$  and  $W$  by a result of Bergman [12, Lemma 2.4]. We claim that  $W_1 W^\omega$  is itself a subword of  $W^\omega$ . To see this, suppose that there is some subword  $V$  of  $W_1 W^\omega$  that does not appear as a subword of  $W^\omega$ . Let  $d$  denote the length of  $W_1$ . Since  $A_U$  is prime we see there exist words  $V_1, V_2, \dots$  such that  $V V_1 V V_2 V \dots V_d V$  is a subword of  $W_1 W^\omega$ . Moreover since  $W_1$  has length  $d$ , we see that the last occurrence of  $V$  in  $V V_1 V V_2 V \dots V_d V$  must be contained entirely in  $W^\omega$ . This is a contradiction, we conclude that  $A_U = A_{W^\omega}$ . The converse follows easily from the fact that  $W^\omega$  has  $O(1)$  subwords of length  $n$  and each of these words occurs infinitely often. The result follows.  $\square$

**Lemma 2.3.** *Let  $k$  be a field and let  $A = k\{x_1, \dots, x_d\}/I$  be a prime monomial algebra of GK dimension at least 2. Let  $W$  be a word on the alphabet  $\{x_1, \dots, x_d\}$  such that every power of  $W$  has non-zero image in  $A$  and let  $n$  be a positive integer. Then there exists a right infinite word  $U$  on  $\{x_1, \dots, x_d\}$  such that every subword of  $W^n U$  has non-zero image in  $A$  and  $U$  does not contain  $W$  as an initial subword.*

**Proof.** Pick a word  $V_1$  that is not a subword of  $W^\omega$ . Since  $A$  is prime there exists a word  $V_2$  such that  $W^n V_2 V_1$  has non-zero image in  $A$ . Since  $A$  is prime there exists an infinite word  $V$  with initial subword  $V_2 V_1$  such that each subword of  $W^n V$  has non-zero image in  $A$ . Since  $V_1$  is not subword of  $W^\omega$ , there is some  $m \geq n$  such that

$$W^n V = W^m U$$

where  $U$  is a right infinite word that does not have  $W$  as an initial subword. This completes the proof.  $\square$

We first show that prime monomial algebras  $A$  of quadratic growth only have finitely many prime monomial ideals  $P$  such that  $A/P$  has GK dimension 1.

**Proposition 2.4.** *Let  $A$  be a prime monomial algebra of quadratic growth. Then  $A$  has only finitely many prime monomial ideals  $P$  such that  $A/P$  has GK dimension 1.*

**Proof.** We may assume that  $A = k\{x_1, \dots, x_d\}/I$ . By assumption there is a constant  $C > 0$  such that there are at most  $Cn^2$  words of length at most  $n$  with non-zero image in  $A$  for every  $n \geq 1$ . Let

$$\mathcal{S} = \{P \mid \text{GKdim}(A/P) = 1, P \text{ a monomial ideal}\}.$$

We show that  $\mathcal{S}$  is finite. By Lemma 2.2, for each  $P \in \mathcal{S}$ , there exists a word  $W$  such that  $A/P \cong A_{W^\omega}$ . If  $\mathcal{S}$  is infinite, we have infinitely many words  $W_1, W_2, \dots$  such that  $W_i^j$  has non-zero image in  $A$  for every  $j \geq 1$  and  $A_{W_i^\omega} \neq A_{W_j^\omega}$  if  $i \neq j$ . Pick integers  $d_{i,j}$  such that  $W_i^{d_{i,j}}$  is not a subword of  $W_j^\omega$ . Pick an integer  $m > 2C$  and let

$$D = \max_{1 \leq i, j \leq m} d_{i,j}.$$

Let

$$n > D \max_{i \leq m} \text{length}(W_i) + D$$

be a positive integer. By Lemma 2.3, there exist right infinite words  $U_1, U_2, \dots$  such that each subword of  $W_i^n U_i$  has non-zero image in  $A$  and  $U_i$  does not have  $W_i$  as an initial subword for  $i \leq m$ . Consider the words

$$Y_i = W_i^{\lfloor n/\text{length}(W_i) \rfloor + D+1} U_i$$

for  $1 \leq i \leq m$ . Given a word  $W$  with at least  $b$  letters, we let  $W(a, b)$  denote the subword of  $W$  formed by taking the word whose initial position occurs at the  $a$ th spot and whose final position occurs at the  $b$ th spot. Consider the set of words

$$\{Y_i(j, j+n-1) \mid 1 \leq i \leq m, D \cdot \text{length}(W_i) \leq j \leq n\}.$$

These words all have non-zero image in  $A$ . We claim that the words in this set are distinct. Notice that if  $Y_i(k, k+n-1) = Y_j(\ell, \ell+n-1)$  then since the first  $D \cdot \text{length}(W_i)$  letters of  $Y_i(k, k+n-1)$  is a subword of  $W_i^\omega$  and the first  $D \cdot \text{length}(W_j)$  letters of  $Y_j(\ell, \ell+n-1)$  is a subword of  $W_j^\omega$ , we see that  $i = j$  by definition of  $D$ . But this can only occur if  $k = \ell$  by definition of the words  $U_1, \dots, U_m$ . Thus there are at least

$$(n - D \max_i \text{length}(W_i))m$$

words of length  $n$  for all  $n$  sufficiently large. Consequently, there are at least

$$\sum_{i=0}^n im + O(n) = m \binom{n}{2} + O(n)$$

words of length at most  $n$ . Dividing this by  $n^2$  and taking the limit as  $n \rightarrow \infty$  and then using the quadratic growth hypothesis, we see

$$m/2 \leq C,$$

a contradiction.  $\square$

**Proposition 2.5.** *Let  $k$  be a field and let  $A$  be a prime monomial  $k$ -algebra of quadratic growth. Then every prime homomorphic image of  $A$  of GK dimension 1 is also a monomial algebra.*

**Proof.** Let  $N$  be the intersection of all prime monomial ideals  $P$  in  $A$  such that  $A/P$  has GK dimension 1. (If  $A$  has no such prime ideals, we take  $N$  to be  $A$ .) Note that  $N \neq (0)$  since we are taking a finite intersection of non-zero ideals and  $A$  is prime (by Proposition 2.4). Observe that any word in  $N$  is nilpotent. To see this, suppose that there is some word  $W \in N$  such that each subword of  $W^\omega$  has non-zero image in  $A$ . Then  $A_{W^\omega}$  is a homomorphic image of  $A$  of GK dimension 1. By Lemma 2.2, there exists some prime monomial ideal  $Q$  such that  $A/Q \cong A_{W^\omega}$ . Notice that  $W$  has non-zero image in  $A/Q$ . But this contradicts the fact that  $W \in N \subseteq Q$ . Let  $P$  be a prime ideal such that  $A/P$  has GK dimension 1. If  $P$  is not a monomial ideal, then  $N \not\subseteq P$ . Let  $\bar{N}$  denote the image of  $N$  in  $A/P$ . We now show that  $\bar{N}$  is an algebraic ideal; that is, every element of  $\bar{N}$  is algebraic over  $k$ . Suppose that  $\bar{N}$  is not algebraic. Then there is a non-algebraic element  $x \in \bar{N}$ ; this element is the image of a linear combination of words  $W_1, \dots, W_d$ , all of whose images are in  $\bar{N}$ . By the remarks above, every element in the semigroup generated by the

images of  $W_1, \dots, W_d$  is nilpotent. It follows from Shirshov's theorem [2], that the subalgebra of  $A/P$  generated by the images of  $W_1, \dots, W_d$  is finite-dimensional as a  $k$ -vector space. It follows that  $x$  is algebraic since it lies in this subalgebra. Thus  $\bar{N}$  is an algebraic ideal. But  $A/P$  is a prime Goldie ring of GK dimension 1 and  $\bar{N}$  is non-zero in  $A/P$  and hence  $A/(P + N)$  has GK dimension 0. In particular, it is finite-dimensional as a  $k$ -vector space. It follows that  $A/P$  must be algebraic as a ring. But a finitely generated algebra of GK dimension one is PI by the Small–Warfield theorem and hence cannot be algebraic. Thus we obtain a contradiction. We conclude that  $P$  is a monomial ideal.  $\square$

To complete the proof of Theorem 1.1, we need a result of small about graded algebras.

**Lemma 2.6.** *Let  $A$  be a finitely generated prime graded algebra of quadratic growth and let  $Q$  be a non-zero prime ideal of  $A$  such that  $A/Q$  is finite-dimensional. Then either  $Q$  is the maximal homogeneous ideal of  $A$  or  $Q$  contains a prime  $P$  such that  $A/P$  has GK dimension 1.*

**Proof.** Let  $\mathcal{S}$  denote the collection of homogeneous ideals contained in  $Q$ . Since  $(0) \in \mathcal{S}$ , we see that  $\mathcal{S}$  is non-empty. A standard argument using Zorn's lemma shows that  $\mathcal{S}$  has a maximal element  $P$ . We claim that  $P$  is prime. To see this, suppose there are non-zero  $a$  and  $b$  in  $A$  such that  $aAb \subseteq P$ . Since  $P$  is homogeneous, it is no loss of generality to assume that  $a$  and  $b$  are homogeneous elements of  $A$  (we can replace  $a$  by the non-zero homogeneous part of  $a$  of highest weight and do the same for  $b$ ). But this says that  $(AaA + P)(AbA + P) \subseteq Q$  and since  $Q$  is prime, we conclude that either  $AaA + P$  or  $AbA + P$  is contained in  $Q$ , contradicting the maximality of  $P$ .

We next claim that  $A/P$  is PI, using an argument of Small. Suppose that  $A/P$  is not PI. Note  $A/Q$  is PI and hence satisfies a multilinear homogeneous identity; further, by assumption  $A/P$  does not satisfy this identity. Let  $f(x_1, \dots, x_d)$  denote this identity. Since  $f$  is multilinear and homogeneous, there exist homogeneous elements  $a_1, \dots, a_d$  in  $A$  such that  $b := f(a_1, \dots, a_d) \notin P$ . By construction  $b$  is a non-zero homogeneous element of  $Q$  that does not lie in  $P$  and so  $P + AbA$  is a homogeneous ideal contained in  $Q$  that properly contains  $P$ , contradicting the maximality of  $P$ . Thus  $A/P$  is PI.

It remains to show that  $B := A/P$  has GK dimension 1 if  $Q$  is not the maximal homogeneous ideal of  $A$ . If  $Q$  is not the maximal homogeneous ideal, then  $B$  is infinite-dimensional and thus has at GK dimension at least 1. Therefore it is sufficient to show that  $B$  cannot have GK dimension 2. We now consider  $Q$  as a prime ideal of  $B$  of finite codimension. Note that  $B$  is a graded prime PI algebra and hence is Goldie [13, Corollary 13.6.6]. It follows that if  $S$  is the set of regular homogeneous elements of  $B$ , then we can invert the elements of  $S$  to obtain an algebra  $S^{-1}B = R[x, x^{-1}; \sigma]$ , where  $R$  is a simple PI algebra [10]. By assumption,  $Q$  does not contain any non-zero homogeneous elements and hence  $S^{-1}Q$  is a proper ideal of  $S^{-1}B$ .

We note that  $S^{-1}B/S^{-1}Q$  is finite-dimensional since  $B/Q$  is. To see this, let  $m$  denote the dimension of  $B/Q$  and suppose  $x_1, \dots, x_{m+1}$  are elements of  $S^{-1}B$ . Then there exists some  $s \in S$  such that  $sx_1, \dots, sx_{m+1} \in B$ . Since  $B/Q$  is  $m$ -dimensional, some linear combination of  $sx_1, \dots, sx_{m+1}$  lies in  $Q$ . But this says that some linear combination of  $x_1, \dots, x_{m+1}$  lies in  $S^{-1}Q$ . Hence any  $m + 1$  elements of  $S^{-1}B$  are linearly dependent mod  $S^{-1}Q$  and so  $S^{-1}B/S^{-1}Q$  is finite-dimensional.

Since  $R$  is simple, it must embed in any homomorphic image of  $S^{-1}B = R[x, x^{-1}; \sigma]$  and in particular  $R$  embeds in  $S^{-1}B/S^{-1}Q$ . Thus  $R$  is finite-dimensional by the remarks above. It

follows that  $S^{-1}B = R[x, x^{-1}; \sigma]$  has GK dimension exactly 1 since  $R$  has GK dimension 0. Thus  $A/P$  has GK dimension 1. This completes the proof.  $\square$

**Proof of Theorem 1.1.** This follows immediately from Propositions 2.4 and 2.5.  $\square$

### 3. Graded algebras of quadratic growth

In this section, we prove Theorems 1.3 and 1.4 and obtain our results about primitivity for prime monomial algebras of quadratic growth. Small [4, Question 3.2] asks whether a prime affine Noetherian algebra of GK dimension 2 is either primitive or PI. In fact, the additional hypothesis that the algebra be semiprimitive is probably necessary over countable fields to obtain this result. We show that if  $A$  is a semiprimitive prime affine monomial algebra of quadratic growth then  $A$  is primitive and, moreover, it has bounded matrix images. Bergman and Lewin [9, p. 29] introduced the notion of the extended centre of a ring to mean the extended centroid of a ring with unit. In this paper by the extended centre of a ring we mean the extended centroid of this ring (for the definition see [7, p. 68]).

**Lemma 3.1.** *Let  $K$  be a field, let  $A = \bigoplus_{i=0}^{\infty} A_i$  be a graded prime  $K$ -algebra, and let  $Z$  denote the extended centre of  $A$ . Suppose that  $I$  is an ideal in  $A$  that does not contain a non-zero homogeneous element and  $z \in Z$ ,  $x, y \in A$  are such that:*

1.  $x$  is a non-zero homogeneous element;
2.  $y$  is a sum of homogeneous elements of degree smaller than the degree of  $x$ ;
3.  $x + y \in I$ ;
4.  $zx = y$ .

*Then  $z$  is not algebraic over  $K$ .*

**Proof.** Suppose that  $z$  is algebraic over  $K$ . We note that  $z \neq -1$  since  $(1+z)x = x + y \neq 0$  since  $\deg x > \deg y$ . Since the extended centre of a prime ring is a field (cf. Beidar et al. [7, p. 70]), we have  $(1+z)p(z) = 1$  for some polynomial  $p(z) \in K[z]$  of degree  $d$ . Note that for  $j \leq d$  we have

$$z^j(a_1xa_2x \cdots xa_dx) = a_1ya_2y \cdots a_jya_{j+1}x \cdots a_dx.$$

Therefore,  $z^j(a_1xa_2x \cdots xa_dx) \in A$ , for each  $j \leq d$ . In particular,  $p(z)(a_1xa_2 \cdots xa_d) \in A$  for all  $a_1, \dots, a_d \in A$  (because  $\deg p(z) = d$ ). Note that

$$\begin{aligned} xa_1xa_2 \cdots xa_dx &= p(z)(1+z)xa_1xa_2 \cdots xa_dx \\ &= (x+y)p(z)a_1 \cdots xa_dx \\ &\in (x+y)A \\ &\subseteq I. \end{aligned}$$

By assumption,  $I$  does not contain any non-zero homogeneous elements and hence

$$xa_1xa_2 \cdots xa_dx = 0$$



for all homogeneous elements  $a_1, \dots, a_d \in A$ . It follows that  $(x)^{d+1} = (0)$ . This is impossible since  $A$  is a prime algebra. This completes the proof.  $\square$

**Corollary 3.2.** *Let  $K$  be a field and let  $A = \bigoplus_{i=1}^{\infty} A_i$  be a finitely generated prime graded non-PI  $K$ -algebra of quadratic growth. If  $P$  is a non-zero prime ideal of  $A$  then either  $P$  is homogeneous or  $A/P$  is PI.*

**Proof.** Let  $P$  be a non-zero prime ideal  $P$  such that  $A/P$  is not PI. We show that  $P$  must be homogeneous. Suppose not. Let  $Q$  be maximal homogeneous ideal of  $A$  contained in  $P$ . Then  $Q$  is prime. Since  $A/Q$  is not PI, by the Small–Warfield theorem and Bergman’s gap theorem  $A/Q$  has quadratic growth [12,14]. The ideal  $P$  is a non-zero prime ideal in the graded algebra  $A/Q$  and  $P$  does not contain a homogeneous element. Let  $\sum_{i=1}^n a_i \in P$  with  $a_i$  of degree  $i$  and  $a_n \neq 0$ . We may assume that the number of non-zero elements  $a_i$  in this sum is minimal. Then by minimality of the number of non-zero terms we have

$$a_i a a_j = a_j a a_i \quad \text{for } i, j \leq n, \text{ and } a \in A.$$

Beidar et al. [7, Theorem 2.3.4] show that for each  $i < n$ , we have

$$z_i a_n = a_i \quad \text{for some } z_i \in Z.$$

Since  $A$  has quadratic growth, the extended centre of  $A$  is algebraic over  $K$  [6, Theorem 1.3]. Lemma 3.1 then shows  $a_i = 0$  for  $i < n$ . Hence the image of  $P$  in  $A/Q$  contains the non-zero homogeneous element  $a_n$ , a contradiction. Thus  $P = Q$  and so  $P$  is homogeneous.  $\square$

**Proof of Theorem 1.3.** We let  $K(X)$  be the field of rational functions in infinitely many commuting indeterminates  $x_{i,j}$  where  $i, j \in \mathbb{Z}$ , and we let  $A(X) = A \otimes_K K(X)$ . Let  $a_1, \dots, a_n$  be generators of  $A$ . Define

$$c_i = \sum_{j=1}^n x_{i,j} a_j \in A(X) \quad \text{for } i \geq 1 \tag{1}$$

and

$$d_i = \sum_{j=1}^n x_{-i,j} a_j \in A(X) \quad \text{for } i \geq 1. \tag{2}$$

We have two cases to consider.

**Case I.** The set  $S = \{c_1 c_2 \cdots c_i u d_j d_{j-1} \cdots d_1 \mid i, j \geq 0\}$  is linearly independent over  $K(X)$ .

In this case it is easy to see

$$\dim \left( AuA \cap \bigoplus_{i=0}^n A_i \right) \geq \binom{n-m}{2},$$

where  $m$  is the degree of  $u$ .

**Case II.** The set  $S = \{c_1 c_2 \cdots c_i u d_j d_{j-1} \cdots d_1 \mid i, j \geq 0\}$  is linearly dependent over  $K(X)$ .

Since  $A(X)$  is a graded  $K(X)$ -algebra and  $S$  is linearly dependent over  $K(X)$ , there is in fact some natural number  $m$  such that the elements  $c_1 c_2 \cdots c_i u d_j d_{j-1} \cdots d_1$  with  $i + j = m$  are linearly dependent over  $K(X)$ . Hence there is some  $k$  with  $0 \leq k < m$  such that

$$c_1 c_2 \cdots c_k u d_{m-k} d_{m-k-1} \cdots d_1 \in c_1 \cdots c_k \sum_{j=1}^{m-k} K(X) c_{k+1} \cdots c_{k+j} u d_{m-k-j} \cdots d_1. \quad (3)$$

(We note that when  $k = 0$ , we take the empty product  $c_1 \cdots c_k$  to be 1.) From item (3), we see

$$A_m u A_n \subseteq \sum_{i=k}^{m-1} A_i u A_{m+n-i} \quad \text{for } n > m. \quad (4)$$

By comparing the left- and right-hand sides of Eq. (3) we also see

$$(c_1 c_2 \cdots c_k) u K(X) A_{m-k} \subseteq (c_1 c_2 \cdots c_k) \sum_{i=1}^{m-i} K(X) c_{k+1} \cdots c_{i+k} u A_{m-i-k}. \quad (5)$$

(These are both consequences of the fact that we can clear denominators and substitute any values of  $K$  for the variable coefficients  $x_{i,j}$  that occur in the  $d_j$  or  $c_i$  and in this way obtain a spanning set for  $A_j$  from  $d_j \cdots d_1$  or  $c_1 \cdots c_j$ .)

Multiplying both sides of Eq. (5) on the left by  $c_m$ , we see

$$c_m c_1 \cdots c_{k-1} c_k u K[X] A_{m-k} \subseteq (c_m c_1 c_2 \cdots c_k) \sum_{i=1}^{m-k} K(X) c_{k+1} \cdots c_{i+k} u A_{m-i-k}.$$

By applying a  $K$ -automorphism of  $K(X)$  that fixes each  $d_i$  and sends  $c_m \mapsto c_1$  and  $c_j \mapsto c_{j+1}$  for  $j \neq m$  and using Eq. (3), we see

$$c_1 \cdots c_k u K[X] A_{m+1-k} \subseteq (c_1 \cdots c_k c_{k+1}) \sum_{i=1}^{m-k} K(X) c_{k+2} \cdots c_{i+k+1} u A_{m-i-k},$$

where we take  $c_{m+1} = c_1$ . It follows that

$$\dim_{K(X)} c_1 \cdots c_k u K(X) A_{m-k+1} \leq \sum_{i=1}^{m-k} \dim(A_{m-i-k}).$$

Continuing in this manner we see that for any natural number  $n \geq m - k$ ,

$$\dim_{K(X)} (c_1 \cdots c_k K(X) u A_n) \leq \sum_{i=1}^{m-k} \dim(A_{m-i-k}).$$

It follows that

$$\dim_K(A_k u A_n) \leq \dim(A_k) \sum_{i=1}^{m-k} \dim(A_{m-i-k}) \quad \text{for } n \geq m - k.$$

From Eq. (4), however, we have by an easy induction argument that  $A_p u A_n \subseteq \sum_{i=k}^{m-1} A_i u A_{p+n-i}$  whenever  $p \geq m$  and  $n > m$ .

Therefore if  $d$  is the degree of  $u$ , then

$$\begin{aligned} \dim_K A_{m+1} A u A A_{m+1} \cap A_{n+p+d} & \\ & \leq \sum_{i=k}^{m-1} \dim_K A_i u A_{n+p-i} \\ & \leq \sum_{i=0}^{m-1-k} \dim_K A_i (A_k u A_{n+p-i-k}) \\ & \leq \sum_{i=1}^m \dim_K(A_i) \dim_K(A_k) (\dim_K(A_1 + A_2 + \cdots + A_{m-k})). \end{aligned}$$

It follows that there is some constant  $C_1 > 0$  such that

$$\dim_K(A_{m+1} A u A A_{m+1} \cap A_n) \leq C_1,$$

for all  $n \geq 0$ . Finally, we note that if  $A$  is prime then  $\dim_K(A A_{m+1} u A_{m+1} A \cap \bigoplus_{i=0}^n A_i)$  grows at least quadratically with  $n$  [6, Lemma 2.3], and so this case cannot occur if  $A$  is prime. This completes the proof.  $\square$

We are now ready to show that the intersection of the non-zero prime ideals  $P$  such that  $A/P$  has GK dimension 2 is either empty or non-zero in the case that  $A$  is a prime graded algebra of quadratic growth.

**Proof of Theorem 1.4.** Let

$$\alpha = \inf_I \liminf_{n \rightarrow \infty} n^{-2} \dim_K \left( I \cap \bigoplus_{i=0}^n A_i \right),$$

where the infimum is taken over all non-zero homogeneous ideals  $I$  in  $A$ . Since  $A$  has quadratic growth, we see that  $\alpha \geq 1/2$ . It follows that we can find a homogeneous element  $x \in A$  such that

$$\liminf_{n \rightarrow \infty} n^{-2} \dim_K \left( (x) \cap \bigoplus_{i=0}^n A_i \right) \leq \alpha + 1/4.$$

Let  $P$  be a non-zero prime ideal in  $A$  such that  $A/P$  has GK dimension 2. If  $A/P$  is PI, then  $P$  contains a homogeneous element since  $A$  is not PI, using the same argument as in Lemma 2.6; if  $A/P$  is not PI, then  $P$  is homogeneous by Corollary 3.2. Either way, we see that if  $Q$  is the

maximal homogeneous ideal contained in  $P$ , then  $Q$  is a non-zero prime ideal in  $A$ . We claim that  $x \in Q$ . To see this, let  $B = \bigoplus_{i=0}^{\infty} B_i = A/Q$ , and let  $\bar{I} = I + Q$ , be the image of  $I$  in  $B$  and suppose that  $\bar{I}$  is non-zero. Then

$$\dim_K \left( I \cap \bigoplus_{i=0}^n A_i \right) \geq \dim_K \left( (I \cap Q) \cap \bigoplus_{i=0}^n A_i \right) + \dim_K \left( \bar{I} \cap \bigoplus_{i=0}^n B_i \right).$$

By Theorem 1.3, we see that

$$\liminf_{n \rightarrow \infty} \frac{1}{n^2} \dim_K \left( \bar{I} \cap \bigoplus_{i=0}^n B_i \right) \geq 1/2.$$

Moreover,  $I \cap Q \neq (0)$  since  $A$  is prime and hence

$$\begin{aligned} & \alpha + 1/4 \\ & \geq \liminf_{n \rightarrow \infty} \frac{1}{n^2} \dim_K \left( I \cap \bigoplus_{i=0}^n A_i \right) \\ & \geq \liminf_{n \rightarrow \infty} \frac{1}{n^2} \dim_K \left( I \cap Q \cap \bigoplus_{i=0}^n A_i \right) + \liminf_{n \rightarrow \infty} \frac{1}{n^2} \dim_K \left( \bar{I} \cap \bigoplus_{i=0}^n B_i \right) \\ & \geq \alpha + 1/2, \end{aligned}$$

a contradiction. It follows that  $x \in Q$  and the result follows.  $\square$

From this result, we immediately obtain our results about primitivity.

**Proof of Corollary 1.5.** To see that  $A$  has bounded matrix images, let  $Q$  be a non-zero prime ideal of  $A$  such that  $\text{GKdim}(A/Q)$  has GK dimension 0. By Lemma 2.6, if  $Q$  is not the maximal homogeneous ideal, then it contains a prime ideal  $P$  with  $\text{GKdim}(A/P) = 1$ . By Theorem 1.1 there are only finitely many such prime ideals  $P$  and hence if  $I$  is the intersection of all such ideals, we see that  $A/Q$  is a homomorphic image of  $A/I$ . Since  $A/I$  is PI, we see that  $A/Q$  necessarily satisfies the same polynomial identity of  $A/I$  and so the matrix images are necessarily bounded. Let  $I$  denote the intersection of the maximal homogeneous ideal and all prime ideals  $P$  such that  $A/P$  has GK dimension at least one. By Theorems 1.1 and 1.4,  $I$  is non-zero since  $A$  is prime. Moreover, by Lemma 2.6 every non-zero primitive ideal must contain  $I$ . Thus either  $(0)$  is a primitive ideal and so  $A$  is primitive, or the Jacobson radical of  $A$  contains  $I$  and so  $A$  has non-zero Jacobson radical. We note that the Jacobson radical of a monomial algebra is locally nilpotent [3]. This completes the proof.  $\square$

We note that there do exist examples of prime monomial algebras of quadratic growth with non-zero Jacobson radical [15]. If an algebra simply has GK dimension 2 and is not of quadratic growth then the collection of primes  $P$  such that  $A/P$  has GK dimension 2 can be very strange; for example, they need not satisfy the ascending chain condition [5]. We conclude this section by posing the following question.

**Question 3.3** (*Trichotomy question*). Let  $A$  be a prime finitely generated monomial algebra. Is it true that  $A$  is either primitive, PI, or has non-zero Jacobson radical?

#### 4. Construction

We now show that if we replace quadratic growth with GK dimension 2 in Theorem 1.1, then the conclusion of the statement of the theorem may not hold; that is, we can find a prime monomial algebra of GK dimension 2 with unbounded matrix images. We begin with some definitions.

**Definition 4.1.** We say that a right infinite word on a two letter alphabet  $\mathcal{X}$  is *Sturmian* if it has exactly  $n + 1$  subwords of length  $n$ .

We note that if a right infinite word has fewer than  $n + 1$  subwords for some  $n$ , then it is necessarily ultimately periodic. Thus, Sturmian words have the smallest possible subword complexity without being eventually periodic. We refer the reader to Allouche and Shallit [1] for examples of such words.

**Definition 4.2.** We say that a right infinite word on a finite alphabet  $\mathcal{X}$  is *prime* if every subword occurs infinitely often.

We note that any right infinite Sturmian word  $W$  is necessarily prime, since if  $W$  has a subword  $W_1$  of length  $d$  that does not occur infinitely often, then we can write  $W = W'V$  where  $W'$  is a word and  $V$  is a right infinite word with no occurrences of  $W_1$ . Then the number of subwords of  $V$  of length  $d$  is at most  $d$ , since  $W_1$  is not a subword of  $V$ . Consequently  $V$  is ultimately periodic, and thus  $W$  is also ultimately periodic, a contradiction. We note that a doubly infinite Sturmian word need not have the property that every subword occurs infinitely often; for example, consider the word

$$\cdots xxxxyyyyy \cdots$$

and the subword  $xy$ . The reason we use the word *prime* to describe such words comes from the following result.

**Proposition 4.3.** Let  $W$  be a right infinite word on a finite alphabet  $\mathcal{X}$ . Let  $A_W$  denote the algebra obtained by taking the free algebra on  $\mathcal{X}$  and taking any word which is not a subword of  $W$  to be a relation. Then  $A_W$  is prime if and only if  $W$  is prime.

**Proof.** Let  $W_1, W_2$  be words with non-zero image in  $A_W$ . Then  $W_1$  is a subword of  $W$ . Hence  $W = aW_1b$ , where  $a$  is a finite word and  $b$  is a right infinite word. Since  $W_2$  occurs infinitely often as a subword of  $W$ ,  $W_2$  is a subword of  $b$ . Hence  $b = b'W_2b''$  for some word  $b'$  and some right infinite word  $b''$ . It follows that  $W_1b'W_2$  is a subword of  $W$  and so  $W_1b'W_2$  has non-zero image in  $A$ . It follows that  $A_W$  is prime. Conversely if  $A_W$  is not prime, then there exist non-zero words  $W_1$  and  $W_2$  such that  $W_1A_WW_2 = (0)$ . Consequently,  $W_2$  never occurs after the first occurrence of  $W_1$  and so  $W_2$  occurs only finitely many times as a subword of  $W$ .  $\square$

Let  $W$  be a right infinite prime Sturmian word. We pick subwords of  $W$  as follows. Let  $W_1$  be the first letter of  $W$ . Let  $W_2$  denote an initial subword of  $W$  of length at least 2 which ends with

$W_1$  (such a  $W_2$  exists since  $W$  is prime). In general, having chosen  $W_1, \dots, W_{n-1}$ , we choose  $W_n$  to be an initial subword of  $W$  of length at least  $2 \text{length}(W_{n-1})$  that ends with  $W_{n-1}$ . We now define a sequence of words  $U_1, U_2, \dots$  as follows. We take  $U_1 = W_1$  and in general we define

$$a_{i,j} = \lceil \text{length}(W_i) / \text{length}(W_j) \rceil,$$

$$V_n = W_n W_{n-1}^{a_{n,n-1}} \dots W_2^{a_{n,2}} W_1^{a_{n,1}} W_2^{a_{n,2}} \dots W_{n-1}^{a_{n,n-1}} W_n, \quad (6)$$

and

$$U_n = (U_{n-1} V_n)^2. \quad (7)$$

By induction, we see that

$$\begin{aligned} \text{length}(U_d) &\leq 4d \text{length}(W_d) + 8(d-1) \text{length}(W_{d-1}) + \dots + 2^{d+1} \text{length}(W_1) \\ &\leq 4d^2 \text{length}(W_d). \end{aligned} \quad (8)$$

Notice that  $U_n$  is an initial subword of  $U_{n+1}$  for every  $n$  and hence we can define the right infinite word

$$U := \lim_{n \rightarrow \infty} U_n. \quad (9)$$

**Lemma 4.4.** *Let  $W$  be a right infinite prime Sturmian word and let  $V_1, V_2, \dots$  be the words defined in Eq. (6). If  $d$  and  $n$  are positive integers satisfying  $\text{length}(W_d) \leq n < \text{length}(W_{d+1})$ , then there are at most  $12d^2(n+1)$  words of length  $n$  that occur as a subword of  $V_j$  for some  $j > d$ .*

**Proof.** We let  $g(n)$  denote the number of words of length  $n$  that are a subword of  $V_j$  for some  $j > d$ . Consider  $V_j$  for  $j > d$ . If we write it as a word in  $W_1, W_2, \dots$ , then  $W_i$  is adjacent to  $W_j$  only if  $|i-j| \leq 1$ . Moreover, by construction, it can be broken into blocks of the form  $W_i^k$  where  $\text{length}(W_i^k) > n$ . Thus any subword of  $V_j$  is a subword of  $W_i^k W_j^\ell$  with  $|i-j| \leq 1$ . If  $i, j > d$ , then  $W_i$  and  $W_j$  have length at least  $n$  and both have  $W_{d+1}$  as an initial and terminal subword. Since  $W_{d+1}$  has length at least  $n$ , we see that if  $i, j > n$ , a subword of  $W_i^k W_j^\ell$  of length  $n$  is either a subword of  $W_i$ , a subword of  $W_j$ , or a subword of  $W_{d+1}^2$ . Since we have already accounted for such words, we may consider subwords of  $W_i^k W_j^\ell$  with  $|i-j| \leq 1$  and  $i$  or  $j$  less than or equal to  $d$ . There are at most  $\text{length}(W_m)$  subwords of length  $n$  in  $W_m^p$  and at most  $n+1$  subwords formed by taking a terminal subword of  $W_i^k$  and an initial subword of  $W_j^\ell$ . Hence there are at most

$$\text{length}(W_i) + \text{length}(W_j) + n + 1$$

subwords of  $W_i^k W_j^\ell$  of length  $n$ . Thus by considering subwords of  $V_j$  for  $j > d$  we get at most

$$2d(n+1+2\text{length}(W_n)) \leq 6d(n+1)$$

unaccounted for words of length  $n$ . Thus

$$g(n) \leq 6d(n+1) + 2(n+1) + \text{length}(U_d).$$

By Eq. (8), we see that

$$\text{length}(U_d) \leq 4d^2 \text{length}(W_d) \leq 4d^2 n.$$

Thus

$$g(n) \leq (n+1)(2+6d+4d^2) \leq 12(n+1)d^2.$$

The result follows.  $\square$

**Lemma 4.5.** *Let  $n \geq 2$  and let  $U$  be the word defined in Eq. (9). Then the number of subwords of  $U$  of length  $n$  is at most  $100(n+1)(\log_2 n)^2$ .*

**Proof.** Let  $d$  be the largest integer such that  $\text{length}(W_d) \leq n$ . Since the length of  $W_j$  is at least twice the length of  $W_{j-1}$ , we see that

$$1 \leq d \leq \lfloor \log_2 n \rfloor + 1.$$

Observe that

$$U = (U_d V_{d+1} U_d V_{d+1}) V_{d+2} (U_d V_{d+1} U_d V_{d+1}) V_{d+2} \cdots,$$

where  $U_i$  and  $V_i$  are as defined in Eqs. (7) and (6).

Consequently, any subword of  $U$  of length  $n$  is either:

- a subword of  $V_j U_d V_k$  for some  $j, k > d$ ; or
- a subword consisting of a terminal subword of  $V_j$  and an initial subword of  $V_{j+1}$  for some  $j \geq d$ .

Since  $W_{d+1}$  is an initial and terminal subword of  $V_j$  for  $j > d$  and has length at least  $n$ , we see that any subword of  $U$  of length  $n$  is either a subword of  $W_{d+1} U_d W_{d+1}$ , a subword of  $W_{d+1}^2$  or a subword of  $V_j$  for some  $j > d$ .

There are exactly  $n+1$  subwords of  $W$  of length  $n$ . Any subword of  $W_{d+1}^2$  which is not a subword of  $W$  must consist of a terminal subword of  $W_{d+1}$  and an initial subword of  $W_{d+1}$ . Since there are at most  $n+1$  such words, there are at most  $2(n+1)$  words of length  $n$  which are subwords of either  $W$  or  $W_{d+1}^2$ . Any subword of  $W_{d+1} U_d W_{d+1}$  which is not a subword of  $W$  must contain a part of  $U_d$ . Hence there are at most  $\text{length}(U_d)$  such words of length  $n$ . Let  $f(n)$  denote the number of subwords of  $U$  of length  $n$ . Then we have

$$f(n) \leq 2(n+1) + \text{length}(U_d) + \# \text{ of subwords of length } n \text{ of some } V_j, \quad j > d.$$

Using Lemma 4.4 and Eq. (8), we see

$$f(n) \leq 2(n+1) + 16d^2 n.$$

Since  $d \leq \log_2 n + 1$ , we see that

$$f(n) \leq 16(n+1)(\log_2 n + 1)^2 + 2(n+1) \leq 100(n+1)(\log_2 n)^2$$

for  $n > 1$ .  $\square$

Thus we see that the number of subwords of  $U$  of length at most  $n$  does not grow too fast. Our ultimate goal is to show that the algebra  $A_U$  is a prime algebra of GK dimension 2 with unbounded matrix images. We now show that  $A_U$  is prime.

**Lemma 4.6.** *Let  $U$  be the word defined in Eq. (9). Then  $U$  is prime.*

**Proof.** To see that  $U$  is prime, let  $a$  be a subword of  $U$ . Suppose that  $a$  only appears finitely many times as a subword of  $U$ . Then there is some  $m$  such that all occurrences of  $a$  occur inside the finite word  $U_m$ . But  $U_{m+1} = U_m b U_m b$  for some word  $b$  and hence  $a$  appears at least once more in  $U_{m+1}$  than it appears in  $U_m$ . This is a contradiction.  $\square$

We use this last result along with the estimates to describe  $A_U$ .

**Proposition 4.7.** *Let  $U$  be the word defined in Eq. (9). Then  $A_U$  is a finitely generated prime monomial algebra of GK dimension 2.*

**Proof.** Let  $V$  be the subspace of  $A$  spanned by 1 and the elements of the finite alphabet  $\mathcal{X}$ . Then

$$\begin{aligned} \dim(V^n) &= 1 + f(1) + \cdots + f(n) \\ &\leq 1 + f(1) + \sum_{j=2}^n 100(j+1)(\log_2 j)^2 \\ &\leq 1 + f(1) + 100(n+1)^2(\log_2 n)^2. \end{aligned}$$

Thus for every  $\varepsilon > 0$  we have

$$\dim(V^n) < n^{2+\varepsilon} \quad \text{for } n \text{ sufficiently large.}$$

It follows that  $A_U$  has GK dimension at most 2. On the other hand, the images of the distinct subwords of the Sturmian word  $W$  are linearly independent. Hence  $\dim(V^n) \geq \binom{n+1}{2}$  and so  $A_U$  has GK dimension at least 2. The fact that  $A_U$  is prime follows from Lemma 4.6.  $\square$

We now show that the algebra  $A_U$  has unbounded matrix images. To do this we need a simple estimate for the PI degree of a class of monomial algebras.

**Lemma 4.8.** *Let  $T$  be a right infinite periodic word with minimal period  $d$ . Then  $A_T$  is a prime ring of GK dimension 1 satisfying  $S_{2d}$  and satisfying no identity of smaller degree.*

**Proof.** Since  $T$  is periodic, it is prime and hence  $A_T$  is a prime ring. Since  $T$  is periodic and infinite,  $A_T$  has GK dimension 1. It follows from the Small–Warfield theorem [14] that  $A_T$  is PI.



Let  $Y_1, \dots, Y_d$  denote the  $d$  distinct words of  $T$  of length  $d$ . Then it is easy to check that the image of

$$Z := Y_1 + \dots + Y_d$$

is central in  $A_T$ . Observe that the image of  $Y_i Y_j$  is 0 if  $i \neq j$  and  $Y_i^2 = Y_i Z = Z Y_i$  in  $A_T$ . Thus  $Y_1 Z^{-1}, \dots, Y_d Z^{-1}$  is a set of orthogonal idempotents in the quotient ring  $Q(A_T)$  of  $A_T$ . Since  $Q(A_T)$  is a simple Artinian ring,  $A$  cannot satisfy an identity of degree less than  $2d$ . Let  $K$  denote the centre of the quotient ring of  $A_T$ . We claim that the quotient ring of  $A_T$  embeds in  $M_d(\overline{K})$ . For  $1 \leq i, j \leq d$  let  $T_{i,j}$  denote the subword of  $T$  starting at the  $i$ th position of  $T$  and terminating at the  $(2d + j)$ th position of  $T$ . Let  $t \in \overline{K}$  be a primitive  $d$ th root of  $Z$  and let  $e_{i,j} = T_{i,j} t^{i-j-2d}$ . Then  $\{e_{i,j}\}$  is a set of matrix units. Moreover, any element of  $A_T$  is in the  $\overline{K}$ -span of these matrix units. Hence we get the desired embedding. It follows that  $A_T$  satisfies  $S_{2d}$  and no smaller identity.  $\square$

**Proposition 4.9.** *Let  $U$  be the word defined in Eq. (9). Then the algebra  $A_U$  has unbounded matrix images.*

**Proof.** Observe that for each  $j$  and  $m$ ,  $W_j^m$  is a subword of  $U$ . Let  $T_j$  denote the periodic right infinite word  $W_j W_j W_j \dots$ . Observe that we have a homomorphism

$$A_U \rightarrow A_{T_j}$$

since every subword of  $T_j$  is a subword of  $U$ . Let  $d_j$  denote the minimal period of  $T_j$ . Then  $A_{T_j}$  has PI degree  $2d_j$ . To show that  $A_U$  has unbounded matrix images, we must show that  $\lim_{j \rightarrow \infty} d_j = \infty$ . To see this, suppose that this is not the case. Then there exists some  $m$  such that  $d_j = m$  for infinitely many  $j$ . In particular,  $W_j$  is periodic with period  $m$  for infinitely many  $j$ . Since  $W_j$  is an initial subword of  $W$  for every  $j$  and  $\text{length}(W_j) \rightarrow \infty$ , we see that  $W$  must also be periodic and have period at most  $m$ , contradicting the fact that  $W$  has at least  $n + 1$  subwords of length  $n$  for every  $n$ .  $\square$

**Proof of Theorem 1.6.** This follows easily from Propositions 4.7 and 4.9.  $\square$

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