Journal of Algebra 321 (2009) 2943–2961

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Arc valuations on smooth varieties

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ARTICLE INFO

Article history:
Received 6 August 2008
Available online 14 February 2009
Communicated by Michel Broué

Keywords:
Valuation theory
Arc spaces

ABSTRACT

Let X be a nonsingular variety (with $\dim X \geqslant 2$) over an algebraically closed field $\mathbf{k}$ of characteristic zero. Let $\alpha : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X$ be an arc on X, and let $v = \operatorname{ord}_\alpha$ be the valuation given by the order of vanishing along α. We describe the maximal irreducible subset $C(v)$ of the arc space of X such that $\operatorname{val}_{C(v)} = v$. We describe $C(v)$ both algebraically, in terms of the sequence of valuation ideals of v, and geometrically, in terms of the sequence of infinitely near points associated to v. As a corollary, we get that v is determined by its sequence of centers. Also, when X is a surface, our construction also applies to any divisorial valuation v, and in this case $C(v)$ coincides with the one introduced in [L. Ein, R. Lazarsfeld, M. Mustaţă, Contact loci in arc spaces, Compos. Math. 140 (2004) 1229–1244, Example 2.5].

1. Introduction

Let X be a nonsingular variety over a field $\mathbf{k}$. A $\mathbf{k}$-arc γ on X is a morphism of $\mathbf{k}$-schemes $\gamma : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X$. There is a scheme X_∞, called the arc space of X, which parametrizes the arcs on X. We refer the reader to [EM, Section 2] for the construction of X_∞. Denote the closed point of $\operatorname{Spec} \mathbf{k}[\![t]\!]$ by o.

In this paper, I study valuations $\operatorname{ord}_\gamma : \mathcal{O}_{X,\gamma(o)} \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ given by the order of vanishing along a $\mathbf{k}$-arc $\gamma : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X$. Such valuations are precisely the $\mathbb{Z}_{\geqslant 0} \cup \{\infty\}$-valued valuations with transcendence degree zero. I associate to $\operatorname{ord}_\gamma$ several different natural subsets of the arc space X_∞. I prove if γ is a nonsingular arc, then these subsets associated to $\operatorname{ord}_\gamma$ are equal. Furthermore, I show this subset is irreducible, and the valuation given by the order of vanishing along a general arc of this subset is equal to the original valuation $\operatorname{ord}_\gamma$.

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[1] The author is grateful to have been supported by the NSF RTG grant DMS-0502170.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.02.002

The motivation for this project was the discovery by Ein, Lazarsfeld, and Mustaţă [ELM, Theorem C] that divisorial valuations (equivalently, valuations with transcendence degree $\dim X - 1$) correspond to a special class of subsets of the arc space called cylinders. More specifically, for a divisorial valuation val_E given by the order of vanishing along a prime divisor E over X, there is an irreducible cylinder $C_{\mathrm{div}}(E) \subseteq X_\infty$ such that for a general arc $\gamma \in C_{\mathrm{div}}(E)$, we have that the order of vanishing of any rational function $f \in \mathbb{C}(X)$ along γ equals its order of vanishing along E. In symbols, $\mathrm{ord}_\gamma(f) = \mathrm{val}_E(f)$ for all $f \in \mathbb{C}(X)$. Conversely, it is shown in [ELM, Theorem C] that every valuation given by the order of vanishing along a general arc of a cylinder is a divisorial valuation.

The goal of this paper is to investigate whether other types of valuations, besides divisorial ones, have a similar interpretation via the arc space. We find there is a nice answer for valuations given by the order of vanishing along a nonsingular arc on a nonsingular variety X. If X is a surface, all valuations with value group $\mathbb{Z}^r$ (lexicographically ordered) for some r are equivalent to a valuation of this type. One can interpret our results as being complementary to those of Ein et al. as follows. Both say that valuations are encoded in a natural way as closed subsets of the arc space. We address the case when the transcendence degree is zero, whereas Ein et al. study the case of valuations with transcendence degree $\dim X - 1$.

1.1. Valuations and subsets of the arc space

In this section, I begin by explaining the relationship between valuations on a nonsingular variety X over a field $\mathbf{k}$ and subsets of the arc space X_∞ of X. I then construct several natural subsets of the arc space that one might associate to a valuation. One of the main results of this paper is that for a large class of valuations, these different constructions agree, i.e. they define the same subset of the arc space.

We need to introduce some notation. An arc $\gamma : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X$ gives a $\mathbf{k}$-algebra homomorphism $\gamma^* : \widehat{\mathcal{O}}_{X,\gamma(o)} \to \mathbf{k}[\![t]\!]$, where o denotes the closed point of $\operatorname{Spec} \mathbf{k}[\![t]\!]$. We define a valuation $\mathrm{ord}_\gamma : \widehat{\mathcal{O}}_{X,\gamma(o)} \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ by $\mathrm{ord}_\gamma(f) = \mathrm{ord}_t\, \gamma^*(f)$ for $f \in \widehat{\mathcal{O}}_{X,\gamma(o)}$. If $\gamma^*(f) = 0$, we will adopt the convention that $\mathrm{ord}_\gamma(f) = \infty$.

Given an ideal sheaf $\mathfrak{a} \subseteq \mathcal{O}_X$ on X we set $\mathrm{ord}_\gamma(\mathfrak{a}) = \min_{f \in \mathfrak{a}_{\gamma(o)}} \mathrm{ord}_\gamma(f)$. For a nonnegative integer q, we define the qth order *contact locus* of $\mathfrak{a}$ by

$$\mathrm{Cont}^{\geqslant q}(\mathfrak{a}) = \{\gamma : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X \mid \mathrm{ord}_\gamma(\mathfrak{a}) \geqslant q\}. \tag{1}$$

The following definition appears in [ELM, p. 3], and provided, at least for us, the initial link between valuations and arc spaces:

Definition 1.1. Let $C \subseteq X_\infty$ be a nonempty irreducible subset. Assume C is a cylinder [ELM, p. 4]. Define a valuation $\mathrm{val}_C : \mathbf{k}(X) \to \mathbb{Z}$ on the function field $\mathbf{k}(X)$ of X as follows. For $f \in \mathbf{k}(X)$, set

$$\mathrm{val}_C(f) = \mathrm{ord}_\gamma(f)$$

for general $\gamma \in C$. Equivalently, if $\alpha \in C$ is the generic point of C, then $\mathrm{val}_C(f) = \mathrm{ord}_\alpha(f)$. (Caveat: α need not be a $\mathbf{k}$-valued point of X_∞. See Remark 2.3.)

It turns out that the conditions that C is a cylinder and X is nonsingular imply that $\mathrm{val}_C(f)$ is always finite [ELM, Proposition 1.1]. If we drop the assumption that C is a cylinder, then the map ord_α (where α is the generic point of C) is a $\mathbb{Z}_{\geqslant 0} \cup \{\infty\}$-valued valuation on $\mathcal{O}_{X,\alpha(o)}$.

We now describe a way to go from valuations centered on X to subsets of the arc space. Following Ishii [Ishii2, Definition 2.8], we associate to a valuation v a subset $C(v) \subseteq X_\infty$ in the following way.

Definition 1.2. Let $p \in X$ be a (not necessarily closed) point. Let $v : \widehat{\mathcal{O}}_{X,p} \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ be a valuation. Define the *maximal arc set* $C(v)$ by

$$C(v) = \overline{\{\gamma \in X_\infty \mid \operatorname{ord}_\gamma = v,\ \gamma(o) = p\}} \subseteq X_\infty,$$

where the bar denotes closure in X_∞.

If we start with an irreducible subset C, we get a valuation val_C by Definition 1.1. We can then form the subset $C(\operatorname{val}_C)$ as in Definition 1.2. We have $C \subseteq C(\operatorname{val}_C)$ because $C(\operatorname{val}_C)$ contains the generic point of C. In general, we do not have equality.

We can associate another subset of X_∞ to a valuation v on a nonsingular variety X as follows. Let $\{E_q\}_{q \geqslant 1}$ be the sequence of divisors formed by blowing up successive centers of v (see Definition 2.8). Following [ELM, Example 2.5], to each divisor E_q we associate a cylinder $C_q = C_{\mathrm{div}}(E_q) \subseteq X_\infty$. Using notation we will explain in Section 2, we will define $C_q = \mu_{q\infty}(\operatorname{Cont}^{\geqslant 1}(E_q))$. In words, C_q is simply the set of arcs on X whose lift to X_{q-1} (a model of X formed by blowing up $q-1$ successive centers of v) has the same center on X_{q-1} as v. This collection $\{C_q\}_{q \geqslant 1}$ of cylinders forms a decreasing nested sequence. We take their intersection, $\bigcap_{q \geqslant 1} C_q$, to get another subset of X_∞ that is reasonable to associate with v.

On the other hand, another way the valuation v can be studied is through its valuation ideals $\mathfrak{a}_q = \{f \in \widehat{\mathcal{O}}_{X,p} \mid v(f) \geqslant q\}$, where q ranges over the positive integers. The set $\bigcap_{q \geqslant 1} \operatorname{Cont}^{\geqslant q}(\mathfrak{a}_q)$ is yet another reasonable set to associate with v.

Given an arc $\alpha : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X$, we have an induced map $\alpha^* : \widehat{\mathcal{O}}_{X,\alpha(o)} \to \mathbf{k}[\![t]\!]$. We associate to $\operatorname{ord}_\alpha$ the set

$$\mathcal{I} = \{\gamma \in X_\infty \mid \gamma(o) = \alpha(o),\ \ker(\alpha^*) \subseteq \ker(\gamma^*) \subseteq \widehat{\mathcal{O}}_{X,\alpha(o)}\}. \tag{2}$$

In words, $\mathcal{I}$ is the set of arcs γ with $\operatorname{ord}_\gamma(f) = \infty$ for all $f \in \widehat{\mathcal{O}}_{X,\alpha(o)}$ with $\operatorname{ord}_\alpha(f) = \infty$.

Finally, let $R = \{\alpha \circ h \in X_\infty \mid h : \operatorname{Spec} \mathbf{k}[\![t]\!] \to \operatorname{Spec} \mathbf{k}[\![t]\!]\}$. In words, R is the set of $\mathbf{k}$-arcs that are reparametrizations of α.

The main result of this paper is that for valuations $v = \operatorname{ord}_\alpha$, all five of these closed subsets ($C(v)$, $\bigcap_{q \geqslant 1} C_q$, $\bigcap_{q \geqslant 1} \operatorname{Cont}^{\geqslant q}(\mathfrak{a}_q)$, $\mathcal{I}$, R) are equal. Furthermore, this subset is irreducible, and the valuation given by the order of vanishing along a general arc of this subset is equal to v.

For convenience, we will assume the arc α we begin with is normalized, that is, the set $\{v(f) \mid f \in \widehat{\mathcal{O}}_{X,p},\ 0 < v(f) < \infty\}$ (where $v = \operatorname{ord}_\alpha$) is nonempty and the greatest common factor of its elements is 1. Every arc valuation taking some value strictly between 0 and ∞ is a scalar multiple of a normalized valuation.

Also, we restrict ourselves to considering the $\mathbf{k}$-arcs in the sets described above. We denote by $(X_\infty)_0$ the subset of points of X_∞ with residue field equal to $\mathbf{k}$. If $D \subseteq X_\infty$, then we set $D_0 = D \cap (X_\infty)_0$.

Theorem 1.3. *Let $\alpha : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X$ be a normalized arc on a nonsingular variety X ($\dim X \geqslant 2$) over an algebraically closed field $\mathbf{k}$ of characteristic zero. Set $v = \operatorname{ord}_\alpha$. Then the following closed subsets of $(X_\infty)_0$ are equal*:

$$C(v)_0 = \Big(\bigcap_{q \geqslant 1} C_q\Big)_0 = \Big(\bigcap_{q \geqslant 1} \operatorname{Cont}^{\geqslant q}(\mathfrak{a}_q)\Big)_0 = (\mathcal{I})_0 = R.$$

Furthermore, the valuation given by the order of vanishing along a general arc of this subset is equal to v.

As a corollary, we get that $\mathbf{k}$-arc valuations are determined by their sequence of centers.

Corollary 1.4. *Let X be a nonsingular variety over an algebraically closed field $\mathbf{k}$ of characteristic zero. Let $v : \mathcal{O}_{X,p} \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ be a normalized $\mathbf{k}$-arc valuation, where $p \in X$ is the center of v on X. Then v is uniquely determined by its sequence of centers, that is, if $v' : \mathcal{O}_{X,p} \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ is another normalized $\mathbf{k}$-arc valuation with the same sequence of centers as v, then $v = v'$ on $\mathcal{O}_{X,p}$.*

The author suspects the above corollary is known (or can be easily directly proved) by experts in valuation theory, but has been unable a reference for it in the literature. It is a classical fact that valuations on surfaces are determined by their sequence of centers. On the other hand, M. Núñez [Núñez] has given examples (with $\dim X \geqslant 3$) of valuations that are not determined by their sequence of centers. Núñez's examples are valuations defined by power series with fractional exponents.

Remark 1.5. If X is a surface and if v is a divisorial valuation, then $\bigcap_{q>0} C_q$ equals the cylinder C_r associated to v in [ELM, Example 2.5], where r is such that p_r is a divisor.

1.2. Outline of the paper

In Section 2 we recall some basic terminology and results regarding arc spaces. In Section 3 we define arc valuations, and we compare them with other notions of a valuation. In Section 4 we show that **k**-arc valuations can be desingularized. We will need this result in Section 5, where we study **k**-arc valuations on nonsingular varieties. We first study the case of a nonsingular arc valuation. Later we consider more general arc valuations and prove Theorem 1.3.

2. Background on arc spaces

Let X be a variety over a field $\mathbf{k}$. Let $\mathbf{k} \subseteq K$ be a field extension. The arc space X_∞ is a scheme over $\mathbf{k}$ whose K-valued points are morphisms $\operatorname{Spec} K[[t]] \to X$ of $\mathbf{k}$-schemes, since we have

$$\operatorname{Hom}(\operatorname{Spec} K, X_\infty) = \operatorname{Hom}\bigl(\operatorname{Spec} K[[t]], X\bigr). \tag{3}$$

In particular, when X is affine, giving a K-valued point of X_∞ is the same thing as giving a homomorphism of $\mathbf{k}$-algebras $\Gamma(X, \mathcal{O}_X) \to K[[t]]$.

Definition 2.1. Let $\mathbf{k} \subseteq K$ be a field extension. A *K-arc* is a morphism of $\mathbf{k}$-schemes $\operatorname{Spec} K[[t]] \to X$.

If $\mu : X' \to X$ is a morphism of schemes, then we have an induced morphism $\mu_\infty : X'_\infty \to X_\infty$ sending γ to $\mu \circ \gamma$.

Let $\gamma : \operatorname{Spec} K[[t]] \to X$ be a K-arc on X. Let $x = \gamma(o)$. Given an ideal sheaf $\mathfrak{a}$ on X, we define $\operatorname{ord}_\gamma(\mathfrak{a}) = \min_{f \in \mathfrak{a}_x} \operatorname{ord}_\gamma(f)$. For a nonnegative integer p, we define $\operatorname{Cont}^{\geqslant p}(\mathfrak{a})$, the *contact locus* of $\mathfrak{a}$ of order p, to be the closed subscheme of X_∞ whose K-valued points (where $\mathbf{k} \subseteq K$ is an extension of fields) are

$$\operatorname{Cont}^{\geqslant p}(\mathfrak{a})(K) = \bigl\{\gamma : \operatorname{Spec} K[[t]] \to X \bigm| \operatorname{ord}_\gamma(\mathfrak{a}) \geqslant p\bigr\}. \tag{4}$$

If Z is a closed subscheme of X defined by the ideal sheaf $\mathcal{I}$, we write $\operatorname{Cont}^{\geqslant p}(Z)$ for $\operatorname{Cont}^{\geqslant p}(\mathcal{I})$. If a closed subscheme structure on a closed subset of X has not been specified, we implicitly give it the reduced subscheme structure.

For an ideal $\mathfrak{a}$ of $\widehat{\mathcal{O}}_{X,\gamma(o)}$, we define $\operatorname{ord}_\gamma(\mathfrak{a}) = \min_{f \in \mathfrak{a}} \operatorname{ord}_\gamma(f)$. For $x \in X$ and an ideal $\mathfrak{a}$ of $\widehat{\mathcal{O}}_{X,x}$ we have a closed subscheme $\operatorname{Cont}^{\geqslant p}(\mathfrak{a})$ of X_∞ whose K-valued points (where $\mathbf{k} \subseteq K$ is an extension of fields) are

$$\operatorname{Cont}^{\geqslant p}(\mathfrak{a})(K) = \bigl\{\gamma : \operatorname{Spec} K[[t]] \to X \bigm| \gamma(o) = x,\ \operatorname{ord}_\gamma(\mathfrak{a}) \geqslant p\bigr\}. \tag{5}$$

Proposition 2.2. *Let X be a variety over a field $\mathbf{k}$. Let $\gamma : \operatorname{Spec} \mathbf{k}[[t]] \to X$ be a $\mathbf{k}$-arc. Then $\gamma(o) \in X$ is a closed point of X with residue field $\mathbf{k}$.*

Proof. Set $p = \gamma(o)$, and let $\kappa(p)$ denote the residue field of $p \in X$. We have a local $\mathbf{k}$-algebra homomorphism $\gamma^* : \mathcal{O}_{X,p} \to \mathbf{k}[\![t]\!]$. Taking the quotient by the maximal ideals, we get a $\mathbf{k}$-algebra homomorphism $\kappa(p) \hookrightarrow \mathbf{k}$ that is an isomorphism on $\mathbf{k} \subseteq \kappa(p)$. Hence $\kappa(p) = \mathbf{k}$. Since $\mathrm{tr.deg}_{\mathbf{k}}\, \kappa(p) = 0$, it follows that p is a closed point. □

2.1. Points of the arc space

We next make a couple of remarks about the notion of a *point of the arc space*.

Remark 2.3. Let X be a scheme of finite type over a field $\mathbf{k}$. Let $\alpha \in X_\infty$ be a (not necessarily closed) point of the scheme X_∞. That is, in some open affine patch of X_∞, α corresponds to a prime ideal. Let $\kappa(\alpha)$ denote the residue field at the point α of the scheme X_∞. There is a canonical morphism $\Theta_\alpha : \mathrm{Spec}\, \kappa(\alpha) \to X_\infty$ induced by the canonical $\mathbf{k}$-algebra homomorphism $\mathcal{O}_{X_\infty,\alpha} \to \kappa(\alpha)$. By Eq. (3), the morphism Θ_α corresponds to a $\kappa(\alpha)$-arc $\theta_\alpha : \mathrm{Spec}\, \kappa(\alpha)[\![t]\!] \to X$. We will abuse notation and refer to this arc $\theta_\alpha : \mathrm{Spec}\, \kappa(\alpha)[\![t]\!] \to X$ by $\alpha : \mathrm{Spec}\, \kappa(\alpha)[\![t]\!] \to X$. That is, given a point $\alpha \in X_\infty$, we have a canonical $\kappa(\alpha)$-arc $\alpha : \mathrm{Spec}\, \kappa(\alpha)[\![t]\!] \to X$.

Remark 2.4. We now examine the reverse of the construction given in Remark 2.3. Let $\mathbf{k} \subseteq K$ be some extension of fields. Given a K-arc $\theta : \mathrm{Spec}\, K[\![t]\!] \to X$, by Eq. (3), we get a morphism $\Theta : \mathrm{Spec}\, K \to X_\infty$. The image $\Theta(\mathtt{pt})$ of the only point $\mathtt{pt}$ of $\mathrm{Spec}\, K$ is a point of X_∞, call it α. By Remark 2.3, we associate to α a $\kappa(\alpha)$-arc $\Theta_\alpha : \mathrm{Spec}\, \kappa(\alpha)[\![t]\!] \to X$. Note that $\Theta : \mathrm{Spec}\, K \to X_\infty$ factors through $\Theta_\alpha : \mathrm{Spec}\, \kappa(\alpha) \to X_\infty$, since on the level of rings, the $\mathbf{k}$-algebra map $\Theta^* : \mathcal{O}_{X_\infty,\alpha} \to K$ induces a map $\kappa(\alpha) \to K$. Hence $\theta : \mathrm{Spec}\, K[\![t]\!] \to X$ factors through $\theta_\alpha : \mathrm{Spec}\, \kappa(\alpha)[\![t]\!] \to X$. To summarize, K-arcs on X correspond to K-valued points of X_∞. To each K-valued point of X_∞, we can assign a point of X_∞. If we let K range over all field extensions on $\mathbf{k}$, this assignment is surjective onto the set of points of X_∞, but it is not injective. To a point α of X_∞, we assign (as described in Remark 2.3) a canonical $\kappa(\alpha)$-valued point of X_∞. The point of X_∞ that we assign to this $\kappa(\alpha)$-valued point is α.

Remark 2.5. Let p be a closed point of an n-dimensional nonsingular variety X, and fix generators $x_1, \dots, x_n$ of the maximal ideal of $\mathcal{O}_{X,p}$. Let $\mathbf{k} \subseteq K$ be an extension of fields. Giving an arc $\gamma : \mathrm{Spec}\, K[\![t]\!] \to X$ such that $\gamma \in \mathrm{Cont}^{\geqslant 1}(p)(K)$ is equivalent to giving a homomorphism of $\mathbf{k}$-algebras $\widehat{\mathcal{O}}_{X,p} \simeq \mathbf{k}[\![x_1, \dots, x_n]\!] \to K[\![t]\!]$ sending each x_i into $(t)K[\![t]\!]$.

Definition 2.6. We say an arc $\gamma : \mathrm{Spec}\, K[\![t]\!] \to X$ is a *trivial arc* if the maximal ideal of $\widehat{\mathcal{O}}_{X,\gamma(o)}$ equals the kernel of the map $\gamma^* : \widehat{\mathcal{O}}_{X,\gamma(o)} \to K[\![t]\!]$.

We have the following observation (whose proof we leave to the reader).

Lemma 2.7. *Let X be a nonsingular variety. If $\mu : X' \to X$ is the blowup of a closed point $p \in X$, with exceptional divisor E, then*:

(1) *Let $\gamma : \mathrm{Spec}\, K[\![t]\!] \to X$ be an arc such that $\gamma \in \mathrm{Cont}^{\geqslant 1}(p)$, and suppose γ is not the trivial arc. Then there exists a unique arc $\gamma' : \mathrm{Spec}\, K[\![t]\!] \to X'$ lifting γ, i.e. $\gamma = \mu \circ \gamma'$. Furthermore, $\gamma' \in \mathrm{Cont}^{\geqslant 1}(E)$.*
(2) *If γ is as in part (1) and additionally $K = \mathbf{k}$, then the residue field at $\gamma'(o) \in X'$ equals $\mathbf{k}$. Furthermore, if $\mathrm{ord}_\gamma(x_1) \leqslant \mathrm{ord}_\gamma(x_i)$ for all $2 \leqslant i \leqslant n$, then there exist $c_i \in \mathbf{k}$ (for $2 \leqslant i \leqslant n$) such that x_1 and $\frac{x_i}{x_1} - c_i$ for $2 \leqslant i \leqslant n$ are local algebraic coordinates at $\gamma'(o)$.*
(3) *$\mu_\infty(\mathrm{Cont}^{\geqslant 1}(E)) = \mathrm{Cont}^{\geqslant 1}(p)$.*

We now describe a geometric construction, called the sequence of centers of a valuation, that is useful in studying valuations, especially those on smooth surfaces. We give the definition only for valuations given by the order of vanishing along an arc $\gamma : \mathrm{Spec}\, \mathbf{k}[\![t]\!] \to X$, as this is the case we will be interested in. For a general valuation, the definition is similar [H, Exercise II.4.12].

Definition 2.8 *(Sequences of centers of an arc valuation).* Let X be a nonsingular variety over a field $\mathbf{k}$ and let $\gamma : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X$ be an arc that is not a trivial arc. The point $p_0 := \gamma(o)$ is called the *center* of v on X. We blow up p_0 to get a model X_1 with exceptional divisor E_1. By Lemma 2.7 the arc γ has a unique lift to an arc $\gamma_1 : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X_1$. Let p_1 be the closed point $\gamma_1(o)$. We define inductively a sequence of closed points p_i and exceptional divisors E_i on models X_i, and lifts $\gamma_i : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X_i$ of γ as follows. We blow up $p_{i-1} \in X_{i-1}$ to get a model X_i, and let E_i be the exceptional divisor of this blowup. Let $\gamma_i : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X_i$ be the lift of $\gamma_{i-1} : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X_{i-1}$. We denote by p_i the closed point $\gamma_i(o)$, and by $\mu_i : X_i \to X$ the composition of the first i blowups. We call $\{p_i\}_{i \geqslant 0}$ the *sequence of centers* of γ.

3. Arc valuations

In this section, we begin the study of arc valuations, which are the central object of this paper. We begin by defining arc valuations, normalized arc valuations, and nonsingular arc valuations.

Definition 3.1 *(Arc valuations).* Let X be a variety over a field $\mathbf{k}$, and let $p \in X$ be a (not necessarily closed) point. Let $\mathbf{k} \subseteq K$ be an extension of fields. A *K-arc valuation* v on X centered at p is a map $v : \mathcal{O}_{X,p} \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ such that there exists an arc $\gamma : \operatorname{Spec} K[\![t]\!] \to X$ with $\gamma(o) = p$ (where o is the closed point of $\operatorname{Spec} K[\![t]\!]$) and $v(f) = \operatorname{ord}_\gamma(f)$ for $f \in \mathcal{O}_{X,p}$. Since $\operatorname{ord}_\gamma$ extends uniquely to $\widehat{\mathcal{O}}_{X,p}$ (the completion of $\mathcal{O}_{X,p}$ at its maximal ideal), we can extend v to $\widehat{\mathcal{O}}_{X,p}$ as well. This extension does not depend on the choice of arcs γ satisfying $v = \operatorname{ord}_\gamma$ on $\mathcal{O}_{X,p}$. Therefore we will also regard arc valuations as maps $v : \widehat{\mathcal{O}}_{X,p} \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ without additional comment.

It is shown in [Ishii, Proposition 2.11] that every divisorial valuation is an arc valuation.

Definition 3.2 *(Normalized arc valuations).* We call an arc valuation v centered at a point $p \in X$ *normalized* if the set $\{v(f) \mid f \in \widehat{\mathcal{O}}_{X,p},\ 0 < v(f) < \infty\}$ is nonempty and the greatest common factor of its elements is 1. Every arc valuation taking some value strictly between 0 and ∞ is a scalar multiple of a normalized valuation. We say an arc $\gamma : \operatorname{Spec} K[\![t]\!] \to X$ is *normalized* if $\operatorname{ord}_\gamma : \widehat{\mathcal{O}}_{X,\gamma(o)} \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ is a normalized arc valuation.

Notation 3.3. Let X be a nonsingular variety over an algebraically closed field $\mathbf{k}$ of characteristic zero. Let $\gamma : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X$ be an arc centered at $p \in X$ and let $\gamma^* : \widehat{\mathcal{O}}_{X,p} \to \mathbf{k}[\![t]\!]$ be the corresponding $\mathbf{k}$-algebra morphism. Assume γ is not a trivial arc. Define a $\mathbf{k}$-algebra A_γ by $A_\gamma = \widehat{\mathcal{O}}_{X,p} / \ker(\gamma^*)$. Let $\tilde{A_\gamma}$ be the normalization of A_γ. Then γ^* induces an injective $\mathbf{k}$-algebra map $\overline{\gamma}^* : A_\gamma \hookrightarrow \mathbf{k}[\![t]\!]$ which extends to an injective $\mathbf{k}$-algebra homomorphism $\overline{\gamma}^* : \tilde{A_\gamma} \hookrightarrow \mathbf{k}[\![t]\!]$. We denote by $\operatorname{ord}_{\overline{\gamma}}$ the composition $\operatorname{ord}_t \circ \overline{\gamma}^* : \tilde{A_\gamma} \to \mathbb{Z}_{\geqslant 0}$. Note that for $f \in \widehat{\mathcal{O}}_{X,p} \setminus \ker(\gamma^*)$, we have $\operatorname{ord}_\gamma(f) = \operatorname{ord}_{\overline{\gamma}}(\overline{f})$. We will show in Lemma 3.5 that there exists $\phi \in \mathbf{k}[\![t]\!]$ such that the image of $\overline{\gamma}^* : \tilde{A_\gamma} \hookrightarrow \mathbf{k}[\![t]\!]$ equals $\mathbf{k}[\![\phi]\!] \subseteq \mathbf{k}[\![t]\!]$.

Lemma 3.4. *We use the setup described in Notation* 3.3. *The ring homomorphism $\overline{\gamma}^* : A_\gamma \hookrightarrow \mathbf{k}[\![t]\!]$ gives $\mathbf{k}[\![t]\!]$ the structure of a finite A_γ-module. In particular, A_γ has Krull dimension one.*

Proof. Choose local coordinates $x_1, \ldots, x_n$ at p such that $\gamma^*(x_1) \neq 0$. We have $\gamma^*(x_1) = t^r u$ for some positive integer r and unit $u \in \mathbf{k}[\![t]\!]$. Since $\mathbf{k}$ is algebraically closed and has characteristic zero, there exists a unit $v \in \mathbf{k}[\![t]\!]$ such that $v^r = u$. Indeed, we may use the binomial series and take $v = u^{1/r}$ to be an rth root of u.

Let $\tau : \mathbf{k}[\![t]\!] \to \mathbf{k}[\![t]\!]$ be the $\mathbf{k}$-algebra automorphism of $\mathbf{k}[\![t]\!]$ defined by $\tau(t) = tv^{-1}$. Then $\tau(\gamma^*(x_1)) = \tau(t^r u) = t^r v^{-r} u = t^r$. Therefore, we may assume without loss of generality that $\gamma^*(x_1) = t^r$.

I claim $1, t, \ldots, t^{r-1}$ generate $\mathbf{k}[\![t]\!]$ as a module over A_γ. Let $f(t) = \sum_{i \geqslant 0} f_i t^i \in \mathbf{k}[\![t]\!]$ with $f_i \in \mathbf{k}$ for all $i \geqslant 0$. For $0 \leqslant j \leqslant r$, define a power series $p_j(X) \in \mathbf{k}[\![X]\!]$ by $p_j(X) = \sum_{i \geqslant 0} f_{j+ir} X^i$. Then

$$\sum_{j=0}^{j=r-1} \gamma^*\big(p_j(x_1)\big)t^j = \sum_{j=0}^{j=r-1} p_j\big(\gamma^*(x_1)\big)t^j = \sum_{j=0}^{j=r-1} p_j\big(t^r\big)t^j$$
$$= \sum_{j=0}^{j=r-1} \sum_{i\geqslant 0} f_{j+ir}t^{j+ir} = \sum_{i\geqslant 0} f_i t^i = f(t).$$

Hence $1, t, \dots, t^{r-1}$ generate $\mathbf{k}[\![t]\!]$ considered as a module over A_γ via the ring homomorphism $\overline{\gamma}^* : A_\gamma \hookrightarrow \mathbf{k}[\![t]\!]$. Since $\mathbf{k}[\![t]\!]$ has dimension one and module finite ring extensions preserve dimension [Eisenbud, Proposition 9.2], we conclude A_γ has dimension one. $\square$

Lemma 3.5. *We continue using the setup and hypotheses of Lemma* 3.4*. There exists $\phi \in \mathbf{k}[\![t]\!]$ such that the image of $\overline{\gamma}^* : \tilde{A_\gamma} \hookrightarrow \mathbf{k}[\![t]\!]$ equals $\mathbf{k}[\![\phi]\!] \subseteq \mathbf{k}[\![t]\!]$.*

Proof. Since an integral extension of rings preserves dimension [Eisenbud, Proposition 9.2], we have that $\tilde{A}_\gamma$ has dimension one. Since $\mathbf{k}[\![t]\!]$ is normal (in fact it is a DVR), the local $\mathbf{k}$-algebra map $\overline{\gamma}^* : A_\gamma \hookrightarrow \mathbf{k}[\![t]\!]$ extends to a $\mathbf{k}$-algebra map $\overline{\gamma}^* : \tilde{A_\gamma} \hookrightarrow \mathbf{k}[\![t]\!]$.

I claim the ring $\tilde{A_\gamma}$ is a complete local domain. The local ring A_γ is complete since it is the image of a complete local ring. The normalization of an excellent ring A (in our case, the complete local domain A_γ) is module finite over A [Matsumura80, p. 259]. A module finite domain over a complete local domain is local and complete (apply [Eisenbud, Corollary 7.6] and use the domain hypothesis to conclude there is only one maximal ideal). Hence $\tilde{A_\gamma}$ is a complete local domain.

Since $\tilde{A_\gamma}$ is a complete normal 1-dimensional local domain containing the field $\mathbf{k}$, it is isomorphic to a power series over $\mathbf{k}$ in one variable [Matsumura80, Corollary 2, p. 206]. That is, there exists $\phi \in \mathbf{k}[\![t]\!]$ such that the image of $\overline{\gamma}^* : \tilde{A_\gamma} \hookrightarrow \mathbf{k}[\![t]\!]$ equals $\mathbf{k}[\![\phi]\!] \subseteq \mathbf{k}[\![t]\!]$. $\square$

The following result was pointed out to me by Mel Hochster.

Proposition 3.6. *Assume the setup of Notation* 3.3 *and let ϕ be as in Lemma* 3.5*. Let d be the greatest common divisor of the elements of the nonempty set $\{\mathrm{ord}_\gamma(f) \mid f \in \widehat{\mathcal{O}}_{X,p},\ 0 < \mathrm{ord}_\gamma(f) < \infty\}$. Then $d = \mathrm{ord}_t(\phi)$. In particular,* ord_γ *is a normalized arc valuation if and only if* $\mathrm{ord}_t(\phi) = 1$.

Proof. For $f, g \in A_\gamma$ such that $\frac{f}{g} \in \tilde{A_\gamma} \subseteq \mathrm{Frac}(A_\gamma)$, we have $\mathrm{ord}_{\overline{\gamma}}(\frac{f}{g}) = \mathrm{ord}_{\overline{\gamma}}(f) - \mathrm{ord}_{\overline{\gamma}}(g)$, and hence d divides $\mathrm{ord}_{\overline{\gamma}}(\frac{f}{g})$. In particular d divides $\mathrm{ord}_t(\phi)$. We have $\overline{\gamma}^*(A_\gamma) \subseteq \overline{\gamma}^*(\tilde{A_\gamma}) = \mathbf{k}[\![\phi]\!] \subseteq \mathbf{k}[\![t]\!]$ and hence $\mathrm{ord}_t(\phi)$ divides $\mathrm{ord}_\gamma(f)$ for all $f \in A_\gamma$. So $\mathrm{ord}_t(\phi)$ divides d. Hence $d = \mathrm{ord}_t(\phi)$. $\square$

Definition 3.7 *(Nonsingular arc valuations).* Let v be an arc valuation centered at p, and let $\mathfrak{m}_p$ denote the maximal ideal of $\widehat{\mathcal{O}}_{X,p}$. We call v *nonsingular* if

$$\min_{f\in\mathfrak{m}_p} v(f) = 1. \tag{6}$$

If $\gamma \in X_\infty$, then we say γ is nonsingular if ord_γ is a nonsingular valuation.

Let C be an irreducible subset of X_∞, and let α be the generic point of C. Following Ein, Lazarsfeld, and Mustaţă [ELM, p. 3], we define a map $\mathrm{val}_C : \mathcal{O}_{X,\alpha(o)} \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ by setting for $f \in \mathcal{O}_{X,\alpha(o)}$

$$\mathrm{val}_C(f) = \min\big\{\mathrm{ord}_\gamma(f) \;\big|\; \gamma \in C \text{ such that } f \in \mathcal{O}_{X,\gamma(o)}\big\}. \tag{7}$$

Proposition 3.8. *If $C \subseteq X_\infty$ is an irreducible subset with generic point $\alpha : \mathrm{Spec}\, K[\![t]\!] \to X$, then* $\mathrm{val}_C = \mathrm{ord}_\alpha$ *on $\mathcal{O}_{X,\alpha(o)}$. In particular,* val_C *is an arc valuation.*

Proof. Fix $f \in \mathcal{O}_{X,\alpha(o)}$, and let $U \subseteq X$ be the maximal open set on which f is regular. We have $\operatorname{ord}_\alpha(f) \geqslant \operatorname{val}_C(f)$ by Eq. (7). Let $\alpha' \in C$ be such that $\operatorname{val}_C(f) = \operatorname{ord}_{\alpha'}(f)$. Let $\pi : X_\infty \to X$ be the canonical morphism sending $\gamma \to \gamma(o)$. If $\operatorname{ord}_\alpha(f) > \operatorname{val}_C(f)$, then $C \cap \operatorname{Cont}^{\geqslant \operatorname{ord}_\alpha(f)}(f)$ is a closed subset of the irreducible set $C \cap \pi^{-1}(U)$, containing α but not $\alpha' \in C$, contradicting $\overline{\{\alpha\}} = C$. Hence $\operatorname{ord}_\alpha(f) = \operatorname{val}_C(f)$ for all $f \in \mathcal{O}_{X,\alpha(o)}$. $\square$

Next, we show arc valuations are the same as $\mathbb{Z}_{\geqslant 0} \cup \{\infty\}$-valued valuations, which are defined as follows:

Definition 3.9. Let R be a $\mathbf{k}$-algebra. A $\mathbb{Z}_{\geqslant 0} \cup \{\infty\}$-*valued valuation* on R is a map $v : R \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ such that

(1) $v(c) = 0$ for $c \in \mathbf{k}^*$,
(2) $v(0) = \infty$,
(3) $v(xy) = v(x) + v(y)$ for $x, y \in R$,
(4) $v(x + y) \geqslant \min\{v(x), v(y)\}$ for $x, y \in R$,
(5) v is not identically 0 on R^*.

Notation 3.10. Let $p \in X$ be a (not necessarily closed) point of X, and let $v : \mathcal{O}_{X,p} \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ be a $\mathbb{Z}_{\geqslant 0} \cup \{\infty\}$-valued valuation. Set $\mathfrak{p} = \{f \in \mathcal{O}_{X,p} \mid v(f) = \infty\}$. We have an induced valuation $\tilde{v} : \mathcal{O}_{X,p}/\mathfrak{p} \setminus \{0\} \to \mathbb{Z}$ that extends as usual to a valuation $\tilde{v} : \operatorname{Frac}(\mathcal{O}_{X,p}/\mathfrak{p}) \setminus \{0\} \to \mathbb{Z}$. Set $\tilde{v}(0) = \infty$. Let $R_{\tilde{v}} = \{f \in \operatorname{Frac}(\mathcal{O}_{X,p}/\mathfrak{p}) \mid \tilde{v}(f) \geqslant 0\}$ be the valuation ring of $\tilde{v}$. $R_{\tilde{v}}$ is a discrete valuation ring. Set $\kappa(v) = R_{\tilde{v}}/\mathfrak{m}_{\tilde{v}}$.

Remark 3.11. Let R be a discrete valuation ring (in the sense of [Matsumura86, p. 78], so R is Noetherian) with maximal ideal $\mathfrak{m}$. Let $v : R \setminus \{0\} \to \mathbb{Z}$ be a valuation, not identically zero. Let $\operatorname{ord}_\mathfrak{m} : R \setminus \{0\} \to \mathbb{Z}$ be given by $\operatorname{ord}_\mathfrak{m}(f) = \max\{n \mid f \in \mathfrak{m}^n\}$. Then there is a positive integer q such that $v = q \operatorname{ord}_\mathfrak{m}$. Indeed, let t be a generator of $\mathfrak{m}$. Set $q = v(t)$. Fix $f \in R \setminus \{0\}$. Let $n = \operatorname{ord}_\mathfrak{m}(f)$. We can write $f = ut^n$, for a unit $u \in R \setminus \mathfrak{m}$. Then $v(u) = 0$ since u is a unit, and $v(f) = v(ut^n) = v(u) + nv(t) = q \operatorname{ord}_\mathfrak{m}(f)$.

Proposition 3.12. *Let $p \in X$ be a (not necessarily closed) point of X. If $v : \mathcal{O}_{X,p} \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ is a valuation as in Definition* 3.9*, then v is an arc valuation on X. In fact, there exists an arc $\gamma : \operatorname{Spec} \kappa(v)[[t]] \to X$ such that $\gamma(o) = p$ and* $\operatorname{ord}_\gamma = v$ *on $\mathcal{O}_{X,p}$.*

Proof. We use the notation introduced in Notation 3.10. By Remark 3.11, there is a positive integer q such that $\tilde{v} = q \operatorname{ord}_{\mathfrak{m}_{\tilde{v}}}$. The completion $\widehat{R}_{\tilde{v}}$ of $R_{\tilde{v}}$ with respect $\mathfrak{m}_{\tilde{v}}$ is again a discrete valuation ring [Matsumura86, Exercise 11.3]. Let $\widehat{\mathfrak{m}}$ denote the maximal ideal of $\widehat{R}_{\tilde{v}}$. The complete regular local $\mathbf{k}$-algebra $\widehat{R}_{\tilde{v}}$ is isomorphic to the power series ring $\kappa(v)[[T]]$ [Matsumura80, Corollary 2, p. 206]. Identify $\kappa(v)[[T]]$ with the subring $\kappa(v)[[t^q]]$ of $\kappa(v)[[t]]$ via $T \to t^q$. The composition of the canonical homomorphisms

$$\mathcal{O}_{X,p} \to \mathcal{O}_{X,p}/\mathfrak{p} \to R_{\tilde{v}} \to \widehat{R}_{\tilde{v}} = \kappa(v)[[T]] = \kappa(v)\left[\left[t^q\right]\right] \hookrightarrow \kappa(v)[[t]]$$

gives an arc $\gamma : \operatorname{Spec} \kappa(v)[[t]] \to X$. Let $f \in \mathcal{O}_{X,p}$. If $f = 0$, then $v(f) = \operatorname{ord}_\gamma(f) = \infty$. If $f \neq 0$, then (writing f to mean the image of f in the appropriate ring)

$$\operatorname{ord}_\gamma(f) = \operatorname{ord}_t(f) = q \operatorname{ord}_T(f) = q \operatorname{ord}_{\widehat{\mathfrak{m}}}(f) = q \operatorname{ord}_{\mathfrak{m}_{\tilde{v}}}(f) = \tilde{v}(f) = v(f).$$

Thus $\operatorname{ord}_\gamma = v$ on $\mathcal{O}_{X,p}$. $\square$

4. Desingularization of normalized k-arc valuations

In this section, we prove that a normalized $\mathbf{k}$-arc valuation on a nonsingular variety X over a field $\mathbf{k}$ can be desingularized. Specifically, the goal of this section is to prove Proposition 4.5, which says that a normalized $\mathbf{k}$-arc can be lifted after finitely many blowups to a $\mathbf{k}$-arc that is nonsingular. Our proof is based on Hamburger–Noether expansions.

Let X be a nonsingular variety of dimension n ($n \geqslant 2$) over a field $\mathbf{k}$ and let $p_0 \in X$ be a closed point. Let $\gamma : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X$ be an arc such that $\gamma(o) = p_0$ and $v := \operatorname{ord}_\gamma$ is a normalized arc valuation (Definition 3.2). Let $p_i \in X_i$ ($i \geqslant 0$) be the sequence of centers of v, as described in Definition 2.8. If γ_r denotes the unique lift of γ to X_r (by Lemma 2.7), then note that v extends to the valuation $\widehat{\mathcal{O}}_{X_r,p_r} \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ associated to γ_r. Hence for $f \in \widehat{\mathcal{O}}_{X_r,p_r}$, we will write $v(f)$ to mean $\operatorname{ord}_{\gamma_r}(f)$.

4.1. Hamburger–Noether expansions

We will use a list of equations known as Hamburger–Noether expansions (HNEs) to keep track of local coordinates of the sequences of centers of v. We explain HNEs in this section. Our source for this material is [DGN, Section 1], where the presentation is given for arbitrary valuations on a nonsingular surface.

HNEs are constructed by repeated application of Lemma 2.7 part (2), which we recall:

Lemma 4.1. *Let X be a nonsingular variety of dimension n ($n \geqslant 2$) over a field $\mathbf{k}$ and let $p_0 \in X$ be a closed point. Let $\gamma : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X$ be an arc such that $\gamma(o) = p_0$ and $v := \operatorname{ord}_\gamma$ is a normalized arc valuation (Definition 3.2). Let $x_1, x_2, \ldots, x_n$ be local algebraic coordinates at p_0 such that $1 \leqslant v(x_1) \leqslant v(x_i)$ for $2 \leqslant i \leqslant n$. Then for $2 \leqslant i \leqslant n$, there exists $a_{i,1} \in \mathbf{k}$ such that if we let $y_i = \frac{x_i}{x_1} - a_{i,1} \in \mathbf{k}(X)$, then $x_1, y_2, \ldots, y_n$ generate the maximal ideal of $\mathcal{O}_{X_1,p_1} \subseteq \mathbf{k}(X) = \mathbf{k}(X_1)$.*

We now describe how to write down the HNEs, following [DGN, Section 1]. Let $x_i, a_{i,1}, y_i$ be as in Lemma 4.1. We have $x_i = a_{i,1}x_1 + x_1 y_i$. If $v(x_1) \leqslant v(y_i)$ for every $2 \leqslant i \leqslant n$, then with the local algebraic coordinates $x_1, y_2, \ldots, y_n$ at p_1 we are in a similar situation as before, and we repeat the process of applying Lemma 4.1 to get local algebraic coordinates at p_2. Suppose that after h steps we have local algebraic coordinates $x_1, y'_2, \ldots y'_n$ at p_h such that $v(x_1) > v(y'_j)$ for some $2 \leqslant j \leqslant n$. We may choose j such that $v(y'_j) \leqslant v(y'_i)$ for $2 \leqslant i \leqslant n$. There are $a_{i,k} \in \mathbf{k}$ such that

$$x_i = a_{i,1}x_1 + a_{i,2}x_1^2 + \cdots + a_{i,h}x_1^h + x_1^h y'_i \tag{8}$$

for $2 \leqslant i \leqslant n$, $1 \leqslant k \leqslant h$. The assumption that p_h is a closed point implies $v(y'_i) > 0$ for $2 \leqslant i \leqslant n$. Let $z_1 = y'_j$, and we repeat the procedure of applying Lemma 4.1 with the local coordinates $z_1, x_1, y'_2, \ldots, y'_{j-1}, y'_{j+1}, \ldots, y'_n$ (note that we brought z_1 to the front of the list because it is the coordinate with smallest value). We will refer to such a change in the first coordinate (in this case, from x_1 to z_1) of our list as an iteration.

If we do not arrive at a situation where $v(x_1) > v(y'_j)$ for some $2 \leqslant j \leqslant n$, then there exist $a_{i,k} \in \mathbf{k}$ (for $2 \leqslant i \leqslant n$, and all $k \geqslant 1$) such that

$$v\left(\frac{x_i - \sum_{k=1}^N a_{i,k}x_1^k}{x_1^N}\right) \geqslant v(x_1),$$

and hence (since $v(x_1) \geqslant 1$)

$$v\left(x_i - \sum_{k=1}^N a_{i,k}x_1^k\right) > N \tag{9}$$

for all $N > 0$.

Let $z_0 = x_1$, and for $l > 0$ let z_l be the first listed local coordinate at the lth iteration. We have $v(z_l) < v(z_{l-1})$ since an iteration occurs when the smallest value of the local coordinates at the center decreases in value after a blowup. So $\{v(z_l)\}_{l \geqslant 0}$ is a strictly decreasing sequence of positive integers, and hence must be finite, say $v(z_0), v(z_1), \ldots, v(z_L)$.

For notational convenience, redefine $x_1, \ldots, x_n$ to be the local algebraic coordinates after the final iteration, with $x_1 = z_L$. So $x_1, \ldots, x_n$ are local algebraic coordinates centered at p_r on X_r for some r, and Eq. (9) becomes

$$v\left(x_i - \sum_{k=1}^{N} c_{i,k} x_1^k\right) > N \tag{10}$$

for $2 \leqslant i \leqslant n$, $c_{i,k} \in \mathbf{k}$, and all $N > 0$.

Definition 4.2. Let $P_1(t) = t$, and for $2 \leqslant i \leqslant n$ define $P_i(t) \in \mathbf{k}[\![t]\!]$ by $P_i(t) = \sum_{k=1}^{\infty} c_{i,k} t^k$.

Remark 4.3. Eq. (10) implies $v(x_i - P_i(x_1)) = \infty$ for $2 \leqslant i \leqslant n$.

Lemma 4.4. *For every $\psi = \psi(x_1, \ldots, x_n) \in \widehat{\mathcal{O}}_{X_r, p_r} \simeq \mathbf{k}[\![x_1, \ldots, x_n]\!]$, we have $v(\psi) = \operatorname{ord}_t \psi(t, P_2(t), \ldots, P_n(t))$.*

Proof. Since $\mathbf{k}[\![x_1, \ldots, x_n]\!]/(x_2 - P_2(x_1), \ldots, x_n - P_n(x_1)) \simeq \mathbf{k}[\![x_1]\!]$, we may write $\psi(x_1, \ldots, x_n) = q(x_1) + \sum_{i=2}^{n} (x_i - P_i(x_1)) h_i$ for $h_i \in \mathbf{k}[\![x_1, \ldots, x_n]\!]$ and $q(x_1) \in \mathbf{k}[\![x_1]\!]$. Note that $q(x_1) = \psi(x_1, P_2(x_1), \ldots, P_n(x_1))$. We have $v(\psi) \geqslant \min\{v(q), v((x_2 - P_2(x_1))h_2), \ldots, v((x_n - P_n(x_1))h_n)\}$. Since $v((x_i - P_i(x_1))h_i) = \infty$, we have $v(\psi) = v(q)$, since in general, if $v(a) \neq v(b)$, then $v(a+b) = \min\{v(a), v(b)\}$.

Let $n = \operatorname{ord}_{x_1} q(x_1)$. We claim $v(q) = n v(x_1)$. If $n = \infty$, then $q = 0$ and both sides of $v(q) = n v(x_1)$ are ∞. If $n < \infty$, then $q = x_1^n u$ for a unit u in $\mathbf{k}[\![x_1]\!]$. We have $v(u) = 0$, since $0 = v(1) = v(uu^{-1}) = v(u) + v(u^{-1})$ and $v(u), v(u^{-1}) \geqslant 0$. Hence $v(q) = n v(x_1)$.

So we have $v(\psi) = v(q) = (\operatorname{ord}_{x_1} q(x_1)) v(x_1) = \operatorname{ord}_{x_1} \psi(x_1, P_2(x_1) \ldots, P_n(x_1)) \cdot v(x_1)$. Since ψ was arbitrary, we have that the image of $v : \mathbf{k}[\![x_1, \ldots, x_n]\!] \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ equals $\mathbb{Z}_{\geqslant 0} \cdot v(x_1) \cup \{\infty\}$. Since v was normalized so that the image of v had 1 as the greatest common factor of its elements, we have $v(x_1) = 1$ and $v(\psi) = \operatorname{ord}_t \psi(t, P_2(t), \ldots, P_n(t))$. $\square$

Summarizing the discussion so far, we have

Proposition 4.5. *Let v be a normalized $\mathbf{k}$-arc valuation on a nonsingular variety X over a field $\mathbf{k}$. Then there exists a nonnegative integer r and local algebraic coordinates $x_1, \ldots, x_n$ at the center p_r of v on X_r and*

$$P_i(t) \in (t)\mathbf{k}[\![t]\!]$$

for $2 \leqslant i \leqslant n$ such that for every $\psi = \psi(x_1, \ldots, x_n) \in \widehat{\mathcal{O}}_{X_r, p_r} \simeq \mathbf{k}[\![x_1, \ldots, x_n]\!]$, we have

$$v(\psi) = \operatorname{ord}_t \psi\big(t, P_2(t), \ldots, P_n(t)\big).$$

Roughly speaking, this result says that a normalized $\mathbf{k}$-arc valuation can be desingularized. More precisely, a normalized $\mathbf{k}$-valued arc γ can be lifted after finitely many blowups (of its centers) to an arc γ_r that is nonsingular (see Definition 3.7 for the definition of nonsingular arc). Using the notation of Proposition 4.5, the arc $\gamma_r : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X_r$ is given by the $\mathbf{k}$-algebra map $\widehat{\mathcal{O}}_{X_r, p_r} \to \mathbf{k}[\![t]\!]$ with $\operatorname{ord}_{\gamma_r}(x_1) = 1$ and $x_i \to P_i(\gamma_r^*(x_1))$ for $2 \leqslant i \leqslant n$. Since $\operatorname{ord}_{\gamma_r}(x_1) = 1$, we have γ_r is a nonsingular arc.

If the arc γ is nonsingular, we can take $r = 0$ in Proposition 4.5, and we have the following result.

Proposition 4.6. *Let $\gamma : \operatorname{Spec}\mathbf{k}[\![t]\!] \to X$ be a nonsingular* **k***-arc on a nonsingular variety X over a field* **k***. Let $x_1, \dots, x_n$ be local algebraic coordinates at $p = \gamma(o)$ on X with $\operatorname{ord}_\gamma(x_1) = 1$ (Definition 3.7). Then there exists*

$$P_i(t) \in (t)\mathbf{k}[\![t]\!]$$

for $2 \leqslant i \leqslant n$ such that $\gamma^*(x_i) = P_i(\gamma^*(x_1))$ for $2 \leqslant i \leqslant n$. Furthermore, for every $\psi = \psi(x_1, \dots, x_n) \in \widehat{\mathcal{O}}_{X,p} \simeq \mathbf{k}[\![x_1, \dots, x_n]\!]$, we have

$$\operatorname{ord}_\gamma(\psi) = \operatorname{ord}_t \psi\big(t, P_2(t), \dots, P_n(t)\big).$$

Proof. Since $\operatorname{ord}_\gamma(x_1) = 1$, there can be no iterations in the Hamburger–Noether algorithm for $v = \operatorname{ord}_\gamma$. Hence Eq. (10) holds, and in particular, Remark 4.3 applies. That is, if the $P_i(t)$ for $2 \leqslant i \leqslant n$ are as in Definition 4.2, we have $\operatorname{ord}_\gamma(x_i - P_i(x_1)) = \infty$ for $2 \leqslant i \leqslant n$. So $\gamma^*(x_i - P_i(x_1)) = 0$, and therefore $\gamma^*(x_i) = \gamma^*(P_i(x_1)) = P_i(\gamma^*(x_1))$ for $2 \leqslant i \leqslant n$. According to Lemma 4.4, for every $\psi = \psi(x_1, \dots, x_n) \in \widehat{\mathcal{O}}_{X,p} \simeq \mathbf{k}[\![x_1, \dots, x_n]\!]$, we have

$$\operatorname{ord}_\gamma(\psi) = \operatorname{ord}_t \psi\big(t, P_2(t), \dots, P_n(t)\big). \qquad \square$$

We will see in the next section that for a nonsingular **k**-valued arc γ, one can explicitly compute the ideals of $\bigcap_{q\geqslant 1} \overline{\mu_{q\infty}(\operatorname{Cont}^{\geqslant 1}(E_q))}$ and $\bigcap_{q\geqslant 1} \operatorname{Cont}^{\geqslant q}(\mathfrak{a}_q)$, where $\mathfrak{a}_q = \{f \in \widehat{\mathcal{O}}_{X,\gamma(o)} \mid \operatorname{ord}_\gamma(f) \geqslant q\}$. We will see that these ideals are the same, and thus these two sets are equal.

5. Main results

In this section, we present the main results of the paper. Let X be a nonsingular variety of dimension n ($n \geqslant 2$) over a field **k**. Let $\alpha : \operatorname{Spec}\mathbf{k}[\![t]\!] \to X$ be a normalized arc. Set $v = \operatorname{ord}_\alpha$ and $p = \alpha(o)$, where o denotes the closed point of $\operatorname{Spec}\mathbf{k}[\![t]\!]$. We associate to v several different subsets of the arc space X_∞. In notation we will explain later in the section, these subsets are $C(v)$, $\bigcap_{q\geqslant 1} \mu_{q\infty}(\operatorname{Cont}^{\geqslant 1}(E_q))$, $\bigcap_{q\geqslant 1} \operatorname{Cont}^{\geqslant q}(\mathfrak{a}_q)$, $\{\gamma \in X_\infty \mid \gamma(o) = \alpha(o),\ \ker(\alpha^*) \subseteq \ker(\gamma^*) \subseteq \widehat{\mathcal{O}}_{X,\alpha(o)}\}$, and $R = \{\alpha \circ h \in X_\infty \mid h : \operatorname{Spec}\mathbf{k}[\![t]\!] \to \operatorname{Spec}\mathbf{k}[\![t]\!]\}$. Our main result is that these five subsets are all equal. We first analyze the case when v is a nonsingular arc valuation (Definition 3.7). We then consider the general case where we drop the hypothesis of nonsingularity.

5.1. Setup

Throughout this section, we fix the following notation. Let X be a nonsingular variety of dimension n ($n \geqslant 2$) over a field **k**. Let $\alpha : \operatorname{Spec}\mathbf{k}[\![t]\!] \to X$ be a normalized arc valuation on X (see Definition 3.2). Set $v = \operatorname{ord}_\alpha$.

In Definition 2.8, we defined the sequence of centers of a **k**-arc valuation. To set notation for the rest of this section, we recall this definition.

Definition 5.1 *(Sequences of centers of a* **k***-arc valuation).* Let X be a nonsingular variety over a field **k**. Let $\alpha : \operatorname{Spec}\mathbf{k}[\![t]\!] \to X$ be an arc on X. Assume α is not the trivial arc (Definition 2.6). Set $p_0 = \alpha(o)$ (where o is the closed point of $\operatorname{Spec}\mathbf{k}[\![t]\!]$) and $v = \operatorname{ord}_\alpha$. By Proposition 2.2, the point p_0 is a closed point (with residue field **k**) of X. The point p_0 is called the *center* of v on $X_0 := X$. Blow up p_0 to get a model X_1 with exceptional divisor E_1. By Lemma 2.7 the arc α has a unique lift to an arc $\alpha_1 : \operatorname{Spec}\mathbf{k}[\![t]\!] \to X_1$. Let p_1 be the closed point $\alpha_1(o)$. Inductively define a sequence of closed points p_i and exceptional divisors E_i on models X_i and lifts $\alpha_i : \operatorname{Spec}\mathbf{k}[\![t]\!] \to X_i$ of α as follows. Blow up $p_{i-1} \in X_{i-1}$, to get a model X_i. Let E_i be the exceptional divisor of this blowup. Let $\alpha_i : \operatorname{Spec}\mathbf{k}[\![t]\!] \to X_i$ be the lift of $\alpha_{i-1} : \operatorname{Spec}\mathbf{k}[\![t]\!] \to X_{i-1}$. Let p_i be the closed point $\alpha_i(o)$. Let $\mu_i : X_i \to X$ be the composition of the first i blowups. We call $\{p_i\}_{i\geqslant 0}$ the *sequence of centers* of v.

5.2. Simplified situation

We first consider the special case when the arc $\alpha : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X$ is nonsingular (Definition 3.7).

Proposition 5.2. *Let X be a nonsingular variety of dimension n ($n \geqslant 2$) over a field $\mathbf{k}$. Let $\alpha : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X$ be a nonsingular arc (Definition 3.7). Set $v = \operatorname{ord}_\alpha$ and $p_0 = \alpha(o)$. Let $C = \bigcap_{q \geqslant 1} \mu_{q\infty}(\operatorname{Cont}^{\geqslant 1}(E_q))$. Then:*

(1) *C is an irreducible subset of X_∞.*
(2) *Let $\mathfrak{a}_q = \{f \in \widehat{\mathcal{O}}_{X,p_0} \mid v(f) \geqslant q\}$. Then $C = \bigcap_{q \geqslant 1} \operatorname{Cont}^{\geqslant q}(\mathfrak{a}_q)$.*
(3) *$\operatorname{val}_C = v$ on $\widehat{\mathcal{O}}_{X,p_0}$.*

Notation 5.3. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X,p_0}$. Since α is nonsingular, there exists $x_1 \in \mathfrak{m}$ such that $\operatorname{ord}_\alpha(x_1) = 1$. Since $\operatorname{ord}_\alpha(x_1) = 1$, we have $x_1 \in \mathfrak{m} \setminus \mathfrak{m}^2$. Choose $x_2, \ldots, x_n$ in $\mathfrak{m}$ so that $x_1, \ldots, x_n$ are local algebraic coordinates at p_0 (i.e. generators of $\mathfrak{m}$). For $2 \leqslant i \leqslant n$, let $P_i(t) \in (t)\mathbf{k}[\![t]\!]$ be as in Proposition 4.6. Write $P_i(t) = \sum_{j \geqslant 1} c_{i,j} t^j \in (t)\mathbf{k}[\![t]\!]$ for $2 \leqslant i \leqslant n$ and $c_{i,j} \in \mathbf{k}$. By Proposition 4.6, for every $\psi(x_1, \ldots, x_n) \in \widehat{\mathcal{O}}_{X,p_0} \simeq \mathbf{k}[\![x_1, \ldots, x_n]\!]$, we have

$$v(\psi) = \operatorname{ord}_t \psi\bigl(t, P_2(t), \ldots, P_n(t)\bigr). \tag{11}$$

For $2 \leqslant i \leqslant n$, we also have

$$\begin{aligned} \alpha^*(x_i) &= P_i\bigl(\alpha^*(x_1)\bigr) \\ &= \sum_{j \geqslant 1} c_{i,j}\bigl(\alpha^*(x_1)\bigr)^j. \end{aligned} \tag{12}$$

We break up the proof of Proposition 5.2 into several steps. For the remainder of this section, v, $x_1, \ldots, x_n$, $P_2(t), \ldots, P_n(t)$ and $c_{i,j}$ are as in Proposition 5.2 and Notation 5.3.

Lemma 5.4. *With the notation in Definition 5.1, Proposition 5.2, and Notation 5.3, the rational functions x_1 and $\frac{x_i - c_{i,1}x_1 - c_{i,2}x_1^2 - \cdots - c_{i,q-1}x_1^{q-1}}{x_1^{q-1}} \in \mathbf{k}(X)$ for $2 \leqslant i \leqslant n$ form local algebraic coordinates on X_{q-1} centered at p_{q-1}.*

Proof. These n functions are elements of positive value under $\operatorname{ord}_{\alpha_q}$ (by Eq. (12)), and hence lie in the maximal ideal of the n-dimensional regular local ring $\mathcal{O}_{X_{q-1},p_{q-1}}$. The ideal $\mathfrak{n} \subseteq \mathcal{O}_{X_{q-1},p_{q-1}}$ they generate satisfies $\mathcal{O}_{X_{q-1},p_{q-1}}/\mathfrak{n} \simeq \mathbf{k}$, and hence $\mathfrak{n}$ is a maximal ideal. $\square$

5.2.1. Reduction to $X = \mathbb{A}^n$

We denote the affine line $\mathbb{A}^1_{\mathbf{k}} = \operatorname{Spec} \mathbf{k}[T]$ simply by $\mathbb{A}^1$. We show that we may reduce many computations about the arc space of the nonsingular n-dimensional variety X to the case $X = \mathbb{A}^n$.

Proposition 5.5. *Let X be a nonsingular variety and $p \in X$. Let $\pi : X_\infty \to X$ be the canonical morphism sending an arc γ to its center $\gamma(o)$. Then $\pi^{-1}(p) \simeq (\mathbb{A}^n_{\kappa(p)})_\infty$, where $\kappa(p)$ is the residue field at $p \in X$. In particular, if $\kappa(p) = \mathbf{k}$ then $\pi^{-1}(p) \simeq (\mathbb{A}^n)_\infty$.*

Proof. Since X is nonsingular, there exists an open affine neighborhood U of p and an étale morphism $\phi : U \to \operatorname{Spec} \mathbf{k}[X_1, \ldots, X_n] = \mathbb{A}^n$ [Milne, Proposition 3.24b]. We will use the following fact [EM, p. 7]: if $f : X \to Y$ is an étale morphism, then $X_\infty = X \times_Y Y_\infty$. Applied to the open inclusion $U \to X$, we have $U_\infty = U \times_X X_\infty$. Applied to the étale map $U \to \mathbb{A}^n$ we have $U_\infty = U \times_{\mathbb{A}^n} \mathbb{A}^n_\infty$. Hence we have

$$\pi^{-1}(U) = U \times_X X_\infty = U_\infty = U \times_{\mathbb{A}^n} \mathbb{A}^n_\infty.$$

Hence

$$\pi^{-1}(p) = \operatorname{Spec}\kappa(p) \times_U \pi^{-1}(U) = \operatorname{Spec}\kappa(p) \times_{\mathbb{A}^n} \left(\mathbb{A}^n\right)_\infty = \left(\mathbb{A}^n_{\kappa(p)}\right)_\infty. \qquad \square$$

We resume considering Proposition 5.2, where now it is sufficient to assume $X = \mathbb{A}^n = \operatorname{Spec}\mathbf{k}[x_1, \dots, x_n]$, and the $\mathbf{k}$-valued point p_0 corresponds to the maximal ideal $(x_1, \dots, x_n)$. We write $(\mathbb{A}^n)_\infty = (\operatorname{Spec}\mathbf{k}[x_1, \dots, x_n])_\infty = \operatorname{Spec}\mathbf{k}[\{x_{i,j}\}_{1\leqslant i\leqslant n,\, j\geqslant 0}]$, where the last equality comes from parametrizing arcs on $\operatorname{Spec}\mathbf{k}[x_1, \dots, x_n]$ by $x_i \to \sum_{j\geqslant 0} x_{i,j}t^j$ for $1 \leqslant i \leqslant n$. Note that $\pi : X_\infty \to X$ (defined in Proposition 5.5) maps C to p_0. Hence

$$C \subseteq \pi^{-1}(p_0) = \left(\mathbb{A}^n\right)_\infty = \operatorname{Spec} S,$$

where

$$S = \mathbf{k}\left[\{x_{i,j}\}_{1\leqslant i\leqslant n,\, j\geqslant 1}\right]. \tag{13}$$

Definition 5.6. For $2 \leqslant i \leqslant n$ and $q \geqslant 1$, let $f_{i,q}(X_1, \dots, X_q)$ be the polynomial that is the coefficient of t^q in

$$\sum_{j=1}^{q} c_{i,j}\left(X_1 t + X_2 t^2 + \cdots\right)^j.$$

(Recall that the $c_{i,j}$ were defined in Notation 5.3.)

Definition 5.7. For each positive integer q, let I_q be the ideal of S generated by

(1) $x_{i,j} - f_{i,j}(x_{1,1}, \dots, x_{1,j})$ for $2 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant q-1$.

Note that I_q is a prime ideal of S, since $S/I_q = \mathbf{k}[\{x_{1,j}\}_{j\geqslant 1}, \{x_{i,j}\}_{2\leqslant i\leqslant n,\, q\leqslant j}]$.

Notation 5.8. If J is an ideal of S, we denote by $V(J)$ the closed subscheme of $\operatorname{Spec} S$ defined by the ideal J.

Definition 5.9. Let I be the ideal of S defined by $I = \bigcup_{q\geqslant 1} I_q$. Since I is the ideal of S generated by $x_{i,j} - f_{i,j}(x_{1,1}, \dots, x_{1,j})$ for $2 \leqslant i \leqslant n$ and $1 \leqslant j$, we have $S/I = \mathbf{k}[\{x_{1,j}\}_{1\leqslant j}]$. In particular, I is a prime ideal of S.

Lemma 5.10. *For each positive integer q, the ideal of $\overline{\mu_{q\infty}(\mathrm{Cont}^{\geqslant 1}(E_q))}$ in S is I_q. (Note: I_q is defined in Definition 5.7.)*

Proof. Note that $\overline{\mu_{q\infty}(\mathrm{Cont}^{\geqslant 1}(E_q))}$ is irreducible (e.g. [ELM, p. 9]). Since I_q is a prime ideal, we need to show

$$\overline{\mu_{q\infty}\left(\mathrm{Cont}^{\geqslant 1}(E_q)\right)} = V(I_q).$$

First we show $\overline{\mu_{q\infty}(\mathrm{Cont}^{\geqslant 1}(E_q))} \subseteq V(I_q)$ by showing that the generic point of $\overline{\mu_{q\infty}(\mathrm{Cont}^{\geqslant 1}(E_q))}$ lies in $V(I_q)$. Suppose $\beta' : \operatorname{Spec} K[\![t]\!] \to X_q$ is the generic point of $\mathrm{Cont}^{\geqslant 1}(E_q)$. To be precise, β' is the canonical arc (described in Remark 2.3) associated to the generic point of $\mathrm{Cont}^{\geqslant 1}(E_q)$. Also, K is the residue field at the generic point of $\mathrm{Cont}^{\geqslant 1}(E_q)$. By Lemma 2.7 part (3), the pushdown of β' to X_{q-1} is an arc $\beta : \operatorname{Spec} K[\![t]\!] \to X_{q-1}$ that is the generic point of $\mathrm{Cont}^{\geqslant 1}(p_{q-1})$. By the description of local coordinates at p_{q-1} given in Lemma 5.4, the arc β corresponds (by Lemma 2.7) to a map $x_1 \to x_{1,1}t + x_{1,2}t^2 + \cdots$ and $\frac{x_i - c_{i,1}x_1 - c_{i,2}x_1^2 - \cdots - c_{i,q-1}x_1^{q-1}}{x_1^{q-1}} \to a_{i,1}t + a_{i,2}t^2 + \cdots$ for $2 \leqslant i \leqslant n$ and some $a_{i,j} \in K$. The pushdown of β to X is the arc given by $x_1 \to x_{1,1}t + x_{1,2}t^2 + \cdots$ and $x_i \to \sum_{j=1}^{j=q-1} c_{i,j}(x_{1,1}t + x_{1,2}t^2 + \cdots)^j + r(t)$ where $r(t) \in (t^q) \subseteq K[\![t]\!]$. In particular, the pushdown of β' to X corresponds to a prime ideal in S containing the ideal I_q of S generated by $x_{i,j} - f_{i,j}(x_{1,1}, \dots, x_{1,j})$ for $1 \leqslant j \leqslant q-1$ and $2 \leqslant i \leqslant n$. That is, the generic point of $\mu_{q\infty}(\mathrm{Cont}^{\geqslant 1}(E_q))$ lies in $V(I_q)$. Hence $\overline{\mu_{q\infty}(\mathrm{Cont}^{\geqslant 1}(E_q))} \subseteq V(I_q)$.

Conversely, we show that $\overline{\mu_{q\infty}(\mathrm{Cont}^{\geqslant 1}(E_q))} \supseteq V(I_q)$. The generators of I_q listed in Definition 5.7 show that the coordinate ring of $V(I_q)$ is $S/I_q = \mathbf{k}[\{x_{1,j}\}_{j\geqslant 1}, \{x_{i,j}\}_{2\leqslant i\leqslant n,\, q\leqslant j}]$. Let $\beta : \operatorname{Spec} K[\![t]\!] \to X$ be the arc corresponding (see Remark 2.3) to the generic point of $V(I_q)$, where $K = \mathbf{k}(\{x_{1,j}\}_{j\geqslant 1}, \{x_{i,j}\}_{2\leqslant i\leqslant n,\, q\leqslant j})$. We have $\beta^*(x_1) = x_{1,1}t + x_{1,2}t^2 + \cdots$. Since I_q contains $x_{i,j} - f_{i,j}(x_{1,1}, \dots, x_{1,j})$ for $1 \leqslant j \leqslant q-1$ and $2 \leqslant i \leqslant n$, we have that $\beta^*(x_i) = \sum_{j\geqslant 1}^{q-1} f_{i,j}(x_{1,1}, \dots, x_{1,j})t^j + t^q r_i(t)$ for some $r_i(t) \in K[\![t]\!]$ and for each $2 \leqslant i \leqslant n$. Hence $\beta^*(x_i) = \sum_{j\geqslant 1}^{q-1} c_{i,j}(\beta^*(x_1))^j + t^q s_i(t)$ for some $s_i(t) \in K[\![t]\!]$, by Definition 5.6.

Therefore

$$\operatorname{ord}_\beta\bigl(x_i - c_{i,1}x_1 - c_{i,2}x_1^2 - \cdots - c_{i,q-1}x_1^{q-1}\bigr) \geqslant q = \operatorname{ord}_\beta\bigl(x_1^{q-1}\bigr) + 1,$$

where the last equality follows from the fact $\operatorname{ord}_\beta(x_1) = 1$ as $x_{1,1} \neq 0 \in K$. In particular, the unique lift of β to an arc on X_{q-1} has center p_{q-1}, by Lemma 5.4. Hence $\beta \in \mu_{q-1\infty}(\mathrm{Cont}^{\geqslant 1}(p_{q-1})) = \mu_{q\infty}(\mathrm{Cont}^{\geqslant 1}(E_q))$. Hence $V(I_q) = \overline{\{\beta\}} \subseteq \overline{\mu_{q\infty}(\mathrm{Cont}^{\geqslant 1}(E_q))}$. $\square$

Lemma 5.11. *The subset C of X_∞ is closed, and the ideal of C in S is I. (Note: C is defined in Proposition* 5.2*, S is defined in Eq.* (13)*, and I is defined in Definition* 5.9.)

Proof. Since I is a prime ideal, we need to show $C = V(I)$. We have

$$\bigcap_{q\geqslant 1} V(I_q) = V\left(\bigcup_{q\geqslant 1} I_q\right) = V(I)$$

and

$$C = \bigcap_{q\geqslant 1} \mu_{q\infty}\bigl(\mathrm{Cont}^{\geqslant 1}(E_q)\bigr) \subseteq \bigcap_{q\geqslant 1} V(I_q)$$

by Lemma 5.10. It remains to show $\bigcap_{q\geqslant 1} \mu_{q\infty}(\mathrm{Cont}^{\geqslant 1}(E_q)) \supseteq \bigcap_{q\geqslant 1} V(I_q)$.

Let $\beta : \operatorname{Spec} K[\![t]\!] \to X$ be an arc corresponding to a point in $\bigcap_{q\geqslant 1} V(I_q)$. We may assume β is not the trivial arc, since the trivial arc lies in $\bigcap_{q\geqslant 1} \mu_{q\infty}(\mathrm{Cont}^{\geqslant 1}(E_q))$. Say $\beta^*(x_1) = \sum_{j\geqslant 1} a_{1,j}t^j$, where $a_{1,j} \in K$. Since I_q contains $x_{i,j} - f_{i,j}(x_{1,1}, \dots, x_{1,j})$ for $1 \leqslant j \leqslant q-1$ and $2 \leqslant i \leqslant n$, we have that $\beta^*(x_i) = \sum_{j=1}^{\infty} f_{i,j}(a_{1,1}, \dots, a_{1,j})t^j$ for each $2 \leqslant i \leqslant n$. Hence $\beta^*(x_i) = \sum_{j=1}^{\infty} c_{i,j}(\beta^*(x_1))^j$, by Definition 5.6. Hence

$$\operatorname{ord}_\beta\bigl(x_i - c_{i,1}x_1 - c_{i,2}x_1^2 - \cdots - c_{i,q-1}x_1^{q-1}\bigr) = \operatorname{ord}_\beta\left(\sum_{j\geqslant q} c_{i,j}x_1^j\right) = \operatorname{ord}_\beta x_1^q \geqslant \operatorname{ord}_\beta\bigl(x_1^{q-1}\bigr) + 1.$$

In particular, the unique lift of β to an arc on X_{q-1} has center p_{q-1}, by Lemma 5.4. Hence $\beta \in \mu_{q-1\infty}(\mathrm{Cont}^{\geqslant 1}(p_{q-1})) = \mu_{q\infty}(\mathrm{Cont}^{\geqslant 1}(E_q))$. Hence $\bigcap_{q\geqslant 1} V(I_q) \subseteq \bigcap_{q\geqslant 1} \mu_{q\infty}(\mathrm{Cont}^{\geqslant 1}(E_q))$. $\square$

Lemma 5.12. *For a positive integer q, let $\mathfrak{a}_q = \{f \in \widehat{\mathcal{O}}_{X,p_0} \mid v(f) \geqslant q\}$. Set $z_i = x_i - \sum_{j=1}^{q-1} c_{i,j} x_1^j$ for $2 \leqslant i \leqslant n$. Then $\mathfrak{a}_q$ is generated (as an ideal in $\widehat{\mathcal{O}}_{X,p_0}$) by $x_1^q, z_2, \dots, z_n$.*

Proof. By Eq. (11), we have $v(x_1^q), v(z_i) \geqslant q$ for $2 \leqslant i \leqslant n$. Suppose $f \in \mathfrak{a}_q$. Since $\mathbf{k}[\![x_1,\dots,x_n]\!]/(z_2,\dots,z_n) \simeq \mathbf{k}[\![x_1]\!]$, we can write $f = \sum_{i=2}^{i=n} h_i z_i + g(x_1)$, where $h_i \in \mathbf{k}[\![x_1,\dots,x_n]\!]$ and $g(x_1) \in \mathbf{k}[\![x_1]\!]$. Then since $v(f) \geqslant q$, and $v(z_i) \geqslant q$, we must have $v(g) \geqslant q$. By Eq. (11), we conclude x_1^q divides $g(x_1)$ in $\mathbf{k}[\![x_1]\!]$. Hence f is in the ideal generated by $x_1^q, z_2, \dots, z_n$. $\square$

Lemma 5.13. *For every positive integer q, the ideal of $\mathrm{Cont}^{\geqslant q}(\mathfrak{a}_q)$ in S is I_q.*

Proof. First we show $\mathrm{Cont}^{\geqslant q}(\mathfrak{a}_q) \subseteq V(I_q)$. Suppose $\beta : \operatorname{Spec} K[\![t]\!] \to X$ is an arc corresponding (via Remark 2.3) to a generic point of $\mathrm{Cont}^{\geqslant q}(\mathfrak{a}_q)$. Write $\beta^*(x_i) = \bar{x}_{i,1} t + \bar{x}_{i,2} t^2 + \cdots$ for $1 \leqslant i \leqslant n$, where $\bar{x}_{i,j} \in K$ denotes the image in K of $x_{i,j} \in S$. Since $\mathfrak{a}_q$ is generated by $x_1^q, z_2, \dots, z_n$ (Lemma 5.12) (recall that $z_i = x_i - \sum_{j=1}^{q-1} c_{i,j} x_1^j$ for $2 \leqslant i \leqslant n$), we have

$$\bar{x}_{i,1} t + \bar{x}_{i,2} t^2 + \cdots - \sum_{j=1}^{q-1} c_{i,j} \big(\bar{x}_{1,1} t + \bar{x}_{1,2} t^2 + \cdots\big)^j \in \big(t^q\big). \tag{14}$$

The coefficient of t^j in Eq. (14) is $\bar{x}_{i,j} - f_{i,j}(\bar{x}_{1,1}, \dots, \bar{x}_{1,j})$. Hence β corresponds to a prime ideal of S containing the ideal I_q of S generated by $x_{i,j} - f_{i,j}(x_{1,1}, \dots, x_{1,j})$ for $2 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant q-1$. Thus $\mathrm{Cont}^{\geqslant q}(\mathfrak{a}_q) \subseteq V(I_q)$.

Conversely, suppose $\beta : \operatorname{Spec} K[\![t]\!] \to X$ corresponds (via Remark 2.3) to the generic point of $V(I_q)$. The coordinate ring of $V(I_q)$ is $S/I_q = \mathbf{k}[\{x_{1,j}\}_{j \geqslant 1}, \{x_{i,j}\}_{2 \leqslant i \leqslant n, q \leqslant j}]$ (Definition 5.7). Hence K, the residue field at the generic point of $V(I_q)$, equals $K = \mathbf{k}(\{x_{1,j}\}_{j \geqslant 1}, \{x_{i,j}\}_{2 \leqslant i \leqslant n, q \leqslant j})$. We have $\beta^*(x_1) = x_{1,1} t + x_{1,2} t^2 + \cdots \in K[\![t]\!]$. Since I_q contains $x_{i,j} - f_{i,j}(x_{1,1}, \dots, x_{1,j})$ for $1 \leqslant j \leqslant q-1$ and $2 \leqslant i \leqslant n$, we have that $\beta^*(x_i) = \sum_{j=1}^{q-1} f_{i,j}(x_{1,1}, \dots, x_{1,j}) t^j + t^q r_i(t)$ for some $r_i(t) \in K[\![t]\!]$ and for each $2 \leqslant i \leqslant n$. Since $\sum_{j \geqslant 1} c_{i,j} (x_{1,1} t + x_{1,2} t^2 + \cdots)^j = \sum_{j \geqslant 1} f_{i,j}(x_{1,1}, \dots, x_{1,j}) t^j$ for $2 \leqslant i \leqslant n$ (Notation 5.3), we have that β^* maps $x_i - c_{i,1} x_1 - c_{i,2} x_1^2 - \cdots - c_{i,q-1} x_1^{q-1}$ into the ideal $(t^q) \subseteq K[\![t]\!]$. Hence by Lemma 5.12, we have $\beta \in \mathrm{Cont}^{\geqslant q}(\mathfrak{a}_q)$. So $V(I_q) = \overline{\{\beta\}} \subseteq \mathrm{Cont}^{\geqslant q}(\mathfrak{a}_q)$. $\square$

Lemma 5.14. *The ideal of $\bigcap_{q \geqslant 1} \mathrm{Cont}^{\geqslant q}(\mathfrak{a}_q)$ in S is I. (Note: S is defined in Eq. (13), and I is defined in Definition 5.9, and $\mathfrak{a}_q$ is defined in Proposition 5.2(2).)*

Proof. Since I is a prime ideal, it is enough to show $\bigcap_{q \geqslant 1} \mathrm{Cont}^{\geqslant q}(\mathfrak{a}_q) = V(I)$. By Lemma 5.13, we have

$$\bigcap_{q \geqslant 1} \mathrm{Cont}^{\geqslant q}(\mathfrak{a}_q) = \bigcap_{q \geqslant 1} V(I_q) = V\left(\bigcup_{q \geqslant 1} I_q\right) = V(I). \qquad \square$$

We now finish the proof of Proposition 5.2.

Proof of Proposition 5.2. Since $S/I \simeq \mathbf{k}[\{x_{1,j}\}_{j \geqslant 1}]$ is a domain, the ideal I is a prime ideal. By Lemma 5.11, the ideal of C is I. Hence C is irreducible. We have $C = \bigcap_q \mathrm{Cont}^{\geqslant q}(\mathfrak{a}_q)$ because by Lemmas 5.11 and 5.14, their ideals are the same.

It remains to show $\mathrm{val}_C = v$. Let $\gamma : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X$ be the arc centered at p_0 with $\gamma^*(x_1) = t$ and $\gamma^*(x_i) = P_i(t)$ for $2 \leqslant i \leqslant n$. Then $\gamma \in C$ since the ideal in S corresponding to γ, namely the ideal generated by $x_{1,0}$, $x_{1,1} - 1$, $x_{1,m}$, $x_{i,0}$, and $x_{i,j} - c_{i,j}$ for $m \geqslant 2$, $2 \leqslant i \leqslant n$, and $j \geqslant 1$ contains I. Hence for any $f \in \mathcal{O}_{X,p_0}$, we have $\mathrm{val}_C(f) \leqslant \mathrm{ord}_\gamma(f) = v(f)$.

For the reverse inequality, first suppose $f \in \mathcal{O}_{X,p_0}$ is such that $s := \nu(f) < \infty$. Let $\gamma \in C$ be such that $\mathrm{val}_C(f) = \mathrm{ord}_\gamma(f)$. Since $f \in \mathfrak{a}_s$ and $\gamma \in \mathrm{Cont}^{\geqslant s}(\mathfrak{a}_s)$, we have $\mathrm{ord}_\gamma(f) \geqslant s$, i.e. $\mathrm{val}_C(f) \geqslant \nu(f)$.

Next suppose $\nu(f) = \infty$. Set $\phi_i = x_i - P_i(x_1)$ for $2 \leqslant i \leqslant n$. Since

$$\mathbf{k}[\![x_1, \ldots, x_n]\!]/(\phi_2, \ldots, \phi_n) \simeq \mathbf{k}[\![x_1]\!],$$

we can write $f = \sum_{i=2}^n \phi_i h_i + g(x_1)$ for $h_i \in \mathbf{k}[\![x_1, \ldots, x_n]\!]$ and $g \in \mathbf{k}[\![x_1]\!]$. Since $\nu(f) = \infty$, we have $g = 0$ by Eq. (11). Let $\gamma \in C$, and write $\gamma^*(x_1) = \sum_{j \geqslant 1} a_j t^j$. Since $x_{i,j} - f_{i,j}(x_{1,1}, \ldots, x_{1,j}) \in I$ for $2 \leqslant i \leqslant n$ and $j \geqslant 1$, we have $\gamma^*(x_i) = \sum_{j \geqslant 1} f_{i,j}(a_1, \ldots, a_j)t^j = \sum_{j \geqslant 1} c_{i,j}(a_1 t + a_2 t^2 + \cdots)^j = P_i(\gamma^*(x_1)) = \gamma^*(P_i(x_1))$. Hence $\gamma^*(\phi_i) = 0$, and so $\gamma^*(f) = \gamma^*(\sum_{i=2}^n \phi_i h_i) = 0$. So $\mathrm{ord}_\gamma(f) = \infty$. Since $\gamma \in C$ was arbitrary, we have $\mathrm{val}_C(f) = \infty$, as desired. $\square$

5.3. General case

Lemma 5.15. *Let X be a nonsingular variety of dimension n ($n \geqslant 2$) over an algebraically closed field $\mathbf{k}$ of characteristic zero. Let $\alpha : \mathrm{Spec}\,\mathbf{k}[\![t]\!] \to X$ be a normalized arc (Definition 3.2). Set $p_0 = \alpha(o)$. Let $\alpha^* : \widehat{\mathcal{O}}_{X,p_0} \to \mathbf{k}[\![t]\!]$ be the $\mathbf{k}$-algebra homomorphism induced by α. Suppose $\gamma : \mathrm{Spec}\,\mathbf{k}[\![t]\!] \to X$ satisfies $\gamma(o) = p_0$ and $\ker(\alpha^*) \subseteq \ker(\gamma^*)$, where $\gamma^* : \widehat{\mathcal{O}}_{X,p_0} \to \mathbf{k}[\![t]\!]$ is the $\mathbf{k}$-algebra homomorphism induced by γ. Assume γ is not the trivial arc (Definition 2.6). Then*

(1) *There exists a morphism $h : \mathrm{Spec}\,\mathbf{k}[\![t]\!] \to \mathrm{Spec}\,\mathbf{k}[\![t]\!]$ such that $\gamma = \alpha \circ h$, i.e. γ is a reparametrization of α.*
(2) *$h^* : \mathbf{k}[\![t]\!] \to \mathbf{k}[\![t]\!]$ is a local homomorphism.*
(3) *Set $N = \mathrm{ord}_t(h)$. Then $\mathrm{ord}_\gamma = N\,\mathrm{ord}_\alpha$ on $\widehat{\mathcal{O}}_{X,p_0}$. (We use the convention that $\infty = N \cdot \infty$.)*

Proof. (Due to Mel Hochster.) We use Notation 3.3. Suppose γ is not the trivial arc. By Lemma 3.4, A_γ has dimension one, and so $\ker(\gamma^*)$ is a prime ideal of height $n-1$. The same is true for $\ker(\alpha^*)$, and so our assumption $\ker(\alpha^*) \subseteq \ker(\gamma^*)$ implies $\ker(\alpha^*) = \ker(\gamma^*)$. Hence $A_\alpha = A_\gamma$. By Lemma 3.5, the map α^* (resp. γ^*) induces an isomorphism $\overline{\alpha^*} : \tilde{A_\alpha} \to \mathbf{k}[\![\phi_\alpha]\!]$ (resp. $\overline{\gamma^*} : \tilde{A_\gamma} \to \mathbf{k}[\![\phi_\gamma]\!]$) for some $\phi_\alpha \in \mathbf{k}[\![t]\!]$ (resp. $\phi_\gamma \in \mathbf{k}[\![t]\!]$). Since α is normalized, we have $\mathrm{ord}_t(\phi_\alpha) = 1$ by Proposition 3.6.

I claim that the inclusion $\mathbf{k}[\![\phi_\alpha]\!] \subseteq \mathbf{k}[\![t]\!]$ is actually an equality. It suffices to find $a_j \in \mathbf{k}$ such that $t = \sum_{j \geqslant 1} a_j(\phi_\alpha)^j$. Suppose $\phi_\alpha = \sum_{j \geqslant 1} b_j t^j$, where $b_j \in \mathbf{k}$ and $b_1 \neq 0$. We proceed to define a_j by induction on j. Set $a_1 = {b_1}^{-1}$. Suppose $a_1, \ldots, a_{d-1}$ have been specified. The coefficient of t^d in $\sum_{j \geqslant 1} a_j(\phi_\alpha)^j$ is $a_d b_1^d + Q_d(a_1, \ldots, a_{d-1}, b_1, \ldots, b_d)$ for some polynomial Q_d. We require this coefficient to be 0. We can solve the equation

$$a_d b_1^d + Q_d(a_1, \ldots, a_{d-1}, b_1, \ldots, b_d) = 0$$

for a_d since $b_1 \neq 0$. This completes the induction, and we have $t = \sum_{j \geqslant 1} a_j(\phi_\alpha)^j$.

Let $h : \mathrm{Spec}\,\mathbf{k}[\![t]\!] \to \mathrm{Spec}\,\mathbf{k}[\![t]\!]$ be induced by the $\mathbf{k}$-algebra homomorphism $h^* : \mathbf{k}[\![t]\!] \to \mathbf{k}[\![t]\!]$ defined by the composition

$$\mathbf{k}[\![t]\!] = \mathbf{k}[\![\phi_\alpha]\!] \xrightarrow{(\overline{\alpha^*})^{-1}} \tilde{A}_\alpha = \tilde{A}_\gamma \xrightarrow{\overline{\gamma^*}} \mathbf{k}[\![\phi_\gamma]\!] \subseteq \mathbf{k}[\![t]\!].$$

The last inclusion is an inclusion of local $\mathbf{k}$-algebras and all other maps are isomorphisms. Hence h^* is a local homomorphism. For $f \in \widehat{\mathcal{O}}_{X,p_0}$, we have $\gamma^*(f) = \overline{\gamma^*}(f) = h^* \circ \overline{\alpha^*}(f) = h^* \circ \alpha^*(f)$, and hence $\gamma = \alpha \circ h$. If $\mathrm{ord}_t(h) = N$ and $a = \mathrm{ord}_\alpha(f)$, then the order of t in $\gamma^*(f) = h^* \circ \alpha^*(f)$ is Na, i.e. $\mathrm{ord}_\gamma(f) = N\,\mathrm{ord}_\alpha(f)$. $\square$

Notation 5.16. We denote by $(X_\infty)_0$ the subset of points of X_∞ with residue field equal to $\mathbf{k}$. If $D \subseteq X_\infty$, then we set $D_0 = D \cap (X_\infty)_0$.

Here is the main theorem of this paper.

Theorem 5.17. *Let X be a nonsingular variety of dimension n ($n \geqslant 2$) over a field $\mathbf{k}$. Let $\alpha : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X$ be a normalized arc (Definition 3.2). Set $p_0 = \alpha(o)$ and $v = \operatorname{ord}_\alpha$. Let E_i and p_i be the sequence of divisors and centers, respectively, of v (described in Definition 2.8). Let $\mu_q : X_q \to X$ be the composition of the first q blowups of centers of v. Let*

$$C = \bigcap_{q>0} \mu_{q\infty}\big(\operatorname{Cont}^{\geqslant 1}(E_q)\big) \subseteq X_\infty. \tag{15}$$

Let $\mathfrak{a}_q = \{f \in \widehat{\mathcal{O}}_{X,p_0} \mid v(f) \geqslant q\}$. Let

$$C'' = \bigcap_{q \geqslant 1} \operatorname{Cont}^{\geqslant q}(\mathfrak{a}_q) \subseteq X_\infty.$$

Set $C(v) = \overline{\{\gamma \in X_\infty \mid \operatorname{ord}_\gamma = v,\ \gamma(o) = p_0\}} \subseteq X_\infty$.

For an arc $\gamma : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X$, let $\gamma^* : \widehat{\mathcal{O}}_{X,\gamma(o)} \to \mathbf{k}[\![t]\!]$ be the induced $\mathbf{k}$-algebra homomorphism. Set $\mathcal{I} = \{\gamma \in X_\infty \mid \gamma(o) = \alpha(o),\ \ker(\alpha^*) \subseteq \ker(\gamma^*) \subseteq \widehat{\mathcal{O}}_{X,\alpha(o)}\}$.

Let $R = \{\alpha \circ h \in X_\infty \mid h : \operatorname{Spec} \mathbf{k}[\![t]\!] \to \operatorname{Spec} \mathbf{k}[\![t]\!]\}$, where h is a morphism of $\mathbf{k}$-schemes.

Then:

(1) *C is an irreducible subset of X_∞ and $\operatorname{val}_C = v$.*
(2) *Assume $\mathbf{k}$ is algebraically closed and has characteristic zero. The following closed subsets of $(X_\infty)_0$ are equal (we use Notation 5.16):*

$$C(v)_0 = C_0 = C''_0 = (\mathcal{I})_0 = R.$$

Proof of Theorem 5.17. (Part (1)) Let r be a nonnegative integer such that the lift of α to X_r is a nonsingular arc. For $q > r$, let $\mu_{q,r} : X_q \to X_r$ be the composition of the blowups along the centers of v, starting at $X_{r+1} \to X_r$ and ending at the blowup $X_q \to X_{q-1}$. Let

$$C' = \bigcap_{q>r} \mu_{q,r\infty}\big(\operatorname{Cont}^{\geqslant 1}(E_q)\big) \subseteq (X_r)_\infty.$$

Note that

$$C = \mu_{r\infty}(C') \subseteq X_\infty.$$

By Proposition 5.2, C' is irreducible. Hence C is irreducible. Since the generic point of C' maps to the generic point of C, we have that $\operatorname{val}_{C'} = \operatorname{val}_C$, i.e. $\operatorname{val}_{C'}(\mu_r^*(f)) = \operatorname{val}_C(f)$ for $f \in \mathcal{O}_{X,p_0}$. Since $v = \operatorname{val}_{C'}$ by Proposition 5.2, we conclude $v = \operatorname{val}_C$.

(Part (2)) We show $C(v)_0 \subseteq C''_0 \subseteq C_0 \subseteq C(v)_0$. Separately we will establish $C''_0 = \mathcal{I}_0$.

First we check $C(v) \subseteq C''$. If $\gamma \in X_\infty$ is such that $\gamma(o) = p_0$ and $\operatorname{ord}_\gamma = v$, then $\gamma \in \operatorname{Cont}^{\geqslant q}(\mathfrak{a}_q)$ for every $q \geqslant 1$, and so $\gamma \in C''$. Since C'' is closed, we have $C(v) \subseteq C''$.

Now we show $C''_0 \subseteq C_0$. Let $\gamma \in C''_0$, and assume without loss of generality that γ is not the trivial arc. We claim that $\ker(\alpha^*) \subseteq \ker(\gamma^*)$. Let $f \in \ker(\alpha^*)$. Then $v(f) = \infty$, and so $f \in \mathfrak{a}_q$ for every $q \in \mathbb{Z}_{\geqslant 0}$. Hence $\operatorname{ord}_\gamma(f) \geqslant q$ for all $q \in \mathbb{Z}_{\geqslant 0}$. Therefore $\operatorname{ord}_\gamma(f) = \infty$, so $f \in \ker(\gamma^*)$. By Lemma 5.15 there exists $h : \operatorname{Spec} \mathbf{k}[\![t]\!] \to \mathbf{k}[\![t]\!]$ such that $\gamma = \alpha \circ h$. It follows that γ has the same sequence of centers as α. Indeed, if $\alpha_q : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X_q$ is the unique lift of α to an arc on X_q, then $\alpha_q \circ h$ is the unique lift of γ to an arc on X_q. Since h^* is a local homomorphism, we have that h maps the closed point of $\operatorname{Spec} \mathbf{k}[\![t]\!]$ to the closed point of $\operatorname{Spec} \mathbf{k}[\![t]\!]$. Hence the center of α_q is the same as the center of $\alpha_g \circ h$. So α and γ have the same sequence of centers. We conclude $\gamma \in C$. Note that this argument also shows $C''_0 \subseteq R$. For the inclusion $C''_0 \supseteq R$, let $f \in \mathfrak{a}_q$ and $h : \operatorname{Spec} \mathbf{k}[\![t]\!] \to \operatorname{Spec} \mathbf{k}[\![t]\!]$ be a morphism of $\mathbf{k}$-schemes. Then $\operatorname{ord}_{\alpha\circ h}(f) = \operatorname{ord}_t(h^*\alpha^*(f)) \geqslant q$, since $\operatorname{ord}_t \alpha^*(f) \geqslant q$ and h^* is a local homomorphism. Thus $\alpha \circ h \in C_0''$.

To see that $C \subseteq C(v)$, let β be the generic point of C. Note that $\operatorname{ord}_\beta = v$ and $\pi(\beta) = p_0$, and so $\beta \in C(v)$. Hence $C \subseteq C(v)$.

Now we show $C_0'' = (\mathcal{I})_0$. Let J be the kernel of the map $\alpha^* : \widehat{\mathcal{O}}_{X,p_0} \to \mathbf{k}[\![t]\!]$. If $f \in J$, then $\operatorname{ord}_\alpha(f) = \infty$ and hence $f \in \mathfrak{a}_q$ for every $q \geqslant 1$. Let $\gamma \in C_0''$. Since $\mathfrak{a}_1$ is the maximal ideal of $\widehat{\mathcal{O}}_{X,p_0}$, we have $\gamma(o) = p_0$, i.e. $\gamma \in \pi^{-1}(p_0)$. Also, since $\operatorname{ord}_\gamma(f) \geqslant q$ for every $q \geqslant 1$, we have $\operatorname{ord}_\gamma(f) = \infty$. Hence $\gamma \in (\mathcal{I})_0$.

For the reverse inclusion $C_0'' \supseteq (\mathcal{I})_0$, let $\gamma \in (\mathcal{I})_0$. Then $J \subseteq \ker(\gamma^*)$, and hence by Lemma 5.15 we have that either γ is the trivial arc or $\operatorname{ord}_\gamma = N \operatorname{ord}_\alpha$ for some positive integer N. In both cases we have $\gamma \in C_0''$. $\square$

As a corollary, we show that $\mathbf{k}$-arc valuations are determined by their sequence of centers.

Corollary 5.18. *Let X be a nonsingular variety over an algebraically closed field $\mathbf{k}$ of characteristic zero. Let $v : \mathcal{O}_{X,p} \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ be a normalized $\mathbf{k}$-arc valuation, where $p \in X$ is the center of v on X. Then v is uniquely determined by its sequence of centers, that is, if $v' : \mathcal{O}_{X,p} \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$ is another normalized $\mathbf{k}$-arc valuation with the same sequence of centers as v, then $v = v'$ on $\mathcal{O}_{X,p}$.*

Proof. Let $\alpha : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X$ (resp. $\alpha' : \operatorname{Spec} \mathbf{k}[\![t]\!] \to X$) be such that $v = \operatorname{ord}_\alpha$ (resp. $v' = \operatorname{ord}_{\alpha'}$). Let C be as in Eq. (15). We have $\alpha' \in C$ (since the sequence of centers is the same for both v, v'). By Theorem 5.17, we have $C_0 = C_0''$, where $C'' := \bigcap_{q \geqslant 1} \operatorname{Cont}^{\geqslant q}(\mathfrak{a}_q)$ and $\mathfrak{a}_q = \{f \in \widehat{\mathcal{O}}_{X,p} \mid v(f) \geqslant q\}$. Hence $\alpha' \in C''$, which means $\operatorname{ord}_{\alpha'} \geqslant v = \operatorname{ord}_\alpha$ on $\widehat{\mathcal{O}}_{X,p}$. By symmetry, we have $\operatorname{ord}_{\alpha'} \leqslant \operatorname{ord}_\alpha$. Hence $\operatorname{ord}_{\alpha'} = \operatorname{ord}_\alpha$, i.e. $v' = v$. $\square$

Remark 5.19. If X is a surface and if v is a divisorial valuation, then the set

$$C = \bigcap_{q>0} \mu_{q\infty}\bigl(\operatorname{Cont}^{\geqslant 1}(E_q)\bigr)$$

equals the cylinder associated to v in [ELM, Example 2.5], namely $\mu_{r\infty}(\operatorname{Cont}^{\geqslant 1}(E_r))$, where r is such that p_r is a divisor.

Proof. If r is such that $p_r \in X_r$ (Definition 2.8) is a divisor, then $C = \mu_{r\infty}(\operatorname{Cont}^{\geqslant 1}(E_r))$ since $\mu_{q\infty}(\operatorname{Cont}^{\geqslant 1}(E_q)) \supseteq \mu_{q+1\infty}(\operatorname{Cont}^{\geqslant 1}(E_{q+1}))$, and for $q > r$ we have equality since the maps $\mu_{q,r}$ are isomorphisms. Hence $C = \mu_{r\infty}(\operatorname{Cont}^{\geqslant 1}(E_r))$, which is the set in [ELM, Example 2.5]. $\square$

Acknowledgments

This paper is part of my PhD thesis, and I thank my thesis advisor Karen E. Smith for suggesting the problem of valuations in arc spaces and for many helpful discussions throughout the course of working on this problem. I am also indebted to Mircea Mustaţă for many mathematical ideas, as well as for correcting an earlier draft of this paper. I am also grateful to Mel Hochster for providing an alternative characterization of normalized arcs (Proposition 3.6) and a key lemma (Lemma 5.15). Conversations with Mattias Jonsson were also useful. Finally, I wish to thank an anonymous reviewer for a long list of corrections.

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