Journal of Algebra 430 (2015) 191–237

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Free centre-by-(abelian-by-exponent 2) groups

Maria Alexandrou, Ralph Stöhr *

School of Mathematics, University of Manchester, Manchester, M13 9PL, United Kingdom

ARTICLE INFO

Article history:
Received 26 November 2013
Available online xxxx
Communicated by E.I. Khukhro

MSC:
20E22
20J05

Keywords:
Free centre-by-metabelian groups
Homology of groups

ABSTRACT

We study free centre-by-(abelian-by-exponent 2) groups. Our main result is a complete description of the centre. It is isomorphic to a direct sum of a free abelian group and a torsion subgroup. The latter is a direct sum of cyclic groups of order two and cyclic groups of order four. We exhibit a generating set consisting of elements of infinite order, order 2, and order 4, such that the centre is the direct sum of cyclic subgroups generated by those generators. Our approach makes essential use of homological methods.

© 2015 Elsevier Inc. All rights reserved.

1. Introduction

Let F be a free group of rank $n > 1$ on a free generating set $X = \{x_1, \ldots, x_n\}$, R a normal subgroup of F, and let $G = F/R$. Consider the quotient

$$F/[R', F] \tag{1.1}$$

where R' is the commutator subgroup of R. This group is a free central extension of F/R':

$$1 \to R'/[R', F] \to F/[R', F] \to F/R' \to 1.$$

* Corresponding author.
E-mail addresses: maria.alexandrou@postgrad.manchester.ac.uk (M. Alexandrou), Ralph.Stohr@manchester.ac.uk (R. Stöhr).

http://dx.doi.org/10.1016/j.jalgebra.2015.02.014
0021-8693/© 2015 Elsevier Inc. All rights reserved.

While the quotient F/R' is always torsion-free [3], elements of finite order may occur in the central subgroup $R'/[R', F]$. The perhaps most prominent instance where torsion is present is the free centre-by-metabelian group $F/[F'', F]$, that is (1.1) with $R = F'$. C.K. Gupta [1] discovered in 1973 that, if $n \geqslant 4$, then the quotient $F''/[F'', F]$ contains an elementary abelian 2-group of rank $\binom{n}{4}$, an unexpected result at that time. Three years later Yu.V. Kuz'min [5] published an alternative proof using homological methods. Due to the intriguing phenomenon of torsion in such central extensions, and to the connection with homology of groups, in subsequent years quotients of the form (1.1) were studied in a number of papers, see [6–9,11–14,2]. Kuz'min [6] proved that, for any R, the central quotient $R'/[R', F]$ is isomorphic to a direct sum of a free abelian group and an abelian group of exponent dividing four. Moreover, he exhibited an example where the torsion subgroup contains an element of order precisely 4. Thus the bound of 4 on the exponent of the torsion subgroup is the best possible in general. However, in many cases the torsion subgroup of $F/[R', F]$ is actually an elementary abelian 2-group or trivial. In fact, a comprehensive description of the torsion subgroup was given under the condition that the group G has no elements of order 2. It was proved by the second author [12] that in this case the torsion subgroup $t(R'/[R', F])$ is isomorphic to the homology group $H_4(G, \mathbb{Z}_2)$. An easy proof of this result and of Kuz'min's general bound on the exponent of the torsion subgroup can be found in [13]. The remaining challenge is still the case where the group G does contain elements of order 2. Very little is known. There are various characterizations of the torsion subgroup in terms of exact sequences (see [9,13,2]), and there are some conditional results, for example, it was shown in [6] that if the integral homology group $H_4(G)$ has no elements of order 2 then the torsion subgroup $t(R'/[R', F])$ has no elements of order 4. However, the only instance of an explicit element of order 4 in a group of the form (1.1) known up to now is the example given by Kuz'min in 1982. He showed [6, Section 4] that if F has rank 2, and R is the normal closure of x_1^2, x_2^2 and $[x_1, x_2]$, i.e. $G = F/R$ is a Klein four group, then the elements $\big[x_1^2, [x_1, x_2]\big]$ and $\big[x_2^2, [x_2, x_1]\big]$ have order 4 in (1.1).

In this paper we study free central extensions of the form (1.1) in the case where $R = F^2$, the subgroup generated by all squares in F. Thus G is an elementary abelian 2-group of rank n, and (1.1) is the free centre-by-(abelian-by-exponent 2) group. Our main result is a complete description of the central subgroup $R'/[R', F]$. We prove that if $R = F^2$, then

$$R'/[R', F] \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{a} \oplus \underbrace{\mathbb{Z}_4 \oplus \cdots \oplus \mathbb{Z}_4}_{b} \oplus \underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{c}$$

where

$$a = \binom{n-1}{2} 2^n + (n-1), \quad b = 2\binom{n+1}{3}, \quad c = n2^n - \frac{n}{2}(n^2 + n + 2)$$

with the convention that $\binom{n}{i} = 0$ for $i > n$. Moreover, we exhibit a generating set for $R'/[R', F]$ such that this group is the direct product of the cyclic subgroups generated by the elements of that generating set. The generators of order 4 are the elements

$$\big[x_i^2, [x_i, x_j]\big], \qquad \big[x_j^2, [x_j, x_i]\big]$$

with $1 \leqslant i < j \leqslant n$, that were already discovered by Kuz'min [6], plus the elements

$$\big[x_i^2, [x_j, x_k]\big] \ \big[[x_i, x_k], [x_i, x_j]\big] \ \big[x_j^2, [x_i, x_k]\big] \ \big[[x_j, x_k], [x_j, x_i]\big]$$

and

$$\big[x_j^2, [x_k, x_i]\big] \ \big[[x_j, x_i], [x_j, x_k]\big] \ \big[x_k^2, [x_j, x_i]\big] \ \big[[x_k, x_i], [x_k, x_j]\big]$$

with $1 \leqslant i < j < k \leqslant n$. The complete result is stated in Section 11.

In the next section we introduce some more notation and our main tools, the isomorphism (2.1) and the six-term exact sequence (2.4), both of them valid in general, that is, for an arbitrary normal subgroup R. At the end of this section we turn to the case where $R = F^2$, and give a brief outline of our strategy. This is then carried out in the remaining Sections 3–11. Basic results on homology of groups will be used without special references being given. These, however, can easily be found in [4].

2. The 6-term exact sequence

In this section F is the free group with free generating set $X = \{x_1, \ldots, x_n\}$ for $n > 1$, R is a normal subgroup of F, and $G = F/R$. The relation module of $G = F/R$ is the free abelian group $M = R/R'$ with right G-action induced by conjugation in F. For an arbitrary G-module A, the trivialization of A is $A_G = A \otimes_G \mathbb{Z}$, the largest quotient of A upon which G acts trivially. The exterior and symmetric squares of A will be denoted by $A \wedge A$ and $A \circ A$, respectively, and both of them as well as the tensor square $A \otimes A$ will be regarded as G-modules with diagonal action. It will be convenient to introduce some special notation for the trivialization of these modules, and also of arbitrary tensor products. We will denote the canonical image of an element $a_1 \wedge a_2$ with $a_1, a_2 \in A$ under the trivialization homomorphism $A \wedge A \to (A \wedge A)_G$ by $a_1 \wedge_* a_2$. Similarly, we write $a_1 \circ_* a_2$ for the canonical image of $a_1 \circ a_2 \in A \circ A$ in $(A \circ A)_G$, and, if B is another G-module, $a \in A$ and $b \in B$, we write $a \otimes_* b$ for the image of $a \otimes b \in A \otimes B$ in the trivialization $(A \otimes B)_G$. The ring of integers $\mathbb{Z}$ will be regarded as a trivial G-module and we set $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. The quotient $R/[R', R]$ is a free nilpotent group of class 2. For its commutator subgroup there is an isomorphism of G-modules

$$R'/[R', R] \cong R/R' \wedge R/R' = M \wedge M$$

where the G-action on $R'/[R', R]$ is induced by conjugation in F, and the G-action on $M \wedge M$ is the diagonal action. In view of the canonical isomorphism $(R'/[R', R])_G \cong R'/[R', F]$, trivializing the action on both sides yields an isomorphism

$$R'/[R', F] \cong (M \wedge M)_G. \tag{2.1}$$

Explicitly, the isomorphism (2.1) acts as follows. For $r_1, r_2 \in R$,

$$[r_1, r_2][R', F] \leftrightarrow r_1 R' \wedge_* r_2 R'.$$

Throughout this paper we will examine the group on the right hand side of (2.1), and only in the final section we will use the isomorphism to restate our findings in the language of group theory.

The relation module M fits into a short exact sequence

$$0 \to M \xrightarrow{\mu} P \xrightarrow{\sigma} IG \to 0 \tag{2.2}$$

where $P = IF \otimes_F \mathbb{Z}G$ is a free G-module of rank n with free generators $e_i = (x_i - 1) \otimes 1$ $(1 \leqslant i \leqslant n)$, IG is the augmentation ideal of $\mathbb{Z}G$, and the maps are given by $rR' \mapsto (r-1) \otimes 1$ $(r \in R)$ and $e_i \mapsto x_i R - 1$, see [4, Section VI.6]. The injection μ is usually called the Magnus embedding. The augmentation ideal IG fits into the short exact sequence

$$0 \to IG \xrightarrow{\iota} P \xrightarrow{\epsilon} \mathbb{Z} \to 0 \tag{2.3}$$

where ι is the inclusion map and ϵ is the augmentation map. The short exact sequences (2.2) and (2.3) yield a 6-term exact sequence of G-modules

$$0 \to M \wedge M \xrightarrow{\alpha_4} P \wedge P \xrightarrow{\alpha_3} P \otimes IG \xrightarrow{\alpha_2} \mathbb{Z}G \circ \mathbb{Z}G \xrightarrow{\alpha_1} \mathbb{Z}G \xrightarrow{\tilde{\epsilon}} \mathbb{Z}_2 \to 0 \tag{2.4}$$

see [12, Section 7.1] or [10, Lemma 5.1]. The maps in (2.4) are as follows. The map on the right, $\tilde{\epsilon} : \mathbb{Z}G \to \mathbb{Z}_2$, is the canonical epimorphism defined by $a\tilde{\epsilon} = a\epsilon + 2\mathbb{Z}$ for $a \in \mathbb{Z}G$. The kernel of this map is the ideal $\tilde{I}G$ generated by all elements $g + h \in \mathbb{Z}G$ where $g, h \in G$. The next map α_1 is the composite of the embedding $\mathbb{Z}G \circ \mathbb{Z}G \to \mathbb{Z}G \otimes \mathbb{Z}G$ defined by $a_1 \circ a_2 \mapsto a_1 \otimes a_2 + a_2 \otimes a_1$ and the epimorphism $\epsilon \otimes 1 : \mathbb{Z}G \otimes \mathbb{Z}G \to \mathbb{Z} \otimes \mathbb{Z}G (= \mathbb{Z}G)$, thus

$$(a_1 \circ a_2)\alpha_1 = (a_1\epsilon)a_2 + (a_2\epsilon)a_1 \quad \text{for} \quad a_1, a_2 \in \mathbb{Z}G.$$

The map α_2 is the composite of $\sigma \otimes 1 : P \otimes IG \to IG \otimes IG$, the canonical projection $IG \otimes IG \to IG \circ IG$, and the canonical injection map $IG \circ IG \to \mathbb{Z}G \circ \mathbb{Z}G$, so $(p \otimes q)\alpha_2 = p\sigma \circ q$ for $p \in P$, $q \in IG$.

The map α_3 is the composite of the embedding $P \wedge P \to P \otimes P$ that is given by $p_1 \wedge p_2 \mapsto p_1 \otimes p_2 - p_2 \otimes p_1$ and the map $1 \otimes \sigma : P \otimes P \to P \otimes IG$, thus

$$(p_1 \wedge p_2)\alpha_3 = p_1 \otimes p_2\sigma - p_2 \otimes p_1\sigma \quad \text{for} \quad p_1, p_2 \in P.$$

Finally, the map α_4 is the map $\mu \wedge \mu$ for the Magnus embedding $\mu : M \to P$, so $(m_1 \wedge m_2)\alpha_4 = m_1\mu \wedge m_2\mu$ for $m_1, m_2 \in M$.

Next we record some easy facts about the modules in the sequence (2.4). For simplicity we now assume that G is an elementary abelian 2-group. Part (ii) of the following lemma is well-known (see, for example, [4, Chapter VI.11]), for Parts (i) and (iii) we refer to [13, Lemmas 4.1 and 6.1].

Lemma 2.1. *If $G = F/R$ is an elementary abelian 2-group, then*

(i) *the exterior square $P \wedge P$ is a direct sum of a free G-module that is freely generated by the elements $e_i \wedge e_j g$ with $1 \leqslant i < j \leqslant n$ and $g \in G$, and a direct sum of the cyclic G-modules generated by the elements $e_i \wedge e_i g$ with $1 \leqslant i \leqslant n$ and $g \in G$, $g \neq 1$. Each of the latter is isomorphic to the ideal $(g-1)\mathbb{Z}G$.*
(ii) *The tensor product $P \otimes IG$ is a free G-module that is freely generated by the elements $e_i \otimes (g-1)$ with $1 \leqslant i \leqslant n$ and $g \in G$, $g \neq 1$.*
(iii) *The symmetric square $P \circ P$ is a direct sum of a free G-module that is freely generated by the elements $e_i \circ e_j g$ with $1 \leqslant i < j \leqslant n$ and $g \in G$, and the elements $e_i \circ e_i$ with $1 \leqslant i \leqslant n$, and a direct sum of the cyclic G-modules generated by the elements $e_i \circ e_i g$ with $1 \leqslant i \leqslant n$ and $g \in G$, $g \neq 1$. Each of the latter is isomorphic to the ideal $(g+1)\mathbb{Z}G$.*

This yields the following for the respective trivializations.

Corollary 2.2. *(i) If $G = F/R$ is an elementary abelian 2-group, then*

(i) *the trivialization $(P \wedge P)_G$ is a direct sum of a free abelian group that is freely generated by the elements $e_i \wedge_* e_j g$ with $1 \leqslant i < j \leqslant n$ and $g \in G$, and a direct sum of the cyclic groups of order 2 generated by the elements $e_i \wedge_* e_i g$ with $1 \leqslant i \leqslant n$ and $g \in G$, $g \neq 1$;*
(ii) *the group $(P \otimes IG)_G$ is a free abelian group that is freely generated by the elements $e_i \otimes_* (g-1)$ with $1 \leqslant i \leqslant n$ and $g \in G$, $g \neq 1$;*
(iii) *the trivialization $(\mathbb{Z}G \circ \mathbb{Z}G)_G$ is a free abelian group that is freely generated by the elements $1 \circ_* g$ with $g \in G$.*

We conclude this section with a brief outline of our approach. Recall that our aim is to determine the structure of the abelian group $(M \wedge M)_G$ in the case where $R = F^2$, and hence $G = F/R$ is an elementary abelian 2-group. To this end we consider the homomorphism

$$(M \wedge M)_G \to (P \wedge P)_G$$

that is induced by the homomorphism α_4 in (2.4). Clearly, our group $(M \wedge M)_G$ is an extension of the kernel of this homomorphism by its image in $(P \wedge P)_G$. In view of Corollary 2.2(i), this image is a direct sum of a free abelian group and an elementary abelian 2-group. We shall exploit the six-term exact sequence (2.4) to determine the kernel of this homomorphism. It turns out that the kernel is a homomorphic image of the homology group $H_4(G, \mathbb{Z}_2)$, an elementary abelian 2-group. Consequently, the kernel belongs to the torsion subgroup T of $(M \wedge M)_G$. Moreover, as we will see in the penultimate Section 10, the kernel actually coincides with $2T$.

3. The relation module of $G = F/F^2$

For the rest of this paper $R = F^2$, the subgroup of the free group F that is generated by all squares, and hence $G = F/R$ is an elementary abelian 2-group of rank n. For $x_i \in X$ we set $y_i = x_i R \in G = F/R$. Thus G is the direct product of the cyclic subgroups of order 2 that are generated by the elements y_i,

$$G = \langle y_1 \rangle \times \langle y_2 \rangle \times \cdots \times \langle y_n \rangle.$$

The relation module $M = R/R'$ will be written additively with the right G-action denoted by "$\cdot$". Thus, for $r \in R$ and $1 \leqslant i \leqslant n$ we have $rR' \cdot y_i = x_i^{-1} r x_i R'$. The normal subgroup R is generated by all squares in F. Alternatively, R is the normal closure in F of the squares x_i^2 $(1 \leqslant i \leqslant n)$ and the commutators $[x_i, x_j]$ $(1 \leqslant j < i \leqslant n)$. Hence $M = R/R'$ is as a G-module generated by the elements $x_i^2 R'$ $(1 \leqslant i \leqslant n)$ and the commutators $[x_i, x_j]R'$ $(1 \leqslant j < i \leqslant n)$. Note that the following holds in M for all $r \in R$ and $1 \leqslant i \leqslant n$

$$rR' \cdot (y_i - 1) = [r, x_i]R'.$$

Indeed, bearing in mind that the module M is written additively, we have

$$rR' \cdot (y_i - 1) = x_i^{-1} r x_i R' - rR' = r^{-1} x_i^{-1} r x_i R' = [r, x_i]R'.$$

This identity will be used throughout without reference. In what follows we will abuse notation by suppressing the R' when working in the relation module. For example, the above identity will be written as

$$r \cdot (y_i - 1) = [r, x_i].$$

One easily calculates that the images of the generators of M under the embedding μ are as follows:

$$x_i^2 \mu = e_i(y_i + 1), \qquad [x_i, x_j]\mu = e_i(y_j - 1) - e_j(y_i - 1). \tag{3.1}$$

Our next aim is to find a free generating set for M as a $\mathbb{Z}$-module. The following relations will be useful.

Lemma 3.1. *In the relation module M we have, for $1 \leqslant i, j, k \leqslant n$,*

(i) $x_i^2 \cdot y_i = x_i^2$
(ii) $[x_i, x_j] \cdot (y_i + 1) = x_i^2 \cdot (y_j - 1)$
(iii) $[x_i, x_j] \cdot (y_k - 1) + [x_j, x_k] \cdot (y_i - 1) + [x_k, x_i] \cdot (y_j - 1) = 0$ (Jacobi identity).

Proof. Part (i) is obvious, and the relations (ii) and (iii) can easily be verified by using the embedding μ. For example,

$$\begin{aligned}([x_i, x_j] \cdot (y_i + 1))\mu &= [x_i, x_j]\,\mu(y_i + 1) \\ &= (e_i(y_j - 1) - e_j(y_i - 1))(y_i + 1) \\ &= e_i(y_i + 1)(y_j - 1) \\ &= x_i^2 \mu(y_j - 1) \\ &= (x_i^2 \cdot (y_j - 1))\mu,\end{aligned}$$

which proves (ii). □

Now we derive a generating set for M as a $\mathbb{Z}$-module. First observe that the set

$$\mathcal{B} = \{(y_{i_1} - 1)(y_{i_2} - 1) \cdots (y_{i_s} - 1);\ \ 1 \leqslant i_1 < i_2 < \cdots < i_s \leqslant n,\ 1 \leqslant s \leqslant n\} \cup \{1\}$$

is a $\mathbb{Z}$-basis of the additive group of the group ring $\mathbb{Z}G$. Since M as a G-module is generated by the elements x_k^2 $(1 \leqslant k \leqslant n)$ and $[x_k, x_l]$ $(1 \leqslant l < k \leqslant n)$, it follows that the elements

$$[x_k, x_l] \cdot b \qquad 1 \leqslant l < k \leqslant n,\ b \in \mathcal{B} \tag{3.2}$$

together with the elements

$$x_k^2 \cdot b \qquad 1 \leqslant k \leqslant n,\ b \in \mathcal{B} \tag{3.3}$$

form a generating set for M as a $\mathbb{Z}$-module. In view of Lemma 3.1 (ii), the elements (3.2) where b involves either $y_k - 1$ or $y_l - 1$ are redundant. Indeed, if, for example, $b = (y_k - 1)b'$ where $b' \in \mathcal{B}$ and b' does not involve the factor $y_k - 1$, then, using Lemma 3.1 (ii),

$$\begin{aligned}[x_k, x_l] \cdot b = [x_k, x_l] \cdot (y_k - 1)b' &= [x_k, x_l] \cdot (-2 + (y_k + 1))b' \\ &= -2[x_k, x_l] \cdot b' + x_k^2 \cdot (x_l - 1)b'\end{aligned}$$

where $[x_k, x_l] \cdot b'$ is of the form (3.2) with b' not involving $y_k - 1$, and $x_k^2 \cdot (x_l - 1)b'$ can be written as a linear combination of the elements of the form (3.3). Moreover, of the remaining elements of the form (3.2) the ones where $b = (y_{i_1} - 1) \cdots (y_{i_s} - 1)$ with $i_1 < l$ are redundant. Indeed, write $b = (y_{i_1} - 1)b'$ where $b' \in \mathcal{B}$ does not involve $y_k - 1$ and $y_l - 1$, then, by using Lemma 3.1 (iii) we have

$$[x_k, x_l] \cdot (y_{i_1} - 1)b' = -[x_l, x_{i_1}] \cdot (y_k - 1)b' + [x_k, x_{i_1}] \cdot (y_l - 1)b',$$

where the terms on the right hand side are (up to sign) of the form (3.2) with $b = (y_{i_1} - 1) \cdots (y_{i_s} - 1)$, $l < i_1$ and $k, l \neq i_1, \ldots, i_2$. Finally, in view of Lemma 3.1(i), the generators of the form (3.3) where b involves the factor $y_k - 1$ are zero. Consequently, M is generated by the elements

$$\begin{aligned} &x_k^2 \cdot (y_{i_1} - 1) \cdots (y_{i_s} - 1) = [x_k^2, x_{i_1}, \ldots, x_{i_s}] \\ &\text{with } 1 \leqslant k \leqslant n,\ 1 \leqslant i_1 < \cdots < i_s \leqslant n,\ k \neq i_1, \ldots, i_s,\ 0 \leqslant s \leqslant n-1 \end{aligned} \tag{3.4}$$

together with the elements

$$\begin{aligned} &[x_k, x_l] \cdot (y_{i_1} - 1) \cdots (y_{i_s} - 1) = [x_k, x_l, x_{i_1}, \ldots, x_{i_s}] \\ &\text{with } 1 \leqslant l < k \leqslant n,\ 1 \leqslant l < i_1 < \cdots < i_s \leqslant n,\ k \neq i_1, \ldots, i_s,\ 0 \leqslant s \leqslant n-2. \end{aligned} \tag{3.5}$$

We claim that this generating set is linearly independent over $\mathbb{Z}$. To verify the claim we count the number of elements in it. The number of elements of the form (3.5) is

$$\sum_{s=2}^{n} \binom{n}{s}(s-1) = \sum_{s=1}^{n} \binom{n}{s} s - n - \sum_{s=0}^{n} \binom{n}{s} + 1 + n = n2^{n-1} - 2^n + 1,$$

and the number of elements of the form (3.4) is

$$n \sum_{r=0}^{n-1} \binom{n-1}{r} = n2^{n-1}.$$

So altogether we have $(n-1)2^n + 1$ elements in this generating set. But this is precisely the rank of $M = R/R'$ as a free abelian group. Indeed, this rank is equal to the rank of the free subgroup group R of F, and the latter is $(n-1)2^n + 1$ by Schreier's formula. It follows that this generating set is a $\mathbb{Z}$-basis. We summarize our discussion as follows.

Lemma 3.2. *The relation module M is a free $\mathbb{Z}$-module of rank $(n-1)2^n + 1$, and the elements (3.4) together with the elements (3.5) form a $\mathbb{Z}$-basis for it.*

4. A generating set for $(M \wedge M)_G$

Let $\mathcal{M}$ denote the $\mathbb{Z}$-basis of the relation module M given in Lemma 3.2. Then the exterior square $M \wedge M$ is, as a $\mathbb{Z}$-module generated by the elements $m_1 \wedge m_2$ with $m_1, m_2 \in \mathcal{M}$, and so is its quotient $(M \wedge M)_G$. In fact, if $<$ is a linear order on $\mathcal{M}$, then the elements $m_1 \wedge m_2$ with $m_1, m_2 \in \mathcal{M}$ and $m_1 < m_2$ form a $\mathbb{Z}$-basis of $M \wedge M$. In order to get the trivialization $(M \wedge M)_G$ from $M \wedge M$, we need to factor out the subgroup generated by the elements

$$(m_1 \wedge m_2)(g-1) \qquad (m_1, m_2 \in M,\ g \in G \setminus \{1\}).$$

In other words, in $(M \wedge M)_G$ we have the relations $(m_1 \wedge_* m_2)(g-1) = 0$, and these are equivalent to the relations $m_1 \cdot g \wedge_* m_2 = m_1 \wedge_* m_2 \cdot g$, which, in turn, are equivalent to

$$m_1 \cdot r \wedge_* m_2 = m_1 \wedge_* m_2 \cdot r \qquad (m_1, m_2 \in M,\ r \in \mathbb{Z}G). \tag{4.1}$$

An obvious consequence of these relations is

$$m_1 \wedge_* m_2 \cdot r = -m_2 \wedge_* m_1 \cdot r \qquad (m_1, m_2 \in M,\ r \in \mathbb{Z}G), \tag{4.2}$$

which will often be used in what follows. Using (4.1), the generating set of $(M \wedge M)_G$ consisting of the elements $m_1 \wedge_* m_2$ with $m_1, m_2 \in \mathcal{M}$ can obviously be reduced to consist of the elements

$$x_i^2 \wedge_* m \qquad (1 \leqslant i \leqslant n,\ m \in \mathcal{M} \setminus \{x_i^2\}) \tag{4.3}$$

and

$$[x_i, x_j] \wedge_* m \qquad (1 \leqslant j < i \leqslant n,\ m \in \mathcal{M} \setminus \{[x_i, x_j]\}). \tag{4.4}$$

Our aim is to further reduce this generating set. First of all, if m in (4.3) is of the form (3.4) with $k \neq i$, we may assume that $i > k$ as by (4.2) we have

$$x_i^2 \wedge_* x_k^2 \cdot (y_{i_1} - 1) \cdots (y_{i_s} - 1) = -x_k^2 \wedge_* x_i^2 \cdot (y_{i_1} - 1) \cdots (y_{i_s} - 1).$$

We also may assume that $i \neq i_1, \ldots, i_s$ as if $y_{i_r} = y_i$ for some $1 \leqslant r \leqslant s$ (using (4.1) and Lemma 3.1(i))

$$\begin{aligned} &x_i^2 \wedge_* x_k^2 \cdot (y_{i_1} - 1) \cdots (y_{i_s} - 1) \\ &= x_i^2 \cdot (y_i - 1) \wedge_* x_k^2 \cdot (y_{i_1} - 1) \cdots (y_{i_{r-1}} - 1)(y_{i_{r+1}} - 1) \cdots (y_{i_s} - 1) \\ &= 0. \end{aligned}$$

If m in (4.3) is of the form (3.4) with $k = i$ and $s \geqslant 1$ then using Lemma 3.1(ii) and (i) and the relation (4.1) we get:

$$\begin{aligned} & x_i^2 \wedge_* x_i^2 \cdot (y_{i_1} - 1) \cdots (y_{i_s} - 1) \\ & \quad = x_i^2 \wedge_* [x_i, x_{i_1}] \cdot (y_i + 1)(y_{i_2} - 1) \cdots (y_{i_s} - 1) \\ & \quad = x_i^2.(y_i + 1) \wedge_* [x_i, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\ & \quad = 2x_i^2 \wedge_* [x_i, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1). \end{aligned}$$

Hence these elements are redundant in our generating set. To summarize, of the elements of the form (4.3) with m of the form (3.4) we keep the elements

$$\begin{aligned} & x_i^2 \wedge_* x_j^2 \cdot (y_{i_1} - 1) \cdots (y_{i_s} - 1) = x_i^2 \wedge_* [x_j^2, x_{i_1}, \ldots, x_{i_s}] \\ & \text{with } 1 \leqslant j < i \leqslant n,\ 1 \leqslant i_1 < \cdots < i_s \leqslant n,\ i, j \neq i_1, \ldots, i_s,\ 0 \leqslant s \leqslant n - 2. \end{aligned} \tag{4.5}$$

Next consider the elements (4.3) where m is of the form (3.5). Suppose that $k = i$ in (3.5) (and so $i \neq i_1, \ldots, i_s$). For $1 \leqslant i, l, q \leqslant n$, and $r \in \mathbb{Z}G$, we have

$$\begin{aligned} x_i^2 \wedge_* [x_i, x_l] \cdot (y_q - 1)r &= -x_i^2 \wedge_* [x_l, x_q] \cdot (y_i - 1)r - x_i^2 \wedge_* [x_q, x_i] \cdot (y_l - 1)r \\ &= -x_i^2 \cdot (y_i - 1) \wedge_* [x_l, x_q] \cdot r + x_i^2 \wedge_* [x_i, x_q] \cdot (y_l - 1)r \\ &= x_i^2 \wedge_* [x_i, x_q] \cdot (y_l - 1)r. \end{aligned}$$

In other words, the relation

$$x_i^2 \wedge_* [x_i, x_l] \cdot (y_q - 1)r = x_i^2 \wedge_* [x_i, x_q] \cdot (y_l - 1)r$$

holds for all $x_i, x_l, x_q \in X$ and $r \in \mathbb{Z}G$. In view of this relation, each element (4.3) where m is of the form (3.5) with $k = i$ is (up to sign) equal to an element of the form

$$\begin{aligned} & x_i^2 \wedge_* [x_i, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) = x_i^2 \wedge_* [x_i, x_{i_1}, x_{i_2}, \ldots, x_{i_s}] \\ & \text{with } 1 \leqslant i \leqslant n,\ 1 \leqslant i_1 < \cdots < i_s \leqslant n,\ i \neq i_1, \ldots, i_s,\ 1 \leqslant s \leqslant n - 1. \end{aligned} \tag{4.6}$$

If we have an element of the form (4.3) where m is of the form (3.5) with $l = i$, anti-commutativity allows us to interchange l with k and turn it into an element of the form (4.6) (possibly with a minus sign) using the same relation. To summarize, of the elements (4.3) where m is of the form (3.5) with $k = i$ or $l = i$ we keep the elements (4.6).

Consider the elements (4.3) with m of the form (3.5) and $i \neq k, l$. Clearly, these are zero if one of the subscripts $i_1, \ldots, i_s$ is equal to i. Thus it remains to deal with those elements where $i \neq k, l, i_1, \ldots, i_s$. None of these are redundant, we keep them all in our generating set. In other words, we keep all the elements

$$x_i^2 \wedge_* [x_{i_1}, x_{i_2}] \cdot (y_{i_3} - 1) \cdots (y_{i_s} - 1) = x_i^2 \wedge_* [x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_s}]$$
$$\text{with } 1 \leqslant i \leqslant n,\ i_1, \ldots, i_s \in \{1, \ldots, n\} \backslash \{i\},\ i_1 > i_2 < i_3 \ldots < i_s,$$
$$i_1 \neq i_3, \ldots, i_s,\ 2 \leqslant s \leqslant n-1 \tag{4.7}$$

in our generating set.

Consider now the elements of the form (4.4). If m is of the form (3.4) the relation (4.2) converts these elements to elements of the form (4.3) (with a minus sign) with m of the form (3.5) which have been discussed above.

Thus we turn to the elements of the form (4.4) where m is of the form (3.5). Firstly we are concerned with a more general set of elements which contains the elements we are interested in. More precisely we examine the elements of the form

$$[x_{j_1}, x_{j_2}] \wedge_* [x_{j_3}, x_{j_4}] \cdot (y_{j_5} - 1) \cdots (y_{j_s} - 1), \qquad j_1, \ldots, j_s \in \{1, \ldots, n\},\ s \geqslant 4, \tag{4.8}$$

where we do not impose any conditions on the subscripts $j_1, \ldots, j_s$. We require the following lemma.

Lemma 4.1. *Any element of the form (4.8) such that two of the subscripts $j_1, \ldots, j_s$ are equal to some i with $1 \leqslant i \leqslant n$ belongs to the span of the elements*

$$[x_i, x_{i_1}] \wedge_* [x_i, x_{i_2}] \cdot (y_{i_3} - 1) \cdots (y_{i_s} - 1), \qquad i_1 \leqslant i_2,\ s \geqslant 2 \tag{4.9}$$

involving the same subscripts (counting multiplicities).

Proof. The arguments used in [10, Section 3, Lemmas 3.1–3.3] are easily adapted to the current situation. □

In order to further reduce the set (4.9) we need the following result.

Lemma 4.2. *For all $1 \leqslant i, j, k, q \leqslant n$ and all $r \in \mathbb{Z}G$ we have*

$$\begin{aligned}[x_i, x_j] \wedge_* [x_i, x_k] \cdot (y_q - 1)r = {} & [x_i, x_j] \wedge_* [x_i, x_q] \cdot (y_k - 1)r \\ & - 2[x_i, x_j] \wedge_* [x_q, x_k] \cdot r \\ & + x_i^2 \wedge_* [x_q, x_k] \cdot (y_j - 1)r.\end{aligned}$$

Proof. By using the Jacobi identity, Lemma 3.1 (ii) and (4.1) we obtain

$$\begin{aligned}& [x_i, x_j] \wedge_* [x_i, x_k] \cdot (y_q - 1)r \\ = {} & [x_i, x_j] \wedge_* [x_i, x_q] \cdot (y_k - 1)r + [x_i, x_j] \wedge_* [x_q, x_k] \cdot (y_i - 1)r \\ = {} & [x_i, x_j] \wedge_* [x_i, x_q] \cdot (y_k - 1)r - 2[x_i, x_j] \wedge_* [x_q, x_k] \cdot r \\ & + [x_i, x_j] \cdot (y_i + 1) \wedge_* [x_q, x_k] \cdot r\end{aligned}$$

$$
\begin{aligned}
&= [x_i, x_j] \wedge_* [x_i, x_q] \cdot (y_k - 1)r - 2[x_i, x_j] \wedge_* [x_q, x_k] \cdot r \\
&\quad + x_i^2 \cdot (y_j - 1) \wedge_* [x_q, x_k] \cdot r \\
&= [x_i, x_j] \wedge_* [x_i, x_q] \cdot (y_k - 1)r - 2[x_i, x_j] \wedge_* [x_q, x_k] \cdot r \\
&\quad + x_i^2 \wedge_* [x_q, x_k] \cdot (y_j - 1)r
\end{aligned}
$$

as required. □

The next result will enable us to further reduce the elements of the form (4.8) in our generating set.

Lemma 4.3. *Any element of the form (4.8) such that two of the subscripts $j_1, \dots, j_s$ are equal to some i with $1 \leqslant i \leqslant n$ belongs to the span of the elements (4.9) satisfying the additional condition that $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_s$, elements of the form (4.3) with m of the form (3.4) and (3.5) and elements of the form (4.8) in which all the subscripts $j_1, \dots, j_s$ are distinct.*

Proof. We use finite induction on s. The claim is clearly true for $s = 4$. Now consider an element (4.8) with $s > 4$. By Lemma 4.1 we may assume that the element is actually of the form (4.9) with $s > 2$. By applying Lemma 4.2 and possibly (4.2) repeatedly, we get that our element is (up to sign) equal to an element of the form (4.9) with $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_s$ plus elements of the form (4.3) with m of the form (3.4) and (3.5) (using Lemma 3.1(ii)) and elements of the form (4.8) with smaller s. If the latter involves two equal subscripts, induction applies, and if not, then all subscripts are distinct as required. □

Lemma 4.4. *Any element of the form*

$$[x_i, x_{i_1}] \wedge_* [x_i, x_{i_2}] \cdot (y_{i_3} - 1) \cdots (y_{i_s} - 1), \qquad s \geqslant 2$$

in which two of the subscripts $i_1, \dots, i_s$ are equal or one of them is equal to i belongs to the span of elements of the form (4.6) and elements of the form (4.9) in which all the subscripts $i_1, \dots, i_s$ are distinct and not equal to i.

Proof. If $i_1 = i$ or $i_2 = i$, the element is clearly zero. If one of $i_3, \dots i_s$ is equal to i we may assume that $i_3 = i$, and then apply Lemma 4.2 with $j = i_1$, $k = i_2$, $q = i_3 = i$, and $r = (y_{i_4} - 1) \cdots (y_{i_s} - 1)$. The lemma then gives that our element is equal to a sum of three elements, the first of which is zero, the second of which is of the same form with fewer of the i_j being equal to i (so induction applies), and the third of the form $x_i^2 \wedge_* [x_i, x_{i_2}] \cdot (y_{i_1} - 1)r$ which is in the span of the elements of the form (4.6) (by (4.1) and Lemma 3.1(ii)) as required. If two of the subscripts $i_1, \dots, i_s$ are equal to j, say, we use Lemma 4.2 and (4.2) to write the element as

$$[x_i, x_j] \wedge_* [x_i, x_j] \cdot (y_{i_1} - 1) \cdots (y_{i_s} - 1)$$

plus elements with fewer of the i_j being equal between then and elements of the form $x_i^2 \wedge_* [x_i, x_{i_2}] \cdot (y_{i_1} - 1)r$. It turns out that the last displayed element is zero. Once this will be established, the proof of the lemma will be complete. So, to finish the lemma off, consider the element $2[x_i, x_j] \wedge_* [x_i, x_k] \cdot (y_{i_1} - 1) \cdots (y_{i_s} - 1)$. We have

$$\begin{aligned}
&2[x_i, x_j] \wedge_* [x_i, x_k] \cdot (y_{i_1} - 1) \cdots (y_{i_s} - 1) \\
&= [x_i, x_j] \cdot (y_{i_1} - 1) \wedge_* [x_i, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad - [x_i, x_j] \wedge_* [x_k, x_{i_1}] \cdot (y_i - 1)(y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad - [x_i, x_j] \wedge_* [x_{i_1}, x_i] \cdot (y_k - 1)(y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&= -[x_j, x_{i_1}] \cdot (y_i - 1) \wedge_* [x_i, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad - [x_{i_1}, x_i] \cdot (y_j - 1) \wedge_* [x_i, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad - [x_i, x_j] \wedge_* [x_k, x_{i_1}] \cdot (y_i + 1)(y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + 2[x_i, x_j] \wedge_* [x_k, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + [x_j, x_k] \cdot (y_i - 1) \wedge_* [x_{i_1}, x_i] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + [x_k, x_i] \cdot (y_j - 1) \wedge_* [x_{i_1}, x_i] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&= -[x_j, x_{i_1}] \cdot (y_i + 1) \wedge_* [x_i, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + 2[x_j, x_{i_1}] \wedge_* [x_i, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad - x_i^2 \cdot (y_j - 1) \wedge_* [x_k, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + 2[x_i, x_j] \wedge_* [x_k, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + [x_j, x_k] \cdot (y_i + 1) \wedge_* [x_{i_1}, x_i] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad - 2[x_j, x_k] \wedge_* [x_{i_1}, x_i] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&= -x_i^2 \cdot (y_k - 1) \wedge_* [x_{i_1}, x_j] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + 2[x_k, x_i] \wedge_* [x_j, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad - x_i^2 \cdot (y_j - 1) \wedge_* [x_k, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + 2[x_i, x_j] \wedge_* [x_k, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + x_i^2 \cdot (y_{i_1} - 1) \wedge_* [x_j, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + 2[x_j, x_k] \wedge_* [x_i, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&= 2\Big(x_i^2 \wedge_* [x_j, x_k] \cdot (y_{i_1} - 1)(y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\qquad + [x_i, x_j] \wedge_* [x_k, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\qquad + [x_j, x_k] \wedge_* [x_i, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\qquad + [x_k, x_i] \wedge_* [x_j, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \Big).
\end{aligned} \tag{4.10}$$

We now return to the elements of the form $[x_i, x_j] \wedge_* [x_i, x_j] \cdot (y_{i_1} - 1) \cdots (y_{i_r} - 1)$. We have

$$\begin{aligned}
&[x_i, x_j] \wedge_* [x_i, x_j] \cdot (y_{i_1} - 1) \cdots (y_{i_s} - 1) \\
&= -[x_i, x_j] \wedge_* [x_j, x_{i_1}] \cdot (y_i - 1)(y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad - [x_i, x_j] \wedge_* [x_{i_1}, x_i] \cdot (y_j - 1)(y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&= -[x_i, x_j] \cdot (y_i + 1) \wedge_* [x_j, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + 2[x_i, x_j] \wedge_* [x_j, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad - [x_i, x_j] \cdot (y_j + 1) \wedge_* [x_{i_1}, x_i] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + 2[x_i, x_j] \wedge_* [x_{i_1}, x_i] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&= -x_i^2 \cdot (y_j - 1) \wedge_* [x_j, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + 2[x_i, x_j] \wedge_* [x_j, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + x_j^2 \cdot (y_i - 1) \wedge_* [x_{i_1}, x_i] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + 2[x_i, x_j] \wedge_* [x_{i_1}, x_i] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&= -x_i^2 \cdot \wedge_* [x_j, x_{i_1}] \cdot (y_j + 1)(y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + 2x_i^2 \cdot \wedge_* [x_j, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + 2[x_i, x_j] \wedge_* [x_j, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + x_j^2 \wedge_* [x_{i_1}, x_i] \cdot (y_i + 1)(y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad - 2x_j^2 \wedge_* [x_{i_1}, x_i] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + 2[x_i, x_j] \wedge_* [x_{i_1}, x_i] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&= -x_i^2 \wedge_* x_j^2 \cdot (y_{i_1} - 1)(y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + 2x_i^2 \wedge_* [x_j, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + 2[x_j, x_{i_1}] \wedge_* [x_j, x_i] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad - x_j^2 \wedge_* x_i^2 \cdot (y_{i_1} - 1)(y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + 2x_j^2 \wedge_* [x_i, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + 2[x_i, x_{i_1}] \wedge_* [x_i, x_j] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&= 2\Big(x_i^2 \wedge_* [x_j, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\qquad + [x_i, x_{i_1}] \wedge_* [x_i, x_j] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\qquad + x_j^2 \wedge_* [x_i, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\qquad + [x_j, x_{i_1}] \wedge_* [x_j, x_i] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \Big).
\end{aligned}$$

Using (4.10) we get

$$\begin{aligned}
&2\Big(x_i^2 \wedge_* [x_j, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + [x_i, x_{i_1}] \wedge_* [x_i, x_j] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + x_j^2 \wedge_* [x_i, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + [x_j, x_{i_1}] \wedge_* [x_j, x_i] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1)\Big) \\
&= 2\Big(x_i^2 \wedge_* [x_j, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + x_i^2 \wedge_* [x_{i_1}, x_j] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + [x_i, x_{i_1}] \wedge_* [x_j, x_{i_2}] \cdot (y_{i_3} - 1) \cdots (y_{i_s} - 1) \\
&\quad + [x_{i_1}, x_j] \wedge_* [x_i, x_{i_2}] \cdot (y_{i_3} - 1) \cdots (y_{i_s} - 1) \\
&\quad + [x_j, x_i] \wedge_* [x_{i_1}, x_{i_2}] \cdot (y_{i_3} - 1) \cdots (y_{i_s} - 1) \\
&\quad + x_j^2 \wedge_* [x_i, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + x_j^2 \wedge_* [x_{i_1}, x_i] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + [x_j, x_{i_1}] \wedge_* [x_i, x_{i_2}] \cdot (y_{i_3} - 1) \cdots (y_{i_s} - 1) \\
&\quad + [x_{i_1}, x_i] \wedge_* [x_j, x_{i_2}] \cdot (y_{i_3} - 1) \cdots (y_{i_s} - 1) \\
&\quad + [x_i, x_j] \wedge_* [x_{i_1}, x_{i_2}] \cdot (y_{i_3} - 1) \cdots (y_{i_s} - 1)\Big) \\
&= 0.
\end{aligned}$$

This completes the proof of the lemma. □

This lemma together with Lemma 4.3 tells us that all elements of the form (4.8) with two of the subscripts equal between them are redundant in our generating set, except the elements of the form (4.9) with the additional conditions that $i \neq i_1, \ldots, i_s$ and $i_1 < i_2 < \cdots < i_s$. In other words, the elements we keep are

$$\begin{aligned}
&[x_i, x_{i_1}] \wedge_* [x_i, x_{i_2}] \cdot (y_{i_3} - 1) \cdots (y_{i_s} - 1) = [x_i, x_{i_1}] \wedge_* [x_i, x_{i_2}, x_{i_3}, \ldots, x_{i_s}] \\
&\text{with } 1 \leqslant i \leqslant n,\ 1 \leqslant i_1 < i_2 < \cdots < i_s \leqslant n,\ i \neq i_1, \ldots, i_s,\ 2 \leqslant s \leqslant n-1. \qquad (4.11)
\end{aligned}$$

Finally, we are left with reducing the set of elements (4.8) in which all subscripts are distinct.

Lemma 4.5. *Any element of the form (4.8) such that the subscripts $j_1, \ldots, j_s$ are mutually distinct is in the span of the elements*

$$[x_{i_1}, x_{i_2}] \wedge_* [x_{i_3}, x_{i_4}] \cdot (y_{i_5} - 1) \cdots (y_{i_s} - 1), \qquad 4 \leqslant s \leqslant n \qquad (4.12)$$

such that $\{i_1, \dots, i_s\} = \{j_1, \dots, j_s\}$ *with* $i_1 > i_2$, $i_3 > i_4$, $i_1 > i_3$, $i_2 > i_4$, $i_2 < i_5 < \cdots < i_s$, *and the element*

$$[x_{k_3}, x_{k_2}] \wedge_* [x_{k_4}, x_{k_1}] \cdot (y_{k_5} - 1) \cdots (y_{k_s} - 1) \tag{4.13}$$

where $\{j_1, \dots, j_s\} = \{k_1, \dots, k_s\}$ *with* $k_1 < k_2 < \cdots < k_s$.

Proof. We refer to the proof of Lemma 3.3 in [10]. The arguments used there in the context of Lie rings are easily adapted to the group commutators in the present paper. □

Following [10], we call elements of the form (4.12) Kuz'min elements. We summarize the result of our discussion so far as follows. The group $(M \wedge M)_G$ is generated by the elements (4.5), (4.6), (4.7), (4.11), and, for all subsets $\{j_1, \dots, j_s\}$, $4 \leqslant s \leqslant n$ of mutually distinct elements in $\{1, 2, \dots, n\}$, the Kuz'min elements (4.12) and the elements (4.13) as specified in Lemma 4.5.

We conclude this section by making some further modifications to this generating set. Consider the elements (4.11) with $s = 2$, that is elements of the form

$$[x_i, x_{i_1}] \wedge_* [x_i, x_{i_2}] \quad \text{with } 1 \leqslant i \leqslant n,\ 1 \leqslant i_1 < i_2 \leqslant n,\ i \neq i_1, i_2.$$

For each triple i, j, k with $1 \leqslant i < j < k \leqslant n$, the following elements of type (4.11) with $s = 2$ are in the generating set of $(M \wedge M)_G$:

$$[x_k, x_i] \wedge_* [x_k, x_j],\ [x_i, x_j] \wedge_* [x_i, x_k],\ [x_j, x_i] \wedge_* [x_j, x_k].$$

We keep the first of those, and replace the second and third by

$$x_i^2 \wedge_* [x_j, x_k] + [x_i, x_k] \wedge_* [x_i, x_j] + x_j^2 \wedge_* [x_i, x_k] + [x_j, x_k] \wedge_* [x_j, x_i]$$

and

$$x_j^2 \wedge_* [x_k, x_i] + [x_j, x_i] \wedge_* [x_j, x_k] + x_k^2 \wedge_* [x_j, x_i] + [x_k, x_i] \wedge_* [x_k, x_j],$$

respectively.

Now consider the generators (4.11) with $s \geqslant 3$, that is generators of the form

$$[x_i, x_j] \wedge_* [x_i, x_k] \cdot (y_{i_1} - 1) \cdots (y_{i_s} - 1), \tag{4.14}$$

with $1 \leqslant i \leqslant n$, $1 \leqslant j < k < i_1 < \cdots < i_s \leqslant n$, $i \neq j, k, i_1, \dots, i_s$, $1 \leqslant s \leqslant n - 3$.

We replace (4.14) by the following element

$$\begin{aligned} &x_i^2 \wedge_* [x_k, x_j] \cdot (y_{i_1} - 1) \cdots (y_{i_s} - 1) + [x_i, x_j] \wedge_* [x_i, x_k] \cdot (y_{i_1} - 1) \cdots (y_{i_s} - 1) \\ &+ [x_i, x_{i_1}] \wedge_* [x_j, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \end{aligned}$$

$$+ [x_{i_1}, x_j] \wedge_* [x_i, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1)$$
$$+ [x_i, x_j] \wedge_* [x_{i_1}, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1).$$

It is clear that the resulting set is again a generating set for $(M \wedge M)_G$. Now we are able to state the main result of this section.

Proposition 4.1. *The group $(M \wedge M)_G$ is generated by the following elements.*

(i) *The elements*

$$x_i^2 \wedge_* [x_j^2, x_{i_1}, \ldots, x_{i_s}]$$

with $1 \leqslant j < i \leqslant n$, $1 \leqslant i_1 < \cdots < i_s \leqslant n$, $i, j \neq i_1, \ldots, i_s$, $0 \leqslant s \leqslant n-2$,

(ii) *the elements*

$$x_i^2 \wedge_* [x_i, x_{i_1}, x_{i_2}, \ldots, x_{i_s}]$$

with $1 \leqslant i \leqslant n$, $1 \leqslant i_1 < \cdots < i_s \leqslant n$, $i \neq i_1, \ldots, i_s$, $1 \leqslant s \leqslant n-1$,

(iii) *the elements*

$$x_i^2 \wedge_* [x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_s}]$$

with $1 \leqslant i \leqslant n$, $i_1, \ldots, i_s \in \{1, \ldots, n\} \backslash \{i\}$, $i_1 > i_2 < i_3 \ldots < i_s$, $i_1 \neq i_3, \ldots, i_s$, $2 \leqslant s \leqslant n-1$,

(iv) *the elements*

(a) $[x_k, x_i] \wedge_* [x_k, x_j]$ *with $1 \leqslant i < j < k \leqslant n$,*

(b) $x_i^2 \wedge_* [x_j, x_k] + [x_i, x_k] \wedge_* [x_i, x_j] + x_j^2 \wedge_* [x_i, x_k] + [x_j, x_k] \wedge_* [x_j, x_i]$ *and* $x_j^2 \wedge_* [x_k, x_i] + [x_j, x_i] \wedge_* [x_j, x_k] + x_k^2 \wedge_* [x_j, x_i] + [x_k, x_i] \wedge_* [x_k, x_j]$ *with $1 \leqslant i < j < k \leqslant n$,*

(c) $x_i^2 \wedge_* [x_k, x_j, x_{i_1}, \ldots, x_{i_s}] + [x_i, x_j] \wedge_* [x_i, x_k, x_{i_1}, \ldots, x_{i_s}]$
$+ [x_i, x_{i_1}] \wedge_* [x_j, x_k, x_{i_2}, \ldots, x_{i_s}] + [x_{i_1}, x_j] \wedge_* [x_i, x_k, x_{i_2}, \ldots, x_{i_s}]$
$+ [x_i, x_j] \wedge_* [x_{i_1}, x_k, x_{i_2}, \ldots, x_{i_s}]$
with $1 \leqslant i \leqslant n$, $1 \leqslant j < k < i_1 < \cdots < i_s \leqslant n$, $i \neq j, k, i_1, \ldots, i_s$, $1 \leqslant s \leqslant n-3$,

(v) *for all subsets $\{j_1, \ldots, j_s\}$, $4 \leqslant s \leqslant n$, of mutually distinct elements in $\{1, 2, \ldots, n\}$, the Kuz'min elements*

$$[x_{i_1}, x_{i_2}] \wedge_* [x_{i_3}, x_{i_4}, x_{i_5}, \ldots, x_{i_s}], \qquad 4 \leqslant s \leqslant n,$$

with $\{i_1, \ldots, i_s\} = \{j_1, \ldots, j_s\}$ and $i_1 > i_2$, $i_3 > i_4$, $i_1 > i_3$, $i_2 > i_4$, $i_2 < i_5, \ldots < i_s$, and the element

$$[x_{k_3}, x_{k_2}] \wedge_* [x_{k_4}, x_{k_1}, x_{k_5}, \ldots, x_{k_s}]$$

where $\{j_1, \ldots, j_s\} = \{k_1, \ldots, k_s\}$ with $k_1 < k_2 < \cdots < k_s$.

5. Homology of elementary abelian 2-groups

In this section we exhibit a free resolution for the trivial G-module $\mathbb{Z}$, and record some facts about the homology of the group G. Recall that $G = G_1 \times \cdots \times G_n$ is a direct product of cyclic groups of order 2. For each $G_i = \langle y_i \mid y_i^2 = 1 \rangle$, the complex

$$\mathbf{P}^i : \cdots \to v_3^i \mathbb{Z}G_i \xrightarrow{\partial_3^i} v_2^i \mathbb{Z}G_i \xrightarrow{\partial_2^i} v_1^i \mathbb{Z}G_i \xrightarrow{\partial_1^i} v_0^i \mathbb{Z}G_i \xrightarrow{\epsilon^i} \mathbb{Z} \to 0$$

with

$$v_j^i \partial_j^i = \begin{cases} v_{j-1}^i (y_i - 1), & \text{for odd } j \geqslant 1; \\ v_{j-1}^i (y_i + 1), & \text{for even } j \geqslant 2 \end{cases}$$

is a periodic free resolution for the trivial G_i-module $\mathbb{Z}$. Then the tensor product $\mathbf{V} = \mathbf{P}^1 \otimes \cdots \otimes \mathbf{P}^n$ is a free resolution of the trivial G-module $\mathbb{Z}$. The terms V_m of this resolution are free G-modules with free generators

$$v_{(s_1,\ldots,s_n)} = v_{s_1}^1 \otimes \cdots \otimes v_{s_n}^n$$

with $s_1, \ldots, s_n \geqslant 0$ and $s_1 + \cdots + s_n = m$, and the differentials $\partial_m : V_m \to V_{m-1}$ are given by

$$v_{(s_1,\ldots,s_n)} \partial_m = \sum_{k=1}^{n} (-1)^{s_1 + \cdots + s_{k-1}} v_{(s_1,\ldots,(s_k-1),\ldots,s_n)} r_k$$

where $r_k \in \mathbb{Z}G$ is defined as

$$r_k = \begin{cases} y_k - 1, & \text{for odd } s_k \geqslant 1; \\ y_k + 1, & \text{for even } s_k \geqslant 2. \end{cases}$$

Here we use the convention that $v_{(s_1,\ldots,s_n)} = 0$ if one of the subscripts is negative, and that $s_1 + \cdots + s_{k-1} = 0$ for $k = 1$. The rank of the free G-module V_m is equal to the number of degree m monomials in n commuting variables, that is $\operatorname{rk} V_m = \binom{m+n-1}{m}$. The homology groups $H_m(G, \mathbb{Z}_2)$ can instantly be obtained from the resolution $\mathbf{V}$. They are the homology groups of the complex $\mathbb{Z}_2 \otimes_G \mathbf{V}$. All the differentials in this complex are zero maps, and this gives the following result.

Lemma 5.1. *The homology groups $H_m(G, \mathbb{Z}_2)$ for $m \geqslant 1$ are elementary abelian 2-groups of rank $\binom{m+n-1}{m}$. As homology groups of the complex $\mathbb{Z}_2 \otimes_G \mathbf{V}$ they are generated by the cycles $1 \otimes v_{(s_1,\ldots,s_n)}$ with $s_1 + \cdots + s_n = m$.*

We also need to know the homology of G with coefficients in the exterior and symmetric squares of free G-modules. By Lemma 2.1, this amounts to determining the homology

of G with coefficients in the cyclic modules $(1 \circ g)\mathbb{Z}G$ and $(1 \wedge g)\mathbb{Z}G$. It is easily seen that the complex

$$\mathbf{P}_g : \cdots \xrightarrow{g-1} e\mathbb{Z}G \xrightarrow{g+1} f\mathbb{Z}G \xrightarrow{g-1} e\mathbb{Z}G \to (1 \circ g)\mathbb{Z}G, \tag{5.1}$$

is a free resolution of the cyclic module $(1 \circ g)\mathbb{Z}G$. Hence the homology groups $H_m(G, (1 \circ g)\mathbb{Z}G)$ can be obtained as the homology of the complex $\mathbf{P}_g \otimes_G \mathbb{Z}$. The terms of this complex are infinite cyclic groups, and the differentials are zero maps in odd dimensions, and multiplication by 2 in even dimensions. This yields the following result for $H_*(G, (1 \circ g)\mathbb{Z}G)$, and the corresponding result for $H_*(G, (1 \wedge g)\mathbb{Z}G)$ can be obtained similarly.

Lemma 5.2. *Let $g \in G$, $g \neq 1$. Then*

$$H_m(G, (1 \circ g)\mathbb{Z}G) = \begin{cases} \mathbb{Z}_2, & \text{for odd } m \geqslant 1; \\ 0, & \text{for even } m \geqslant 2. \end{cases}$$

$$H_m(G, (1 \wedge g)\mathbb{Z}G) = \begin{cases} 0, & \text{for odd } m \geqslant 1; \\ \mathbb{Z}_2, & \text{for even } m \geqslant 2. \end{cases}$$

As cycles in the complex $\mathbf{P}_g \otimes_G \mathbb{Z}$ the nontrivial elements of $H_*(G, (1 \circ g)\mathbb{Z}G)$ are generated by the cycles $f \otimes 1$.

Of course, the groups $H_m(G, (1 \circ g)\mathbb{Z}G)$ can also be obtained as the homology groups of the complex $(1 \circ g)\mathbb{Z}G \otimes_G \mathbf{V}$. One of the technically most challenging tasks in this paper will be to determine the homology groups $H_m(G, (1 \circ g)\mathbb{Z}G)$ in terms of cycles in that complex. This will be carried out in Section 9.

By combining Lemma 5.2 with Corollary 2.2 we get the following.

Corollary 5.3. *We have*

(i) $H_m(G, P \wedge P) = 0$ *for all odd* $m \geqslant 1$,
(ii) $H_m(G, \mathbb{Z}G \circ \mathbb{Z}G) = 0$ *for all even* $m \geqslant 2$.

6. The kernel of the homomorphism $(M \wedge M)_G \to (P \wedge P)_G$

In this section we explore the six-term exact sequence (2.4) to obtain information about the kernel of the homomorphism $(M \wedge M)_G \to (P \wedge P)_G$. For brevity, we denote this kernel by T_2, and we also suppress the group G in the standard notation for homology, i.e. we write $H_m(A)$ rather than $H_m(G, A)$ for the m-th homology group with coefficients in a G-module A. First of all, we break up (2.4) into four short exact sequences as follows. Firstly, we have

$$0 \to M \wedge M \to P \wedge P \to Q \to 0 \tag{6.1}$$

where $Q = \ker \alpha_2$, then we have

$$0 \to Q \to P \otimes IG \to IG \circ IG \to 0, \tag{6.2}$$

furthermore we have

$$0 \to IG \circ IG \to \mathbb{Z}G \circ \mathbb{Z}G \to \tilde{I}G \to 0, \tag{6.3}$$

and finally we have

$$0 \to \tilde{I}G \to \mathbb{Z}G \to \mathbb{Z}_2 \to 0. \tag{6.4}$$

Consider the long exact homology sequence stemming from (6.1)

$$\cdots \to H_1(P \wedge P) \to H_1(Q) \to (M \wedge M)_G \to (P \wedge P)_G \to Q_G \to 0.$$

Since $H_1(P \wedge P) = 0$ by Corollary 5.3, this yields an isomorphism

$$T_2 \cong H_1(Q).$$

Dimension shifting using the short exact sequence (6.2) whose middle term is a free G-module gives $H_1(Q) \cong H_2(IG \circ IG)$, and hence we have an isomorphism

$$T_2 \cong H_2(IG \circ IG). \tag{6.5}$$

The long exact homology sequence stemming from (6.3) yields an exact sequence

$$\cdots \to H_3(\mathbb{Z}G \circ \mathbb{Z}G) \to H_3(\tilde{I}G) \to H_2(IG \circ IG) \to 0, \tag{6.6}$$

where the zero on the right is actually $H_2(\mathbb{Z}G \circ \mathbb{Z}G)$ which is zero by Corollary 5.3. Finally, dimension shifting using (6.4) gives an isomorphism $H_3(\tilde{I}G) \cong H_4(\mathbb{Z}_2)$. By combining this with (6.5) and (6.6) we get the following.

Lemma 6.1. *There is an exact sequence*

$$\cdots \to H_3(\mathbb{Z}G \circ \mathbb{Z}G) \to H_4(\mathbb{Z}_2) \to T_2 \to 0.$$

In particular, T_2 is an elementary abelian 2-group, a quotient of the homology group $H_4(\mathbb{Z}_2)$.

7. Torsion elements in the generating set

In this section we examine the images in $(P \wedge P)_G$ of some of the elements in the generating set of $(M \wedge M)_G$ in Proposition 4.1.

Lemma 7.1. *The images in* $(P \wedge P)_G$ *of the elements* (ii), (iv.b) *and* (iv.c) *in Proposition 4.1 are of order* 2, *and, moreover, they are linearly independent over* $\mathbb{Z}_2$.

Proof. We know from Corollary 2.2 (i) that $(P \wedge P)_G$ is a direct sum of a free abelian group and its torsion subgroup $t((P \wedge P)_G)$, an elementary abelian 2-group. The latter is (as a $\mathbb{Z}_2$-module) freely generated by the elements $e_i \wedge_* e_i g$ with $1 \leqslant i \leqslant n$ and $g \in G \setminus \{1\}$. It is easily seen that the elements

$$e_i \wedge_* e_i(y_{i_1} - 1) \cdots (y_{i_s} - 1) \tag{7.1}$$

with $1 \leqslant i \leqslant n$, $1 \leqslant i_1 < \cdots < i_s \leqslant n$ and $1 \leqslant s \leqslant n$ also form a $\mathbb{Z}_2$-basis of $t((P \wedge P)_G)$. Now, the images of the module generators x_i^2 and $[x_j, x_i]$ of M under the Magnus embedding μ have been recorded in (3.1). Using this, one calculates that the images of the elements (ii), (iv.b) and (iv.c) in Proposition 4.1 in $(P \wedge P)_G$ are, respectively, as follows:

$$\begin{aligned}
\text{(ii)}' \quad & e_i \wedge_* e_i(y_i - 1)(y_{i_1} - 1)(y_{i_2} - 1) \cdots (y_{i_s} - 1), \\
\text{(iv.b)}' \quad & e_i \wedge_* e_i(y_j - 1)(y_k - 1) + e_j \wedge_* e_j(y_i - 1)(y_k - 1), \\
& e_j \wedge_* e_j(y_i - 1)(y_k - 1) + e_k \wedge_* e_k(y_i - 1)(y_j - 1), \\
\text{(iv.c)}' \quad & e_i \wedge_* e_i(y_j - 1)(y_k - 1)(y_{i_1} - 1) \cdots (y_{i_s} - 1).
\end{aligned}$$

We observe that the images of all those elements are in $t((P \wedge P)_G)$. The elements (ii)′ and (iv.c)′ are distinct basis elements of the form (7.1): the former involve both e_i and y_i but the latter don't. Moreover, for each triple i, j, k with $1 \leqslant i < j < k \leqslant n$, the images (iv.b)′ involving the subscripts i, j, k span a 2-dimensional subspace in the 3-dimensional subspace of $t((P \wedge P)_G)$ that is spanned by the basis elements

$$e_i \wedge_* e_i(y_j - 1)(y_k - 1), \quad e_j \wedge_* e_j(y_i - 1)(y_k - 1), \quad e_k \wedge_* e_k(y_i - 1)(y_j - 1).$$

These elements are not of type (ii)′ (since those elements involve both e_i and y_i) and they are not of type (iv.c)′ (since for those elements $s \geqslant 1$). Hence all these 2-dimensional subspaces (for distinct triples i, j, k) and the span of the elements (ii)′ and (iv.c)′ generate their direct sum in $t((P \wedge P)_G)$. It follows that the elements (ii)′, (iv.b)′, and (iv.c)′ are linearly independent over $\mathbb{Z}_2$ and, consequently, the elements (ii), (iv.b) and (iv.c) with a range of subscripts as in Proposition 4.1 are also linearly independent over $\mathbb{Z}_2$. □

Lemma 7.1 tells us that, modulo T_2, the elements (ii), (iv.b) and (iv.c) in Proposition 4.1 have order 2, and since T_2 is an elementary abelian 2-group, these elements have

order 2 or order 4 in $(M \wedge M)_G$. We conclude this section by showing that some of these elements are actually of order 2.

Lemma 7.2. *Of the elements in the generating set for $(M \wedge M)_G$ as given in Proposition 4.1, the elements* (ii) *with $s > 1$ and the elements* (iv.c) *have order* 2.

Proof. Consider the element $x_i^2 \wedge_* [x_i, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1)$ with $s > 1$. We have

$$\begin{aligned}
&x_i^2 \wedge_* [x_i, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&= x_i^2 \cdot (y_{i_2} - 1) \wedge_* [x_i, x_{i_1}] \cdot (y_{i_3} - 1) \cdots (y_{i_s} - 1) \qquad \text{by (4.1)} \\
&= [x_i, x_{i_2}] \cdot (y_i + 1) \wedge_* [x_i, x_{i_1}] \cdot (y_{i_3} - 1) \cdots (y_{i_s} - 1) \qquad \text{by Lemma 3.1 (ii)} \\
&= [x_i, x_{i_2}] \wedge_* [x_i, x_{i_1}] \cdot (y_i + 1)(y_{i_3} - 1) \cdots (y_{i_s} - 1) \qquad \text{by (4.1)} \\
&= [x_i, x_{i_2}] \wedge_* x_i^2 \cdot (y_{i_1} - 1)(y_{i_3} - 1) \cdots (y_{i_s} - 1) \qquad \text{by Lemma 3.1 (ii)} \\
&= x_i^2 \wedge_* [x_{i_2}, x_i] \cdot (y_{i_1} - 1)(y_{i_3} - 1) \cdots (y_{i_s} - 1).
\end{aligned}$$

Hence we can write

$$\begin{aligned}
&2x_i^2 \wedge_* [x_i, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&= x_i^2 \wedge_* \big([x_{i_2}, x_i] \cdot (y_{i_1} - 1)(y_{i_3} - 1) \cdots (y_{i_s} - 1) + [x_i, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1)\big) \\
&= -x_i^2 \wedge_* \big([x_{i_1}, x_{i_2}] \cdot (y_i - 1)(y_{i_3} - 1) \cdots (y_{i_s} - 1)\big) \qquad \text{by the Jacobi identity} \\
&= -x_i^2 \cdot (y_i - 1)(y_{i_3} - 1) \cdots (y_{i_s} - 1) \wedge_* [x_{i_1}, x_{i_2}] \qquad \text{by (4.1)} \\
&= 0.
\end{aligned}$$

This shows that the element $x_i^2 \wedge_* [x_i, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1)$ has order 2. For the elements (iv.c) we have

$$\begin{aligned}
&2\Big(x_i^2 \wedge_* [x_k, x_j] \cdot (y_{i_1} - 1) \cdots (y_{i_s} - 1) \\
&\quad + [x_i, x_j] \wedge_* [x_i, x_k] \cdot (y_{i_1} - 1) \cdots (y_{i_s} - 1) \\
&\quad + [x_i, x_{i_1}] \wedge_* [x_j, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + [x_{i_1}, x_j] \wedge_* [x_i, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + [x_i, x_j] \wedge_* [x_{i_1}, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1)\Big) \\
&= 2\big(x_i^2 \wedge_* [x_k, x_j] \cdot (y_{i_1} - 1) \cdots (y_{i_s} - 1) \\
&\quad + [x_i, x_{i_1}] \wedge_* [x_j, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1)\big) \\
&\quad - [x_i, x_j] \wedge_* \big([x_k, x_{i_1}] \cdot (y_i - 1) + [x_{i_1}, x_i] \cdot (y_k - 1)\big)(y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&\quad + [x_i, x_j] \cdot (y_{i_1} - 1) \wedge_* [x_i, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1)
\end{aligned}$$

$$
\begin{aligned}
&+ 2[x_{i_1}, x_j] \wedge_* [x_i, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&- 2[x_i, x_j] \wedge_* [x_k, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&= 2\big(x_i^2 \wedge_* [x_k, x_j] \cdot (y_{i_1} - 1) \cdots (y_{i_s} - 1) \\
&\qquad + [x_i, x_{i_1}] \wedge_* [x_j, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1)\big) \\
&+ [x_i, x_j] \cdot (y_i + 1) \wedge_* [x_{i_1}, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&+ [x_i, x_j] \cdot (y_k - 1) \wedge_* [x_i, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&- \big([x_j, x_{i_1}] \cdot (y_i - 1) + [x_{i_1}, x_i] \cdot (y_j - 1)\big) \wedge_* [x_i, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&- 2[x_j, x_{i_1}] \wedge_* [x_i, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&= 2\big(x_i^2 \wedge_* [x_k, x_j] \cdot (y_{i_1} - 1) \cdots (y_{i_s} - 1) \\
&\qquad + [x_i, x_{i_1}] \wedge_* [x_j, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1)\big) \\
&+ x_i^2 \cdot (y_j - 1) \wedge_* [x_{i_1}, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&+ \big([x_k, x_i] \cdot (y_j - 1) + [x_i, x_j] \cdot (y_k - 1)\big) \wedge_* [x_i, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&+ [x_i, x_k] \cdot (y_i + 1) \wedge_* [x_j, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&= 2\big(x_i^2 \wedge_* [x_k, x_j] \cdot (y_{i_1} - 1) \cdots (y_{i_s} - 1) \\
&\qquad + [x_i, x_{i_1}] \wedge_* [x_j, x_k] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1)\big) \\
&- x_i^2 \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \wedge_* \big([x_k, x_j] \cdot (y_{i_1} - 1) + [x_j, x_{i_1}] \cdot (y_k - 1)\big) \\
&- [x_j, x_k] \cdot (y_i - 1) \wedge_* [x_i, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&+ x_i^2 \cdot (y_k - 1) \wedge_* [x_j, x_{i_1}] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&= \big(x_i^2 \cdot (y_{i_1} - 1) - [x_i, x_{i_1}] \cdot (y_i + 1)\big) \wedge_* [x_k, x_j] \cdot (y_{i_2} - 1) \cdots (y_{i_s} - 1) \\
&= 0.
\end{aligned}
$$

This shows that these elements have order 2. □

The remaining torsion elements in our generating set, that is the elements (ii) with $s = 1$ and the elements (iv.b) have order 4. This will be shown in Section 10. But first we turn to the free abelian part of $(M \wedge M)_G$.

8. The free abelian part of $(M \wedge M)_G$

In this section we deal with the elements of our generating set from Proposition 4.1 that have not been discussed in Section 7. We will show that these elements freely generate a free abelian group. In order to prove this, we first determine the number of these elements, and then we exploit the exact sequence (2.4) to show that this number is precisely the rank of the free part of $(M \wedge M)_G$.

When counting the various sets of elements, we will use the identities

$$\sum_{k=0}^{n}\binom{n}{k}=2^n,\quad \sum_{k=q}^{n}\binom{n}{k}\binom{k}{q}=2^{n-q}\binom{n}{q}\quad\text{and}\quad \sum_{k=0}^{n}k^2\binom{n}{k}=(n+n^2)2^{n-2}.$$

The number of elements of type (i) is

$$\binom{n}{2}\sum_{m=0}^{n-2}\binom{n-2}{m}=\binom{n}{2}2^{n-2}=n^2 2^{n-3}-n2^{n-3}.$$

As to type (iii), for a fixed set of distinct free generators $\{x_{i_1},\dots,x_{i_s}\}$ the number of elements of the form $[x_{i_1},x_{i_2},x_{i_3},\dots,x_{i_s}]$ with $i_1>i_2<i_3\dots<i_s$ is $s-1$. Consequently, the number of elements of type (iii) is

$$\begin{aligned}n\sum_{s=2}^{n-1}\binom{n-1}{s}(s-1)&=n\sum_{s=2}^{n-1}\binom{n-1}{s}s-n\sum_{s=2}^{n-1}\binom{n-1}{s}\\&=n2^{n-2}(n-1)-n(n-1)-n2^{n-1}+n+n(n-1)\\&=n^2 2^{n-2}-n2^{n-2}-n2^{n-1}+n.\end{aligned}$$

The number of elements of type (iv.a) is clearly $\binom{n}{3}$. The number of Kuz'min elements with distinct entries $\{x_{i_1},\dots,x_{i_s}\}$, $4\leqslant s\leqslant n$ is $\frac{1}{2}s(s-3)$ (see [10, Lemma 7.1(ii)] where this is proved for $s\geqslant 5$, and the case $s=4$ is easily verified directly). Hence the total number of Ku'min elements of type (v) is

$$\begin{aligned}\frac{1}{2}\sum_{m=4}^{n}\binom{n}{m}m(m-3)&=\frac{1}{2}\sum_{m=4}^{n}\binom{n}{m}m^2-\frac{3}{2}\sum_{m=4}^{n}\binom{n}{m}m\\&=\frac{1}{2}(n+n^2)2^{n-2}-\frac{1}{2}n-\frac{1}{2}\binom{n}{2}4-\frac{1}{2}\binom{n}{3}9\\&\quad-\frac{3}{2}n2^{n-1}+\frac{3}{2}n+\frac{3}{2}\binom{n}{2}2+\frac{3}{2}\binom{n}{3}3\\&=n2^{n-3}+n^2 2^{n-3}+n+\binom{n}{2}-3n2^{n-2}.\end{aligned}$$

Also, for each set of distinct entries $\{x_{i_1},\dots,x_{i_s}\}$, $4\leqslant s\leqslant n$ we have precisely one extra element of type (v), i.e. we altogether have

$$\sum_{m=4}^{n}\binom{n}{m}=2^n-1-n-\binom{n}{2}-\binom{n}{3}$$

such elements. Hence the total number of elements of types (i), (iii), (iv.a) and (v) is

$$\begin{aligned}
&\left(n^2 2^{n-3} - n2^{n-3}\right) + \left(n^2 2^{n-2} - n2^{n-2} - n2^{n-1} + n\right) + \binom{n}{3} \\
&\quad + \left(n2^{n-3} + n^2 2^{n-3} + n + \binom{n}{2} - 3n2^{n-2}\right) + \left(2^n - 1 - n - \binom{n}{2} - \binom{n}{3}\right) \\
&= n^2 2^{n-1} - n2^{n-1} - n2^n + 2^n + n - 1 \\
&= n(n-1)2^{n-1} - (n-1)2^n + (n-1) \\
&= \left[\binom{n}{2} - \binom{n-1}{1}\right] 2^n + (n-1) \\
&= \binom{n-1}{2} 2^n + (n-1).
\end{aligned}$$

We record the result as follows.

Lemma 8.1. *The total number of elements of types* (i), (iii), (iv.a) *and* (v) *in the generating set for* $(M \wedge M)_G$ *in Proposition 4.1 is*

$$\binom{n-1}{2} 2^n + (n-1).$$

Now we work out the rank of the free abelian part of $(M \wedge M)_G$. It is plain that this rank coincides with the dimension of $((M \wedge M) \otimes \mathbb{Q})_G$. Tensoring the six-term exact sequence (2.4) with $\mathbb{Q}$ gives an exact sequence

$$\begin{aligned}
0 \to (M \wedge M) \otimes \mathbb{Q} \to (P \wedge P) \otimes \mathbb{Q} \to (P \otimes IG) \otimes \mathbb{Q} \\
\to (\mathbb{Z}G \circ \mathbb{Z}G) \otimes \mathbb{Q} \to \mathbb{Z}G \otimes \mathbb{Q} \to 0,
\end{aligned}$$

which remains exact upon trivializing the G-action:

$$\begin{aligned}
0 \to ((M \wedge M) \otimes \mathbb{Q})_G \to ((P \wedge P) \otimes \mathbb{Q})_G \to ((P \otimes IG) \otimes \mathbb{Q})_G \\
\to ((\mathbb{Z}G \circ \mathbb{Z}G) \otimes \mathbb{Q})_G \to (\mathbb{Z}G \otimes \mathbb{Q})_G \to 0.
\end{aligned}$$

This exact sequence yields

$$\begin{aligned}
\dim((M \wedge M) \otimes \mathbb{Q})_G = \dim((P \wedge P) \otimes \mathbb{Q})_G - \dim((P \otimes IG) \otimes \mathbb{Q})_G \\
+ \dim((\mathbb{Z}G \circ \mathbb{Z}G) \otimes \mathbb{Q})_G - \dim(\mathbb{Z}G \otimes \mathbb{Q})_G. \qquad (8.1)
\end{aligned}$$

In view of Corollary 2.2, the free part of $(P \wedge P)_G$ is a vector space with basis $e_i \wedge_* e_j g$ with $1 \leqslant i < j \leqslant n$ and $g \in G$. Hence

$$\dim((P \wedge P) \otimes \mathbb{Q})_G = \binom{n}{2} 2^n.$$

The tensor product $(P \otimes IG)_G$ is free on $e_i \otimes_* (g-1)$ with $1 \leqslant i \leqslant n$ and $g \in G \backslash \{1\}$, so

$$\dim((P \otimes IG) \otimes \mathbb{Q})_G = n(2^n - 1),$$

$(\mathbb{Z}G \circ \mathbb{Z}G)_G$ is free abelian on $1 \circ_* g$ for $g \in G$, so

$$\dim((\mathbb{Z}G \circ \mathbb{Z}G) \otimes \mathbb{Q})_G = 2^n$$

and, finally, $(\mathbb{Z}G \otimes \mathbb{Q}) \cong \mathbb{Q}G$, so

$$\dim(\mathbb{Z}G \otimes \mathbb{Q})_G = 1.$$

Substituting into (8.1) we get

$$\begin{aligned} \dim((M \wedge M) \otimes \mathbb{Q})_G &= \binom{n}{2} 2^n - n(2^n - 1) + 2^n - 1 \\ &= \binom{n-1}{2} 2^n + (n-1). \end{aligned}$$

Since this number is equal to the rank of the free abelian part of $(M \wedge M)_G$, we can now state the following.

Lemma 8.2. *The rank of the free abelian part of the tensor product* $(M \wedge M)_G$ *is* $\binom{n-1}{2} 2^n + (n-1)$.

Lemmas 8.1 and 8.2 together imply the final result of this section:

Proposition 8.1. *The elements of types* (i), (iii), (iv.a) *and* (v) *in the generating set for* $(M \wedge M)_G$ *in Proposition* 4.1 *freely generate a free abelian group of rank* $\binom{n-1}{2} 2^n + (n-1)$.

Let us summarize what we have got so far. Modulo T_2, the tensor product $(M \wedge M)_G$ is a direct sum of a free abelian group, freely generated by the elements specified in Proposition 8.1, and an elementary abelian 2-group, say T_1, generated by the elements specified in Lemma 7.1. Moreover, by Lemma 7.2 some of the generators of T_1 have actually order 2 in $(M \wedge M)_G$ itself (rather than modulo T_2). By Lemma 6.1, the kernel T_2 is also an elementary abelian 2-group. In other words, $(M \wedge M)_G$ is a direct sum of a free abelian group and a torsion subgroup T, and the latter is an extension of T_2 by T_1. In the next two sections we work out the nature of this extension. This requires a closer examination of the kernel T_2. Note that all the generators of T have non-zero images in T_1 so none of them belongs to T_2. If $\alpha \in T$ we can only have that $2\alpha = 0$ or $0 \neq 2\alpha \in T_2$. In the latter case $2(2\alpha) = 0$ which implies that α has order 4 in T. Thus the rank of T_2 is the number of the linearly independent generators of T of order 4.

9. The homomorphism $H_3(G, \mathbb{Z}G \circ \mathbb{Z}G) \to H_4(G, \mathbb{Z}_2)$

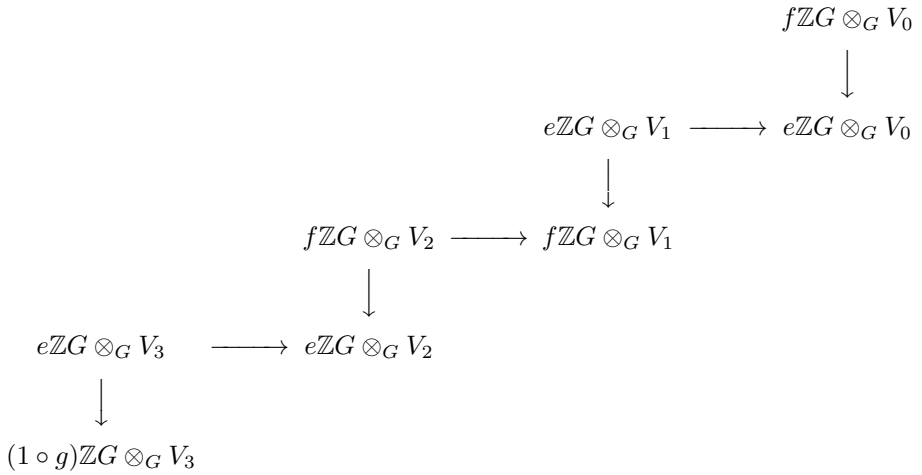
By Lemma 6.1, T_2 is a homomorphic image of the homology group $H_4(\mathbb{Z}_2)$, more precisely, it is isomorphic to the quotient of $H_4(\mathbb{Z}_2)$ by the image of $H_3(\mathbb{Z}G \circ \mathbb{Z}G)$. In this section we work out that image. The discussion in Section 6 shows that this image can be obtained as follows. First we need to work out the image of the homomorphism

$$H_3(\mathbb{Z}G \circ \mathbb{Z}G) \to H_3(\tilde{I}G) \tag{9.1}$$

in (6.6) that is induced by the epimorphism $\mathbb{Z}G \circ \mathbb{Z}G \to \tilde{I}G$ in (6.3), and then we need to apply the inverse of the isomorphism

$$H_4(\mathbb{Z}_2) \to H_3(\tilde{I}G), \tag{9.2}$$

which is a connecting homomorphism stemming from the short exact sequence (6.4), to the result. The group $H_3(\mathbb{Z}G \circ \mathbb{Z}G)$ is direct sum of the groups $H_3((1 \circ g)\mathbb{Z}G)$ where $g \in G \setminus \{1\}$. Each of those is isomorphic to $\mathbb{Z}_2$ and generated by the cycle $f \otimes 1$ in the complex $\mathbf{P}_g \otimes_G \mathbb{Z}$. As mentioned at the end of Section 5, in order to compute the homomorphism (9.1), we need to find a cycle that generates this group in $(1 \circ g)\mathbb{Z}G \otimes_G \mathbf{V}$. This can be done via the double complex $\mathbf{P}_g \otimes_G \mathbf{V}$. We start with the cycle $f \otimes 1 \in f\mathbb{Z}G \otimes_G V_0$, and then alternating with applying differentials and taking inverse images along the following staircase diagram until we get a cycle in $(1 \circ g)\mathbb{Z}G \otimes V_3$.



It is plainly sufficient to perform this calculation for group elements of the form $g = y_1 y_2 \cdots y_r$ with $1 \leqslant r \leqslant n$. The result for an arbitrary group element $g = y_{i_1} y_{i_2} \cdots y_{i_r}$ with $1 \leqslant i_1 < \cdots < i_r \leqslant n$ can be obtained from the former by a suitable substitution. Note that we write $v_{(1,\cdot)}$ for $v_{(1,0,\ldots,0)} \in V_1$ and similarly for any other element in the modules $V_1, V_2, \ldots$, the dot will imply that the remaining entries in the index vector of v are zero.

So, let $g = y_1 y_2 \cdots y_r$ with $1 \leqslant r \leqslant n$ and consider the cycle $f \otimes 1$ in $f\mathbb{Z}G \otimes_G V_0$. This element is mapped to $e(g-1) \otimes 1$, and by using the identity

$$y_1 y_2 \dots y_r - 1 = (y_1 - 1)y_2 \dots y_r + (y_2 - 1)y_3 \dots y_r + \cdots + (y_r - 1),$$

this can be rewritten as

$$e(y_1 y_2 \dots y_r - 1) \otimes_G 1 = \sum_{i=1}^{r} e y_{i+1} \dots y_r \otimes_G (y_i - 1).$$

An inverse image of this element in $e\mathbb{Z}G \otimes_G V_1$ is

$$\sum_{i=1}^{r} e y_{i+1} \dots y_r \otimes_G v_{(0,\dots,0,1_i,\cdot)}$$

where 1_i indicates that 1 occurs in the ith position of the subscript vector. This element is mapped to the element

$$\begin{aligned}
&\sum_{i=1}^{r} f(y_1 \dots y_r + 1) y_{i+1} \dots y_r \otimes_G v_{(0,\dots,0,1_i,\cdot)} \\
&= \sum_{i=1}^{r} f(y_1 \dots y_i + y_{i+1} \dots y_r) \otimes_G v_{(0,\dots,0,1_i,\cdot)} \\
&= f \sum_{i=1}^{r} \Big(- \sum_{s=1}^{i-1} (y_s - 1) y_1 \dots y_{s-1} y_{i+1} \dots y_r \\
&\quad + (y_i + 1) y_1 \dots y_{i-1} \\
&\quad + \sum_{s=i+1}^{r} (y_s - 1) y_1 \dots y_{i-1} y_{s+1} \dots y_r \Big) \otimes_G v_{(0,\dots,0,1_i,\cdot)} \\
&= \sum_{i=1}^{r} f y_1 \dots y_{i-1} \otimes_G v_{(0,\dots,0,1_i,\cdot)} (y_i + 1) \\
&\quad - \sum_{1 \leqslant i < j \leqslant r} f y_1 \dots y_{i-1} y_{j+1} \dots y_r \otimes_G \\
&\quad [v_{(0,\dots,0,1_j,\cdot)} (y_i - 1) - v_{(0,\dots,0,1_i,\cdot)} (y_j - 1)]
\end{aligned}$$

in $f\mathbb{Z}G \otimes_G V_1$. An inverse image of this element in $f\mathbb{Z}G \otimes_G V_2$ is

$$\sum_{i=1}^{r} f y_1 \dots y_{i-1} \otimes_G v_{(0,\dots,0,2_i,\cdot)} - \sum_{1 \leqslant i < j \leqslant r} f y_1 \dots y_{i-1} y_{j+1} \dots y_r \otimes_G v_{(0,\dots,0,1_i,0,\dots,0,1_j,\cdot)}.$$

This element is mapped to the following element in $e\mathbb{Z}G \otimes_G V_2$

$$\begin{aligned}
&\sum_{i=1}^{r} e(y_1 \dots y_r - 1)y_1 \dots y_{i-1} \otimes_G v_{(0,\dots,0,2_i,\cdot)} \\
&\quad - \sum_{1\leqslant i<j\leqslant r} e(y_1 \dots y_r - 1)y_1 \dots y_{i-1}y_{j+1} \dots y_r \otimes_G v_{(0,\dots,0,1_i,0,\dots,0,1_j,\cdot)} \\
&= \sum_{i=1}^{r} e(y_i \dots y_r - y_1 \dots y_{i-1}) \otimes_G v_{(0,\dots,0,2_i,\cdot)} \\
&\quad - \sum_{1\leqslant i<j\leqslant r} e(y_i \dots y_j - y_1 \dots y_{i-1}y_{j+1} \dots y_r) \otimes_G v_{(0,\dots,0,1_i,0,\dots,0,1_j,\cdot)} \\
&= e\sum_{i=1}^{r} \Big(\sum_{s=1}^{i-1} (y_s - 1)y_s \dots y_{i-1} + (y_i - 1)y_{i+1} \dots y_r \\
&\quad + \sum_{s=i+1}^{r} (y_s - 1)y_{s+1} \dots y_r \Big) \otimes_G v_{(0,\dots,0,2_i,\cdot)} \\
&\quad + e \sum_{1\leqslant i<j\leqslant r} \Big(- \sum_{s=1}^{i-1} (y_s - 1)y_s \dots y_{i-1}y_{j+1} \dots y_r + (y_i + 1)y_{j+1} \dots y_r \\
&\quad + \sum_{s=i+1}^{j-1} (y_s - 1)y_i \dots y_{s-1}y_{j+1} \dots y_r - (y_j + 1)y_i \dots y_{j-1} \\
&\quad - \sum_{s=j+1}^{r} (y_s - 1)y_i \dots y_{j-1}y_{s+1} \dots y_r \Big) \otimes_G v_{(0,\dots,0,1_i,0,\dots,0,1_j,\cdot)} \\
&= \sum_{i=1}^{r} ey_{i+1} \dots y_r \otimes_G v_{(0,\dots,0,2_i,\cdot)}(y_i - 1) \\
&\quad + \sum_{1\leqslant i<j\leqslant r} ey_{j+1} \dots y_r \otimes_G [v_{(0,\dots,0,1_i,0,\dots,0,1_j,\cdot)}(y_i + 1) + v_{(0,\dots,0,2_i,\cdot)}(y_j - 1)] \\
&\quad + \sum_{1\leqslant i<j\leqslant r} ey_i \dots y_{j-1} \otimes_G [v_{(0,\dots,0,2_j,\cdot)}(y_i - 1) - v_{(0,\dots,0,1_i,0,\dots,0,1_j,\cdot)}(y_j + 1)] \\
&\quad - \sum_{1\leqslant i<j\leqslant r} ey_i \dots y_{j-1}y_{k+1} \dots y_r \otimes_G [v_{(0,\dots,0,1_j,0,\dots,0,1_k,\cdot)}(y_i - 1) \\
&\quad - v_{(0,\dots,0,1_i,0,\dots,0,1_k,\cdot)}(y_j - 1) + v_{(0,\dots,0,1_i,0,\dots,0,1_j,\cdot)}(y_k - 1)].
\end{aligned}$$

An inverse image of this element in $e\mathbb{Z}G \otimes_G V_3$ is

$$\sum_{i=1}^{r} ey_{i+1} \dots y_r \otimes_G v_{(0,\dots,0,3_i,\cdot)}$$

$$
\begin{aligned}
&+ \sum_{1\leqslant i<j\leqslant r} ey_{j+1}\dots y_r \otimes_G v_{(0,\dots,0,2_i,0,\dots,0,1_j,\cdot)}\\
&+ \sum_{1\leqslant i<j\leqslant r} ey_i\dots y_{j-1} \otimes_G v_{(0,\dots,0,1_i,0,\dots,0,2_j,\cdot)}\\
&- \sum_{1\leqslant i<j\leqslant r} ey_i\dots y_{j-1}y_{k+1}\dots y_r \otimes_G v_{(0,\dots,0,1_i,0,\dots,0,1_j,0,\dots,0,1_k,\cdot)}.
\end{aligned}
$$

Finally, this element is mapped to the following element in $(1\circ g)\mathbb{Z}G\otimes_G V_3$

$$
\begin{aligned}
&\sum_{i=1}^{r}(1\circ y_1\dots y_r)y_{i+1}\dots y_r \otimes_G v_{(0,\dots,0,3_i,\cdot)}\\
&+ \sum_{1\leqslant i<j\leqslant r} (1\circ y_1\dots y_r)y_{j+1}\dots y_r \otimes_G v_{(0,\dots,0,2_i,0,\dots,0,1_j,\cdot)}\\
&+ \sum_{1\leqslant i<j\leqslant r} (1\circ y_1\dots y_r)y_i\dots y_{j-1} \otimes_G v_{(0,\dots,0,1_i,0,\dots,0,2_j,\cdot)}\\
&- \sum_{1\leqslant i<j\leqslant r} (1\circ y_1\dots y_r)y_i\dots y_{j-1}y_{k+1}\dots y_r \otimes_G v_{(0,\dots,0,1_i,0,\dots,0,1_j,0,\dots,0,1_k\cdot)}\\
=&\sum_{i=1}^{r} y_{i+1}y_{i+2}\dots y_r\circ y_1y_2\dots y_i \otimes_G v_{(0,\dots,0,3_i,\cdot)}\\
&+ \sum_{1\leqslant i<j\leqslant r} y_{j+1}y_{j+2}\dots y_r\circ y_1y_2\dots y_j \otimes_G v_{(0,\dots,0,2_i,0\dots,0,1_j,\cdot)}\\
&+ \sum_{1\leqslant i<j\leqslant r} y_iy_{i+1}\dots y_{j-1}\circ y_1y_2\dots y_{i-1}y_jy_{j+1}\dots y_r \otimes_G\\
&\qquad v_{(0,\dots,0,1_i,0\dots,0,2_j,\cdot)}\\
&- \sum_{1\leqslant i<j<k\leqslant r} y_iy_{i+1}\dots y_{j-1}y_{k+1}y_{k+2}\dots y_r\circ y_1y_2\dots y_{i-1}y_jy_{j+1}\dots y_k \otimes_G\\
&\qquad v_{(0,\dots,0,1_i,0\dots,0,1_j0,\dots,0,1_k,\cdot)}.
\end{aligned}
$$

Lemma 9.1. *The cyclic group $H_3((1\circ g)\mathbb{Z}G)$ with $g = y_1y_2\cdots y_r$ with $1\leqslant r\leqslant n$ is, as a homology group in the complex $(1\circ g)\mathbb{Z}G\otimes_G \mathbf{V}$, generated by the cycle*

$$
\begin{aligned}
&\sum_{i=1}^{r} y_{i+1}y_{i+2}\dots y_r\circ y_1y_2\dots y_i \otimes_G v_{(0,\dots,0,3_i,\cdot)}\\
&+ \sum_{1\leqslant i<j\leqslant r} y_{j+1}y_{j+2}\dots y_r\circ y_1y_2\dots y_j \otimes_G v_{(0,\dots,0,2_i,0\dots,0,1_j,\cdot)}\\
&+ \sum_{1\leqslant i<j\leqslant r} y_iy_{i+1}\dots y_{j-1}\circ y_1y_2\dots y_{i-1}y_jy_{j+1}\dots y_r \otimes_G\\
&\qquad v_{(0,\dots,0,1_i,0\dots,0,2_j,\cdot)}
\end{aligned}
$$

$$- \sum_{1\leqslant i<j<k\leqslant r} y_i y_{i+1}\dots y_{j-1}y_{k+1}y_{k+2}\dots y_r \circ y_1 y_2 \dots y_{i-1}y_j y_{j+1}\dots y_k \otimes_G v_{(0,\dots,0,1_i,0\dots,0,1_j 0,\dots,0,1_k,\cdot)}. \tag{9.3}$$

Next we calculate the image of (9.3) under the homomorphism (9.1). We get the following cycle in $\tilde{I}G \otimes_G V_3$

$$\begin{aligned}
&\sum_{i=1}^{r}(y_1y_2\dots y_i + y_{i+1}y_{i+2}\dots y_r)\otimes_G v_{(0,\dots,0,3_i,\cdot)}\\
&+\sum_{1\leqslant i<j\leqslant r}(y_1y_2\dots y_j + y_{j+1}y_{j+2}\dots y_r)\otimes_G v_{(0,\dots,0,2_i,0\dots,0,1_j,\cdot)}\\
&+\sum_{1\leqslant i<j\leqslant r}(y_1y_2\dots y_{i-1}y_jy_{j+1}\dots y_r + y_iy_{i+1}\dots y_{j-1})\otimes_G v_{(0,\dots,0,1_i,0\dots,0,2_j,\cdot)}\\
&-\sum_{1\leqslant i<j<k\leqslant r}(y_1y_2\dots y_{i-1}y_jy_{j+1}\dots y_k + y_iy_{i+1}\dots y_{j-1}y_{k+1}y_{k+2}\dots y_r)\otimes_G\\
&\quad v_{(0,\dots,0,1_i,0\dots,0,1_j0,\dots,0,1_k,\cdot)}\\
=&\sum_{1\leqslant i<j<k\leqslant r}\Bigg(\Big(\sum_{s=1}^{i-1}(y_s-1)y_{s+1}\dots y_r + (y_i+1)y_{i+1}\dots y_r\\
&\qquad+\sum_{s=i+1}^{r}(y_s-1)y_1\dots y_s\Big)\otimes_G v_{(0,\dots,0,3_i,\cdot)}\\
&+\Big(\sum_{s=1}^{i-1}(y_s-1)y_{s+1}\dots y_{i-1}y_{i+1}\dots y_j - (y_i-1)y_1\dots y_j\\
&\qquad-\sum_{s=i+1}^{j-1}(y_s-1)y_{i+1}\dots y_{s-1}y_{j+1}\dots y_r\\
&\qquad+(y_j+1)y_{i+1}\dots y_{j-1}+\sum_{s=j+1}^{r}(y_s-1)y_{i+1}\dots y_{j-1}y_{s+1}\dots y_r\Big)\otimes_G\\
&\quad v_{(0,\dots,0,2_i,0\dots,0,1_j,\cdot)}\\
&+\Big(\sum_{s=1}^{i-1}(y_s-1)y_{s+1}\dots y_{i-1}y_j\dots y_r + (y_i+1)y_{i+1}\dots y_{j-1}\\
&\qquad-\sum_{s=i+1}^{j-1}(y_s-1)y_{i+1}\dots y_{s-1}y_j\dots y_r\\
&\qquad-(y_j-1)y_{i+1}\dots y_r-\sum_{s=j+1}^{r}(y_s-1)y_{i+1}\dots y_{j-1}y_{j+1}\dots y_s\Big)\otimes_G\\
&\quad v_{(0,\dots,0,1_i,0\dots,0,2_j,\cdot)}
\end{aligned}$$

$$
\begin{aligned}
&+\Big(-\sum_{s=1}^{i-1}(y_s-1)y_{s+1}\dots y_{i-1}y_j\dots y_k-(y_i+1)y_{i+1}\dots y_{j-1}y_{k+1}\dots y_r\\
&+\sum_{s=i+1}^{j-1}(y_s-1)y_{i+1}\dots y_{s-1}y_j\dots y_k-(y_j+1)y_{i+1}\dots y_{j-1}y_{j+1}\dots y_k\\
&-\sum_{s=j+1}^{k-1}(y_s-1)y_{i+1}\dots y_{j-1}y_s\dots y_k+(y_k+1)y_{i+1}\dots y_{j-1}y_k\dots y_r\\
&+\sum_{s=k+1}^{r}(y_s-1)y_{i+1}\dots y_{j-1}y_k\dots y_s\Big)\otimes_G v_{(0,\dots,0,1_i,0\dots,0,1_j0,\dots,0,1_k,\cdot)}\Big)\\
=&\sum_{a=1}^{r}(y_a+1)\otimes_G v_{(0,\dots,0,3_a,\cdot)}y_{a+1}\dots y_r\\
&-\sum_{1\leqslant a<b\leqslant r}\big[(y_a-1)\otimes_G v_{(0,\dots,0,2_a,0\dots,0,1_b,\cdot)}-(y_b-1)\otimes_G v_{(0,\dots,0,3_a,\cdot)}\big]y_1\dots y_b\\
&+\sum_{1\leqslant a<b\leqslant r}\big[(y_a+1)\otimes_G v_{(0,\dots,0,1_a,0\dots,0,2_b,\cdot)}\\
&\qquad+(y_b+1)\otimes_G v_{(0,\dots,0,2_a,0\dots,0,1_b,\cdot)}\big]y_{a+1}\dots y_{b-1}\\
&+\sum_{1\leqslant a<b\leqslant r}\big[(y_a-1)\otimes_G v_{(0,\dots,0,3_b,\cdot)}\\
&\qquad-(y_b-1)\otimes_G v_{(0,\dots,0,1_a,0\dots,0,2_b,\cdot)}\big]y_{a+1}\dots y_r\\
&-\sum_{1\leqslant a<b<c\leqslant r}\big[(y_a+1)\otimes_G v_{(0,\dots,0,1_a,0\dots,0,1_b,0\dots,0,1_c,\cdot)}\\
&\qquad+(y_b-1)\otimes_G v_{(0,\dots,0,2_a,0\dots,0,1_c,\cdot)}\\
&\quad-(y_c-1)\otimes_G v_{(0,\dots,0,2_a,0,\dots,0,1_b,\cdot)}\big]y_{a+1}\dots y_{b-1}y_{c+1}\dots y_r\\
&+\sum_{1\leqslant a<b<c\leqslant r}\big[(y_a-1)\otimes_G v_{(0,\dots,0,2_b,0\dots,1_c,\cdot)}\\
&\qquad-(y_b+1)\otimes_G v_{(0,\dots,0,1_a,0\dots,0,1_b,0\dots,0,1_c,\cdot)}\\
&\qquad-(y_c-1)\otimes_G v_{(0,\dots,0,1_a,0,\dots,0,2_b,\cdot)}\big]y_{a+1}\dots y_{b-1}y_{b+1}\dots y_c\\
&+\sum_{1\leqslant a<b<c\leqslant r}\big[(y_a-1)\otimes_G v_{(0,\dots,0,1_b,0\dots,0,2_c,\cdot)}-(y_b-1)\otimes_G v_{(0,\dots,0,1_a,0\dots,0,2_c,\cdot)}\\
&\qquad+(y_c+1)\otimes_G v_{(0,\dots,0,1_a,0,\dots,0,1_b,0,\dots,0,1_c,\cdot)}\big]y_{a+1}\dots y_{b-1}y_c\dots y_r
\end{aligned}
$$

$$
\begin{aligned}
-\sum_{1\leqslant a<b<c<d\leqslant r} \big[&(y_a-1)\otimes_G v_{(0,\dots,0,1_b,0\dots,0,1_c,0\dots,0,1_d,\cdot)}\\
&-(y_b-1)\otimes_G v_{(0,\dots,0,1_a,0\dots,0,1_c,0\dots,0,1_d,\cdot)}\\
&+(y_c-1)\otimes_G v_{(0,\dots,0,1_a,0,\dots,0,1_b,0,\dots,0,1_d,\cdot)}\\
&-(y_d-1)\otimes_G v_{(0,\dots,0,1_a,0,\dots,0,1_b,0,\dots,0,1_c,\cdot)}\big] y_{a+1}\cdots y_{b-1}y_c\cdots y_d.
\end{aligned}
$$

Now we apply the inverse to the isomorphism (9.2) to this element. In order to compute the images of the generators of $H_4(\mathbb{Z}_2)$ under this isomorphism we use the diagram

$$
\begin{array}{ccc}
\mathbb{Z}_2\otimes_G V_4 & & \\
\uparrow & & \\
\mathbb{Z}G\otimes_G V_4 & \longrightarrow & \mathbb{Z}G\otimes_G V_3\\
 & & \uparrow\\
 & & \tilde{I}G\otimes_G V_3
\end{array}
$$

which is part of the exact sequence of complexes obtained by tensoring the short exact sequence (6.4) with the resolution $\mathbf{V}$. The image of a generator of $H_4(\mathbb{Z}_2)$ is obtained by taking inverse images and applying differentials as indicated in the diagram. For the five principal types of generators of $H_4(\mathbb{Z}_2)$ this works as follows:

$$
\begin{aligned}
1\otimes_G v_{(4,\cdot)} &\mapsto (y_1+1)\otimes_G v_{(3,\cdot)}\\
1\otimes_G v_{(3,1,\cdot)} &\mapsto (y_1-1)\otimes_G v_{(2,1,\cdot)}-(y_2-1)\otimes_G v_{(3,\cdot)}\\
1\otimes_G v_{(2,2,\cdot)} &\mapsto (y_1+1)\otimes_G v_{(1,2,\cdot)}+(y_2+1)\otimes_G v_{(2,1,\cdot)}\\
1\otimes_G v_{(2,1,1,\cdot)} &\mapsto (y_1+1)\otimes_G v_{(1,1,1,\cdot)}+(y_2-1)\otimes_G v_{(2,0,1,\cdot)}\\
&\quad -(y_3-1)\otimes_G v_{(2,1,\cdot)}\\
1\otimes_G v_{(1,1,1,1,\cdot)} &\mapsto (y_1-1)\otimes_G v_{(0,1,1,1,\cdot)}-(y_2-1)\otimes_G v_{(1,0,1,1,\cdot)}\\
&\quad +(y_3-1)\otimes_G v_{(1,1,0,1,\cdot)}-(y_4-1)\otimes_G v_{(1,1,1,\cdot)}
\end{aligned}
$$

Now that we have the images of these five elements we can obtain the image of a general cycle of the form $1\otimes_G v_{(q_1,\dots,q_n)}\in\mathbb{Z}_2\otimes V_4$ by a suitable substitution. In order to apply the inverse isomorphism of (9.2) to the image of the element (9.3), note that this element is a sum of cycles in $\tilde{I}G\otimes_G V_3$, and for each of them the image in $\mathbb{Z}_2\otimes_G V_4$ can easily be read of using the calculation above. The image of the element (9.3) in $\mathbb{Z}_2\otimes V_4$ is the cycle

$$
\sum_{q\in I_r} 1\otimes_G v_q \tag{9.4}
$$

where I_r denotes the set of all subscript vectors $q = (q_1, q_2, \ldots, q_n)$ with $q_1 + \cdots + q_r = 4$ and $q_{r+1} = \cdots = q_n = 0$. This allows us to describe the image of the homomorphism $H_3(G, \mathbb{Z}G \circ \mathbb{Z}G) \to H_4(G, \mathbb{Z}_2)$ as follows.

Lemma 9.2. *The image of the homomorphism $H_3(G, \mathbb{Z}G \circ \mathbb{Z}G) \to H_4(G, \mathbb{Z}_2)$ is generated by the elements*

(i) $1 \otimes_G v_{(4,\cdot)}$
(ii) $1 \otimes_G v_{(3,1,\cdot)} + 1 \otimes_G v_{(2,2,\cdot)} + 1 \otimes_G v_{(1,3,\cdot)}$
(iii) $1 \otimes_G v_{(2,1,1,\cdot)} + 1 \otimes_G v_{(1,2,1,\cdot)} + 1 \otimes_G v_{(1,1,2,\cdot)}$
(iv) $1 \otimes_G v_{(1,1,1,1,\cdot)}$

and all elements that can be obtained from them by a place permutation of the subscript vectors. In particular, the rank of this image is $\binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \binom{n}{4}$.

Lemma 9.3. *T_2 is an elementary abelian 2-group of rank $2\binom{n}{2} + 2\binom{n}{3}$.*

Proof. By Lemma 6.1, T_2 is isomorphic to the quotient of $H_4(G, \mathbb{Z}_2)$ by the image of the homomorphism $H_3(G, \mathbb{Z}G \circ \mathbb{Z}G) \to H_4(G, \mathbb{Z}_2)$. By Lemma 5.1, $H_4(G, \mathbb{Z}_2)$ is an elementary abelian 2-group of rank

$$\binom{n+3}{4} = \binom{n}{1} + 3\binom{n}{2} + 3\binom{n}{3} + \binom{n}{4}.$$

Combining this with Lemma 9.2, we get the result. □

10. The homomorphism $H_4(G, \mathbb{Z}_2) \to (M \wedge M)_G$

In order to get T_2, it remains to work out the image of the homology group $H_4(\mathbb{Z}_2)$ in $(M \wedge M)_G$. In fact, it is sufficient to find the images of the elements $1 \otimes_G v_{(3,1,\cdot)}$ and $1 \otimes_G v_{(2,1,1,\cdot)}$ since, by Lemma 9.2, these elements and all elements obtained from them by a place permutation of the multidegree vectors, generate $H_4(G, \mathbb{Z}_2)$ modulo the kernel of $H_4(G, \mathbb{Z}_2) \to (M \wedge M)_G$. The discussion leading up to Lemma 6.1 shows that the homomorphism $H_4(G, \mathbb{Z}_2) \to (M \wedge M)_G$ is a composite of connecting homomorphisms stemming from the four short exact sequences obtained from breaking up the six-term exact sequence (2.4). To work out this homomorphism, we tensor (2.4) with the resolution $\mathbf{V}$. This results in a double complex, and we can obtain the image in $(M \wedge M)_G$ of an element from $H_4(G, \mathbb{Z}_2)$ by starting with a cycle in $\mathbb{Z}_2 \otimes_G V_4$ (see Lemma 5.1), and then alternating with taking inverse images and applying differentials along the following staircase diagram

$$\begin{array}{ccccc}
\mathbb{Z}_2 \otimes_G V_4 & & & & \\
\uparrow & & & & \\
\mathbb{Z}G \otimes_G V_4 & \longrightarrow & \mathbb{Z}G \otimes_G V_3 & & \\
 & & \uparrow & & \\
 & & \dots & \longrightarrow & P \otimes IG \otimes_G V_1 \\
 & & & & \uparrow \\
 & & & & P \wedge P \otimes_G V_1 \\
 & & & & \uparrow \\
 & & & & (M \wedge M) \otimes_G V_0
\end{array}$$

until we get to $(M \wedge M) \otimes_G V_0$, and then on to $(M \wedge M)_G$. The result of this calculation for the relevant elements is given in the following lemma.

Lemma 10.1. *The connecting homomorphism $H_4(G, \mathbb{Z}_2) \to (M \wedge M)_G$ has the following effect on the elements of $H_4(G, \mathbb{Z}_2)$, represented by cycles in $\mathbb{Z}_2 \otimes V_4$:*

$$\begin{aligned}
1 \otimes_G v_{(3,1,\cdot)} &\mapsto 2x_1^2 \wedge_* [x_1, x_2] \\
1 \otimes_G v_{(2,1,1,\cdot)} &\mapsto 2\big(x_2^2 \wedge_* [x_3, x_1] + [x_2, x_1] \wedge_* [x_2, x_3] \\
&\qquad + x_3^2 \wedge_* [x_2, x_1] + [x_3, x_1] \wedge_* [x_3, x_2]\big)
\end{aligned}$$

Proof. First we deal with $v_{(3,1,\cdot)} \in \mathbb{Z}_2 \otimes_G V_4$. A preimage of $1 \otimes_G v_{(3,1,\cdot)} \in \mathbb{Z}_2 \otimes_G V_4$ in $\mathbb{Z}G \otimes_G V_4$ is $1 \otimes_G v_{(3,1,\cdot)}$. This has the following image in $\mathbb{Z}G \otimes_G V_3$

$$1 \otimes_G [v_{(2,1,\cdot)}(y_1 - 1) - v_{(3,\cdot)}(y_2 - 1)] = (y_1 - 1) \otimes_G v_{(2,1,\cdot)} - (y_2 - 1) \otimes_G v_{(3,\cdot)}.$$

A preimage of $(y_1 - 1) \otimes_G v_{(2,1,\cdot)} - (y_2 - 1) \otimes_G v_{(3,\cdot)}$ in $\mathbb{Z}G \circ \mathbb{Z}G \otimes_G V_3$ is

$$(1 \circ y_1 - 1 \circ 1) \otimes_G v_{(2,1,\cdot)} - (1 \circ y_2 - 1 \circ 1) \otimes_G v_{(3,\cdot)}.$$

This maps to the following element in $\mathbb{Z}G \circ \mathbb{Z}G \otimes_G V_2$

$$\begin{aligned}
&(1 \circ y_1 - 1 \circ 1) \otimes_G [v_{(1,1,\cdot)}(y_1 + 1) + v_{(2,\cdot)}(y_2 - 1)] \\
&\quad - (1 \circ y_2 - 1 \circ 1) \otimes_G v_{(2,\cdot)}(y_1 - 1) \\
&= (y_1 \circ 1 - y_1 \circ y_1 + 1 \circ y_1 - 1 \circ 1) \otimes_G v_{(1,1,\cdot)} \\
&\quad + (y_2 \circ y_1 y_2 - y_2 \circ y_2 - 1 \circ y_1 + 1 \circ 1 - y_1 \circ y_1 y_2 \\
&\quad + y_1 \circ y_1 + 1 \circ y_2 - 1 \circ 1) \otimes_G v_{(2,\cdot)} \\
&= -(y_1 - 1) \circ (y_1 - 1) \otimes_G v_{(1,1,\cdot)}
\end{aligned}$$

$$
\begin{aligned}
&+\Big[(y_2-1)\circ(y_1y_2-1)+y_2\circ 1+1\circ y_1y_2-(y_2-1)\circ(y_2-1)\\
&\quad-1\circ y_2-(y_1-1)\circ(y_1y_2-1)-y_1\circ 1-1\circ y_1y_2+1\circ 1\\
&\quad+(y_1-1)\circ(y_1-1)+1\circ y_1-1\circ 1\Big]\otimes_G v_{(2,\cdot)}\\
&=-(y_1-1)\circ(y_1-1)\otimes_G v_{(1,1,\cdot)}\\
&\quad+\Big[(y_2-1)\circ[(y_1-1)y_2+(y_2-1)]-(y_2-1)\circ(y_2-1)\\
&\quad-(y_1-1)\circ[(y_2-1)y_1+(y_1-1)]+(y_1-1)\circ(y_1-1)\Big]\otimes_G v_{(2,\cdot)}\\
&=-(y_1-1)\circ(y_1-1)\otimes_G v_{(1,1,\cdot)}\\
&\quad-(y_2-1)\circ(y_1-1)\otimes_G v_{(2,\cdot)}y_2+(y_2-1)\circ(y_2-1)\otimes_G v_{(2,\cdot)}\\
&\quad-(y_2-1)\circ(y_2-1)\otimes_G v_{(2,\cdot)}+(y_1-1)\circ(y_2-1)\otimes_G v_{(2,\cdot)}y_1\\
&\quad-(y_1-1)\circ(y_1-1)\otimes_G v_{(2,\cdot)}+(y_1-1)\circ(y_1-1)\otimes_G v_{(2,\cdot)}\\
&=-(y_1-1)\circ(y_1-1)\otimes_G v_{(1,1,\cdot)}-(y_1-1)\circ(y_2-1)\otimes_G v_{(2,\cdot)}y_2\\
&\quad+(y_1-1)\circ(y_2-1)\otimes_G v_{(2,\cdot)}y_1.
\end{aligned}
$$

A preimage of this element in $P\otimes IG\otimes_G V_2$ is

$$
\begin{aligned}
&-e_1\otimes(y_1-1)\otimes_G v_{(1,1,\cdot)}-e_2\otimes(y_1-1)\otimes_G v_{(2,\cdot)}y_2+e_1\otimes(y_2-1)\otimes_G v_{(2,\cdot)}y_1\\
&=-e_1\otimes(y_1-1)\otimes_G v_{(1,1,\cdot)}-e_2y_2\otimes(y_1-1)y_2\otimes_G v_{(2,\cdot)}\\
&\quad+e_1y_1\otimes(y_2-1)y_1\otimes_G v_{(2,\cdot)}.
\end{aligned}
$$

The image of this element in $P\otimes IG\otimes_G V_1$ is

$$
\begin{aligned}
&-e_1\otimes(y_1-1)\otimes_G[v_{(0,1,\cdot)}(y_1-1)-v_{(1,\cdot)}(y_2-1)]\\
&\quad+\Big(-e_2y_2\otimes(y_1-1)y_2+e_1y_1\otimes(y_2-1)y_1\Big)\otimes_G v_{(1,\cdot)}(y_1+1)\\
&=\Big(e_1y_1\otimes(y_1-1)+e_1\otimes(y_1-1)\Big)\otimes_G v_{(0,1,\cdot)}\\
&\quad+\Big(e_1y_2\otimes(y_1-1)y_2-e_1\otimes(y_1-1)+e_2y_1y_2\otimes(y_1-1)y_2\\
&\quad-e_2y_2\otimes(y_1-1)y_2+e_1\otimes(y_2-1)+e_1y_1\otimes(y_2-1)y_1\Big)\otimes_G v_{(1.\cdot)}\\
&=\Big(e_1y_1\otimes(y_1-1)-e_1\otimes(y_1-1)y_1\Big)\otimes_G v_{(0,1,\cdot)}\\
&\quad+\Big(e_1y_2\otimes y_1y_2-e_1y_2\otimes y_2-e_1\otimes y_1+e_1\otimes 1\\
&\quad+e_2y_1y_2\otimes y_1y_2-e_2y_1y_2\otimes y_2-e_2y_2\otimes y_1y_2+e_2y_2\otimes y_2\\
&\quad+e_1\otimes y_2-e_1\otimes 1+e_1y_1\otimes y_1y_2-e_1y_1\otimes y_1\Big)\otimes_G v_{(1,\cdot)}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(e_1y_1 \otimes (y_1-1) - e_1 \otimes (y_1-1)y_1\Big) \otimes_G v_{(0,1,\cdot)} \\
&\quad + \Big(e_1y_2 \otimes (y_2-1)y_2 - e_2y_2 \otimes (y_1-1)y_2 - e_1y_2 \otimes 1 + e_2y_1y_2 \otimes (y_1-1)y_2 \\
&\qquad - e_1y_2 \otimes (y_2-1)y_1y_2 + e_1y_2 \otimes y_1 \\
&\qquad + e_1y_2 \otimes (y_1-1) - e_1 \otimes (y_1-1)y_2 - e_1y_2 \otimes y_1 + e_1y_2 \otimes 1 + e_1 \otimes y_1y_2 \\
&\quad + e_1 \otimes (y_2-1)y_1 - e_2y_1 \otimes (y_1-1) - e_1 \otimes y_1y_2 + e_2y_1 \otimes y_1 - e_2y_1 \otimes 1 \\
&\quad + e_1y_1 \otimes (y_2-1)y_1 - e_2y_1 \otimes (y_1-1)y_1 + e_2y_1 \otimes 1 - e_2y_1 \otimes y_1\Big) \otimes_G v_{(1,\cdot)}.
\end{aligned}
$$

A preimage of this element in $P \wedge P \otimes_G V_1$ is

$$
\begin{aligned}
&e_1y_1 \wedge e_1 \otimes_G v_{(0,1,\cdot)} \\
&+ \big(e_1y_2 \wedge e_2y_2 + e_2y_1y_2 \wedge e_1y_2 + e_1y_2 \wedge e_1 + e_1 \wedge e_2y_1 + e_1y_1 \wedge e_2y_1\big) \otimes_G v_{(1,\cdot)}.
\end{aligned}
$$

The image of this element in $P \wedge P \otimes_G V_0$ is

$$
\begin{aligned}
&e_1y_1 \wedge e_1 \otimes_G (y_2-1) \\
&+ \big(e_1y_2 \wedge e_2y_2 + e_2y_1y_2 \wedge e_1y_2 + e_1y_2 \wedge e_1 + e_1 \wedge e_2y_1 \\
&\qquad + e_1y_1 \wedge e_2y_1\big) \otimes_G (y_1-1) \\
&= \big(e_1y_1y_2 \wedge e_1y_2 - e_1y_1 \wedge e_1 + e_1y_1y_2 \wedge e_2y_1y_2 \\
&\qquad + e_2y_2 \wedge e_1y_1y_2 + e_1y_1y_2 \wedge e_1y_1 + e_1y_1 \wedge e_2 + e_1 \wedge e_2 \\
&\qquad - e_1y_2 \wedge e_2y_2 - e_2y_1y_2 \wedge e_1y_2 - e_1y_2 \wedge e_1 - e_1 \wedge e_2y_1 - e_1y_1 \wedge e_2y_1\big) \otimes_G 1 \\
&= \Big(e_1y_1 \wedge [e_1(y_2-1) - e_2(y_1-1)] - e_1y_1 \wedge e_1y_2 \\
&\qquad + e_1 \wedge [e_1(y_2-1) - e_2(y_1-1)] + e_1 \wedge e_1(y_1+1)y_2 \\
&\qquad - e_1 \wedge e_1y_1y_2 - e_1 \wedge e_1y_2 - e_1y_2 \wedge e_1(y_1+1)y_2 \\
&\qquad - e_2y_1y_2 \wedge e_1(y_1+1)y_2 + e_2y_2 \wedge e_1(y_1+1)y_2 + e_1y_1y_2 \wedge e_1(y_1+1) \\
&\qquad - e_1y_1y_2 \wedge e_1 + e_1y_2 \wedge e_1(y_1+1) - e_1y_2 \wedge e_1y_1 - e_1y_2 \wedge e_1\Big) \otimes_G 1 \\
&= \Big(e_1(y_1+1) \wedge [e_1(y_2-1) - e_2(y_1-1)] \\
&\qquad + [e_1(y_2-1) - e_2(y_1-1)]y_2 \wedge e_1(y_1+1)y_2 \\
&\qquad + e_1(y_1+1)y_2 \wedge e_1(y_1+1)\Big) \otimes_G 1.
\end{aligned}
$$

A preimage of this element in $(M \wedge M) \otimes_G V_0$ is

$$
\big(x_1^2 \wedge [x_1,x_2] + [x_1,x_2] \cdot y_2 \wedge x_1^2 \cdot y_2 + x_1^2 \cdot y_2 \wedge x_1^2\big) \otimes_G 1.
$$

The image of this element in $(M \wedge M)_G$ is

$$\begin{aligned}
&x_1^2 \wedge_* [x_1, x_2] + [x_1, x_2] \cdot y_2 \wedge_* x_1^2 \cdot y_2 + x_1^2 \cdot y_2 \wedge_* x_1^2 \\
&= x_1^2 \wedge_* [x_1, x_2] - x_1^2 \cdot y_2 \wedge_* [x_1, x_2] \cdot y_2 + \big(x_1^2 + [x_1, x_2] \cdot (y_1 + 1)\big) \wedge_* x_1^2 \\
&= -x_1^2 \cdot (y_1 + 1) \wedge_* [x_1, x_2] \\
&= -2x_1^2 \wedge_* [x_1, x_2] \\
&= 2x_1^2 \wedge_* [x_1, x_2]
\end{aligned}$$

using the fact that $x_1^2 \wedge_* [x_1, x_2]$ belongs to the torsion subgroup of $(M \wedge M)_G$ which has exponent dividing 4.

Now we deal with the element $1 \otimes_G v_{(2,1,1,\cdot)} \in \mathbb{Z}_2 \otimes_G V_4$. A preimage of $1 \otimes_G v_{(2,1,1,\cdot)} \in \mathbb{Z}_2 \otimes_G V_4$ in $\mathbb{Z}G \otimes_G V_4$ is $1 \otimes_G v_{(2,1,1,\cdot)}$. This has the following image in $\mathbb{Z}G \otimes_G V_3$

$$\begin{aligned}
&1 \otimes_G [v_{(1,1,1,\cdot)}(y_1 + 1) + v_{(2,0,1,\cdot)}(y_2 - 1) - v_{(2,1,\cdot)}(y_3 - 1)] \\
&= (y_1 + 1) \otimes_G v_{(1,1,1,\cdot)} + (y_2 - 1) \otimes_G v_{(2,0,1,\cdot)} - (y_3 - 1) \otimes_G v_{(2,1,\cdot)}.
\end{aligned}$$

A preimage of $(y_1+1)\otimes_G v_{(1,1,1,\cdot)}+(y_2-1)\otimes_G v_{(2,0,1,\cdot)}-(y_3-1)\otimes_G v_{(2,1,\cdot)}$ in $\mathbb{Z}G\circ\mathbb{Z}G\otimes_G V_3$ is

$$1 \circ y_1 \otimes_G v_{(1,1,1,\cdot)} + (1 \circ y_2 - 1 \circ 1) \otimes_G v_{(2,0,1,\cdot)} - (1 \circ y_3 - 1 \circ 1) \otimes_G v_{(2,1,\cdot)}.$$

This maps to the following element in $\mathbb{Z}G \circ \mathbb{Z}G \otimes_G V_2$

$$\begin{aligned}
&1 \circ y_1 \otimes_G \big[v_{(0,1,1,\cdot)}(y_1 - 1) - v_{(1,0,1,\cdot)}(y_2 - 1) + v_{(1,1,\cdot)}(y_3 - 1)\big] \\
&+ (1 \circ y_2 - 1 \circ 1) \otimes_G \big[v_{(1,0,1,\cdot)}(y_1 + 1) + v_{(2,\cdot)}(y_3 - 1)\big] \\
&- (1 \circ y_3 - 1 \circ 1) \otimes_G \big[v_{(1,1,\cdot)}(y_1 + 1) + v_{(2,\cdot)}(y_2 - 1)\big] \\
&= \big(- y_2 \circ y_1 y_2 + 1 \circ y_1 + y_1 \circ y_1 y_2 - y_1 \circ y_1 + 1 \circ y_2 - 1 \circ 1\big) \otimes_G v_{(1,0,1,\cdot)} \\
&+ \big(y_3 \circ y_1 y_3 - 1 \circ y_1 - y_1 \circ y_1 y_3 + y_1 \circ y_1 - 1 \circ y_3 + 1 \circ 1\big) \otimes_G v_{(1,1,\cdot)} \\
&+ \big(y_3 \circ y_2 y_3 - y_3 \circ y_3 - 1 \circ y_2 + 1 \circ 1 - y_2 \circ y_2 y_3 + y_2 \circ y_2 + 1 \circ y_3 \\
&\qquad - 1 \circ 1\big) \otimes_G v_{(2,\cdot)} \\
&= \Big(- (y_2 - 1) \circ (y_1 y_2 - 1) - 1 \circ y_1 y_2 + (y_1 - 1) \circ (y_1 y_2 - 1) + y_1 \circ 1 \\
&\qquad + 1 \circ y_1 y_2 - 1 \circ 1 - (y_1 - 1) \circ (y_1 - 1) - 1 \circ y_1 + 1 \circ 1\Big) \otimes_G v_{(1,0,1,\cdot)} \\
&+ \Big((y_3 - 1) \circ (y_1 y_3 - 1) + 1 \circ y_1 y_3 - (y_1 - 1) \circ (y_1 y_3 - 1) - y_1 \circ 1 \\
&\qquad - 1 \circ y_1 y_3 + 1 \circ 1 + (y_1 - 1) \circ (y_1 - 1) + 1 \circ y_1 - 1 \circ 1\Big) \otimes_G v_{(1,1,\cdot)} \\
&+ \Big((y_3 - 1) \circ (y_2 y_3 - 1) + y_3 \circ 1 + 1 \circ y_2 y_3
\end{aligned}$$

$$
\begin{aligned}
&\quad - (y_3-1)\circ(y_3-1) - 1\circ y_3 + (y_2-1)\circ(y_2-1)\\
&\quad + 1\circ y_2 - 1\circ 1 - (y_2-1)\circ(y_2y_3-1) - y_2\circ 1 - 1\circ y_2y_3 + 1\circ 1\Big)\otimes_G v_{(2,\cdot)}\\
&= \Big(- (y_2-1)\circ\big[(y_1-1)y_2+(y_2-1)\big] + (y_1-1)\circ\big[(y_2-1)y_1+(y_1-1)\big]\\
&\quad - (y_1-1)\circ(y_1-1)\Big)\otimes_G v_{(1,0,1,\cdot)}\\
&\quad + \Big((y_3-1)\circ\big[(y_1-1)y_3+(y_3-1)\big] - (y_1-1)\circ\big[(y_3-1)y_1+(y_1-1)\big]\\
&\quad + (y_1-1)\circ(y_1-1)\Big)\otimes_G v_{(1,1,\cdot)}\\
&\quad + \Big((y_3-1)\circ\big[(y_2-1)y_3+(y_3-1)\big] - (y_3-1)\circ(y_3-1)\\
&\quad + (y_2-1)\circ(y_2-1) - (y_2-1)\circ\big[(y_3-1)y_2+(y_2-1)\big]\Big)\otimes_G v_{(2,\cdot)}\\
&= (y_1-1)\circ(y_2-1)\otimes_G v_{(1,0,1,\cdot)}y_2 - (y_2-1)\circ(y_2-1)\otimes_G v_{(1,0,1,\cdot)}\\
&\quad - (y_1-1)\circ(y_2-1)\otimes_G v_{(1,0,1,\cdot)}y_1 - (y_1-1)\circ(y_3-1)\otimes_G v_{(1,1,\cdot)}y_3\\
&\quad + (y_3-1)\circ(y_3-1)\otimes_G v_{(1,1,\cdot)} + (y_1-1)\circ(y_3-1)\otimes_G v_{(1,1,\cdot)}y_1\\
&\quad - (y_2-1)\circ(y_3-1)\otimes_G v_{(2,\cdot)}y_3 + (y_2-1)\circ(y_3-1)\otimes_G v_{(2,\cdot)}y_2.
\end{aligned}
$$

A preimage of this element in $P\otimes IG\otimes_G V_2$ is

$$
\begin{aligned}
&\quad e_1\otimes(y_2-1)\otimes_G v_{(1,0,1,\cdot)}y_2 - e_2\otimes(y_2-1)\otimes_G v_{(1,0,1,\cdot)}\\
&\quad - e_2\otimes(y_1-1)\otimes_G v_{(1,0,1,\cdot)}y_1 - e_1\otimes(y_3-1)\otimes_G v_{(1,1,\cdot)}y_3\\
&\quad + e_3\otimes(y_3-1)\otimes_G v_{(1,1,\cdot)} + e_3\otimes(y_1-1)\otimes_G v_{(1,1,\cdot)}y_1\\
&\quad - e_2\otimes(y_3-1)\otimes_G v_{(2,\cdot)}y_3 + e_3\otimes(y_2-1)\otimes_G v_{(2,\cdot)}y_2\\
&= \Big(- e_1y_2\otimes(y_2-1) - e_2\otimes(y_2-1) + e_2y_1\otimes(y_1-1)\Big)\otimes_G v_{(1,0,1,\cdot)}\\
&\quad + \Big(e_1y_3\otimes(y_3-1) + e_3\otimes(y_3-1) - e_3y_1\otimes(y_1-1)\Big)\otimes_G v_{(1,1,\cdot)}\\
&\quad + \Big(e_2y_3\otimes(y_3-1) - e_3y_2\otimes(y_2-1)\Big)\otimes_G v_{(2,\cdot)}.
\end{aligned}
$$

The image of this element in $P\otimes IG\otimes_G V_1$ is

$$
\begin{aligned}
&\Big(- e_1y_2\otimes(y_2-1) - e_2\otimes(y_2-1) + e_2y_1\otimes(y_1-1)\Big)\otimes_G\\
&[v_{(0,0,1,\cdot)}(y_1-1) - v_{(1,\cdot)}(y_3-1)]\\
&+ \Big(e_1y_3\otimes(y_3-1) + e_3\otimes(y_3-1) - e_3y_1\otimes(y_1-1)\Big)\otimes_G\\
&[v_{(0,1,\cdot)}(y_1-1) - v_{(1,\cdot)}(y_2-1)]\\
&+ \Big(e_2y_3\otimes(y_3-1) - e_3y_2\otimes(y_2-1)\Big)\otimes_G v_{(1,\cdot)}(y_1+1)
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(- e_1y_1y_2 \otimes (y_2-1)y_1 - e_2y_1 \otimes (y_2-1)y_1 - e_2 \otimes (y_1-1) \\
&\quad + e_1y_2 \otimes (y_2-1) + e_2 \otimes (y_2-1) - e_2y_1 \otimes (y_1-1)\Big) \otimes_G v_{(0,0,1,\cdot)} \\
&\quad + \Big(e_1y_2y_3 \otimes (y_2-1)y_3 + e_2y_3 \otimes (y_2-1)y_3 - e_2y_1y_3 \otimes (y_1-1)y_3 \\
&\qquad - e_1y_2 \otimes (y_2-1) - e_2 \otimes (y_2-1) + e_2y_1 \otimes (y_1-1) \\
&\qquad - e_1y_2y_3 \otimes (y_3-1)y_2 - e_3y_2 \otimes (y_3-1)y_2 + e_3y_1y_2 \otimes (y_1-1)y_2 \\
&\qquad + e_1y_3 \otimes (y_3-1) + e_3 \otimes (y_3-1) - e_3y_1 \otimes (y_1-1) \\
&\qquad + e_2y_1y_3 \otimes (y_3-1)y_1 - e_3y_1y_2 \otimes (y_2-1)y_1 \\
&\qquad + e_2y_3 \otimes (y_3-1) - e_3y_2 \otimes (y_2-1)\Big) \otimes_G v_{(1,\cdot)} \\
&\quad + \Big(e_1y_1y_3 \otimes (y_3-1)y_1 + e_3y_1 \otimes (y_3-1)y_1 + e_3 \otimes (y_1-1) \\
&\qquad - e_1y_3 \otimes (y_3-1) - e_3 \otimes (y_3-1) + e_3y_1 \otimes (y_1-1)\Big) \otimes_G v_{(0,1,\cdot)} \\
&= \big(- e_1y_1y_2 \otimes y_1y_2 + e_1y_1y_2 \otimes y_1 - e_2y_1 \otimes y_1y_2 + e_2y_1 \otimes y_1 \\
&\qquad - e_2 \otimes y_1 + e_2 \otimes 1 + e_1y_2 \otimes y_2 - e_1y_2 \otimes 1 + e_2 \otimes y_2 \\
&\qquad - e_2 \otimes 1 - e_2y_1 \otimes y_1 + e_2y_1 \otimes 1\big) \otimes_G v_{(0,0,1,\cdot)} \\
&\quad + \big(e_1y_2y_3 \otimes y_2y_3 - e_1y_2y_3 \otimes y_3 + e_2y_3 \otimes y_2y_3 - e_2y_3 \otimes y_3 - e_2y_1y_3 \otimes y_1y_3 \\
&\qquad + e_2y_1y_3 \otimes y_3 - e_1y_2 \otimes y_2 + e_1y_2 \otimes 1 - e_2 \otimes y_2 + e_2 \otimes 1 + e_2y_1 \otimes y_1 \\
&\qquad - e_2y_1 \otimes 1 - e_1y_2y_3 \otimes y_2y_3 + e_1y_2y_3 \otimes y_2 - e_3y_2 \otimes y_2y_3 + e_3y_2 \otimes y_2 \\
&\qquad + e_3y_1y_2 \otimes y_1y_2 - e_3y_1y_2 \otimes y_2 + e_1y_3 \otimes y_3 - e_1y_3 \otimes 1 + e_3 \otimes y_3 - e_3 \otimes 1 \\
&\qquad - e_3y_1 \otimes y_1 + e_3y_1 \otimes 1 + e_2y_1y_3 \otimes y_1y_3 - e_2y_1y_3 \otimes y_1 - e_3y_1y_2 \otimes y_1y_2 \\
&\qquad + e_3y_1y_2 \otimes y_1 + e_2y_3 \otimes y_3 - e_2y_3 \otimes 1 - e_3y_2 \otimes y_2 + e_3y_2 \otimes 1\big) \otimes_G v_{(1,\cdot)} \\
&\quad + \big(e_1y_1y_3 \otimes y_1y_3 - e_1y_1y_3 \otimes y_1 + e_3y_1 \otimes y_1y_3 - e_3y_1 \otimes y_1 + e_3 \otimes y_1 - e_3 \otimes 1 \\
&\qquad - e_1y_3 \otimes y_3 + e_1y_3 \otimes 1 - e_3 \otimes y_3 + e_3 \otimes 1 + e_3y_1 \otimes y_1 \\
&\qquad - e_3y_1 \otimes 1\big) \otimes_G v_{(0,1,\cdot)} \\
&= \Big(e_1y_2 \otimes (y_1-1)y_1y_2 - e_1y_1y_2 \otimes (y_1-1)y_2 + e_1y_2 \otimes y_1y_2 \\
&\qquad - e_1y_1y_2 \otimes y_2 + e_2 \otimes (y_1-1)y_1y_2 - e_1y_1y_2 \otimes (y_2-1) + e_2 \otimes y_1y_2 \\
&\qquad + e_1y_1y_2 \otimes y_2 - e_1y_1y_2 \otimes 1 + e_1y_1y_2 \otimes (y_1-1) - e_1 \otimes (y_1-1)y_1y_2 \\
&\qquad + e_1y_1y_2 \otimes 1 + e_1 \otimes y_2 - e_1 \otimes y_1y_2 + e_1y_2 \otimes (y_2-1)y_1 - e_2y_1 \otimes (y_1-1)y_2 \\
&\qquad - e_1y_2 \otimes y_1y_2 + e_1y_2 \otimes y_1 - e_2y_1 \otimes y_2 + e_2 \otimes (y_2-1)y_1 - e_2y_1 \otimes (y_2-1) \\
&\qquad - e_2 \otimes y_1y_2 + e_2y_1 \otimes y_2 + e_1y_2 \otimes (y_1-1) - e_1 \otimes (y_1-1)y_2 - e_1y_2 \otimes y_1 \\
&\qquad + e_1 \otimes y_1y_2 - e_1 \otimes y_2\Big) \otimes_G v_{(0,0,1,\cdot)}
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(e_3\otimes(y_1-1)y_2y_3-e_1y_2y_3\otimes(y_3-1)-e_3\otimes y_1y_2y_3+e_3\otimes y_2y_3\\
&\quad-e_1y_2y_3\otimes 1+e_1y_2y_3\otimes(y_2-1)-e_2\otimes(y_1-1)y_2y_3+e_1y_2y_3\otimes 1\\
&\quad+e_2\otimes y_1y_2y_3-e_2\otimes y_2y_3+e_2y_1y_3\otimes(y_3-1)-e_3\otimes(y_2-1)y_1y_3\\
&\quad+e_2y_1y_3\otimes 1+e_3\otimes y_1y_2y_3-e_3\otimes y_1y_3+e_1\otimes(y_2-1)y_1y_3\\
&\quad-e_2y_1y_3\otimes(y_1-1)-e_1\otimes y_1y_2y_3+e_1\otimes y_1y_3-e_2y_1y_3\otimes 1\\
&\quad+e_2\otimes(y_3-1)y_1y_2-e_3y_1y_2\otimes(y_2-1)-e_2\otimes y_1y_2y_3+e_2\otimes y_1y_2\\
&\quad-e_3y_1y_2\otimes 1+e_3y_1y_2\otimes(y_1-1)-e_1\otimes(y_3-1)y_1y_2+e_3y_1y_2\otimes 1\\
&\quad+e_1\otimes y_1y_2y_3-e_1\otimes y_1y_2+e_2y_3\otimes(y_3-1)y_2-e_3y_2\otimes(y_2-1)y_3\\
&\quad+e_2y_3\otimes y_2-e_3y_2\otimes y_3+e_2y_3\otimes(y_2-1)-e_2\otimes(y_2-1)y_3-e_2y_3\otimes y_2\\
&\quad+e_2\otimes y_2y_3-e_2\otimes y_3+e_2\otimes(y_1-1)y_2-e_1y_2\otimes(y_2-1)-e_2\otimes y_1y_2\\
&\quad+e_3\otimes(y_3-1)y_2-e_3y_2\otimes(y_3-1)-e_3\otimes y_2y_3+e_3\otimes y_2+e_3y_2\otimes y_3\\
&\quad+e_1y_3\otimes(y_3-1)-e_3\otimes(y_1-1)y_3+e_3\otimes y_1y_3\\
&\quad+e_2y_1\otimes(y_1-1)-e_1\otimes(y_2-1)y_1+e_1\otimes y_1y_2-e_1\otimes y_1\\
&\quad+e_1\otimes(y_3-1)y_1-e_3y_1\otimes(y_1-1)-e_1\otimes y_1y_3+e_1\otimes y_1\\
&\quad+e_3\otimes(y_2-1)-e_2\otimes(y_3-1)-e_3\otimes y_2+e_2\otimes y_3\Big)\otimes_G v_{(1,\cdot)}\\
&+\Big(e_1y_1y_3\otimes(y_1-1)y_3-e_1y_3\otimes(y_1-1)y_1y_3+e_1y_1y_3\otimes y_3-e_1y_3\otimes y_1y_3\\
&\quad+e_1y_1y_3\otimes(y_3-1)-e_3\otimes(y_1-1)y_1y_3-e_1y_1y_3\otimes y_3+e_1y_1y_3\otimes 1\\
&\quad-e_3\otimes y_1y_3+e_1\otimes(y_1-1)y_1y_3-e_1y_1y_3\otimes(y_1-1)-e_1\otimes y_3\\
&\quad+e_1\otimes y_1y_3-e_1y_1y_3\otimes 1+e_3y_1\otimes(y_1-1)y_3-e_1y_3\otimes(y_3-1)y_1\\
&\quad+e_3y_1\otimes y_3+e_1y_3\otimes y_1y_3-e_1y_3\otimes y_1+e_1\otimes(y_1-1)y_3\\
&\quad-e_1y_3\otimes(y_1-1)-e_1\otimes y_1y_3+e_1\otimes y_3+e_1y_3\otimes y_1\\
&\quad+e_3y_1\otimes(y_3-1)-e_3\otimes(y_3-1)y_1-e_3y_1\otimes y_3+e_3\otimes y_1y_3\Big)\otimes_G v_{(0,1,\cdot)}.
\end{aligned}
$$

A preimage of this element in $P\wedge P\otimes_G V_1$ is

$$
\begin{aligned}
&\big(e_1y_2\wedge e_1y_1y_2+e_2\wedge e_1y_1y_2+e_1y_1y_2\wedge e_1+e_1y_2\wedge e_2y_1\\
&\quad+e_2\wedge e_2y_1+e_1y_2\wedge e_1\big)\otimes_G v_{(0,0,1,\cdot)}\\
&+\big(e_3\wedge e_1y_2y_3+e_1y_2y_3\wedge e_2+e_2y_1y_3\wedge e_3+e_1\wedge e_2y_1y_3+e_2\wedge e_3y_1y_2\\
&\quad+e_3y_1y_2\wedge e_1+e_2y_3\wedge e_3y_2+e_2y_3\wedge e_2+e_2\wedge e_1y_2+e_3\wedge e_3y_2\\
&\quad+e_1y_3\wedge e_3+e_2y_1\wedge e_1+e_1\wedge e_3y_1+e_3\wedge e_2\big)\otimes_G v_{(1,\cdot)}\\
&+\big(e_1y_1y_3\wedge e_1y_3+e_1y_1y_3\wedge e_3+e_1\wedge e_1y_1y_3+e_3y_1\wedge e_1y_3\\
&\quad+e_1\wedge e_1y_3+e_3y_1\wedge e_3\big)\otimes_G v_{(0,1,\cdot)}.
\end{aligned}
$$

The image of this element in $P \wedge P \otimes_G V_0$ is

$$
\begin{aligned}
&\big(e_1y_2 \wedge e_1y_1y_2 + e_2 \wedge e_1y_1y_2 + e_1y_1y_2 \wedge e_1 + e_1y_2 \wedge e_2y_1 \\
&\quad + e_2 \wedge e_2y_1 + e_1y_2 \wedge e_1\big) \otimes_G (y_3 - 1) \\
&+ \big(e_3 \wedge e_1y_2y_3 + e_1y_2y_3 \wedge e_2 + e_2y_1y_3 \wedge e_3 + e_1 \wedge e_2y_1y_3 \\
&\quad + e_2 \wedge e_3y_1y_2 + e_3y_1y_2 \wedge e_1 + e_2y_3 \wedge e_3y_2 \\
&\quad + e_2y_3 \wedge e_2 + e_2 \wedge e_1y_2 + e_3 \wedge e_3y_2 + e_1y_3 \wedge e_3 \\
&\quad + e_2y_1 \wedge e_1 + e_1 \wedge e_3y_1 + e_3 \wedge e_2\big) \otimes_G (y_1 - 1) \\
&+ \big(e_1y_1y_3 \wedge e_1y_3 + e_1y_1y_3 \wedge e_3 + e_1 \wedge e_1y_1y_3 + e_3y_1 \wedge e_1y_3 \\
&\quad + e_1 \wedge e_1y_3 + e_3y_1 \wedge e_3\big) \otimes_G (y_2 - 1) \\
= &\big(e_1y_2y_3 \wedge e_1y_1y_2y_3 + e_2y_3 \wedge e_1y_1y_2y_3 + e_1y_1y_2y_3 \wedge e_1y_3 + e_1y_2y_3 \wedge e_2y_1y_3 \\
&\quad + e_2y_3 \wedge e_2y_1y_3 + e_1y_2y_3 \wedge e_1y_3 - e_1y_2 \wedge e_1y_1y_2 - e_2 \wedge e_1y_1y_2 \\
&\quad - e_1y_1y_2 \wedge e_1 - e_1y_2 \wedge e_2y_1 - e_2 \wedge e_2y_1 - e_1y_2 \wedge e_1 + e_3y_1 \wedge e_1y_1y_2y_3 \\
&\quad + e_1y_1y_2y_3 \wedge e_2y_1 + e_2y_3 \wedge e_3y_1 + e_1y_1 \wedge e_2y_3 + e_2y_1 \wedge e_3y_2 \\
&\quad + e_3y_2 \wedge e_1y_1 + e_2y_1y_3 \wedge e_3y_1y_2 + e_2y_1y_3 \wedge e_2y_1 + e_2y_1 \wedge e_1y_1y_2 \\
&\quad + e_3y_1 \wedge e_3y_1y_2 + e_1y_1y_3 \wedge e_3y_1 + e_2 \wedge e_1y_1 + e_1y_1 \wedge e_3 + e_3y_1 \wedge e_2y_1 \\
&\quad - e_3 \wedge e_1y_2y_3 - e_1y_2y_3 \wedge e_2 - e_2y_1y_3 \wedge e_3 - e_1 \wedge e_2y_1y_3 - e_2 \wedge e_3y_1y_2 \\
&\quad - e_3y_1y_2 \wedge e_1 - e_2y_3 \wedge e_3y_2 - e_2y_3 \wedge e_2 - e_2 \wedge e_1y_2 - e_3 \wedge e_3y_2 \\
&\quad - e_1y_3 \wedge e_3 - e_2y_1 \wedge e_1 - e_1 \wedge e_3y_1 - e_3 \wedge e_2 + e_1y_1y_2y_3 \wedge e_1y_2y_3 \\
&\quad + e_1y_1y_2y_3 \wedge e_3y_2 + e_1y_2 \wedge e_1y_1y_2y_3 + e_3y_1y_2 \wedge e_1y_2y_3 + e_1y_2 \wedge e_1y_2y_3 \\
&\quad + e_3y_1y_2 \wedge e_3y_2 - e_1y_1y_3 \wedge e_1y_3 - e_1y_1y_3 \wedge e_3 \\
&\quad - e_1 \wedge e_1y_1y_3 - e_3y_1 \wedge e_1y_3 - e_1 \wedge e_1y_3 - e_3y_1 \wedge e_3\big) \otimes_G 1 \\
= &\Big(e_1y_1y_2 \wedge [e_1(y_2 - 1) - e_2(y_1 - 1)] + e_1y_2 \wedge [e_1(y_2 - 1) - e_2(y_1 - 1)] \\
&\quad - e_1y_1y_3 \wedge [e_1(y_3 - 1) - e_3(y_1 - 1)] - e_1y_3 \wedge [e_1(y_3 - 1) - e_3(y_1 - 1)] \\
&\quad - e_1y_1y_2y_3 \wedge [e_1(y_2 - 1) - e_2(y_1 - 1)] - e_1y_1y_2y_3 \wedge e_1 + e_1y_1y_2y_3 \wedge e_2 \\
&\quad - e_1y_2y_3 \wedge [e_1(y_2 - 1) - e_2(y_1 - 1)] - e_1y_2y_3 \wedge e_1 - e_1y_2y_3 \wedge e_2y_1 \\
&\quad + e_1y_1y_2y_3 \wedge [e_1(y_3 - 1) - e_3(y_1 - 1)] + e_1y_1y_2y_3 \wedge e_1 - e_1y_1y_2y_3 \wedge e_3 \\
&\quad + e_1y_2y_3 \wedge [e_1(y_3 - 1) - e_3(y_1 - 1)] + e_1y_2y_3 \wedge e_1 + e_1y_2y_3 \wedge e_3y_1 \\
&\quad - e_1y_1y_2y_3 \wedge [e_2(y_3 - 1) - e_3(y_2 - 1)] - e_1y_1y_2y_3 \wedge e_2 + e_1y_1y_2y_3 \wedge e_3 \\
&\quad - e_1y_2y_3 \wedge [e_2(y_3 - 1) - e_3(y_2 - 1)] + e_1y_2y_3 \wedge e_2y_3 - e_1y_2y_3 \wedge e_2 \\
&\quad - e_1y_2y_3 \wedge e_3y_2 + e_1y_2y_3 \wedge e_3 + e_1y_2y_3 \wedge [e_2(y_3 - 1) - e_3(y_2 - 1)]y_1 \\
&\quad + e_1y_2y_3 \wedge e_2y_1 - e_1y_2y_3 \wedge e_3y_1 - e_3y_1y_2 \wedge [e_2(y_3 - 1) - e_3(y_2 - 1)]y_1
\end{aligned}
$$

$$
\begin{aligned}
&- e_3y_1y_2 \wedge e_2y_1 - e_1y_2 \wedge [e_2(y_3-1) - e_3(y_2-1)]y_1 + e_1y_2 \wedge e_2y_1y_3 \\
&- e_1y_2 \wedge e_2y_1 - e_1y_2 \wedge e_3y_1y_2 + e_1y_2 \wedge e_3y_1 \\
&+ e_3y_2 \wedge [e_2(y_3-1) - e_3(y_2-1)]y_1 - e_3y_2 \wedge e_2y_1y_3 - e_3y_2 \wedge e_3y_1 \\
&- e_2y_1y_3 \wedge [e_2(y_3-1) - e_3(y_2-1)] - e_2y_1y_3 \wedge e_2 - e_2y_1y_3 \wedge e_3y_2 \\
&+ e_2y_3 \wedge [e_2(y_3-1) - e_3(y_2-1)] - e_2y_3 \wedge e_3 \\
&+ e_1y_2y_3 \wedge [e_2(y_3-1) - e_3(y_2-1)] - e_1y_2y_3 \wedge e_2y_3 + e_1y_2y_3 \wedge e_2 \\
&+ e_1y_2y_3 \wedge e_3y_2 - e_1y_2y_3 \wedge e_3 - e_1y_3 \wedge [e_2(y_3-1) - e_3(y_2-1)] \\
&+ e_1y_3 \wedge e_2y_3 - e_1y_3 \wedge e_2 - e_1y_3 \wedge e_3y_2 + e_1y_3 \wedge e_3 \\
&+ e_3y_1y_2 \wedge [e_1(y_2-1) - e_2(y_1-1)] - e_3y_1y_2 \wedge e_1y_2 + e_3y_1y_2 \wedge e_2y_1 \\
&- e_3y_1 \wedge [e_1(y_2-1) - e_2(y_1-1)] + e_3y_1 \wedge e_1y_2 + e_3y_1 \wedge e_2 \\
&- e_2y_1y_3 \wedge [e_1(y_2-1) - e_2(y_1-1)] + e_2y_1y_3 \wedge e_1y_2 + e_2y_1y_3 \wedge e_2 \\
&+ e_2y_1 \wedge [e_1(y_2-1) - e_2(y_1-1)] - e_2y_1 \wedge e_1y_2 \\
&- e_2y_3 \wedge [e_1(y_3-1) - e_3(y_1-1)] + e_2y_3 \wedge e_1y_3 - e_2y_3 \wedge e_1 + e_2y_3 \wedge e_3 \\
&+ e_2 \wedge [e_1(y_3-1) - e_3(y_1-1)] - e_2 \wedge e_1y_3 + e_2 \wedge e_1 + e_2 \wedge e_3y_1 \\
&+ e_3y_2 \wedge [e_1(y_3-1) - e_3(y_1-1)] - e_3y_2 \wedge e_1y_3 + e_3y_2 \wedge e_1 + e_3y_2 \wedge e_3y_1 \\
&- e_3 \wedge [e_1(y_3-1) - e_3(y_1-1)] + e_3 \wedge e_1y_3 - e_3 \wedge e_1 \\
&+ e_1y_1 \wedge [e_2(y_3-1) - e_3(y_2-1)] + e_1 \wedge [e_2(y_3-1) - e_3(y_2-1)] \\
&- e_1 \wedge e_2y_3 + e_1 \wedge e_2 + e_1 \wedge e_3y_2 - e_1 \wedge e_3\Big) \otimes_G 1
\end{aligned}
$$

$$
\begin{aligned}
= \Big(&e_1(y_1+1)y_2 \wedge [e_1(y_2-1) - e_2(y_1-1)] \\
&- e_1(y_1+1)y_3 \wedge [e_1(y_3-1) - e_3(y_1-1)] \\
&- e_1(y_1+1)y_2y_3 \wedge [e_1(y_2-1) - e_2(y_1-1)] \\
&+ e_1(y_1+1)y_2y_3 \wedge [e_1(y_3-1) - e_3(y_1-1)] \\
&- e_1(y_1+1)y_2y_3 \wedge [e_2(y_3-1) - e_3(y_2-1)] \\
&+ [e_1(y_3-1) - e_3(y_1-1)]y_2 \wedge [e_2(y_3-1) - e_3(y_2-1)]y_1 \\
&+ [e_1(y_2-1) - e_2(y_1-1)]y_3 \wedge [e_2(y_3-1) - e_3(y_2-1)] \\
&- [e_2(y_3-1) - e_3(y_2-1)]y_1 \wedge [e_1(y_2-1) - e_2(y_1-1)] \\
&- [e_2(y_3-1) - e_3(y_2-1)] \wedge [e_1(y_3-1) - e_3(y_1-1)] \\
&+ e_1(y_1+1) \wedge [e_2(y_3-1) - e_3(y_2-1)]\Big) \otimes_G 1.
\end{aligned}
$$

A preimage of this element in $(M \wedge M) \otimes_G V_0$ is

$$\big(x_1^2 \cdot y_2 \wedge [x_1, x_2] - x_1^2 \cdot y_3 \wedge [x_1, x_3] - x_1^2 \cdot y_2y_3 \wedge [x_1, x_2] + x_1^2 \cdot y_2y_3 \wedge [x_1, x_3]$$

$$- x_1^2 \cdot y_2y_3 \wedge [x_2,x_3] + [x_1,x_3] \cdot y_2 \wedge [x_2,x_3] \cdot y_1 + [x_1,x_2] \cdot y_3 \wedge [x_2,x_3]$$
$$- [x_2,x_3] \cdot y_1 \wedge [x_1,x_2] - [x_2,x_3] \wedge [x_1,x_3] + x_1^2 \wedge [x_2,x_3]\big) \otimes_G 1.$$

The image of this element in $(M \wedge M)_G$ is

$$\begin{aligned}
&x_1^2 \cdot y_2 \wedge_* [x_1,x_2] - x_1^2 \cdot y_3 \wedge_* [x_1,x_3] - x_1^2 \cdot y_2y_3 \wedge_* [x_1,x_2] \\
&+ x_1^2 \cdot y_2y_3 \wedge_* [x_1,x_3] - x_1^2 \cdot y_2y_3 \wedge_* [x_2,x_3] + [x_1,x_3] \cdot y_2 \wedge_* [x_2,x_3] \cdot y_1 \\
&+ [x_1,x_2] \cdot y_3 \wedge_* [x_2,x_3] - [x_2,x_3] \cdot y_1 \wedge_* [x_1,x_2] \\
&- [x_2,x_3] \wedge_* [x_1,x_3] + x_1^2 \wedge_* [x_2,x_3] \\
= {} & -x_1^2 \wedge_* [x_2,x_1] \cdot y_2 + x_1^2 \wedge_* [x_3,x_1] \cdot y_3 + x_1^2 \cdot y_3 \wedge_* [x_2,x_1] \cdot y_2 \\
&- x_1^2 \cdot y_2 \wedge_* [x_3,x_1] \cdot y_3 - x_1^2 \wedge_* [x_2,x_3] \cdot y_2y_3 + [x_1,x_3] \cdot y_1 \wedge_* [x_2,x_3] \cdot y_2 \\
&- [x_1,x_2] \wedge_* [x_3,x_2] \cdot y_3 - [x_2,x_3] \wedge_* [x_1,x_2] \cdot y_1 + [x_3,x_1] \wedge_* [x_3,x_2] \\
&+ x_1^2 \wedge_* [x_2,x_3] \\
= {} & x_1^2 \wedge_* [x_2,x_1] - x_1^2 \wedge_* x_2^2 \cdot y_1 + x_1^2 \wedge_* x_2^2 - x_1^2 \wedge_* [x_3,x_1] + x_1^2 \wedge_* x_3^2 \cdot y_1 \\
&- x_1^2 \wedge_* x_3^2 - x_1^2 \cdot y_3 \wedge_* [x_2,x_1] + x_1^2 \cdot y_3 \wedge_* x_2^2 \cdot y_1 - x_1^2 \cdot y_3 \wedge_* x_2^2 \\
&+ x_1^2 \cdot y_2 \wedge_* [x_3,x_1] - x_1^2 \cdot y_2 \wedge_* x_3^2 \cdot y_1 + x_1^2 \cdot y_2 \wedge_* x_3^2 + x_1^2 \wedge_* [x_2,x_3] \cdot y_3 \\
&- x_1^2 \wedge_* x_2^2 + x_1^2 \wedge_* x_2^2 \cdot y_3 - [x_1,x_3] \wedge_* \big(- [x_2,x_3] + x_2^2 \cdot y_3 - x_2^2\big) \\
&+ x_1^2 \cdot y_3 \wedge_* \big(- [x_2,x_3] + x_2^2 \cdot y_3 - x_2^2\big) \\
&- x_1^2 \wedge_* \big(- [x_2,x_3] + x_2^2 \cdot y_3 - x_2^2\big) + [x_1,x_2] \wedge_* [x_3,x_2] - [x_1,x_2] \wedge_* x_3^2 \cdot y_2 \\
&+ [x_1,x_2] \wedge_* x_3^2 + [x_2,x_3] \wedge_* [x_1,x_2] - [x_2,x_3] \wedge_* x_1^2 \cdot y_2 + [x_2,x_3] \wedge_* x_1^2 \\
&+ [x_3,x_1] \wedge_* [x_3,x_2] + x_1^2 \wedge_* [x_2,x_3] \\
= {} & x_1^2 \wedge_* [x_2,x_1] - x_1^2 \wedge_* [x_3,x_1] + x_1^2 \wedge_* [x_1,x_2] \cdot y_3 + x_1^2 \wedge_* [x_3,x_1] \cdot y_2 \\
&- x_1^2 \wedge_* [x_2,x_3] - x_1^2 \wedge_* x_3^2 \cdot y_2 + x_1^2 \wedge_* x_3^2 - x_1^2 \wedge_* x_2^2 + x_1^2 \wedge_* x_2^2 \cdot y_3 \\
&+ [x_1,x_3] \wedge_* [x_2,x_3] - [x_3,x_1] \wedge_* x_2^2 + x_3^2 \cdot y_1 \wedge_* x_2^2 - x_3^2 \wedge_* x_2^2 \\
&+ [x_1,x_3] \wedge_* x_2^2 - x_1^2 \wedge_* [x_3,x_2] + x_1^2 \wedge_* x_3^2 \cdot y_2 - x_1^2 \wedge_* x_3^2 + x_1^2 \wedge_* x_2^2 \\
&- x_1^2 \cdot y_3 \wedge_* x_2^2 + x_1^2 \wedge_* [x_2,x_3] - x_1^2 \wedge_* x_2^2 \cdot y_3 + x_1^2 \wedge_* x_2^2 + [x_1,x_2] \wedge_* [x_3,x_2] \\
&- [x_2,x_1] \wedge_* x_3^2 + x_2^2 \cdot y_1 \wedge_* x_3^2 - x_2^2 \wedge_* x_3^2 + [x_1,x_2] \wedge_* x_3^2 \\
&+ [x_2,x_3] \wedge_* [x_1,x_2] + [x_2,x_3] \wedge_* x_1^2 - x_2^2 \cdot y_3 \wedge_* x_1^2 \\
&+ x_2^2 \wedge_* x_1^2 + [x_3,x_1] \wedge_* [x_3,x_2] \\
= {} & x_1^2 \wedge_* \big([x_3,x_1] \cdot (y_2-1) + [x_1,x_2] \cdot (y_3-1) + [x_2,x_3] \cdot (y_1-1)\big) \\
&+ 2\big(x_2^2 \wedge_* [x_3,x_1] + [x_2,x_1] \wedge_* [x_2,x_3] + x_3^2 \wedge_* [x_2,x_1] + [x_3,x_1] \wedge_* [x_3,x_2]\big) \\
= {} & 2\big(x_2^2 \wedge_* [x_3,x_1] + [x_2,x_1] \wedge_* [x_2,x_3] + x_3^2 \wedge_* [x_2,x_1] + [x_3,x_1] \wedge_* [x_3,x_2]\big). \qquad \square
\end{aligned}$$

Corollary 10.2. *The elements* (ii) *with* $s = 1$ *and* (iv.b) *in Proposition 4.1 have order* 4 *and generate their direct sum in* $(M \wedge M)_G$.

Proof. Let T_4 denote the group generated by those elements. In view of Lemma 10.1, $2T_4$ coincides with the image of $H_4(\mathbb{Z}_2)$ in $(M \wedge M)_G$ (which is T_2). The latter is an elementary abelian 2-group of rank $2\binom{n}{2} + 2\binom{n}{3} = 2\binom{n+1}{3}$ (see Lemma 9.3) which is precisely the number of those generators of T_4. The result follows. □

11. The main result

It remains to put the results of the previous sections together to obtain the main result of this paper.

Theorem. *Let* F *be a free group of rank* n *with free generating set* $X = \{x_1, \dots, x_n\}$ *for* $n > 1$, *and let* $R = F^2$ *be the verbal subgroup of* F *that is generated by all squares. The quotient* $R'/[R', F]$ *is the direct product of the cyclic groups generated by*

(i) *the elements*

$$\big[x_i^2, [x_j^2, x_{i_1}, \dots, x_{i_s}]\big]$$

with $1 \leqslant j < i \leqslant n$, $1 \leqslant i_1 < \cdots < i_s \leqslant n$, $i, j \neq i_1, \dots, i_s$, $0 \leqslant s \leqslant n-2$,

(ii) *the elements*

$$\big[x_i^2, [x_i, x_{i_1}, x_{i_2}, \dots, x_{i_s}]\big]$$

with $1 \leqslant i \leqslant n$, $1 \leqslant i_1 < \cdots < i_s \leqslant n$, $i \neq i_1, \dots, i_s$, $1 \leqslant s \leqslant n-1$,

(iii) *the elements*

$$\big[x_i^2, [x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_s}]\big]$$

with $1 \leqslant i \leqslant n$, $i_1, \dots, i_s \in \{1, \dots, n\} \backslash \{i\}$, $i_1 > i_2 < i_3 \dots < i_s$, $i_1 \neq i_3, \dots, i_s$, $2 \leqslant s \leqslant n-1$,

(iv) *the elements*

(a) $\big[[x_k, x_i], [x_k, x_j]\big]$ *with* $1 \leqslant i < j < k \leqslant n$,

(b) $\big[x_i^2, [x_j, x_k]\big]$ $\big[[x_i, x_k], [x_i, x_j]\big]$ $\big[x_j^2, [x_i, x_k]\big]$ $\big[[x_j, x_k], [x_j, x_i]\big]$ *and*
$\big[x_j^2, [x_k, x_i]\big]$ $\big[[x_j, x_i], [x_j, x_k]\big]$ $\big[x_k^2, [x_j, x_i]\big]$ $\big[[x_k, x_i], [x_k, x_j]\big]$
with $1 \leqslant i < j < k \leqslant n$,

(c) $\big[x_i^2, [x_k, x_j, x_{i_1}, \dots, x_{i_s}]\big]$
$\big[[x_i, x_j], [x_i, x_k, x_{i_1}, \dots, x_{i_s}]\big]$ $\big[[x_i, x_{i_1}], [x_j, x_k, x_{i_2}, \dots x_{i_s}]\big]$
$\big[[x_{i_1}, x_j], [x_i, x_k, x_{i_2}, \dots, x_{i_s}]\big]$ $\big[[x_i, x_j], [x_{i_1}, x_k, x_{i_2}, \dots, x_{i_s}]\big]$
with $1 \leqslant i \leqslant n$, $1 \leqslant j < k < i_1 < \cdots < i_s \leqslant n$, $i \neq j, k, i_1, \dots, i_s$, $1 \leqslant s \leqslant n-3$,

(v) *for all subsets* $\{j_1, \dots, j_s\}$, $4 \leqslant s \leqslant n$, *of mutually distinct elements in* $\{1, 2, \dots, n\}$, *the Kuz'min elements*

$$\big[[x_{i_1}, x_{i_2}], [x_{i_3}, x_{i_4}, x_{i_5}, \dots, x_{i_s}]\big], \qquad 4 \leqslant s \leqslant n,$$

with $\{i_1, \dots, i_s\} = \{j_1, \dots, j_s\}$ *and* $i_1 > i_2$, $i_3 > i_4$, $i_1 > i_3$, $i_2 > i_4$, $i_2 < i_5, \dots <$ i_s, *and the element*

$$\big[[x_{k_3}, x_{k_2}], [x_{k_4}, x_{k_1}, x_{k_5}, \dots, x_{k_s}]\big]$$

where $\{j_1, \dots, j_s\} = \{k_1, \dots, k_s\}$ *with* $k_1 < k_2 < \cdots < k_s$.

Of these, the elements (i), (iii), (iv.a) *and* (v) *have infinite order, the elements* (ii) *with* $s = 1$ *and* (iv.b) *have order* 4, *and the elements* (ii) *with* $s > 1$ *and the elements* (iv.c) *have order* 2.

Proof. We use the isomorphism (2.1) to translate the results obtained for the tensor product $(M \wedge M)_G$ into the setting of the quotient $R'/[R', F]$. By Proposition 4.1 it is generated by the elements (i)–(v). In Section 7 we have shown that the elements (ii), (iv.b) and (iv.c) are torsion elements. Then, in Section 8, we have shown that the elements (i), (iii), (iv.a) and (v) freely generate a free $\mathbb{Z}$-module. By Corollary 10.2, the elements (ii) with $s = 1$ and (iv.b) have order 4 and generate their direct product. Finally, by Lemma 7.2 the elements (ii) with $s > 1$ and the elements (iv.c) have order 2, and by Lemma 7.1 they are linearly independent over $\mathbb{Z}_2$. The theorem follows. □

For completeness we add that the number of infinite cyclic factors in $R'/[R', F]$ is $\binom{n-1}{2}2^n + (n-1)$, the number of cyclic factors of order 4 is $2\binom{n+1}{3}$, and the number of cyclic factors of order 2 is $n\sum_{m=2}^{n-1}\binom{n-1}{m} + n\sum_{m=3}^{n-1}\binom{n-1}{m} = n2^n - \frac{n}{2}(n^2 + n + 2)$.

References

[1] Chander Kanta Gupta, The free centre-by-metabelian groups, J. Aust. Math. Soc. 16 (1973) 294–299.
[2] B. Hartley, R. Stöhr, Homology of higher relation modules and torsion in free central extensions of groups, Proc. Lond. Math. Soc. (3) 62 (2) (1991) 325–352.
[3] Graham Higman, Finite groups having isomorphic images in every finite group of which they are homomorphic images, Q. J. Math. Oxford Ser. (2) 6 (1955) 250–254.
[4] P.J. Hilton, U. Stammbach, A Course in Homological Algebra, Springer, Berlin, 1971.
[5] Yu.V. Kuz'min, Free center-by-metabelian groups, Lie algebras and $\mathcal{D}$-groups, Izv. Akad. Nauk SSSR Ser. Mat. 41 (1) (1977) 3–33, 231 (in Russian); English translation: Math. USSR Izv. 11 (1) (1977) 1–30.
[6] Yu.V. Kuz'min, On elements of finite order in free groups of some varieties, Mat. Sb. (N.S.) 119(161) (1) (1982) 119–131, 160 (in Russian); English translation: Math. USSR Sb. 47 (1) (1984) 115–126.
[7] Yu.V. Kuz'min, Elements of order 2 in a free centrally solvable group, Mat. Zametki 37 (5) (1985) 643–652 (in Russian); English translation: Math. Notes 37 (5) (1985) 350–356.
[8] Yu.V. Kuz'min, The structure of free groups of certain varieties, Mat. Sb. (N.S.) 125 (1) (1984) 128–142 (in Russian); English translation: Math. USSR Sb. 43 (1) (1986) 131–145.

[9] Yu.V. Kuz'min, Homology theory of free abelianized extensions, Comm. Algebra 16 (12) (1988) 2447–2533.
[10] Nil Mansuroğlu, Ralph Stöhr, Free centre-by-metabelian Lie rings, Q. J. Math. 65 (2) (2014) 555–579.
[11] Ralph Stöhr, On Gupta representations of central extensions, Math. Z. 187 (2) (1984) 259–267.
[12] Ralph Stöhr, On torsion in free central extensions of some torsion-free groups, J. Pure Appl. Algebra 46 (2–3) (1987) 249–289.
[13] Ralph Stöhr, On elements of order four in free central extensions of groups, Math. Proc. Cambridge Philos. Soc. 106 (1) (1989) 13–28.
[14] Rainer Zerck, On the homology of the higher relation modules, J. Pure Appl. Algebra 58 (3) (1989) 305–320.