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Categories parametrized by schemes and representation theory in complex rank

Akhil Mathew

Madison High School, 170 Ridgedale Avenue, Madison, NJ 07940, USA

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ABSTRACT

Many key invariants in the representation theory of classical groups (symmetric groups S_n , matrix groups GL_n , O_n , Sp_{2n}) are polynomials in n (e.g., dimensions of irreducible representations). This allowed Deligne to extend the representation theory of these groups to complex values of the rank n . Namely, Deligne defined generically semisimple families of tensor categories parametrized by $n \in \mathbb{C}$, which at positive integer n specialize to the classical representation categories. Using Deligne's work, Etingof proposed a similar extrapolation for many *non-semisimple* representation categories built on representation categories of classical groups, e.g., degenerate affine Hecke algebras (dAHA). It is expected that for generic $n \in \mathbb{C}$ such extrapolations behave as they do for large integer n ("stabilization").

The goal of our work is to provide a technique to prove such statements. Namely, we develop an algebro-geometric framework to study categories indexed by a parameter n , in which the set of values of n for which the category has a given property is constructible. This implies that if a property holds for integer n , it then holds for generic complex n . We use this to give a new proof that Deligne's categories are generically semisimple. We also apply this method to Etingof's extrapolations of dAHA, and prove that when n is transcendental, "finite-dimensional" simple objects are quotients of certain standard induced objects, extrapolating Zelevinsky's classification of simple dAHA-modules for $n \in \mathbb{N}$. Finally, we obtain similar results for the extrapolations of categories associated to wreath products of the symmetric group with associative algebras.

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E-mail address: akhilmathew1@gmail.com.

1. Introduction

1.1. Interpolation of categories

The representation theory of the classical groups yields families of semisimple symmetric tensor categories $\text{Rep}^{\text{ord}}(S_n)$, $\text{Rep}^{\text{ord}}(GL_n)$, etc. These categories, which depend on the parameter n called the *rank*, have a rich and well-studied structure, e.g., parametrizations of the simple objects and tensor product formulas. It turns out, however, that many of these parametrizations can be described polynomially in n . As a result of this, Deligne [4,5] constructed interpolations of the categories of representations of classical groups to complex rank. These form families of symmetric tensor categories that when specialized to positive integral parameters, reduce modulo a tensor ideal to the classical representation-theoretic categories. These constructions, together with Deligne's proof of semisimplicity, have been extended in [12,13] to the representation categories of other families of groups such as the semidirect products $S_n \ltimes G^n$ for finite groups G and $GL_n(\mathbb{F}_q)$.

Many families of algebraic objects of interest in representation theory (e.g. affine Hecke algebras, Cherednik algebras, etc.) are built out of the classical groups, and thus depend on a rank $n \in \mathbb{Z}_{\geq 0}$. In many cases, the additional generators and relations of such objects can be stated in a uniform “category-friendly” framework independent of n . This allowed Etingof to use Deligne's categories to define non-semisimple categories of representations of algebraic objects in complex rank. In [6] and [7], a general program has been proposed to study non-semisimple Deligne categories that interpolate the categories of representations of algebraic objects derived from these groups. It is of interest to study the structure of these categories for their own sake; one might also hope that the theory of categorification will find in them a convenient setting.

In this paper, we begin investigating the categories of Etingof's program and show, loosely speaking, that the categories are generically similar to the classical categories. The paper is structured as follows. Section 2 surveys the work of Deligne and Etingof. In Section 3, we build a method, typified by Theorem 3.9 and Proposition 3.7, of deducing such properties based on a framework of categories parametrized by schemes. In Section 4, we apply this method to the categories of representations of the degenerate affine Hecke algebra in complex rank. Section 5 studies the category of representations of semidirect products $S_n \ltimes A^{\otimes n}$ for n complex and A an associative algebra; we obtain a generic classification of simple objects. In addition, we introduce tensor structures on these categories for A a Hopf algebra. In Section 6, we demonstrate that these categories are generically tensor equivalent to Knop's categories of wreath product representations. We expect that these techniques will be useful in the study of the other categories introduced in [6,7], which we plan to investigate in the future.

We remark that the use of the language of schemes is largely for convenience, and that none of the results depend upon it. The essential idea is the observation that many elementary categorical properties can be described using first-order logic. When one has an “algebraically varying” family of categories \mathcal{C}_t , $t \in \mathbb{C}$, the locus of such a property is described using the first-order logic of one's ground field. It is known that the sets definable in affine space \mathbb{C}^d using first-order logic are precisely the *constructible sets*. In particular, the locus of the parameter t such that a certain categorical property of \mathcal{C}_t holds will often be constructible, which implies that if it contains \mathbb{N} , it must contain all transcendental numbers.

2. Preliminaries

2.1. Deligne's construction

We now quickly review Deligne's construction, which begins with a combinatorial parametrization of certain representations of the symmetric group. Define a *recollement* of a family of sets $\{U_i, i \in A\}$ as a family of injections $u_i : U_i \rightarrow C$ such that $C = \bigcup_i u_i(U_i)$. Two recollements C, D of $\{U_i\}$ are *equivalent* if there is a bijection $C \rightarrow D$ commuting with the maps $U_i \rightarrow C, U_i \rightarrow D$.

Now fix $n \in \mathbb{Z}_{\geq 0}$ and define $I = \{1, 2, \dots, n\}$. For a finite set U , let $\text{Inj}(U, I)$ be the permutation representation on the S_n -set $\text{Inj}(U, I)$ of injections $U \rightarrow I$. It is easy to see that every S_n -representation is a direct factor of a direct sum of representations of this form. Given an injection

2.2. The dAHA

The degenerate affine Hecke algebra (dAHA) of type A is the associative unital algebra \mathcal{H}_n over \mathbb{C} generated by variables s_i corresponding to the transpositions $(i \ i+1) \in S_n$ for $1 \leq i < n$, and commuting variables x_1, \dots, x_n , with relations $s_i s_j = s_j s_i$ if $|i-j| \geq 2$; $(s_i s_{i+1})^3 = 1$; $s_i^2 = 1$; $s_j x_i = x_i s_j$ if $j \neq i-1, i$; and $s_i x_i - x_{i+1} s_i = 1$. Another presentation of the dAHA can be given via generators $\sigma \in S_n$ and y_1, \dots, y_n with the relations

$$\sigma y_i \sigma^{-1} = y_{\sigma(i)}, \quad \sigma \in S_n \quad \text{and} \quad [y_i, y_j] = \frac{1}{4} \sum_{k \neq i, j} (ijk) - (jik) \quad \text{if } i \neq j. \quad (4)$$

The two presentations can be shown equivalent via the transformation $s_i \rightarrow (i \ i+1)$, $x_i \rightarrow y_i - (1/2) \sum_{l=1}^n \text{sgn}(i-l)(i \ l)$, where $\text{sgn } x = x/|x|$ if $x \neq 0$ and zero otherwise. See [11].

2.3. The categories $\text{Rep}(\mathcal{H}_t)$

The definition of an \mathcal{H}_n -module can be rewritten in terms of S_n -modules, the *permutation representation* $\mathfrak{h} = \text{Inj}(\text{One}, I)$ for a fixed one-element set One , and the central *Jucys–Murphy element* $\Omega = \sum_{i < j} (i \ j) \in \text{Cent } \mathbb{C}[S_n]$. There is a corresponding endomorphism Ω of the functor $\text{Id}_{\text{Rep}^{\text{ord}}(S_n)}$. Fix $M \in \text{Rep}^{\text{ord}}(S_n)$. On $\mathfrak{h} \otimes \mathfrak{h} \otimes M \in \text{Rep}^{\text{ord}}(S_n)$, let $P_{1,2} = \sigma_{\mathfrak{h}, \mathfrak{h}} \otimes \text{Id}_M$ where $\sigma_{\mathfrak{h}, \mathfrak{h}} : \mathfrak{h} \otimes \mathfrak{h} \rightarrow \mathfrak{h} \otimes \mathfrak{h}$ is the permutation morphism. Define the endomorphisms of $\mathfrak{h} \otimes \mathfrak{h} \otimes M$: $\Omega^{2,3} = \text{Id} \otimes \Omega_{\mathfrak{h} \otimes M}$ and $\Omega^{1,3} = P_{1,2} \circ \Omega^{2,3} \circ P_{1,2}$. Also, let $B_{\mathfrak{h}} : \mathfrak{h} \otimes \mathfrak{h} \rightarrow \mathbf{1}$ be the evaluation map. Then:

Proposition 2.2. (See Etingof [6,7].) To give a module over \mathcal{H}_n is equivalent to giving an S_n -representation M with an S_n -morphism $y : \mathfrak{h} \otimes M \rightarrow M$ satisfying

$$y \circ (\text{Id} \otimes y) \circ (\text{Id} - P_{1,2}) = (B_{\mathfrak{h}} \otimes \text{Id}) \circ [\Omega^{1,3}, \Omega^{2,3}] : \mathfrak{h} \otimes \mathfrak{h} \otimes M \rightarrow M. \quad (5)$$

A morphism between \mathcal{H}_n -modules is an S_n -morphism commuting with the y -maps.

The terms $P_{1,2}, B_{\mathfrak{h}}, \Omega^{1,3}, \Omega^{2,3}$ in (5) can be interpolated to $t \notin \mathbb{Z}_{\geq 0}$ because Deligne’s categories are symmetric monoidal categories, and each object is self-dual. The object \mathfrak{h} is defined as $[\text{One}]$. The interpolation of the Jucys–Murphy endomorphism is due to [3] and will be discussed below.

Note that where [7] uses $\text{Rep}(S_t)$, we use $\text{Rep}(S_t)$; our definitions are thus slightly different, but they coincide outside of $\mathbb{Z}_{\geq 0}$. Our convention will be more convenient for the method developed in this paper.

Definition 2.3. (See Etingof [6,7].) For $t \in \mathbb{C}$, objects in $\text{Rep}(\mathcal{H}_t)$ are objects M in Deligne’s category $\text{Rep}(S_t)$ equipped with $\text{Rep}(S_t)$ -morphisms $y : \mathfrak{h} \otimes M \rightarrow M$ satisfying (5). Morphisms in $\text{Rep}(\mathcal{H}_t)$ are morphisms in $\text{Rep}(S_t)$ commuting with the y maps.

Remark. The definition in [6,7] actually used ind-objects. This suggests further research, as we only study the finite-dimensional theory.

3. Categories depending on a parameter

3.1. Definitions

First, we review some relevant algebro-geometric facts. Let $\mathbf{Sch}_{\mathcal{A}}$ denote the category of noetherian schemes. For $X \in \mathbf{Sch}_{\mathcal{A}}$, let $\text{Const}(X)$ denote the collection of its constructible subsets. If $f : X \rightarrow Y$ is a morphism, then there exists $f^{-1} : \text{Const}(Y) \rightarrow \text{Const}(X)$ sending $C \rightarrow f^{-1}(C)$; this makes Const a contravariant functor $\mathbf{Sch}_{\mathcal{A}} \rightarrow \mathbf{Bool}$, where \mathbf{Bool} is the category of Boolean algebras.

Similarly, by a special case of Chevalley's theorem, if $S \in \mathbf{Sch}_{\mathcal{N}}$ and $(\mathbf{Sch}/S)_f$ denotes the category of S -schemes of finite type, we can make \mathbf{Const} a covariant functor $(\mathbf{Sch}/S)_f \rightarrow \mathbf{Poset}$, for \mathbf{Poset} the category of posets, where $f : X \rightarrow Y$ defines the map $f_* : \mathbf{Const}(X) \rightarrow \mathbf{Const}(Y)$, $D \mapsto f(D)$; by abuse of notation, we often write f for f_* .

Let X, Y, T be S -schemes and $X_T = X \times_S T$, $Y_T = Y \times_S T$. The diagram

$$\begin{array}{ccc} X_T & \xrightarrow{p_X} & X \\ f_T \downarrow & & \downarrow f \\ Y_T & \xrightarrow{p_Y} & Y \end{array}$$

is cartesian, so for any $C \subset X$, we have $p_Y^{-1}(f(C)) = f_T(p_X^{-1}(C))$ by [10], I.3.4.5.

Given an S -scheme X , $G \subset X$, and a point $s \in S$, we define $G_{\bar{s}} = f_{\bar{s}}'^{-1}(G) \subset X_{\bar{s}}$, where $f_{\bar{s}} : \mathrm{Spec} \bar{k}(s) \rightarrow S$ is the canonical morphism and $f_{\bar{s}}' : X_{\bar{s}} = X \times_S \mathrm{Spec} \bar{k}(s) \rightarrow X$ is the base extension to X . Here $\bar{k}(s)$ is an algebraic closure of $k(s)$; when used, the choice of $\bar{k}(s)$ will be irrelevant.

Definition 3.1. A category over S is a category internal to the category \mathbf{Sch}/S of S -schemes and S -morphisms. There is a notion of a functor over S , or even a natural transformation over S , so we may consider the 2-category \mathbf{Cat}_S of categories over S .

A category over S is thus a four-tuple $(\mathrm{Ob}, \mathrm{Mor}, \mathrm{Comp}, \mathrm{Id})$, where Ob is an S -scheme corresponding to the objects; Mor is an S -scheme with an S -morphism $\mathrm{Mor} \rightarrow \mathrm{Ob} \times_S \mathrm{Ob}$, whose first and second projections are denoted dom and cod respectively; Comp is an S -morphism $\mathrm{Mor} \times_{\mathrm{Ob}} \mathrm{Mor} \rightarrow \mathrm{Mor}$; and $\mathrm{Id} : \mathrm{Ob} \rightarrow \mathrm{Mor}$ is an S -morphism. There are certain conditions on these that need to be satisfied, cf. [14]. A category over S is said to be of finite type when the morphisms $\mathrm{Ob} \rightarrow S$, $\mathrm{Mor} \rightarrow S$ are of finite type, or if it is internal to $(\mathbf{Sch}/S)_f$.

Given a category \mathcal{C} over S , one may construct a category \mathcal{C} by the Yoneda process. The objects are the S -sections $S \rightarrow \mathrm{Ob}$; the morphisms from $a : S \rightarrow \mathrm{Ob}$, $b : S \rightarrow \mathrm{Ob}$ are defined as the S -maps $h : S \rightarrow \mathrm{Mor}$ such that $\mathrm{dom} \circ h = a$, $\mathrm{cod} \circ h = b$. Composition of appropriate $h : S \rightarrow \mathrm{Mor}$, $g : S \rightarrow \mathrm{Mor}$ is defined as $\mathrm{Comp} \circ (h \times_S g)$. There is thus a 2-functor $\mathcal{R}_S : \mathbf{Cat}_S \rightarrow \mathbf{Cat}$, where \mathbf{Cat} is the 2-category of categories, which we often abbreviate by \mathcal{R} .

Next, since the functor $\mathbf{Sch}/S \rightarrow \mathbf{Sch}/T$ defined by $X \mapsto X \times_S T$ preserves finite limits, we can perform base extension of S -categories; hence, given $T \rightarrow S$, there is a 2-functor $B_S^T : \mathbf{Cat}_S \rightarrow \mathbf{Cat}_T$. We abbreviate $\mathcal{C}_T = B_S^T(\mathcal{C})$, $\mathcal{C}_T = \mathcal{R}_T(B_S^T(\mathcal{C}))$.

3.2. Constructibility of categorical properties

For an S -scheme Y , by abuse of notation denote by $\Delta_Y \subset Y \times_S Y$ the locally closed subscheme that is the image of the diagonal map $\Delta_Y : Y \rightarrow Y \times_S Y$. Let $\mathcal{C} = (\mathrm{Ob}, \mathrm{Mor}, \mathrm{Comp}, \mathrm{Id})$ be a category of finite type over the noetherian scheme S . Consider the S -scheme $C = (\mathrm{Mor} \times_{\mathrm{Ob} \times_S \mathrm{Ob}} \mathrm{Mor}) \times_{\mathrm{Ob}} \mathrm{Mor}$, where the map $(\mathrm{Mor} \times_{\mathrm{Ob} \times_S \mathrm{Ob}} \mathrm{Mor}) \rightarrow \mathrm{Ob}$ comes from cod , and $\mathrm{Mor} \rightarrow \mathrm{Ob}$ is dom . There is a map $f : C \rightarrow \mathrm{Mor} \times_{\mathrm{Ob} \times_S \mathrm{Ob}} \mathrm{Mor}$ obtained by composition on each factor: $f = (\mathrm{Comp} \circ (p_3, p_1), \mathrm{Comp} \circ (p_3, p_2))$, where p_1, p_2, p_3 are the projections on the respective factors of C . Now consider $f^{-1}(\Delta_{\mathrm{Mor}}) = (\Delta_{\mathrm{Mor}} \times_{\mathrm{Ob}} \mathrm{Mor}) \subset C$; this is a constructible set X . Set $\mathrm{Mon} = \mathrm{Mor} - p_3(X) \subset \mathrm{Mor}$. The formation of Mon commutes with base extension, and Mon is constructible by Chevalley's theorem. Finally, when $S = \mathrm{Spec} k$ for k an algebraically closed field, then the closed (i.e., k -valued) points of Mon correspond precisely to monomorphisms in the category $\mathcal{C}_k = \mathcal{C}_{\mathrm{Spec} k}$. This discussion and its dual imply the following result.

Proposition 3.2. Suppose the scheme S is noetherian and \mathcal{C} is of finite type. Then there is a constructible set $\mathrm{Mon} \subset \mathrm{Mor}$ (resp. $\mathrm{Epi} \subset \mathrm{Mor}$) such that for each $s \in S$, $\mathrm{Mon}_{\bar{s}} \subset \mathrm{Mor}_{\bar{s}}$ (resp. $\mathrm{Epi}_{\bar{s}} \subset \mathrm{Mor}_{\bar{s}}$) has closed points corresponding to the monic (resp. epic) maps in $\mathcal{C}_{\bar{s}} = \mathcal{C}_{\bar{k}(s)}$.

Next, we study additive structures on S -categories.

Definition 3.3. An *Ab-category* over S is an S -category together with the structure of an $\text{Ob} \times_S \text{Ob}$ group scheme on $\mathcal{M}or$, with an addition $\text{add} : \mathcal{M}or \times_{\text{Ob} \times_S \text{Ob}} \mathcal{M}or \rightarrow \mathcal{M}or$ that commutes with composition. An *additive functor* over S is one which commutes with the group structures. There is thus a 2-category \mathbf{AbCat}_S of Ab-categories over S , additive functors, and natural transformations. Base extension preserves additivity.

The 2-functor $\mathcal{R} : \mathbf{Cat}_S \rightarrow \mathbf{Cat}$ defined earlier induces a 2-functor from \mathbf{AbCat}_S to \mathbf{AbCat} for \mathbf{AbCat} the 2-category of Ab-categories, additive functors, and natural transformations.

If A is an object in an ordinary Ab-category \mathcal{C} , say that A is *simple* if every map $A \rightarrow B$ is either zero or monic, or equivalently if any map $C \rightarrow A$ is either zero or epic, i.e., if for all $B, C \in \mathcal{C}$ and nonzero $f : C \rightarrow A$ and $g : A \rightarrow B$, the composition $g \circ f$ is nonzero.

Proposition 3.4. Suppose S is noetherian and \mathcal{C} is an Ab-category of finite type over S . Then there is a constructible set $\text{Sim} \subset \text{Ob}$ such that for each $s \in S$, $\text{Sim}_s \subset \text{Ob}_s$ has closed points corresponding precisely to the simple objects in \mathcal{C}_s .

Proof. Let $z : \text{Ob} \times_S \text{Ob} \rightarrow \mathcal{M}or$ be the section corresponding to the zero morphism. Consider the constructible set $R = \mathcal{M}or - \text{Mon} - z(\text{Ob} \times_S \text{Ob}) \subset \mathcal{M}or$; R loosely speaking parametrizes nonzero non-monomorphic morphisms. Set $\text{Sim} = \text{Ob} - \text{dom}(R)$, and one may argue as in the discussion preceding Proposition 3.2. \square

A simple example of this phenomenon is as follows. Fix a finitely generated but not necessarily commutative k -algebra A for k an algebraically closed field and a cut-off $N \in \mathbb{Z}_{\geq 0}$. Then the category of k -valued matrix representations of A of dimension at most N and intertwining maps can be obtained by applying \mathcal{R}_k to a category of finite type over $\text{Spec } k$. These propositions then reduce to well-known facts, e.g., that injective (or surjective) homomorphisms are described by a constructible set. Indeed, the set of injective homomorphisms is even open by linear algebra, because it can be described by the nonvanishing of certain minors. One can also apply these results to the categories of finite-dimensional representations of algebras depending on a parameter, such as quantum \mathfrak{sl}_2 , or affine or finite Hecke algebras; these form categories over open subsets of $\mathbb{A}_{\mathbb{C}}^1$.

Given a scheme X , let $\text{cl}(X)$ denote the subspace of closed points. The following lemma is elementary.

Lemma 3.5. Let $f : X \rightarrow Y$ be an S -morphism for X, Y of finite type over the noetherian scheme S . Then the set of $s \in S$ with the map $\text{cl}(f_s) : \text{cl}(X_s) \rightarrow \text{cl}(Y_s)$ bijective is constructible.

Proposition 3.6. Let $\mathcal{C} = (\text{Ob}, \mathcal{M}or, \text{Comp}, \text{Id})$ be an S -category of finite type, with S noetherian. There is a constructible set $\text{In} \subset \text{Ob}$ (resp. $\text{Fi} \subset \text{Ob}$) such that, for $s \in S$, the closed points in In_s (resp. Fi_s) correspond to the initial (resp. final) objects in \mathcal{C}_s .

Proof. We treat the initial case. An object A in a category \mathcal{C} is initial if and only if for all $B \in \mathcal{C}$, the diagonal map $\text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(A, B)$ is bijective. Hence, consider $\Delta_{\mathcal{M}or} : \mathcal{M}or \rightarrow \mathcal{M}or \times_{\text{Ob} \times_S \text{Ob}} \mathcal{M}or$. Let $T \subset \text{Ob} \times_S \text{Ob}$ denote the constructible set of $t \in \text{Ob} \times_S \text{Ob}$ such that $(\Delta_{\mathcal{M}or})_t = \Delta_{\mathcal{M}or_t}$ is bijective. Then let $\text{In} = \text{Ob} - p_1(\text{Ob} \times_S \text{Ob} - T)$, where $p_1 : \text{Ob} \times_S \text{Ob} \rightarrow \text{Ob}$ is projection on the first factor. \square

Given an S -category \mathcal{C} and a finite (ordinary) category \mathcal{Z} , we can consider the functor category $\mathcal{C}^{\mathcal{Z}}$, whose object scheme E is constructed so as to parametrize \mathcal{Z} -diagrams of morphisms, and whose morphism scheme is constructed to parametrize morphisms of diagrams. Specifically, E is constructed from the scheme $\prod_{f:A \rightarrow A', A, A' \in \mathcal{Z}} \mathcal{M}or$ via a suitable intersection of equalizers corresponding to composability and the relations defining \mathcal{Z} , for instance. Finally, we can consider the

comma category $\mathcal{C}om$ as a category over E ; indeed, there is a functor over S from $\mathcal{C}om$ to \mathcal{C}^Z . Let F be the object scheme. Then, since the universal objects in a comma category consist of diagram schemes and their limits, Proposition 3.6 implies the following corollary. It can also be dualized for colimits.

Corollary 3.7. *Notation as above, if \mathcal{C} is of finite type and S noetherian, there is a constructible subset $L \subset F$ such that for each $s \in S$, L_s has closed points corresponding to pairs of C_s valued Z -diagram schemes with limits in C_s , and those limits.*

3.3. Envelopes of categories

Recall that the *Karoubi envelope* of a category is formed by adding formal images of idempotent elements. The construction, presented, e.g., in [2], is as follows. Given \mathcal{C} , an object in $\text{Kar}(\mathcal{C})$ is a pair (A, e) where $A \in \mathcal{C}$ and $e \in \text{Hom}_{\mathcal{C}}(A, A)$ is idempotent. A morphism between (A, e) and (B, f) is a \mathcal{C} -morphism $h : A \rightarrow B$ with $h = h \circ e = f \circ h$. Note incidentally that the identity endomorphism of (A, e) is given by $h = e$. The Karoubi envelope of an Ab-category is an Ab-category too. An Ab-category equivalent to its Karoubi envelope, or equivalently such that every idempotent has an image, is called *Karoubian*. Next, the *additive envelope* of an Ab-category \mathcal{A} has objects consisting of formal direct sums $\bigoplus_{i=1}^n A_i$ with $A_i \in \mathcal{A}$, with morphisms between $\bigoplus_{i=1}^n A_i, \bigoplus_{j=1}^m B_j$ defined as appropriate m -by- n matrices, and composition as matrix multiplication. Finally, the *pseudo-abelian envelope* is the Karoubi envelope of the additive envelope. A *pseudo-abelian category* is an additive category where every idempotent endomorphism is a projection, or alternatively one equivalent to its pseudo-abelian envelope.

Proposition 3.8. *There is a 2-functor $\text{Kar} : \mathbf{AbCat}_S \rightarrow \mathbf{AbCat}_S$ preserving the property of finite type such that $\mathcal{R}(\text{Kar}(\mathcal{C}))$ is equivalent to the Karoubi envelope of $\mathcal{R}(\mathcal{C})$. There are similar 2-functors $\text{Add}, \text{Ps} : \mathbf{AbCat}_S \rightarrow \mathbf{AbCat}_S$, though these do not necessarily preserve the property of finite type. $\text{Kar}, \text{Add}, \text{Ps}$ all commute with base extension.*

As the proof will show, this is true for internal categories in a category with arbitrary fibered products. Even so, we cannot find this result in a reference.

Proof. All these categories are constructed to parametrize suitable diagrams. Given $\mathcal{C} = (\text{Ob}, \text{Mor}, \text{Comp}, \text{Id})$, we take for the object scheme Ob_K of $\text{Kar}(\mathcal{C})$ the intersection of the equalizer of $\text{Ob} \times_S \text{Mor}$ under the three maps $\{p_1, \text{dom} \circ p_2, \text{cod} \circ p_2\}$ and the equalizer of the maps $\{\text{Comp} \circ \Delta_{\text{Mor}} \circ p_2, p_2\}$, where p_1, p_2 are projections on the first and second factor. The morphism scheme is taken to be the intersection of the equalizers of $\text{Ob}_K \times_S \text{Ob}_K \times_S \text{Mor}$ under $\{p_1 \circ q_1, \text{dom} \circ q_3\}, \{p_1 \circ q_2, \text{cod} \circ q_3\}, \{\text{Comp} \circ (q_3, p_2 \circ q_1), \text{Comp} \circ (p_2 \circ q_2, q_3), q_3\}$, where q_1, q_2, q_3 are the projections from $\text{Ob}_K \times_S \text{Ob}_K \times_S \text{Mor}$ onto the first, second, and third factors. Then the first part of the result follows from the definitions, since base extension commutes with S -products. The proof for Add is similar, though because of the infinite coproduct involved over direct sums of larger sizes, Add does not preserve the property of finite type. Then $\text{Ps} = \text{Kar} \circ \text{Add}$. \square

3.4. Semisimplicity

We shall now demonstrate a method of proving generic semisimplicity of a family of categories.

Theorem 3.9. *Let \mathcal{C} be a finite type Ab-category over S for S a noetherian scheme. Suppose $Z \subset S$ is a Zariski dense subset such that for $s \in Z$ we have $\text{Ps}(C_s)$ semisimple abelian with endomorphism rings either zero or infinite. Then, for ξ in a dense open subset of S , $\text{Ps}(C_\xi)$ is semisimple abelian.*

Proof. Without loss of generality, assume C_s Karoubian for $s \in Z$. Recall that $\text{Sim} \subset \text{Ob}$ is constructible, for Ob the object scheme of \mathcal{C} . By Propositions 3.4 and 3.7, there are constructible subsets

$\text{Sim}^1, \text{Sim}^2, \dots \subset \text{Ob}$ with $(\text{Sim}^n)_{\bar{s}}$ having closed points corresponding to objects in $\mathcal{C}_{\bar{s}}$ which are the direct sum of at most n simple objects, for each s, n .

Let $M = \sup_{x \in \text{Ob} \times_S \text{Ob}} \dim(\text{Mor}_x) = \sup_{x \in \text{Ob} \times_S \text{Ob}} \dim(\text{Mor}_{\bar{x}})$. We show that for $s \in Z$, any object in $\mathcal{C}_{\bar{s}}$ is a direct summand of at most \sqrt{M} simple objects. Indeed, if $X \in \mathcal{C}_{\bar{s}}$ is nonzero and simple, then $\dim \text{Mor}_{(\bar{X}, \bar{X})} \geq 1$ since $\text{Hom}(X, X)$ is infinite. If $Y = X_1 \oplus \dots \oplus X_m$ for X_i nonzero and simple in $\mathcal{C}_{\bar{s}}$, then $m^2 \leq \dim \text{Mor}_{(\bar{Y}, \bar{Y})} \leq M$, since from projection, inclusion, and Comp, there is an isomorphism of schemes $\text{Mor}_{(\bar{Y}, \bar{Y})} \simeq \prod_{i,j} \text{Mor}_{(\bar{X}_i, \bar{X}_j)}$. In particular, we see that $(\text{Ob} - \text{Sim}^N)_{\bar{s}} = \emptyset$ if $N > \sqrt{M}$ and $s \in Z$. Hence $\text{Ob}_{\bar{s}} = (\text{Sim}^N)_{\bar{s}}$ for $\bar{s} \in U$ for $U \subset S$ open and dense. By shrinking U if necessary and using Proposition 3.7, we may assume $\mathcal{C}_{\bar{s}}$ admits kernels and cokernels for $\bar{s} \in U$. The additive envelope of $\mathcal{C}_{\bar{s}}$ for $\bar{s} \in U$ is thus semisimple abelian and consequently is equivalent to the pseudo-abelian envelope. \square

3.5. Deligne's categories as $\mathbb{A}_{\mathbb{C}}^1$ -categories

We now incorporate Deligne's construction into the above framework. It follows that there is a category $\mathfrak{Rep}'(S_T)$ over \mathbb{A}^1 with $\mathcal{R}(\mathfrak{Rep}'(S_T)) \simeq \text{Rep}'(S_T)$; the object scheme is an infinite coproduct $\coprod_{j \in \mathbb{Z}_{\geq 0}} \mathbb{A}^1$ while $\text{Mor} = \coprod_{j,k} \mathbb{A}^1 \times \mathbb{A}^{\text{Rec}(j,k)}$, where $\text{Rec}(j, k)$ denotes the number of equivalence classes of recollements of a set with j elements and one with k elements.

Let $\mathfrak{Rep}''(S_T) = \text{Ps}(\mathfrak{Rep}'(S_T))$; this admits a bifunctor over \mathbb{A}^1 corresponding to the tensor product. By base extension of $\mathfrak{Rep}''(S_T)$ via $\mathbb{Z}[T] \rightarrow R$, $T \rightarrow t$, we find:

Proposition 3.10. *There is a Spec R -category $\mathfrak{Rep}(S)_t^R$ with $\mathcal{R}(\mathfrak{Rep}(S)_t^R) \simeq \text{Rep}(S_t, R)$.*

It is clear that the tensor structure on $\mathfrak{Rep}(S)_t^R$ is a bifunctor T over $\text{Spec } R$. Moreover, if D is the “diagonal functor” $\mathfrak{Rep}(S)_t^R \rightarrow \mathfrak{Rep}(S)_t^R \times_R \mathfrak{Rep}(S)_t^R$, then there are natural transformations over $\text{Spec } R$, $\text{Id} \rightarrow B(D)$, $B(D) \rightarrow \text{Id}$ from the self-duality in $\text{Rep}(S_t, R)$, which satisfy the appropriate axioms for evaluations and coevaluations.

We let $\mathfrak{Rep}(S) = \mathfrak{Rep}(S)_T^{\mathbb{C}[T]}$; this is a category over $\mathbb{A}_{\mathbb{C}}^1$. For $t \in \mathbb{A}_{\mathbb{C}}^1$, let $\mathfrak{Rep}(S)_t$ be the fiber of $\mathfrak{Rep}(S)$ at t . There is a similar situation for the $\mathfrak{Rep}(S)^{(N)}$, which correspond to the categories $\text{Rep}(S_t)^{(N)}$ defined in 4.1 of [5]; these are pseudo-abelian envelopes of $\mathbb{A}_{\mathbb{C}}^1$ -categories of finite type. Finally, we let $\mathfrak{Rep}(S)^{(M,N)}$ correspond to the category $\text{Rep}(S_t)^{(M,N)}$, which we define as the Karoubi envelope of the category obtained by formal direct sums with at most M elements of objects in $\text{Rep}'(S_t)^{(N)}$; then $\mathfrak{Rep}(S)^{(M,N)}$ is an approximation to $\mathfrak{Rep}(S)^{(N)}$ and, most importantly, is of finite type.

There is a similar situation for the interpolations of $\text{Rep}^{\text{ord}}(GL_n)$, $\text{Rep}^{\text{ord}}(O_n)$, and $\text{Rep}^{\text{ord}}(Sp_{2n})$, in view of their definitions in [4,5]. We omit the details. We thus obtain a new proof of the following result.

Theorem 3.11. *(See Deligne [5], contained in Theorem 2.18.) Let F be an algebraically closed field of characteristic zero. Then $\text{Rep}(S_t, F)^{(N)}$ is semisimple abelian for all but finitely many $t \in F$, which are algebraic over \mathbb{Q} . $\text{Rep}(S_t, F)$ is semisimple abelian for transcendental t . Similarly for the other Deligne categories associated to the general linear, orthogonal, and symplectic groups.*

Proof. The statement about $\text{Rep}(S_t, F)^{(N)}$ follows from Theorem 3.9 applied to $(\mathfrak{Rep}(S)_T^{[T]})^{(1,N)}$ and Maschke's theorem. The statement about algebraicity follows because all the schemes in question are actually defined over \mathbb{Q} . This implies the rest of the corollary for $\text{Rep}(S_t, F)$ since $\text{Rep}(S_t, F)^{(N)} \subset \text{Rep}(S_t, F)$ is a full subcategory and $\text{Rep}(S_t, F) = \bigcup_N \text{Rep}(S_t, F)^{(N)}$. The other cases are handled similarly. \square

Unfortunately, this method does not address the rationality questions that arise when F is not algebraically closed. For a different method that yields the generic result using the specific properties

of the partition algebra, see [3]. For results on the generic semisimplicity of *tensor* categories obtained using the calculus of relations on regular categories, cf. [13]; this method uses the specific properties of degree functions.

By base extension, there is a *specialization functor*

$$\mathrm{Sp}_t : \mathrm{Rep}(S_T, \mathbb{C}[T])^{(M,N)} \simeq \mathcal{R}_{\mathbb{A}_\mathbb{C}^1}(\mathfrak{Rep}(S)^{(M,N)}) \rightarrow \mathrm{Rep}(S_t)^{(M,N)}$$

for $t \in \mathbb{C}$. In simple terms, this corresponds to substituting a value in for a polynomial. When $n \in \mathbb{Z}_{\geq 0}$, we may also define $\mathbf{Sp} : \mathrm{Rep}(S_T, \mathbb{C}[T]) \rightarrow \mathrm{Rep}^{\mathrm{ord}}(S_n)$.

For a given object or morphism a in $\mathrm{Rep}(S_T, \mathbb{C}(T))$, we can define almost all specializations $\mathrm{Sp}_t a$. Indeed, if Y is either the object or morphism scheme, then the map $\mathrm{Spec} \mathbb{C}(T) \rightarrow Y$ factors into a map of the form $\mathrm{Spec} \mathbb{C}(T) \rightarrow \mathrm{Spec} \mathbb{C}[T]_{P(T)} \rightarrow Y$, where $P(T)$ is a suitable polynomial.

For almost all t , specialization sends the simple objects X_π in Deligne's category $\mathrm{Rep}(S_T, \mathbb{C}(T))^{(N)}$ indexed by the partition $\pi : l_2 + l_3 + \cdots + l_k = l$ to those of $\mathrm{Rep}(S_t)^{(N)}$ similarly indexed, by the proofs in [5], §5. Deligne constructs the relevant objects by using certain “Young projectors” on objects $[U]$.

Finally, for $n > 2N + 1$, the simple object X_π of $\mathrm{Rep}(S_n)^{(N)}$, which is a full subcategory of $\mathrm{Rep}^{\mathrm{ord}}(S_n)$, corresponds to the classical Specht module associated to the partition $n = n - l + l_2 + \cdots + l_k$, in view of 6.4 in [5].

4. The dAHA categories

We shall now apply the methods of the previous section to the Etingof categories associated to the dAHA.

4.1. $\mathrm{Rep}(\mathcal{H}_t)$ in the present framework

It is not immediate that the Etingof categories form an instance of “categories depending on a parameter.” This follows from the next result, essentially due to Comes and Ostrik.

Proposition 4.1. (See Comes and Ostrik [3].) *On the $\mathbb{A}_\mathbb{C}^1$ -category $\mathfrak{Rep}(S)$, there exists an endomorphism Ω of the identity functor $\mathrm{Id}_{\mathfrak{Rep}(S)}$ interpolating the action of the Jucys–Murphy element in the integer case.*

Consequently, there is an $\mathbb{A}_\mathbb{C}^1$ -category $\mathfrak{Rep}(\mathcal{H})$ with $\mathcal{R}(\mathfrak{Rep}(\mathcal{H})_t) \simeq \mathrm{Rep}(\mathcal{H}_t)$. There is a similar situation for $\mathfrak{Rep}(\mathcal{H})^{(N)}$ and $\mathfrak{Rep}(\mathcal{H})^{(M,N)}$, which correspond to the categories where the underlying object in $\mathrm{Rep}(S_t)$ belongs to $\mathrm{Rep}(S_t)^{(N)}$ (resp. $\mathrm{Rep}(S_t)^{(M,N)}$).

4.2. The classical dAHA representation theory

We state the classification of simple \mathcal{H}_n -modules. Given $a \in \mathbb{C}$, there exists an *evaluation homomorphism* $\mathcal{H}_n \rightarrow \mathbb{C}[S_n]$ (see [1], Lemma 1.5.3) that is the identity on $\mathbb{C}[S_n]$ and sends $x_i \rightarrow a + \sum_{j < i} (i \ j)$. Alternatively, the homomorphism sends $y_i \rightarrow a + \frac{1}{2} \sum_{j \neq i} (i \ j)$. There is a corresponding evaluation functor $\mathrm{Ev}_a : \mathrm{Rep}^{\mathrm{ord}} S_n \rightarrow \mathrm{Rep}^{\mathrm{ord}} \mathcal{H}_n$. Next, there is a homomorphism $\mathcal{H}_m \otimes \mathcal{H}_n \rightarrow \mathcal{H}_{m+n}$ inducing the canonical maps $\mathbb{C}[S_m] \otimes \mathbb{C}[S_n] \rightarrow \mathbb{C}[S_{m+n}]$ and $\mathbb{C}[x_1, \dots, x_m] \otimes \mathbb{C}[x_{m+1}, \dots, x_{m+n}] \rightarrow \mathbb{C}[x_1, \dots, x_{m+n}]$. They lead to induction functors $\mathrm{Rep}^{\mathrm{ord}} \mathcal{H}_m \times \mathrm{Rep}^{\mathrm{ord}} \mathcal{H}_n \rightarrow \mathrm{Rep}^{\mathrm{ord}} \mathcal{H}_{m+n}$ sending A, B to $\mathcal{H}_{m+n} \otimes_{\mathcal{H}_m \otimes \mathcal{H}_n} (A \otimes B)$, which can be iterated. Cf. [11].

Fix an unordered partition $\pi : n = l_1 + \cdots + l_k$. Let $\mathfrak{t}_k = \mathbb{C}^k$, spanned by a basis $\epsilon_1^\vee, \dots, \epsilon_k^\vee$; this corresponds to the abelian Lie subalgebra of \mathfrak{gl}_k consisting of diagonal matrices. Let \mathfrak{t}_k^* be its dual, and $\epsilon_1, \dots, \epsilon_k \in \mathfrak{t}_k^*$ be the dual basis to $\epsilon_1^\vee, \dots, \epsilon_k^\vee$. Set $\rho = \frac{1}{2} \sum_{i < j} (\epsilon_i - \epsilon_j)$. Let

$$D_k = \{ \lambda \in \mathfrak{t}_k^* : (\lambda + \rho)(\epsilon_i^\vee - \epsilon_j^\vee) \notin \mathbb{Z} \cap (-\infty, 0), \ 1 \leq i < j \leq k \}.$$

Fix $\mu = \sum_i \mu_i \epsilon_i \in \mathfrak{t}_k^*$. Define

$$M^n(\mu)_\pi = \text{Ind}_{l_1, \dots, l_k}^n (\text{Ev}_{\mu_1 + \rho_1} \mathbf{1}, \dots, \text{Ev}_{\mu_k + \rho_k} \mathbf{1}) \in \text{Rep}^{\text{ord}}(\mathcal{H}_n).$$

If $\mu + l_1 \epsilon_1 + l_2 \epsilon_2 + \dots + l_k \epsilon_k \in D_k$, then $M^n(\mu)_\pi$ is called a *standard module*.

Theorem 4.2. (See Zelevinsky [16].)

1. Each standard module $M^n(\mu)_\pi$ has a unique simple quotient $L^n(\mu)_\pi$. Every simple representation of \mathcal{H}_n is isomorphic to some $L^n(\mu)_\pi$.
2. $L^n(\mu)_\pi$ contains as an S_n -submodule the irreducible S_n -representation corresponding to the partition π .

Zelevinsky proved this in [16] in the language of p -adic groups. The statement above is from Theorems 2.3.1 and 2.3.3 of [15]. We will interpolate the standard modules to complex rank.

4.3. Interpolation of Ev_a

The evaluation homomorphisms themselves do not make sense in complex rank, but the next result shows that the associated functors do.

Proposition 4.3. *There is a functor over $\mathbb{A}_{\mathbb{C}}^1$, $\text{Ev} : B_{\mathbb{A}_{\mathbb{C}}^1}^{\mathbb{A}_{\mathbb{C}}^2}(\mathfrak{Rep}(S)) \rightarrow \mathfrak{Rep}(\mathcal{H})$ inducing by fibers for $a \in \mathbb{C}$, functors $\text{Ev}_a : \mathfrak{Rep}(S) \rightarrow \mathfrak{Rep}(\mathcal{H})$ that interpolate the evaluation functors defined above.*

For finite sets U_1, \dots, U_m , the space $\bigotimes_i \text{Inj}(U_i, I)$ has a $\mathbb{C}[S_n]$ -basis indexed naturally by recollements C of U_1, \dots, U_m such that $|C| \leq n$. This is a generalization of Proposition 2.8 of [5] and is proved precisely the same way. We shall use this in the proof.

Proof. In the sequel, we shall abuse notation and occasionally blur the distinction between $[U]$ and $\text{Inj}(U, I)$. Consider the integral case. Let $I = \{1, 2, \dots, n\}$. From the Ev_a functor, we have an S_n -morphism $[\text{One}] \otimes [U] \rightarrow [U]$, which corresponds to an element $e \in [\text{One}] \otimes [U] \otimes [U]$. Let $y_i, i \in I$ denote the canonical basis for $[\text{One}]$. Then

$$e = \sum_{i \in I, f \in \text{Inj}(U, I)} y_i \otimes f \otimes \left(af + \frac{1}{2} \sum_{j \neq i} (i \ j) \circ f \right).$$

We compute the coefficients of e in terms of the canonical basis $y_i \otimes f \otimes g$, $i \in I$, $f, g \in \text{Inj}(U, I)$. Fix i, f, g . Then $y_i \otimes f \otimes g$ will occur in e precisely if $f = g$ or if f, g differ by a transposition starting at i , which corresponds to the image of One in the recollement induced by i, f, g .

If $f \neq g$, then the coefficient of $y_i \otimes f \otimes g$ is 1. If $f = g$, and $i \notin \text{Im}(f)$, the coefficient is $a + \text{Card } I - \text{Card } U - 1$; if $f = g$ and $i \in \text{Im}(f)$, the coefficient is a . Otherwise, it is zero.

Thus, translating the basis elements into recollements as in Proposition 4.1, we can describe $[\text{One}] \otimes [U] \rightarrow [U]$ by the sum $\sum_{D \in \text{Rec}(\text{One}, U, U)} Q^D(n, a)(D)$ (which by itself is a map $[\emptyset] \rightarrow [\text{One}] \otimes [U] \otimes [U]$, but one can apply self-duality), where the Q^D are polynomials. In general, in $\text{Rep}'(S_T, \mathbb{C}[T])$, the morphism $\mathfrak{h} \otimes M \rightarrow M$ for $M = [U]$ is defined through $\sum_D Q_D(T, a)(D)$. We can check that the dAHA identity (5) holds by specializing $T \rightarrow n$ for n large, and similarly for functoriality. The functors then extend to the pseudo-abelian envelope, and one then applies base extension to $\text{Spec } \mathbb{C}[T]$. \square

4.4. Interpolation of induction functors

We now construct the induction functors necessary to interpolate the standard modules.

Lemma 4.4. *Let I, J be finite sets, and let S_I, S_J be the corresponding permutation groups. Then $\text{Ind}_{S_I \times S_J}^{S_{I \sqcup J}}(\mathbf{Inj}(V, I) \otimes \mathbf{Inj}(J, J)) \simeq \mathbf{Inj}(V \sqcup J, I \sqcup J)$, where \sqcup denotes the disjoint union of sets.*

Proof. We have an $S_I \times S_J$ -morphism $\mathbf{Inj}(V, I) \otimes \mathbf{Inj}(J, J) \rightarrow \text{Res}_{S_I \times S_J}^{S_{I \sqcup J}} \mathbf{Inj}(V \sqcup J, I \sqcup J)$ that sends $e_f \otimes e_\sigma$ for $f: V \rightarrow I$ and $\sigma: J \rightarrow J$ to $e_{f \sqcup \sigma}$. By Frobenius reciprocity, we get an $S_{I \sqcup J}$ -map $\text{Ind}_{S_I \times S_J}^{S_{I \sqcup J}}(\mathbf{Inj}(V, I) \otimes \mathbf{Inj}(J, J)) \rightarrow \mathbf{Inj}(V \sqcup J, I \sqcup J)$, which is seen to be surjective as follows. When $|V| < |I|$, the image contains at least one basis element e_h for some $h: V \sqcup J \rightarrow I \sqcup J$ gluing together some $f: V \rightarrow I, \text{Id}: J \rightarrow J$, and the span of the corresponding $S_{I \sqcup J}$ orbit is seen to be the full space. Since the dimensions of the two vector spaces are easily checked to be equal, the map is bijective. \square

Proposition 4.5. *Fix N, n . Then for all t , there is a functor*

$$\text{Ind}: \text{Rep}(S_t)^{(N)} \times \text{Rep}^{\text{ord}}(S_n) \rightarrow \text{Rep}(S_{t+n})^{(N+n)}$$

such that if $A \in \text{Rep}(S_t, \mathbb{C}(T))$, $B \in \text{Rep}^{\text{ord}}(S_n)$, then for almost all t , we have $\text{Ind}(\text{Sp}_t(A), B) \simeq \text{Sp}_t(\text{Ind}(A, B)) \in \text{Rep}(S_{t+n})$ and for almost all $m \in \mathbb{Z}_{\geq 0}$, $\text{Ind}_{S_m \times S_n}^{S_{m+n}}(\mathbf{Sp}_m(A), B) \simeq \mathbf{Sp}_m(\text{Ind}(A, B)) \in \text{Rep}^{\text{ord}}(S_{m+n})$.

This could have been stated more closely in line with the framework developed in the previous section, but we thought it would only obscure the meaning.

Proof. We first describe the functoriality of induction via recollements. As usual, we start with the integral case. Assume $|J| = n$ and $|I| = m$. Given $C \in \text{Rec}(V, U)$ and $D \in \text{Rec}(J, J)$ with $|D| = n$, then

$$\text{Ind}_{S_I \times S_J}^{S_{I \sqcup J}}((C), (D)) = (C \sqcup D): \mathbf{Inj}(V \sqcup J, I \sqcup J) \rightarrow \mathbf{Inj}(U \sqcup J, I \sqcup J),$$

i.e., $(C \sqcup D) \in \text{Rec}(V \sqcup J, U \sqcup J)$ is induced by $(C), (D)$. Note that such $(C), (D)$ form a basis for the relative Hom-spaces if m is large enough (and always span them). In view of this, we now construct the interpolated Ind on the full subcategory of $\text{Rep}(S_t) \times \text{Rep}^{\text{ord}}(S_n)$ consisting of objects of the form $([V], [J])$, by sending $([V], [J]) \rightarrow [V \sqcup J]$ and morphisms $(C), (D)$ to $(C \sqcup D)$; functoriality is a polynomial condition in terms of recollements and holds everywhere by testing at large integers. The functor extends by additivity to the Karoubi envelope of this full subcategory, which up to equivalence contains as a full subcategory $\text{Rep}(S_t) \times \text{Rep}^{\text{ord}}(S_n)$. The specialization assertion follows from Lemma 4.4. \square

Remark. It does not seem possible to make more than one parameter non-integral in the above functor, because induction from $S_m \times S_n \rightarrow S_{m+n}$ multiplies the dimension by $\frac{(m+n)!}{m!n!}$, which is not a polynomial in n, m ; if, however, one variable is fixed, then the expression becomes a polynomial in the other. Nevertheless, it is possible to define restriction functors $\text{Rep}(S_{t+u}) \rightarrow \text{Rep}(S_t) \boxtimes \text{Rep}(S_u)$, by sending $h \rightarrow h \otimes \mathbf{1} \otimes h$ and extending tensorially. Consequently one may define consequently a left adjoint (at least generically, when the categories are semisimple) from $\text{Rep}(S_t) \boxtimes \text{Rep}(S_u)$ into the ind-completion of $\text{Rep}(S_{t+u})$. When one parameter is in $\mathbb{Z}_{\geq 0}$ and we use the ordinary category, we do not need ind-objects.

In the next result, we do not obtain an everywhere defined induction functor. It is an open question to determine “degeneracy phenomena” where it cannot be defined.

Proposition 4.6. Fix $n, N \in \mathbb{N}$. Then for transcendental t , there is a functor

$$\text{Ind} : \text{Rep}(\mathcal{H}_t)^{(N)} \times \text{Rep}^{\text{ord}}(\mathcal{H}_n) \rightarrow \text{Rep}(\mathcal{H}_{t+n})^{(N+n)}$$

such that $A \in \text{Rep}(\mathcal{H}_t)$, $B \in \text{Rep}^{\text{ord}}(\mathcal{H}_n)$ imply that for almost all $m \in \mathbb{Z}_{\geq 0}$, $\text{Ind}_{\mathcal{H}_m \otimes \mathcal{H}_n}^{\mathcal{H}_{m+n}}(\text{Sp}_m(A), B) \simeq \text{Sp}_m(\text{Ind}(A, B)) \in \text{Rep}^{\text{ord}}(\mathcal{H}_{m+n})$.

Here T is an indeterminate, and the category $\text{Rep}(\mathcal{H}_t, \mathbb{C}(T))$ is defined in the analogous way.

Proof. The Poincaré–Birkhoff–Witt basis theorem for the dAHA (cf. [11]) implies that for $A \in \text{Rep}^{\text{ord}}(\mathcal{H}_m)$, $B \in \text{Rep}^{\text{ord}}(\mathcal{H}_n)$, we have

$$\text{Res}_{S_{m+n}}^{\mathcal{H}_{m+n}} \text{Ind}_{\mathcal{H}_m \otimes \mathcal{H}_n}^{\mathcal{H}_{m+n}}(A, B) \simeq \text{Ind}_{S_m \times S_n}^{S_{m+n}}(\text{Res}_{S_m}^{\mathcal{H}_m}(A), \text{Res}_{S_n}^{\mathcal{H}_n}(B)).$$

By Proposition 4.5, we need only describe the y -morphism.

First, we describe the functors appropriately in the integral case, as usual. Let $I = \{1, 2, \dots, m\}$, $J = \{m+1, \dots, m+n\}$. Consider $A \in \text{Rep}^{\text{ord}}(\mathcal{H}_m)$, $B \in \text{Rep}^{\text{ord}}(\mathcal{H}_n)$. By adding a direct summand with dAHA-action coming from Ev if necessary, we may assume that as an S_m (resp. S_n) module, $A = \bigoplus_{s \in S} [U_s] \in \text{Rep}^{\text{ord}}(\mathcal{H}_m)$, $B = \bigoplus_{s' \in S'} [J] \in \text{Rep}^{\text{ord}}(\mathcal{H}_n)$ for some finite sets U_s, S, S' . Now $\text{Ind}_{S_m \times S_n}^{S_{m+n}}(\bigoplus_S [U_s], \bigoplus_{S'} [J]) = \bigoplus_{S \times S'} [U_s + J]$.

For simplicity, we assume S and S' consist of one element; the general case is handled similarly. The y -map $[U] \otimes [\text{One}] \rightarrow [U]$ is described by some element $\sum_{f,i,g} C_{f,i}^g e_f \otimes e_i \otimes e_g \in [U] \otimes [\text{One}] \otimes [U]$. Similarly, the map $[J] \otimes [\text{One}] \rightarrow [J]$ is described by $\sum_{h,j,l} D_{h,j}^l e_h \otimes e_j \otimes e_l$, and the map $[U + J] \otimes [\text{One}] \rightarrow [U + J]$ can be described by some $\sum_{p,k,q} E_{p,k}^q e_p \otimes e_k \otimes e_q$. Given $C_{f,i}^g, D_{h,j}^l$, we need to compute $E_{p,k}^q$. Given the action of y_i ($i \in I$) on $[U]$, to get an action of y_i ($i \in I$) on $[U + J]$ we must add $-\frac{1}{2}$ times the action of the cycles $(i \ j')$ for $j' \in [m+1, m+n]$. We similarly get the action of y_j ($j \in J$) on $[U + J]$ using the action of y_j ($j \in J$) on $[U]$ and additional 2-cycles, by the definition of the homomorphism $\mathcal{H}_m \otimes \mathcal{H}_n \rightarrow \mathcal{H}_{m+n}$, which preserves the x_i generators.

Given $k \in I + J$ corresponding to the image of One , $p \in \text{Inj}(U + J, I + J)$, $q \in \text{Inj}(U + J, I + J)$, we compute $E_{p,k}^q$. By S_{I+J} -invariance, assume p comes from pasting $p' : U \rightarrow I$, $p'' = \text{Id}_J : J \rightarrow J$. Suppose first $k \notin \text{Im}(p'') = J \subset I + J$. Then $E_{p,k}^q = C_{p',k}^{q|_J}$ if $q|_J = \text{Id}_J$. If $q|_J$ varies from the identity by a transposition starting at k , then $E_{p,k}^q = -\frac{1}{2}$, and otherwise $E_{p,k}^q = 0$. Now suppose $k \in \text{Im}(p'') = J$. Then $E_{p,k}^q = D_{\text{Id}_J,k}^{q|_J}$ if $q|_U = p|_U$, $E_{p,k}^q = \frac{1}{2}$ if $q|_U, p|_U$ differ by a transposition starting at k and $q|_J = \text{Id}_J$, and $E_{p,k}^q = 0$ otherwise. Thus, we have described the induced y -morphism in terms of recollements, and interpolation can proceed.

Finally, the dAHA identity (5) on each factor describes closed subschemes V_1, V_2 of some coproduct of affine spaces over $\mathbb{C}[T]$, and induction is a morphism f into some coproduct of affine spaces, which has a closed subscheme V_3 from (5) too. Now, $(f(V_1 \times V_2))_r \subset (V_3)_r$ for $r \in \mathbb{Z}_{\geq 0}$ large, so we can apply the usual constructibility argument to see that the induced y -morphism satisfies (5) for all but possibly finitely many algebraic t . \square

This functor can be iterated to multiple factors, a fact that we shall use below. We shall keep the same notation.

4.5. Simple objects in $\text{Rep}(\mathcal{H}_t)$

Fix N . Let $\pi : l_2 + \dots + l_k$ be an unordered partition of $l \leq N$. Then for $\mathbf{a} = (a_1, \dots, a_k) \in \mathfrak{t}_k^*$, set $M^t(\mathbf{a})\pi = \text{Ind}_{t-l, l_2, \dots, l_k}^t(\text{Ev}_{a_1+\rho_1} \mathbf{1}, \dots, \text{Ev}_{a_k+\rho_k} \mathbf{1}) \in \text{Rep}(\mathcal{H}_t)$, where $\mathbf{1} = [\emptyset]$. This initially appears to be defined for t transcendental, but in general we may define

$$M^t(\mathbf{a})_\pi = \mathrm{Sp}_t(\mathrm{Ind}_{T^{-l, l_2, \dots, l_k}}^T(\mathrm{Ev}_{a_1 + \rho_1} \mathbf{1}, \dots, \mathrm{Ev}_{a_k + \rho_k} \mathbf{1})) \in \mathrm{Rep}(\mathcal{H}_t).$$

By the construction of Proposition 4.6, there is a morphism $s: \mathbb{A}_{\mathbb{C}}^1 \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^k \rightarrow \mathrm{Ob}$, where Ob is the object scheme of $\mathfrak{Rep}(\mathcal{H})^{(M, N)}$ for M, N large, such that $s(t, a_1, \dots, a_k) = M^t(a_1, \dots, a_k)_\pi$.

Theorem 4.7.

1. For all transcendental t , every simple object in $\mathrm{Rep}(\mathcal{H}_t)$ is a quotient of some $M^t(\mathbf{a})_\pi$.
2. Given an unordered partition π and t transcendental, the set of $\mathbf{a} \in \mathbb{C}^k$ with $M^t(\mathbf{a})_\pi$ not simple is contained in a finite union of hypersurfaces.

Proof. We shall prove this in steps. The first part will follow from the next lemma.

Lemma 4.8. If $N \in \mathbb{Z}_{\geq 0}$, then outside a finite set $S_N \subset \mathbb{C}$ of algebraic numbers, any simple object in $\mathrm{Rep}(\mathcal{H}_t, \mathbb{C})^{(N, N)}$ is a quotient of some $M^t(\mathbf{a})_\pi$ with $|\pi| \leq N$.

Proof. It is enough to prove this for each N . Let Sim be the constructible subset of the object scheme Ob of $\mathfrak{Rep}(\mathcal{H})^{(N, N)}$ as in Proposition 3.4, and let $\mathrm{Ind} \subset \mathrm{Ob}$ be the constructible set parametrizing the images of $M^t(\mathbf{a})_\pi$ for $|\pi| \leq N$, $\mathbf{a} \in \mathfrak{t}_k^*$, $t \in \mathbb{A}_{\mathbb{C}}^1$. Using Proposition 3.2, we see that there is a constructible $Q \subset \mathrm{Ob}$ with $Q_{\bar{s}}$ consisting of quotients of objects in $\mathrm{Ind}_{\bar{s}}$. Now $Q_{\bar{s}} \supset \mathrm{Sim}_{\bar{s}}$ for $s \in \mathbb{Z}_{\geq 0}$ sufficiently large by parts (1) and (2) of Theorem 4.2, so the statement holds generically. Thus it holds at least one transcendental t , so in view of the action of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, at all transcendental t . \square

For the next part, we prove:

Lemma 4.9. Let π be an unordered partition of $N < n$. Then the set of $\mathbf{a} \in \mathbb{C}^k$ with $M^n(\mathbf{a})_\pi \in \mathrm{Rep}^{\mathrm{ord}}(\mathcal{H}_n)$ not simple is contained in a finite union of hyperplanes.

First, we review some known results. Fix n . Given $\lambda \in \mathfrak{t}_k^*$, there is an *Arakawa–Suzuki functor*, constructed in [1] for $\mathcal{O}(\mathfrak{gl}_k)$ and described slightly more conveniently for us in [15], $F_\lambda: \mathcal{O}(\mathfrak{gl}_k) \rightarrow \mathrm{Rep}^{\mathrm{ord}}(\mathcal{H}_n)$, where $\mathcal{O}(\mathfrak{gl}_k)$ is the *BGG category* associated to the Lie algebra \mathfrak{gl}_k . This is defined similarly to the category \mathcal{O} associated to a semisimple Lie algebra, cf., e.g., [9]. In detail, let $\mathfrak{t}_k, \mathfrak{n}_k$ be the subalgebras of upper-triangular and diagonal matrices, respectively. An object $X \in \mathcal{O}(\mathfrak{gl}_k)$ is a representation of \mathfrak{gl}_k which is finitely generated over the enveloping algebra of \mathfrak{gl}_k , semisimple as a representation of \mathfrak{t}_k , and locally finite under the action of \mathfrak{n}_k .

For $\lambda \in \mathfrak{t}_k^*$, let $V(\lambda) \in \mathcal{O}(\mathfrak{gl}_k)$ be the Verma module and $W(\lambda)$ its unique simple quotient. If $\lambda = \mu + (n-l)\epsilon_1 + l_2\epsilon_2 + \dots + l_k\epsilon_k$ is dominant, then by Theorems 3.2.1 and 3.2.2 of [15] we have $F_\lambda(V(\mu)) = M^n(\mu)_\pi$ and $F_\lambda(W(\mu)) = L^n(\mu)_\pi$ or 0. Recall that $L^n(\mu)_\pi$ is the unique simple quotient, as in Theorem 4.2.

Proof of Lemma 4.9. Consider the set of \mathbf{a} such that $\mathbf{a} + (n-l, l_2, \dots, l_k)$ is dominant and \mathbf{a} is antidominant. The set of such \mathbf{a} is a countable union of hyperplanes in $\mathbb{A}_{\mathbb{C}}^k = \mathfrak{t}_k^*$. For such \mathbf{a} , the Verma module $V(\mathbf{a})$ is irreducible (cf. [9]) and $W(\mathbf{a}) \simeq V(\mathbf{a})$, so it follows that

$$F_{\mathbf{a} + (n-l, l_2, \dots, l_k)}(W(\mathbf{a})) \simeq F_{\mathbf{a} + (n-l, l_2, \dots, l_k)}(V(\mathbf{a})) \simeq M^n(\mathbf{a})_\pi \neq 0.$$

Hence $M^n(\mathbf{a})_\pi \simeq L^n(\mathbf{a})_\pi$, which is simple. Thus the set of such \mathbf{a} is contained in a countable union of hyperplanes. However, in view of Proposition 3.4, the set of such \mathbf{a} is constructible and thus contained in a finite such union. \square

We now finish the proof of Theorem 4.7. Given π , there is a constructible set $\text{Const} \subset \mathbb{A}_{\mathbb{C}}^1 \times_{\mathbb{C}} \mathbb{A}_{\mathbb{C}}^k$ such that the fiber over $t \in \mathbb{C}$ corresponds to $\mathbf{a} \in \mathbb{C}^k$ with $M^t(\mathbf{a})_{\pi} \in \text{Rep}(\mathcal{H}_t)^{(N,N)}$ simple; this is a consequence of Proposition 3.4. Then $\dim(\overline{\text{Const}}_n) \leq k-1$ when n is a large integer by the previous lemma. The fibers of Const have constant dimension on an open dense subset of $\mathbb{A}_{\mathbb{C}}^1$, so it follows that Const_t is contained in a subvariety of proper codimension for almost all t . In particular, this must include one transcendental t , and we can use the same Galois group argument as before. \square

In the above proof, the second part of Theorem 4.2 was crucial, because it gave a classification of the irreducibles in the truncated categories $\text{Rep}^{\text{ord}}(\mathcal{H}_n)^{(N)}$ consisting of modules whose S_n -restrictions contain irreducibles associated to Young diagrams with at most N squares below the first row.

5. Wreath product categories

5.1. Definition of $\text{Rep}(\mathbf{A}_t)$

Let A be a unital associative algebra over an algebraically closed field k of characteristic zero. Let $r: \mathfrak{h} \rightarrow \mathfrak{h} \otimes \mathfrak{h}$ be the dual to the algebra bilinear form $\rho: \mathfrak{h} \otimes \mathfrak{h} \rightarrow \mathfrak{h}$, where \mathfrak{h} is the permutation representation of S_n . We shall abbreviate $S_n \ltimes A^{\otimes n}$ to \mathbf{A}_n for ease of notation.

Consider the definition of a module M over \mathbf{A}_n : M is a representation of S_n , together with linear maps $y_i: A \otimes M \rightarrow M$ for $1 \leq i \leq n$ that send $a \otimes m$ to $(1 \otimes 1 \cdots \otimes a \otimes \cdots \otimes 1)m$, where the a is in the i -th place. If $a \in A$, write $y_{i,a} \in \text{Hom}_{\mathbb{C}}(M, M)$ for the map $m \rightarrow y_i(a \otimes m)$.

Then by definition, $\sigma y_{i,a} \sigma^{-1} = y_{\sigma(i),a}$ for $\sigma \in S_n$ and $y_{i,a} \circ y_{i,b} = y_{i,ab}$. Also, $y_{i,1} = \text{Id}_M$. Finally, $[y_{i,a}, y_{j,b}] = 0$ if $i \neq j$ and $[y_{i,a}, y_{i,b}] = y_{i,[a,b]}$. Conversely, a system of maps $y_{i,a}$ for $1 \leq i \leq n$, $a \in A$ satisfying the above conditions on an S_n -representation M yields an action of \mathbf{A}_n . The next result now follows.

Proposition 5.1. *To give an \mathbf{A}_n -module M is equivalent to giving an S_n -representation M together with maps $y_a: \mathfrak{h} \otimes M \rightarrow M$ for $a \in A$ such that the association $a \rightarrow y_a$ is k -linear, $y_1 = s \otimes \text{Id}_M$ for $s: \mathfrak{h} \rightarrow \mathbf{1}$ the standard morphism, and such that y satisfies the following two further conditions:*

$$y_a \circ (\text{Id}_{\mathfrak{h}} \otimes y_b) \circ r = y_{ab}: \mathfrak{h} \otimes M \rightarrow M, \quad (6)$$

$$y_a \circ (\text{Id}_{\mathfrak{h}} \otimes y_b) - y_b \circ (\text{Id}_{\mathfrak{h}} \otimes y_a) \circ P_{1,2} = y_{[a,b]} \circ (\rho \otimes \text{Id}_{\mathfrak{h}}): \mathfrak{h} \otimes \mathfrak{h} \otimes M \rightarrow M. \quad (7)$$

All this is in a category-friendly stage, and it is now clear how to make the following definition:

Definition 5.2. (P. Etingof, personal communication.) The category $\text{Rep}(\mathbf{A}_t)$ is defined as follows. Objects in $\text{Rep}(\mathbf{A}_t)$ are objects $M \in \text{Rep}(S_t)$ endowed with a morphism $y: A \otimes \mathfrak{h} \otimes M \rightarrow M$, where A is viewed as a vector space, such that the induced morphisms $y_a: \mathfrak{h} \otimes M \rightarrow M$ satisfy (6) and (7) above, and $y_1 = s \otimes \text{Id}_M$. Morphisms in $\text{Rep}(\mathbf{A}_t)$ are morphisms in $\text{Rep}(S_t)$ commuting with the y maps.

Again, it is enough to specify y_a for a generating A .

Proposition 5.3. *If A is finitely generated, there is a category over \mathbb{A}_k^1 , $\mathfrak{Rep}(A)$, whose fiber at each $t \in k$ is $\text{Rep}(\mathbf{A}_t)$ as defined above. If $R, S \in \mathbb{Z}_{\geq 0}$, there are subcategories $\mathfrak{Rep}(A)^{(R,S)}$ of finite type.*

Proof. The result is now evident from the remark preceding the proposition when A is finitely presented. However, when M is of fixed size, e.g., is an object of $\text{Rep}(S_t)^{(R,S)}$, then the y -morphism is described by certain parameters in k depending on recollements whose number is bounded in terms of R and S . The relations imposed on y are simply polynomial relations in these parameters, and by the noetherian property of a polynomial ring, it is clear that we need only impose finitely many relations on y (where finite depends on R and S). \square

We now carry out the interpolation of certain important structures.

5.2. V_χ functors

Fix $n \in \mathbb{Z}_{\geq 0}$. Let A be a finitely generated (hence finitely presented) commutative algebra over k . In this case, we can get the simple objects in $\text{Rep}^{\text{ord}}(\mathbf{A}_n)$ by a certain process of parabolic induction, which begins with an analog of the evaluation homomorphism for \mathcal{H}_n .

Let $\text{Char}(A)$ be the collection of characters of A , i.e., algebra-homomorphisms $A \rightarrow k$. For $\chi \in \text{Char}(A)$, we define the functor $V_\chi : \text{Rep}^{\text{ord}}(S_n) \rightarrow \text{Rep}^{\text{ord}}(\mathbf{A}_n)$ as follows. Choose $M \in \text{Rep}^{\text{ord}}(S_n)$. Define $V_\chi(M)$ to be M as an S_n -module, and for $a \in A$, let $1 \otimes \cdots \otimes a \otimes \cdots \otimes 1 \in A^{\otimes n}$ act on M via scalar multiplication by $\chi(a)$. If $f : M \rightarrow N$ is an S_n -homomorphism, then $V_\chi(f) : V_\chi(M) \rightarrow V_\chi(N)$ is set-theoretically the same function f .

There is a scheme of finite type over k , which by abuse of notation we denote by $\text{Char}(A)$, parametrizing the characters of A .

Proposition 5.4. *There is a functor over \mathbb{A}_k^1 , $V : \text{Char}(A) \times_k \mathfrak{Rep}(S) \rightarrow \mathfrak{Rep}(\mathbf{A})$, that interpolates the previous functors V_χ at natural number points and $\chi \in \text{Char}(A)$.*

Proof. We start by working in the category $\text{Rep}'(\mathbf{A}_T, k[T])$ defined in an obvious way. Let a_1, \dots, a_r be a generating set for A . Let $s : \mathfrak{h} \rightarrow \mathbf{1}$ be the standard map. Then if $M \in \text{Rep}'(S_T)$, define $y_{a_i} : \mathfrak{h} \otimes M \rightarrow M$ by $\chi(a_i)(s \otimes \text{Id})$. We claim that this defines a functor $\text{Rep}'(S_T) \rightarrow \text{Rep}'(\mathbf{A}_T)$. Indeed, the fact that (6) and (7) hold can be checked by specialization $T \rightarrow n$ for n large, since it is easy to see that the following definition of y_{a_i} is just a category-friendly version of the usual one for the V_χ functor. It is similarly checked that for a morphism $(C) : [U] \rightarrow [V]$, the diagram

$$\begin{array}{ccc} \mathfrak{h} \otimes [U] & \xrightarrow{y_{a_i}} & [U] \\ \downarrow \text{Id}_{\mathfrak{h}} \otimes (C) & & \downarrow (C) \\ \mathfrak{h} \otimes [V] & \xrightarrow{y_{a_i}} & [V] \end{array}$$

commutes for each i , so functoriality follows. Then we can specialize the functor appropriately, $T \rightarrow t$, and extend to the pseudo-abelian envelope. \square

5.3. Induction functors

We momentarily drop the hypothesis that A be commutative. Now fix $n_2, n_3, \dots, n_l \in \mathbb{Z}_{\geq 0}$. Let $N = \sum n_s$.

Proposition 5.5. *Fix a cutoff R . For transcendental t , there is a functor*

$$\text{Ind} : \text{Rep}(\mathbf{A}_{t-N})^{(R)} \times \prod_{s=2}^l \text{Rep}^{\text{ord}}(\mathbf{A}_{n_s}) \rightarrow \text{Rep}(\mathbf{A}_t)^{(R+N)}$$

interpolating the usual functors.

Proof. We handle the case $l = 2$, because the general case can be obtained by iteration. Fix $a \in A$. Let $t \in \mathbb{Z}_{\geq 0}$. Assume for simplicity that the two objects restrict as S_t -objects to $[U] \in \text{Rep}^{\text{ord}}(S_{t-N})$, $[J] \in \text{Rep}^{\text{ord}}(S_N)$, where $|J| = N$. Then there are maps $y : [\text{One}] \otimes [U] \rightarrow [U]$, $y' : [\text{One}] \otimes [J] \rightarrow [J]$ that describe the action of a ; we have dropped the subscript for convenience. Let y be described by $\sum_{f,i,g} C_{f,i}^g e_f \otimes e_i \otimes e_g \in [U] \otimes [\text{One}] \otimes [U]$. Similarly, the map $y' : [J] \otimes [\text{One}] \rightarrow [J]$ is described

by $\sum_{h,j,d} D_{h,j}^l e_h \otimes e_j \otimes e_d$. Finally, the map $[U + J] \otimes [\text{One}] \rightarrow [U + J]$ can be described by some $\sum_{p,m,q} E_{p,m}^q e_p \otimes e_m \otimes e_q$. Given $C_{f,i}^g, D_{h,j}^d$, we need to compute $E_{p,m}^q$.

Given $m \in I + J$ corresponding to the image of One, $p \in \text{Inj}(U + J, I + J), q \in \text{Inj}(U + J, I + J)$, we compute $E_{p,m}^q$. By S_{I+J} -invariance, assume p comes from pasting $p' : U \rightarrow I, p'' = \text{Id}_J : J \rightarrow J$. Suppose first $k \notin \text{Im}(p'') = J \subset I + J$. Then $E_{p,m}^q = C_{p',m}^{q|_U}$ if $q|_J = \text{Id}_J$. Otherwise $E_{p,m}^q = 0$.

We do a similar procedure when $m \in \text{Im}(p'')$. Then, $E_{p,m}^q = D_{\text{Id}_J,m}^{q|_J}$ if $q(J) \subset J$ and $p|_U = q|_U$, and $E_{p,m}^q = 0$ otherwise.

Using this, we can express the functor in terms of recollements (recall that $E_{p,m}^q$ becomes the coefficient for the recollement defined by p, q, m), and argue as in the proof of induction for the dAHA (Proposition 4.6). \square

5.4. Some combinatorics

We start by describing the construction of simple objects in the standard case. Fix $n \in \mathbb{Z}_{\geq 0}$.

Theorem 5.6 (Classical). *Suppose A is commutative and finitely generated. Every simple object in $\text{Rep}^{\text{ord}}(\mathbf{A}_n)$ can be obtained from a partition $n = n_1 + \cdots + n_l$, characters χ_1, \dots, χ_l , objects $M_j \in \text{Rep}^{\text{ord}}(S_{n_j})$ for $1 \leq j \leq l$, via $\text{Ind}_{\mathbf{A}_{n_1} \otimes \cdots \otimes \mathbf{A}_{n_l}}^{\mathbf{A}_n} (V_{\chi_1}(M_1), \dots, V_{\chi_l}(M_l))$ where the $M_j \in \text{Rep}(S_{n_j})$ are simple, and $\chi_j \in \text{Char}(A)$.*

For a proof when A is the group algebra of a finite group, cf. [8]. We shall derive a similar classification in transcendental rank. In order to apply the machinery developed, we will need corresponding results on the filtration of these categories.

Corollary 5.7. *Suppose $n > 2N$. Then every simple object in $\text{Rep}^{\text{ord}}(\mathbf{A}_n)^{(N)}$ is of the form $\text{Ind}(V_{\chi_1}(M_1), \dots, V_{\chi_l}(M_l))$ for $n_2 + \cdots + n_l \leq N$ and $M_1 \in \text{Rep}^{\text{ord}}(S_{n_1})^{(N)}$ (all M_j simple).*

The corollary is a consequence of Theorem 5.6 and the next lemma.

Lemma 5.8. *Let $n > 2N$. Suppose $\mu : n_1 + n_2 + \cdots + n_l = n$ is an ordered partition of n , $V_j \in \text{Rep}^{\text{ord}}(S_j)$ is nonzero, and $\text{Ind}_{S_{n_1} \times \cdots \times S_{n_l}}^{S_n} (\otimes V_j) \in \text{Rep}^{\text{ord}}(S_n)^{(N)}$. Then $n_2 + \cdots + n_l \leq N$ and $V_1 \in \text{Rep}^{\text{ord}}(S_{n_1})^{(N)}$.*

Proof. Without loss of generality, we may assume each V_j simple. Recall what it means for an irreducible object in $\text{Rep}^{\text{ord}}(S_n)$, parametrized by a Young diagram, to belong to $\text{Rep}^{\text{ord}}(S_n)^{(N)}$. It means that below the first row, there are at most N squares. In particular, if λ, μ are Young diagrams and $\lambda > \mu$ in the lexicographical order, then $\lambda \in \text{Rep}^{\text{ord}}(S_n)^{(N)}$ if $\mu \in \text{Rep}^{\text{ord}}(S_n)^{(N)}$.

Now, by the following lemma, the lowest simple object in the S_n -representation $\text{Ind}_{S_{n_1} \times \cdots \times S_{n_l}}^{S_n} (V_1 \otimes \cdots \otimes V_l)$, with respect to the lexicographical order, has a top row in the associated Young diagram of length at most n_1 . In particular, the rows below it have total length at least $n_2 + \cdots + n_l$, so this is at most N . Using similar reasoning, the number of boxes in the Young diagram of V_1 below the first row can be at most N . \square

Lemma 5.9. *Let $n = n_1 + \cdots + n_l$, let $V_{\lambda_i} \in \text{Rep}^{\text{ord}}(S_{n_i})$ be the irreducible object associated to the partition λ_i of n_i , and let $V = \text{Ind}_{S_{n_1} \times \cdots \times S_{n_l}}^{S_n} (\otimes_i V_{\lambda_i})$. Then for some $C_\mu, V = V_\lambda \oplus \bigoplus_{\mu: \mu > \lambda} C_\mu V_\mu$ where λ is the partition of n obtained by splicing together all the λ_i (and possibly reordering) and $V_\lambda \in \text{Rep}^{\text{ord}}(S_n)$ the associated simple representation.*

This is essentially the branching rule for the symmetric group.

5.5. Generic classification of simple objects

Using the machinery developed, we can now extend this result to complex rank generically:

Theorem 5.10. *For t transcendental, every simple object in $\text{Rep}(\mathbf{A}_t)$ is of the form $\text{Ind}(V_{\chi_1}(M_1), V_{\chi_2}(M_2), \dots, V_{\chi_l}(M_l))$ where $M_1 \in \text{Rep}(S_{t-N})$ is simple, $M_j \in \text{Rep}^{\text{ord}}(S_{n_j})$ is simple for $j > 1$, $n_2 \geq n_3 \geq \dots$, $N = \sum_2^l n_j$, and each $\chi_j \in \text{Char}(A)$. Moreover, different choices yield non-isomorphic objects.*

In other words, the simple objects are indexed by families of Young diagrams of arbitrary size, together with the assignment of a character to each diagram. To the top one, one can append a “very long line” as usual.

The result follows from the next proposition, since the difference between objects associated to different Young diagrams or character arrangements is implied by the classification at the integers.

Proposition 5.11. *If W is fixed, then for all but finitely many t the simple objects in $\text{Rep}(\mathbf{A}_t)^{(W,W)}$ are of the form $\text{Ind}(V_{\chi_1}(M_1), V_{\chi_2}(M_2), \dots, V_{\chi_k}(M_k))$ with $M_1 \in \text{Rep}(S_{t-N})^{(N)}$, $M_j \in \text{Rep}^{\text{ord}}(S_{n_j})$ simple, and $N \leq W$.*

The object $\text{Ind}(V_{\chi_1}(M_1), V_{\chi_2}(M_2), \dots, V_{\chi_k}(M_k))$ may be defined by choosing $M'_1 \in \text{Rep}(\mathbf{A}_{T-N}, \mathbb{C}(T))$ specializing to M_1 , applying the functor of Proposition 5.5, and specializing.

Proof. The images of such objects form a constructible subset T of Ob , the object scheme of $\mathfrak{Rep}(\mathbf{A})^{(W,W)}$, in view of Proposition 3.4. Applying Proposition 3.4 again to $\mathfrak{Rep}(\mathbf{A})^{(W,W)}$, we see that $T_{\bar{t}} \supset \text{Sim}_{\bar{t}}$ for all but finitely many t , since this is true for the (Zariski dense) subset of sufficiently large integers by Corollary 5.7. \square

5.6. The case of A noncommutative

We shall briefly indicate the changes necessary when A is noncommutative. For $n \in \mathbb{Z}_{\geq 0}$, it is known that the finite-dimensional representations of \mathbf{A}_n can be obtained via a similar process of parabolic induction, but where characters χ are replaced with irreducible representations of A . If χ is an irreducible representation of A , define the functor $V_{\chi} : \text{Rep}^{\text{ord}}(S_n) \rightarrow \text{Rep}^{\text{ord}}(\mathbf{A}_n)$ by sending $M \in \text{Rep}^{\text{ord}}(S_n)$ to $M \otimes \chi^{\otimes n}$, where S_n permutes the factors of $\chi^{\otimes n}$. Every finite-dimensional irreducible \mathbf{A}_n representation can be obtained from a partition $n = n_1 + \dots + n_l$, irreducible representations χ_1, \dots, χ_l of A , irreducible S_{n_j} representations M_j for $1 \leq j \leq l$, via $\text{Ind}_{\mathbf{A}_{n_1} \otimes \dots \otimes \mathbf{A}_{n_l}}^{\mathbf{A}_n}(V_{\chi_1}(M_1), \dots, V_{\chi_l}(M_l))$.

Now take $n \gg N$. It follows that the simple \mathbf{A}_n representations that restrict to the truncation $\text{Rep}^{\text{ord}}(S_n)^{(N,N)}$ are obtained via such a procedure with $n_2 + \dots + n_l \leq N$, $M_1 \in \text{Rep}^{\text{ord}}(S_{n_1})^{(N)}$ simple, the dimensions of the χ_j , $1 \leq j \leq l$, bounded by some bound depending on N , and χ_1 a one-dimensional representation (i.e., a character). Indeed, the bound on $n_2 + \dots + n_l$ and the fact that $M_1 \in \text{Rep}^{\text{ord}}(S_{n_1})^{(N)}$ follow from Lemma 5.8. It follows that χ_1 is one-dimensional, or otherwise $V_{\chi_1}(M_1)$ would be too large because of the tensor power and the induced object would not belong to $\text{Rep}^{\text{ord}}(S_n)^{(N,N)}$. Similarly, the dimensions of the χ_j , $2 \leq j \leq l$, must be bounded in terms of N for n sufficiently large.

Thus, in the same way, a version of Proposition 5.11 holds for A noncommutative (but with χ_1 required to be one-dimensional), because the χ_j can all be parametrized by suitable schemes, since their dimensions are bounded depending only on N . It follows that:

Theorem 5.12. *For t transcendental, the simple objects in $\text{Rep}(\mathbf{A}_t)$ for A finitely generated are obtained via $\text{Ind}(V_{\chi_1}(M_1), \dots, V_{\chi_l}(M_l))$ for each M_j simple and χ_1 one-dimensional.*

Curiously, if A has no one-dimensional representations (e.g., is a simple algebra), then $\text{Rep}(\mathbf{A}_t)$ is empty for t transcendental; this is an immediate corollary of Theorem 5.12. The author is indebted to Etingof for pointing out this generalization.

5.7. Tensor structure for A a Hopf algebra

We now prove:

Theorem 5.13. *If A is a Hopf algebra, then $\text{Rep}(\mathbf{A}_t)$ can be made into a tensor category, in a manner interpolating the usual tensor structure (using the Hopf algebra structure on \mathbf{A}_n for $n \in \mathbb{Z}_{\geq 0}$). Indeed, the tensor structure becomes a bifunctor over \mathbb{A}_k^1 .*

For any object $M \in \text{Rep}(S_t)$, set $Q_1(M) = \text{Hom}(\mathfrak{h} \otimes M, M)$, and $Q_2(M) = \text{Hom}(\mathfrak{h} \otimes \mathfrak{h} \otimes M, M)$. Both these can be made into associative rings in view of the map $r : \mathfrak{h} \otimes \mathfrak{h} \rightarrow \mathfrak{h}$ and its dual $\rho : \mathfrak{h} \rightarrow \mathfrak{h} \otimes \mathfrak{h}$. When $t \in \mathbb{Z}_{\geq 0}$ is very large, we can think of $Q_1(M)$ as parametrizing families of maps $f_i : M \rightarrow M$, $1 \leq i \leq n$, such that $\sigma f_i \sigma^{-1} = f_{\sigma(i)}$ for $\sigma \in S_n$. The ring structure comes from composing these families of maps. Similarly, $Q_2(M)$ parametrizes families of maps $f_{ij} : M \rightarrow M$, $1 \leq i, j \leq n$ such that $\sigma f_{ij} \sigma^{-1} = f_{\sigma(i)\sigma(j)}$. To check that the ring is associative in general, one reduces to the case $M = [S]$ for S a finite set. Then associativity is a polynomial condition in t , and its holding at sufficiently large natural numbers implies it at all t . Similarly, it can be checked that there are algebra-homomorphisms $Q_1(X) \otimes_k Q_1(Y) \rightarrow Q_1(X \otimes Y)$, $Q_2(X) \otimes_k Q_2(Y) \rightarrow Q_2(X \otimes Y)$ for all X, Y .

Suppose given a y -map $A \rightarrow Q_1(M)$ for $M \in \text{Rep}(S_t)$. In order that it define a structure of $\text{Rep}(\mathbf{A}_t)$, there are certain conditions it must satisfy. We shall first introduce some notation.

Define a k -linear map $\theta : Q_1(M) \otimes_k Q_1(M) \rightarrow Q_2(M)$ as follows. Given $f, g : \mathfrak{h} \otimes M \rightarrow M$, define $\theta(f \otimes g) : \mathfrak{h} \otimes \mathfrak{h} \otimes M \rightarrow M$ by

$$\theta(f \otimes g) = f \circ (\text{Id}_{\mathfrak{h}} \otimes g) - g \circ (\text{Id}_{\mathfrak{h}} \otimes f) \circ P_{1,2}.$$

Define $\gamma, \gamma' : Q_1(M) \otimes_k Q_1(M) \rightarrow Q_2(M)$ by

$$\gamma(f \otimes g) = f \circ (\text{Id}_{\mathfrak{h}} \otimes g), \quad \gamma'(f \otimes g) = g \circ (\text{Id}_{\mathfrak{h}} \otimes f) \circ P_{1,2}.$$

Note that $\theta = \gamma - \gamma'$. In the integer case, θ sends $\{f_i\} \otimes \{g_j\}$ to $f_i g_j - g_j f_i$, and there are similar computations for γ, γ' . Define $\phi : Q_1(M) \rightarrow Q_2(M)$ by setting $\phi(f) : \mathfrak{h} \otimes \mathfrak{h} \otimes M \rightarrow M$ to be $\phi(f) = f \circ (\rho \otimes \text{Id}_M)$. It is easy to check that ϕ is a ring-homomorphism, by reduction to the integral case.

Finally, we may state the two conditions for when $y : A \rightarrow Q_1(M)$ defines on $M \in \text{Rep}(S_t)$ the structure of an object in $\text{Rep}(\mathbf{A}_t)$. First, y must be a (unital) ring-homomorphism. Second, the following diagram must commute:

$$\begin{array}{ccccc} A \otimes_k A & \xrightarrow{y \otimes y} & Q_1(U) \otimes_k Q_1(U) & \xrightarrow{\theta} & Q_2(U) \\ \downarrow [\cdot, \cdot] & & & \nearrow \phi & \\ A & \xrightarrow{\quad} & Q_1(U) & & \end{array} \quad (8)$$

Here, of course, $[\cdot, \cdot] : A \otimes_k A \rightarrow A$ is the commutator map.

As a result, the following is immediate:

Proposition 5.14. *An algebra homomorphism $A \rightarrow B$ induces a functor $\text{Rep}(\mathbf{B}_t) \rightarrow \text{Rep}(\mathbf{A}_t)$.*

We shall now derive certain commutation relations in $\text{Rep}(S_t)$ necessary to the proof of Theorem 5.13.

Lemma 5.15. *The following diagram is commutative for all X, Y :*

$$\begin{array}{ccc}
 Q_1(X) \otimes_k Q_1(Y) \otimes_k Q_1(X) \otimes_k Q_1(Y) & \xrightarrow{\theta_X \otimes \gamma_Y + \gamma'_X \otimes \theta_Y} & Q_2(X) \otimes_k Q_2(Y) \\
 \downarrow & & \downarrow \\
 Q_1(X \otimes Y) \otimes_k Q_1(X \otimes Y) & \xrightarrow{\theta_{X \otimes Y}} & Q_2(X \otimes Y)
 \end{array} \quad (9)$$

Proof. The result may be seen by reducing to the integral case, since a nonzero 1-variable polynomial has finitely many roots; this argument will recur. Fix $X, Y \in \text{Rep}^{\text{ord}}(S_n)$ and families $f_i, f'_j: X \rightarrow X$, $g_i, g'_j: Y \rightarrow Y$ corresponding to elements f, f', g, g' of $Q_1(X), Q_1(Y)$. Then going down and right in the diagram yields the family of maps

$$\begin{aligned}
 h_{ij} &= (f_i \otimes g_i) \circ (f'_j \otimes g'_j) - (f'_j \otimes g'_j) \circ (f_i \otimes g_i) \\
 &= (f_i f'_j - f'_j f_i) \otimes g_i g'_j - f'_j f_i \otimes (g'_j g_i - g_i g'_j) \\
 &= \theta_X(f \otimes f') \otimes \gamma_Y(g \otimes g') + \gamma'_X(f \otimes f') \otimes \theta_Y(g \otimes g'). \quad \square
 \end{aligned}$$

Lemma 5.16. *For any X, Y , the following diagram is commutative:*

$$\begin{array}{ccccc}
 Q_1(X) \otimes_k Q_1(X) \otimes_k Q_1(Y) & \xrightarrow{\gamma_X \otimes \text{Id}} & Q_2(X) \otimes_k Q_1(Y) & \xrightarrow{\text{Id} \otimes \phi_Y} & Q_2(X) \otimes_k Q_2(Y) \\
 \downarrow m_{Q_1(X)} \otimes \text{Id} & & & & \downarrow \\
 Q_1(X) \otimes_k Q_1(Y) & \xrightarrow{\quad} & Q_1(X \otimes Y) & \xrightarrow{\phi_{X \otimes Y}} & Q_2(X \otimes Y)
 \end{array} \quad (10)$$

Here $m_{Q_1(X)}$ denotes the multiplication map.

Proof. As usual, we need only consider the integral case. Consider $f, f' \in Q_1(X)$, $g \in Q_1(Y)$ parametrized by maps $\{f_i\}, \{f'_i\}, \{g_i\}$. Then going down and right yields the family of maps $h_{ii} = f_i f'_i \otimes g_i$ and $h_{ij} = 0$ if $i \neq j$. Going one step to the right from the top-left square yields $\{f_i f'_j\} \otimes g \in Q_2(X) \otimes_k Q_1(Y)$, where the first $\{f_i f'_j\}$ is interpreted as a family of maps (hence is in $Q_2(X)$). Then, going right again and down yields the same h_{ij} , as is easily checked. \square

Proposition 5.17. *Let $C = A \otimes_k B$. There is a tensor product bifunctor $\text{Rep}(\mathbf{A}_t) \times \text{Rep}(\mathbf{B}_t) \rightarrow \text{Rep}(\mathbf{C}_t)$ sending $X, Y \rightarrow X \otimes Y$ such that if $A \rightarrow Q_1(X)$, $y: B \rightarrow Q_1(Y)$ are the appropriate homomorphisms, then one takes $y: A \otimes_k B \rightarrow Q_1(X) \otimes_k Q_1(Y) \rightarrow Q_1(X \otimes Y)$.*

Proof. It is clear that $A \otimes_k B \rightarrow Q_1(X \otimes Y)$ is a ring-homomorphism. We must show that the second condition is satisfied. All homomorphisms $A \rightarrow Q_1(X), B \rightarrow Q_1(Y), A \otimes_k B \rightarrow Q_1(X \otimes Y)$ will be denoted by y ; this abuse of notation should cause no confusion. Fix $a \otimes b, a' \otimes b' \in A \otimes_k B$. Then, since $[a \otimes b, a' \otimes b'] = [a, a'] \otimes bb' + a'a \otimes [b, b']$,

$$\phi(y_{[a \otimes b, a' \otimes b']}) = \phi(y_{[a, a']} \otimes y_{bb'} + y_{a'a} \otimes y_{[b, b']}). \quad (11)$$

Here we use $y_{[a, a']} \otimes y_{bb'}$ to refer to the image of it in $Q_1(X \otimes Y)$, by abuse of notation.

Next, by (9), (10) and its analog for γ' , and (11),

$$\begin{aligned}\theta(y_{a \otimes b} \otimes y_{a' \otimes b'}) &= \theta_X(y_a \otimes y_{a'}) \otimes \gamma_Y(y_b \otimes y_{b'}) + \gamma'_X(y_a \otimes y_{a'}) \otimes \theta_Y(y_b \otimes y_{b'}) \\ &= \phi_X(y_{[a, a']}) \otimes \gamma_Y(y_b \otimes y_{b'}) + \gamma'_X(y_a \otimes y_{a'}) \otimes \phi_Y(y_{[b, b']}) \\ &= \phi_{X \otimes Y}(y_{[a, a']} \otimes y_{bb'} + y_{a'a} \otimes y_{[b, b']}) \\ &= \phi_{X \otimes Y}(y_{[a \otimes b, a' \otimes b']}) = \phi(y_{[a \otimes b, a' \otimes b']}).\end{aligned}$$

It follows that the homomorphism $y : A \otimes_k B \rightarrow Q_1(X \otimes Y)$ satisfies (8).

Suppose given homomorphisms $f : X \rightarrow X'$, $g : Y \rightarrow Y'$ where $X, Y \in \text{Rep}(\mathbf{A}_t)$, $X', Y' \in \text{Rep}(\mathbf{B}_t)$. Then, we can consider $f \otimes g$ as a morphism in $\text{Rep}(S_t)$ from $X \otimes Y \rightarrow X' \otimes Y'$. It must, however, be checked that the commutation with the y -maps is preserved. This is an easy calculation. We have commutative diagrams:

$$\begin{array}{ccc} \mathfrak{h} \otimes X & \xrightarrow{y_a} & X \\ \downarrow \text{Id}_{\mathfrak{h}} \otimes f & & \downarrow f \\ \mathfrak{h} \otimes X' & \xrightarrow{y_a} & X' \end{array} \quad \begin{array}{ccc} \mathfrak{h} \otimes Y & \xrightarrow{y_b} & Y \\ \downarrow \text{Id}_{\mathfrak{h}} \otimes g & & \downarrow g \\ \mathfrak{h} \otimes Y' & \xrightarrow{y_b} & Y' \end{array} \quad (12)$$

Consider the diagram:

$$\begin{array}{ccccccc} \mathfrak{h} \otimes X \otimes Y & \xrightarrow{\rho \otimes \text{Id}} & \mathfrak{h} \otimes \mathfrak{h} \otimes X \otimes Y & \xrightarrow{\text{Id} \otimes y_a \otimes \text{Id}} & \mathfrak{h} \otimes X \otimes Y & \xrightarrow{P_{1,2} \circ (y_b \otimes \text{Id}) \circ P_{2,3}} & X \otimes Y \\ \downarrow \text{Id} \otimes f \otimes \text{Id} & & \downarrow \text{Id} \otimes f \otimes \text{Id} & & \downarrow \text{Id} \otimes f \otimes \text{Id} & & \downarrow f \otimes \text{Id} \\ \mathfrak{h} \otimes X' \otimes Y & \xrightarrow{\rho \otimes \text{Id}} & \mathfrak{h} \otimes \mathfrak{h} \otimes X' \otimes Y & \xrightarrow{\text{Id} \otimes y_a \otimes \text{Id}} & \mathfrak{h} \otimes X' \otimes Y & \xrightarrow{P_{1,2} \circ (y_b \otimes \text{Id}) \circ P_{2,3}} & X' \otimes Y \\ \downarrow \text{Id} \otimes \text{Id} \otimes g & & \downarrow \text{Id} \otimes \text{Id} \otimes g & & \downarrow \text{Id} \otimes \text{Id} \otimes g & & \downarrow \text{Id} \otimes g \\ \mathfrak{h} \otimes X' \otimes Y' & \xrightarrow{\rho \otimes \text{Id}} & \mathfrak{h} \otimes \mathfrak{h} \otimes X' \otimes Y' & \xrightarrow{\text{Id} \otimes y_a \otimes \text{Id}} & \mathfrak{h} \otimes X' \otimes Y' & \xrightarrow{P_{1,2} \circ (y_b \otimes \text{Id}) \circ P_{2,3}} & X' \otimes Y' \end{array}$$

It is easy to see that each small square is commutative from (12), so the whole square is commutative. This implies that $f \otimes g$ commutes with the y -maps on $X \otimes Y$. Thus, we have a functor indeed. \square

Proof of Theorem 5.13. When A is a Hopf algebra, the comultiplication homomorphism $\Delta : A \rightarrow A \otimes_k A$ yields a functor $\text{Rep}((\mathbf{A} \otimes_k \mathbf{A})_t) \rightarrow \text{Rep}(\mathbf{A}_t)$ by Proposition 5.14, and consequently induces a monoidal structure on $\text{Rep}(\mathbf{A}_t)$ in view of Proposition 5.17. The associativity property of this functor follows from the associativity of the comultiplication, $(\Delta \otimes \text{Id}_A) \circ \Delta = (\text{Id}_A \otimes \Delta) \circ \Delta$, and the pentagon axiom follows since it is true in $\text{Rep}(S_t)$.

Next, we construct the unital object. We take $\mathbf{1} \in \text{Rep}(S_t)$ with the homomorphism $A \rightarrow Q_1(\mathbf{1})$ from $A \rightarrow k \rightarrow Q_1(\mathbf{1})$, where $A \rightarrow k$ is the counit and $k \rightarrow Q_1(\mathbf{1})$ the unit; this is an object of $\text{Rep}(\mathbf{A}_t)$ because $Q_1(\mathbf{1})$, $Q_2(\mathbf{1})$ are commutative. The unital axioms now follow from those of the Hopf algebra, so that $\text{Rep}(\mathbf{A}_t)$ is indeed a monoidal category.

We now show that $\text{Rep}(\mathbf{A}_t)$ admits duals, which will complete the proof of the theorem. First, for all $M \in \text{Rep}(S_t)$, the k -algebra $Q_1(M)$ has an anti-involution τ obtained as follows: if $M = [U]$, then the recollement (C) of One, U , U is sent to the one of One, U , U reversing the order of the U 's. This is to be thought of in the following manner. If $|I| = t$, $\text{Inj}(U, I)$ has an S_t -invariant bilinear form. Given a family $\{f_i\} \in Q_1(M)$, τ sends this to the family $\{f_i^\vee\} : M \rightarrow M$. There is a similar anti-involution on $Q_2(M)$.

Given $M_A \in \text{Rep}(\mathbf{A}_t)$ now, we define the right dual M_A^\vee as follows: as an object of $\text{Rep}(S_t)$, it is the same, which we will denote M to emphasize that M itself has no A -action. If $y : A \rightarrow Q_1(M)$ is the morphism defining the A -structure on M_A , then consider the homomorphism $y' : A \xrightarrow{S} A \rightarrow Q_1(M) \xrightarrow{\tau} Q_1(M)$ as defining the A -structure on M_A^\vee , where $S : A \rightarrow A$ is the antipode map. We now check that the diagram (7) is commutative. Note that $\theta(\tau f \otimes \tau g) = -\tau\theta(f \otimes g)$ and $\phi(\tau f) = \tau\phi(f)$, as are easily checked by reducing to the integral case. Now,

$$\begin{aligned}\theta(y'_a \otimes y'_b) &= \theta(\tau y_a \otimes \tau y_b) = -\tau\theta(y_a \otimes y_b) \\ &= -\tau\phi(y_{[a,b]}) = \phi(y'_{[a,b]}),\end{aligned}$$

so $M_A^\vee \in \text{Rep}(\mathbf{A}_t)$ indeed. Define the evaluation and coevaluation maps $M_A^\vee \otimes M_A \rightarrow \mathbf{1}$, $\mathbf{1} \rightarrow M_A \otimes M_A^\vee$ in $\text{Rep}(\mathbf{A}_t)$ as simply the usual maps in $\text{Rep}(S_t)$; it needs only to be checked that they commute with the y, y' -actions. We will do this for the coevaluation.

Let $u \in Q_1(M) \otimes_k Q_1(M)$. There is a map $w : \mathfrak{h} \rightarrow M \otimes M$ obtained by $\mathfrak{h} \xrightarrow{(\text{Id} \otimes \text{coev})} \mathfrak{h} \otimes M \otimes M \xrightarrow{(\text{Id} \otimes \tau)u} M \otimes M$. Here $(\text{Id} \otimes \tau)u \in Q_1(M) \otimes_k Q_1(M)$, which is regarded as an element of $Q_1(M \otimes M)$. Here $\text{coev} : \text{Id} \rightarrow M \otimes M$ is the coevaluation map in the tensor category $\text{Rep}(S_t)$.

Lemma 5.18. *The map $w : \mathfrak{h} \rightarrow M \otimes M$ is dual to $m_{Q_1(M)}(u) \in Q_1(M)$.*

Proof. We need only prove this in the integral case. Suppose that $u = f \otimes g \in Q_1(M) \otimes_k Q_1(M)$. Then this can be represented as two families $\{f_i\}, \{g_i\}$ of maps $M \rightarrow M$. The map $\mathfrak{h} \rightarrow M \otimes M$ corresponds to a family of maps $\{h_i : M \rightarrow M\}$, which is seen to be $f_i g_i$. \square

Recall that for a Hopf algebra A , the following diagram is commutative:

$$\begin{array}{ccccc} A & \xrightarrow{\Delta} & A \otimes_k A & \xrightarrow{\text{Id} \otimes S} & A \otimes_k A & \xrightarrow{m} & A \\ & \searrow \eta & & & & \nearrow \epsilon & \\ & & k & & & & \end{array}$$

There is a homomorphism $A \rightarrow Q_1(M \otimes M)$ from the action on $M_A \otimes M_A^\vee$. This is given by the composition

$$A \xrightarrow{(\text{Id} \otimes S) \circ \Delta} A \otimes_k A \xrightarrow{y \otimes y} Q_1(M) \otimes_k Q_1(M) \xrightarrow{\text{Id} \otimes \tau} Q_1(M) \otimes_k Q_1(M).$$

Finally, we must show that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\text{Id} \otimes \text{coev}} & \mathfrak{h} \otimes M \otimes M \\ \downarrow \eta(a) & & \downarrow (\text{Id} \otimes \tau) \circ (y \otimes y) \circ ((\text{Id} \otimes S) \circ \Delta(a)) \\ \mathbf{1} & \xrightarrow{\text{coev}} & M \otimes M \end{array}$$

The map obtained by going right and then down is equivalent to the map $\mathfrak{h} \otimes M \rightarrow M$ from $m((y \otimes y)(\text{Id} \otimes S) \circ \Delta(a)) = y_{\epsilon(\eta(a))}$. Since y is a unital homomorphism, this is equivalent to $\eta(a)(s \otimes \text{Id}_M) : \mathfrak{h} \otimes M \rightarrow M$. The map $\mathfrak{h} \rightarrow \mathbf{1}$ is, by definition, $\eta(a)$ times $s : \mathfrak{h} \rightarrow \mathbf{1}$, so the map $\mathfrak{h} \rightarrow M \otimes M$ obtained by going down and then right is equivalent to $\eta(a)(s \otimes \text{Id}_M) : \mathfrak{h} \otimes M \rightarrow M$. The same is done to show that $M_A^\vee \otimes M_A \rightarrow \mathbf{1}$ is a morphism in $\text{Rep}(\mathbf{A}_t)$. Similarly, the existence of left duals can be shown. \square

6. Wreath products with finite groups

Fix a finite group G . We recall the definition of the Knop categories $\text{Rep}(S_t \ltimes G^t)$ in non-integral rank.

First suppose $n \in \mathbb{Z}_{\geq 0}$. Let U be a finite G -set, i.e., one where the stabilizer of each element is trivial, or equivalently one isomorphic to a coproduct of copies of G . To it, we associate a representation $\text{Inj}_G(U, \coprod_n G) \in \text{Rep}^{\text{ord}}(S_n \ltimes G^n)$ by considering the permutation representation of $S_n \ltimes G^n$ on the set $\text{Inj}_G(U, \coprod_n G)$ of G -equivariant injections $U \rightarrow \coprod_n G$. Note that

$$\text{Inj}_G\left(U, \coprod_n G\right) \otimes \text{Inj}_G\left(V, \coprod_n G\right) = \bigoplus_{C \in \text{Rec}_G(U, V)} \text{Inj}_G\left(C, \coprod_n G\right)$$

where C ranges over the set $\text{Rec}_G(U, V)$ of “ G -recollements” of U, V , where a G -recollement is defined as an equivariant analog of the usual notion: one has equivariant injections $U \rightarrow C, V \rightarrow C$ whose images cover C . Next, when n is large relative to U, V , the space $\text{Hom}_{S_n \ltimes G^n}(\text{Inj}_G(U, \coprod_n G), \text{Inj}_G(V, \coprod_n G))$ has a basis of $(C)_G, C \in \text{Rec}_G(U, V)$, where $(C)_G$ is defined in a similar manner as it was in Deligne’s case. Finally, one can find polynomials $Q_{C,D}^E(T)$ for $C \in \text{Rec}_G(U, V), D \in \text{Rec}_G(V, W), E \in \text{Rec}_G(U, W)$ such that

$$(C)_G \circ (D)_G = \sum_{E \in \text{Rec}_G(U, W)} Q_{C,D}^E(n)(E)_G. \quad (13)$$

Therefore, one may define a $\mathbb{Z}[T]$ -linear category $\text{Rep}'(S_T \ltimes G^T)$ whose objects are symbols $[U]_G$ for finite free G -sets U and $\text{Hom}([U]_G, [V]_G)$ is the free module on symbols $(C)_G$ for $C \in \text{Rec}_G(U, V)$. Composition is defined as in (13) with T replacing n . For R a ring and $t \in R$, one specializes to obtain R -linear categories $\text{Rep}'(S_t \ltimes G^t, R)$ and takes the pseudo-abelian envelope to obtain $\text{Rep}(S_t \ltimes G^t, R)$. Henceforth, we take $R = k$ for k an algebraically closed field of characteristic zero unless indicated otherwise.

It is now clear that Knop’s categories fit into the framework developed above. In particular, there is an \mathbb{A}_k^1 -category $\mathfrak{Rep}(S \ltimes G)$ whose fibers at $t \in k$ yield $\text{Rep}(S_t \ltimes G^t)$. Moreover, there are full subcategories $\mathfrak{Rep}(S \ltimes G)^{(M,N)}$ of finite type that filter $\mathfrak{Rep}(S \ltimes G)$, as before.

We shall investigate the relationship between Knop’s categories and Etingof’s categories of semidirect products. In detail, fix a finite group G , and let $A = k[G]$ be the group algebra.

Proposition 6.1. *There is a functor F of \mathbb{A}_k^1 -monoidal categories $\mathfrak{Rep}(S \ltimes G) \rightarrow \mathfrak{Rep}(S \ltimes A^{\otimes})$ interpolating the usual ones.*

Proof. We shall first associate to each object $[U]_G \in \text{Rep}'(S_T \ltimes G^T, k[T])$ an object of $\text{Add}(\text{Rep}'(S_T))$ and appropriate y -morphisms. Indeed, choose a set $u \subset U$ of representatives of the orbits of G in U , and note that for any n , $\text{Inj}_G(U, \coprod_n G) \simeq \bigoplus_{G^u} \text{Inj}(u, I)$. So we define a map $\text{Add Rep}'(S_T \ltimes G^T) \rightarrow \text{Add Rep}'(S_T)$ by sending $[U]_G \rightarrow \bigoplus_{G^u} [u]$. The y -morphism can be described combinatorially. Given $g \in G$ and $\mathbf{g}, \mathbf{g}' \in G^u$ and a recollement C of u, u . One, we must associate a coefficient in such a manner that it describes the homomorphism $y_g : \mathfrak{h} \otimes \bigoplus_{G^u} [u] \rightarrow \bigoplus_{G^u} [u]$. Suppose C is given by $a : u \rightarrow C, b : u \rightarrow C$, and $c \in C$ corresponding to the image of One. Then (C) occurs with coefficient 1 if $a = b$, and \mathbf{g}, \mathbf{g}' differ at one point corresponding to c , by the value g . Otherwise, (C) occurs with coefficient zero. The functoriality of this map can be described combinatorially in terms of recollements in a similar way, and we omit the details. The fact that this is a tensor functor can be checked by specialization. Indeed, it can be checked that the functor just defined actually interpolates the usual tensor functors. \square

Our aim is to prove:

Theorem 6.2. For $t \in k$ transcendental, the categories $\text{Rep}(S_t \ltimes G^t)$ and $\text{Rep}(S_t \ltimes A^{\otimes t})$ for $A = k[G]$ are tensor equivalent.

Lemma 6.3. Let S be a noetherian scheme. Let R, R', T, T' be schemes of finite type over S . Suppose given a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \downarrow & & \downarrow \\ T & \longrightarrow & T' \end{array}$$

Then there is a constructible subset $C \subset T$ consisting of the $t \in T$ such that $R_{\bar{t}} \rightarrow R'_{\bar{t}'}$ is bijective (where $t' \in T'$ is the image of t).

Proof. There is a map $f : R \rightarrow T \times_{T'} R'$ of T -schemes. Now, for $t \in T$,

$$(T \times_{T'} R')_{\bar{t}} \simeq \text{Spec } \overline{k(t)} \times_{T'} R' \simeq \text{Spec } \overline{k(t')} \times_{T'} R' \simeq (R')_{\bar{t}'}$$

Then the set C consists of $t \in T$ with $f_{\bar{t}}$ bijective. Therefore, we can apply Lemma 3.5 to the map f and get constructibility. Note that $\overline{k(t')} \simeq \overline{k(t)}$ by the finite type hypotheses. \square

Lemma 6.4. If \mathcal{C} is a finite-type S -category for S noetherian and $C \subset \text{Ob}$ is constructible, there is a constructible set $D \subset \text{Ob}$ such that if $s \in S$, then the closed points of $D_{\bar{s}}$ corresponds precisely to the objects of $C_{\bar{s}}$ isomorphic to objects in $C_{\bar{s}}$.

Proof. There is a constructible subset $\text{Iso} \subset \text{Mor}$ such that $(\text{Iso})_{\bar{s}}$ consists precisely of the isomorphisms in $C_{\bar{s}}$. This is easy to see because Comp is a morphism and the identities form a constructible subset. Then, the lemma becomes clear. \square

Proposition 6.5. Let \mathcal{C}, \mathcal{D} be finite-type S -categories for S a noetherian scheme and $F : \mathcal{C} \rightarrow \mathcal{D}$ a functor over S . Then the set of $s \in S$ with $F_{\bar{s}} : \mathcal{C}_{\bar{s}} \rightarrow \mathcal{D}_{\bar{s}}$ fully faithful (resp. essentially surjective) is constructible.

Proof. Let $\text{Ob}_{\mathcal{C}}, \text{Mor}_{\mathcal{C}}, \text{Ob}_{\mathcal{D}}, \text{Mor}_{\mathcal{D}}$ be the object and morphism schemes of \mathcal{C}, \mathcal{D} . Denote the induced morphisms of schemes $\text{Ob}_{\mathcal{C}} \rightarrow \text{Ob}_{\mathcal{D}}, \text{Mor}_{\mathcal{C}} \rightarrow \text{Mor}_{\mathcal{D}}$ by $F_{\text{Ob}}, F_{\text{Mor}}$, respectively. Let $T \subset \text{Ob}_{\mathcal{C}}$ be the set of points t such that $F_{\text{Mor}} : (\text{Mor}_{\mathcal{C}})_{\bar{t}} \rightarrow (\text{Mor}_{\mathcal{D}})_{\overline{F_{\text{Ob}}(\bar{t})}}$ is bijective. Let $S_1 \subset S$ be defined as $S_1 = S - f(\text{Ob}_{\mathcal{C}} - T)$, if $f : \text{Ob}_{\mathcal{C}} \rightarrow S$ is the projection. Then S_1 is constructible and corresponds to the set of $s \in S$ such that $F_{\bar{s}} : \mathcal{C}_{\bar{s}} \rightarrow \mathcal{D}_{\bar{s}}$ is fully faithful.

Consider $U_1 = F_{\text{Ob}}(\text{Ob}_{\mathcal{C}})$. Let U be constructed from U_1 as in the previous lemma. Let $S_2 = S - f(\text{Ob}_{\mathcal{D}} - U)$. Then S_2 corresponds to points s with $F_{\bar{s}}$ essentially surjective. \square

Proof of Theorem 6.2. The internal functor F as defined previously induces tensor functors $F_t : \text{Rep}(S_t \ltimes G^t) \rightarrow \text{Rep}(\mathbf{A}_t)$. Then the induced functor $\text{Rep}(S_t \ltimes G^t)^{(N)} \rightarrow \text{Rep}(\mathbf{A}_t)^{(N)}$ is an equivalence when $t \in \mathbb{N}$ and $t \gg N$. More precisely, we can see that the functor is faithful, and is full when considered as a functor from a subcategory of $\text{Rep}(S_t \ltimes G^t)^{(N, N')} \rightarrow \text{Rep}(\mathbf{A}_t)^{(N, N')}$, where $N'' \gg N'$ depends only on N', N . This last fact is thus true generically. The theorem follows. \square

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