



# A Littlewood–Richardson rule for the Macdonald inner product and bimodules over wreath products



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## ABSTRACT

We prove a Littlewood–Richardson type formula for  $(s_{\lambda/\mu}, s_{\nu/\kappa})_{t^k, t}$ , the pairing of two skew Schur functions in the Macdonald inner product at  $q = t^k$  for positive integers  $k$ . This pairing counts graded decomposition numbers in the representation theory of wreath products of the algebra  $\mathbb{C}[x]/x^k$  and symmetric groups.

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## 1. Introduction

Let  $\Lambda$  denote the algebra of symmetric functions, endowed with the standard bilinear form with respect to which the Schur basis  $\{s_{\lambda}\}$  is orthonormal. The Littlewood–Richardson rule gives an enumerative formula for the inner products

$$c_{\mu\nu}^{\lambda} = (s_{\lambda}, s_{\mu}s_{\nu}) = (s_{\nu}^{\perp} s_{\lambda}, s_{\mu}),$$

which are known as Littlewood–Richardson coefficients. Here  $s_{\nu}^{\perp}$  is the linear operator on symmetric functions adjoint to multiplication by the Schur function  $s_{\nu}$ . (We refer to [3,5]

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for detailed treatments of the Littlewood–Richardson rule.) The  $c_{\mu\nu}^\lambda$  are non-negative integers, and they enumerate tableaux satisfying certain conditions. The integrality of the Littlewood–Richardson coefficients is also manifest in their appearance as tensor product multiplicities in the representation theory of  $GL(n)$  and as intersection numbers in the Schubert calculus of Grassmannians. A generalization of the Littlewood–Richardson rule adds a fourth partition to the picture: consider the algebra  $\mathcal{H}_\Lambda$  of operators on symmetric functions generated by the operators  $\{s_\mu, s_\kappa^\perp\}$ , which is spanned as a vector space by  $\{s_\mu s_\kappa^\perp\}_{\mu, \kappa}$ . Define  $c_{\mu\nu}^{\kappa\lambda}$  to be the structure constants in the expansion

$$s_\nu^\perp s_\lambda = \sum_{\mu, \kappa} c_{\mu\nu}^{\kappa\lambda} s_\mu s_\kappa^\perp. \quad (1)$$

For  $\kappa = \emptyset$  these are the ordinary Littlewood–Richardson coefficients; an enumerative formula for the general case was found by Zelevinsky [7] in the language of *pictures*. The algebra  $H_\Lambda$  is a Hopf algebra; in fact it is the Heisenberg double of the Hopf algebra  $\Lambda$ . Thus the Littlewood–Richardson coefficients may also be thought of as structure constants in the canonical basis of the Hopf algebra  $H_\Lambda$ .

Let  $\Lambda_{q,t}$  denote the algebra of symmetric polynomials over the two-variable coefficient ring  $\mathbb{C}(q, t)$ , together with the Macdonald inner product  $(\cdot, \cdot)_{q,t}$ , which specializes to the standard inner product at  $q = t$ . We will be interested in the specialization  $\Lambda_{t^k, t}$  for some integer  $k \geq 1$ . The ring with inner product  $\Lambda_{t^k, t}$  appears in several other mathematical contexts, including the representation theory of quantum affine algebras. In particular, several important bases of  $\Lambda_{t^k, t}$ , such as the Schur basis, should be related to important bases in the representation theory of quantum affine algebras and in the geometry of quiver varieties of affine type. As a result, it is natural to suspect that much of the positive integral structure appearing in the ordinary theory of symmetric functions will admit an interesting generalization from  $\Lambda$  to  $\Lambda_{t^k, t}$ . The first goal of the present paper is to extend (1) to the ring  $\Lambda_{t^k, t}$ , in which the dual is now taken with respect to  $(\cdot, \cdot)_{t^k, t}$  instead of the standard inner product. As the inner product on  $\Lambda_{t^k, t}$  takes values in the ring  $\mathbb{C}(t)$ , the precise statement involves a  $t$ -weighted count of tableaux. In Section 2, we define  $k$ -tableaux, which are fillings of a Young diagram with entries which are monomials of the form  $at^m$ ,  $0 \leq m < k$ . A  $k$ -tableau  $T$  has an associated statistic  $c(T)$ , the degree of the product of all its entries. Our main theorem then states:

**Theorem A.** *We have*

$$s_\nu^* s_\lambda = \sum_{\mu, \kappa} c_{\mu\nu}^{\kappa\lambda}(t) s_\mu s_\kappa^*, \quad c_{\mu\nu}^{\kappa\lambda}(t) = \sum_T c(T).$$

Here  $c(T)$  is the  $t$ -degree of the tableau  $T$ , which ranges over  $k$ -tableaux, and  $f^*$  is the adjoint of the multiplication operator with respect to the Macdonald inner product at  $q = t^k$ . When  $k = 1$ , this theorem reduces to the extended Littlewood–Richardson rule of Zelevinsky [7]. If we then set  $\kappa$  to the empty partition, the above statement

further reduces to the usual Littlewood–Richardson rule. The proof of this theorem is given in Section 2, which deals only with combinatorics. We also show in Proposition 1 that these coefficients are symmetric and unimodal, something that is not obvious from enumerative description of Theorem A.

In Section 3 we identify an integral form  $\Lambda_{t^k,t}^{\mathbb{Z}}$  of  $\Lambda_{t^k,t}$  with the Grothendieck group of graded projective modules over an  $S_n$ -equivariant graded ring. Under this identification, the Schur polynomial  $s_\lambda$  corresponds to a certain indecomposable projective module  $S_\lambda$ , and  $(s_\mu, s_\nu)_{t^k,t}$  measures the graded dimensions of the  $\text{Hom}(S_\mu, S_\lambda)$  up to a grading shift. In particular, this implies that this graded dimension is a Laurent polynomial in  $t$  with nonnegative integer coefficients, explaining the positive-integral structure of the generalized Littlewood–Richardson coefficients  $c_{\mu\nu}^{\kappa\lambda}(t)$ . Specifically, let  $A_k = \mathbb{C}[x]/x^k$ , and let  $A_k^{[n]}$  denote the smash product of  $A_k$  with  $\mathbb{C}[S_n]$ , see Section 3.1 for a definition. The algebras  $A_k^{[n]}$  are graded, and we consider the Grothendieck group  $K(A_k^{[n]}\text{-gmod})$  of finitely generated projective  $A_k^{[n]}$  modules. This space is a free  $\mathbb{Z}[t, t^{-1}]$  module, where multiplication by  $t$  corresponds to a shift in the grading. Since the Hom pairing in the category of graded  $A_k^{[n]}$  modules induces a semi-linear pairing on the Grothendieck group, we slightly modify the bilinear form on  $\Lambda_{t^k,t}^{\mathbb{Z}}$  to be semi-linear in  $t$ . Our second main theorem is then the following.

**Theorem B.** *There is an isometric isomorphism*

$$\Phi : \bigoplus_{n=0}^{\infty} K(A_k^{[n]}\text{-gmod}) \longrightarrow \Lambda_{t^k,t}^{\mathbb{Z}},$$

where the bilinear form on the left hand side is induced from the Hom bifunctor, and  $A_k^{[n]}\text{-gmod}$  is the category of graded modules over  $A_k^{[n]}$ .

As a result, we obtain an interpretation of the polynomials  $c_{\mu\nu}^{\kappa\lambda}(t)$  as decomposition numbers in the convolution product of explicit bimodules over the rings  $A_k^{[n]}$ . When  $k = 1$  there is no interesting grading on the module category, and the identification above reduces to the well-known isomorphism between  $\Lambda$  and the Grothendieck group of representations of all symmetric groups [4].

The graded rings  $A_k^{[n]}$  appear in several other representation theoretic contexts. For example, these algebras for  $k = 2$  play a central role in the categorification [1] of the Heisenberg double of  $\Lambda_{t^2,t}$  and in the level one quantum affine categorifications of [2]; the algebras  $A_k^{[n]}$  for  $k > 2$  should appear in the higher level analogs of those constructions. In fact, part of our original motivation for considering the polynomials  $c_{\mu\nu}^{\kappa\lambda}(t)$  was to combinatorially compute the structure constants for multiplication in the canonical basis of the quantum Heisenberg algebra considered in [1]; these structure constants are given by the  $k = 2$  case of Theorem A. Similarly, the generalized Littlewood–Richardson coefficients  $c_{\mu\nu}^{\kappa\lambda}(t)$  should all appear as structure constants for multiplication in the canonical basis of certain infinite dimensional Hopf algebras. Another closely related appearance

of  $A_k^{[n]}$  involves category  $\mathcal{O}$  for the rational Cherednik algebra of the complex reflection group  $(\mathbb{Z}/k\mathbb{Z}) \wr S_n$  at integral parameters. The algebra  $A_k^{[n]}$  is an example of an Ariki–Koike algebra, and the study of  $\mathcal{O}$  via the KZ functor involves mapping  $\mathcal{O}$  to a category of  $A_k^{[n]}$  modules. From this point of view, [Theorems A and B](#) together have applications to combinatorial description of hom spaces between projective modules in  $\mathcal{O}$ , though we have not fully explored this here. Relationships between higher-level Heisenberg categorification and category  $\mathcal{O}$  for rational Cherednik algebras appear in [\[6\]](#).

The bilinear form on  $\Lambda_{t^k, t}$ , the ring  $A_k^{[n]}$ , and the generalized Littlewood–Richardson coefficients  $c_{\mu\nu}^{\kappa\lambda}(t)$  all depend on the choice of positive integer  $k$ . It might be interesting to formulate versions of [Theorems A and B](#) in a way that does not require  $k$  to be a positive integer and thus recover the variable  $q$  in the Macdonald theory. Notice that a rational function in  $\mathbb{C}(q, t)$  is determined by its values at  $q = t^k$  at all positive integers  $k$ , so in a sense we lose no information by specializing.

## 2. The Littlewood–Richardson rule

### 2.1. Notations

For all notations in this section, we have followed Macdonald’s book [\[5\]](#). Given a skew diagram  $\lambda - \mu$ , a semistandard tableau of shape  $\lambda - \mu$  is a way of filling in the boxes of the skew shape  $\lambda - \mu$  with positive integers in a way that is nonstrictly increasing in each row, and strictly increasing in each column. The weight of a tableau  $T$  is the sequence  $[\pi_1, \pi_2, \dots]$  where  $\pi_i$  is the number of times  $i$  appears in  $T$ . We let  $\text{Tab}(\lambda - \mu, \pi)$  denote the set of all semistandard tableaux of shape  $\lambda - \mu$  and weight  $\pi$ . For instance,

$$\begin{array}{|c|c|c|c|c|c|c|}
 \hline
 & & 1 & 2 & 4 & 4 & 4 \\
 \hline
 & & 2 & & & & \\
 \hline
 1 & 3 & 3 & & & & \\
 \hline
 \end{array} \tag{2}$$

is an element of  $\text{Tab}([7, 3, 3] - [2, 2], [2, 2, 2, 3])$ . The *word*  $w(T)$  of a tableau  $T$  is the set of numbers in the diagram read from right to left, top to bottom. For instance, the word of the above tableau is 444212331. The word of a partition  $w(\lambda)$  is defined as the word of the tableau of shape  $\lambda$  in which row  $i$  is filled with the number  $i$ . Let  $\text{Tab}^0(\lambda - \mu, \pi)$  denote the subset of tableau whose word  $a_1 \cdots a_n$  is a *lattice permutation*, meaning that for each  $i, k$ , the number of occurrences of  $i$  in  $a_1 \cdots a_k$  is greater than or equal to the number of occurrences of  $i + 1$ . More generally, let us say that a tableau has weight  $\nu - \kappa$  for some skew shape  $\nu - \kappa$  if the number of times  $i$  appears is equal to  $\nu_i - \kappa_i$ . Notice that there are many skew shapes with this property, so we are not associating a unique skew shape associated to each tableau. We define  $\text{Tab}^0(\lambda - \mu, \nu - \kappa)$  to be the set of tableaux  $T$  with weight  $\nu - \kappa$  such that the concatenated word  $w(\kappa)w(T)$  is a lattice permutation. Notice that this in fact depends on  $\kappa$ , not just the skew shape  $\nu - \kappa$ .

Fix an integer  $k \geq 1$ , and define a  $k$ -tableau of shape  $\lambda - \mu$  to be a labeling of the boxes of  $\lambda - \mu$  with monomials of the form  $at^b$  for  $a \geq 1$ ,  $0 \leq b \leq k - 1$ . We define a total order on these monomials by

$$at^b \leq ct^d \iff b < d \text{ or } b = d \text{ and } a \leq c, \tag{3}$$

which is the same as the ordering obtained by replacing  $t$  by a large positive number. For any multiset  $\gamma$  of such monomials, we may consider the set  $\text{Tab}(\lambda - \mu, \gamma)$  of semistandard tableaux filled with the elements of  $\gamma$  that are semistandard with respect to this ordering. Call  $T'$  the tableau obtained from  $T$  by setting  $t = 1$ , and let  $\text{Tab}_k(\lambda - \mu, \nu - \kappa)$  be the set of  $k$ -tableaux which are semistandard with respect to (3), such that weight of  $T'$  is  $\nu - \kappa$ . We also define a statistic on  $k$ -Tableau by

$$c(T) = \prod_{at^b \in T} t^b. \tag{4}$$

Any  $k$ -tableau  $T$  corresponds to a sequence of regular tableaux  $T^i$  for  $i \geq 0$ , defined as the subdiagram of coefficients in all boxes containing  $at^i$  for some  $a$ . For instance, if

			3	2t	2t
		1	t	t <sup>2</sup>	
t <sup>2</sup>	t <sup>2</sup>				

,

then

			3		
		1			

,

				2	2
			1		

,

				1	
1	1				

Let  $\text{Tab}_k^0(\lambda - \mu, \nu - \kappa)$  denote the subset of  $k$ -tableaux such that

$$w(\kappa)w(T^0)w(T^1)\cdots$$

is a lattice permutation. For instance,

		2	3	2t <sup>2</sup>
	t	t		
1	2t <sup>2</sup>			

∈

$\text{Tab}_3^0([532] - [21], [541] - [21])$

because 1 123 211 122 is a lattice permutation, and the semistandardness condition is satisfied. The statistic is  $c(T) = t^6$ .

## 2.2. The main theorem

We first recall some definitions about symmetric polynomials, also from in Macdonald's book [5]. Consider the space of symmetric polynomials  $\Lambda$  in infinitely many variables, and let  $p_\mu$ ,  $e_\mu$ ,  $h_\mu$ , and  $s_\mu$  denote the power sum, elementary, complete, and Schur bases respectively. The formula for the standard inner product is given in the Schur and power sum bases by

$$(s_\mu, s_\nu) = \delta_{\mu,\nu}, \quad (p_\mu, p_\nu) = \delta_{\mu,\nu} z_\mu, \quad z_\mu = \prod_{i \geq 1} i^{m_i(\mu)} m_i(\mu)!, \quad (5)$$

where  $m_i(\mu)$  is the number of times  $i$  appears in  $\mu$ . For any dual bases  $u_\mu, v_\mu$  under (5), we have the formula,

$$\sum_{\mu} u_{\mu}(x) v_{\mu}(y) = \prod_{i,j} \frac{1}{1 - x_i y_j} \quad (6)$$

in two different sets of variables  $x, y$ . In particular, the sum

$$\sum_{\mu} u_{\mu} \otimes v_{\mu} \in \Lambda \otimes \Lambda \quad (7)$$

is independent of the choice of dual basis  $(u, v)$ . We also have the expansion

$$s_{\mu} = \sum_{\lambda} \chi_{\lambda}^{\mu} z_{\lambda}^{-1} p_{\lambda}, \quad (8)$$

where  $\chi_{\lambda}^{\mu}$  denotes the value of the irreducible representation of the symmetric group corresponding to  $\mu$  on the conjugacy class with cycle type  $\lambda$ .

The Macdonald inner product on  $\Lambda$  is defined in the power sum basis by

$$(p_{\mu}, p_{\nu})_{q,t} = \delta_{\mu\nu} z_{\mu} \prod_j \frac{1 - q^{\mu_j}}{1 - t^{\mu_j}}.$$

It may also be written as  $(f, g)_{q,t} = (f, \rho_{q,t} g)$ , where  $\rho_{q,t}$  is the ring homomorphism defined on the power sum generators by

$$\rho_{q,t} : p_j \mapsto \frac{1 - q^j}{1 - t^j} p_j. \quad (9)$$

For a fixed integer  $k$ , we will be interested in the inner product  $(f, g)_{t^k, t}$ . This has the important property that it takes values in  $\mathbb{Z}[t]$  on  $\Lambda^{\mathbb{Z}} \times \Lambda^{\mathbb{Z}}$ , where  $\Lambda^{\mathbb{Z}}$  denotes the integral span of the Schur functions in  $\Lambda$ .

For any symmetric polynomial  $f \in \Lambda$ , define a dual multiplication operator by

$$(f^*g, h)_{t^k, t} = (g, fh)_{t^k, t}, \quad (f^\perp g, h) = (g, fh),$$

which may also be written as

$$f^* = \rho_{t^k, t}(f)^\perp = \rho_{t^k, t}^{-1} f^\perp \rho_{t^k, t}. \tag{10}$$

In particular, the  $f^*$  operation agrees with  $f^\perp$  when  $k = 1$ . We also have the skew Schur polynomials, given by

$$s_{\mu/\lambda} = s_\lambda^\perp s_\mu.$$

We may now state the main theorem:

**Theorem 1.** Fix a positive integer  $k$ . There exist unique coefficients  $c_{\mu\nu}^{\kappa\lambda}(t)$  satisfying

$$s_\nu^* s_\lambda = \sum_{\mu, \kappa} c_{\mu\nu}^{\kappa\lambda}(t) s_\mu s_\kappa^*. \tag{11}$$

Furthermore,

$$c_{\mu\nu}^{\kappa\lambda}(t) = \sum_{T \in \text{Tab}_k^0(\lambda - \mu, \nu - \kappa)} c(T), \tag{12}$$

if  $\mu \subset \lambda$  and  $\kappa \subset \nu$ , or zero otherwise.

**Example 1.** Take  $k = 2, \kappa = [1], \lambda = [32], \mu = [1], \nu = [32]$ . Using [Lemma 1](#) below we have

$$c_{\mu\nu}^{\kappa\lambda}(t) = (s_{\lambda/\mu}, s_{\nu/\kappa})_{t^2, t} = 2 + 5t + 7t^2 + 5t^3 + 2t^4$$

which equals 21 at  $t = 1$ . On the other hand, there are 25 elements of  $\text{Tab}_2([32] - [1], [32] - [1])$ . The four that do not satisfy the lattice word condition are

$$\begin{array}{|c|c|c|} \hline & 2 & 2 \\ \hline 1 & t & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline & 2 & 2 \\ \hline t & t & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline & 2 & t \\ \hline 2 & t & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline & 2 & 2t \\ \hline t & t & \\ \hline \end{array}.$$

**Example 2.** If  $\nu = (n), \lambda = (m)$  consist of a horizontal strip, then  $c_{\mu(n)}^{\kappa(m)}(t)$  is zero unless  $\kappa = (n - l), \mu = (m - l)$  for some  $l \geq 0$ . In this case

$$\text{Tab}_k^0(\lambda - \mu, \nu - \kappa) = \text{Tab}_k(\lambda - \mu, \nu - \kappa)$$

because the coefficients of every box are one, so that the lattice word condition is always satisfied. The answer is the weighted count of the ways to insert the monomials  $t^i$  for

$0 \leq i \leq k-1$  in sorted order into a strip of length  $l$ , which is given by the Gaussian binomial coefficient

$$c_{\mu(n)}^{\kappa(m)}(t) = \binom{k+l-1}{l}_t = \frac{(1-t^{k+l-1}) \cdots (1-t^k)}{(1-t) \cdots (1-t^l)}.$$

**Lemma 1.** *The coefficients satisfying (11) are determined by*

$$c_{\mu\nu}^{\kappa\lambda}(t) = (s_{\lambda/\mu}, s_{\nu/\kappa})_{t^k, t}. \quad (13)$$

**Proof.** We first show uniqueness: Suppose there exist coefficients  $a_{\mu}^{\kappa}(t)$  not all zero, such that

$$\Phi = \sum_{\kappa, \mu} a_{\mu}^{\kappa}(t) s_{\mu} s_{\kappa}^* = 0.$$

Let  $\pi$  be any partition of the smallest possible norm such that  $a_{\mu}^{\pi}(t) \neq 0$  for some  $\mu$ . Then since  $s_{\kappa}^{\perp} s_{\pi} = 0$  unless  $\kappa \subset \pi$ , and  $a_{\mu}^{\kappa}(t) = 0$  if  $\kappa \subsetneq \pi$ , we have that

$$\begin{aligned} \Phi(\rho_{t^k, t}^{-1} s_{\pi}) &= \sum_{\kappa, \mu} a_{\mu}^{\kappa}(t) s_{\mu} s_{\kappa}^* \rho_{t^k, t}^{-1} s_{\pi} = \sum_{\kappa, \mu} a_{\mu}^{\kappa}(t) s_{\mu} \rho_{t^k, t}^{-1} s_{\kappa}^{\perp} \rho_{t^k, t}^{-1} \rho_{t^k, t}^{-1} s_{\pi} \\ &= \sum_{\kappa, \mu} a_{\mu}^{\kappa}(t) s_{\mu} \rho_{t^k, t}^{-1} s_{\kappa}^{\perp} s_{\pi} = \sum_{\mu} a_{\mu}^{\pi}(t) s_{\mu}, \end{aligned}$$

so we must have  $a_{\mu}^{\pi}(t) = 0$ , a contradiction.

Next, we may repeatedly use the commutation relation

$$[p_m^*, p_n] = m \delta_{m, n} \frac{1 - t^{km}}{1 - t^m},$$

to obtain the analogous formula for power sums,

$$q_{\nu}^* q_{\lambda} = \sum_{\mu, \kappa} (p_{\mu}^{\perp} q_{\lambda}, p_{\kappa}^{\perp} q_{\nu})_{t^k, t} q_{\mu} q_{\kappa}^*, \quad q_{\mu} = \mathfrak{z}_{\mu}^{-1} p_{\mu}.$$

Now change to the Schur basis using (8) to get

$$\begin{aligned} s_{\nu}^* s_{\lambda} &= \sum_{\alpha, \beta} \chi_{\beta}^{\nu} \chi_{\alpha}^{\lambda} q_{\beta}^* q_{\alpha} = \sum_{\alpha, \beta, \mu, \kappa} \chi_{\alpha}^{\lambda} \chi_{\beta}^{\nu} (p_{\mu}^{\perp} q_{\alpha}, p_{\kappa}^{\perp} q_{\beta})_{t^k, t} q_{\mu} q_{\kappa}^* \\ &= \sum_{\mu, \kappa} (p_{\mu}^{\perp} s_{\lambda}, p_{\kappa}^{\perp} s_{\nu})_{t^k, t} q_{\mu} q_{\kappa}^* = \sum_{\mu, \kappa} (s_{\mu}^{\perp} s_{\lambda}, s_{\kappa}^{\perp} s_{\nu})_{t^k, t} s_{\mu} s_{\kappa}^*, \end{aligned}$$

as desired. In the final equality, we have applied the independence of (7) to replace the dual basis pairs  $(p, q)$  with  $(s, s)$  in both sums.  $\square$



**Lemma 2.** *We have*

$$(s_{\lambda/\mu}, h_\nu)_{t^k, t} = \sum_{T \in \text{Tab}_k(\lambda - \mu, \nu)} c(T).$$

**Proof.** We begin by rewriting

$$(s_{\lambda/\mu}, h_\nu)_{t^k, t} = (h_\nu^* s_{\lambda/\mu}, 1) = (s_{\lambda/\mu}, g_\nu) = (s_\lambda, g_\nu s_\mu), \quad (14)$$

where  $g_\mu = \rho_{t^k, t}(h_\mu)$ , and  $\rho_{q, t}$  is the ring homomorphism defined in (9).

Now let us proceed by induction on  $l$ , the length of  $\nu$ . We start with the base case in which  $\nu$  only has one component, say  $\nu_1 = m$ . We first find the coefficients in the expansion

$$g_m = \sum_{|\pi|=m} a_\pi(t) h_\pi. \quad (15)$$

We claim that

$$a_\pi(t) = \sum_{\beta \rightarrow \pi} \sum_{0 \leq a_1 < \dots < a_\ell \leq k-1} t^{a_1 \beta_1 + \dots + a_\ell \beta_\ell}, \quad (16)$$

where  $\ell = \ell(\pi)$  is the length of  $\pi$ , and  $\beta$  ranges over all rearrangements of  $\pi$ , i.e. over the orbit of  $\pi$  under  $S_\ell$ . The easiest way to see this is to notice that

$$g_\mu(x_1, x_2, \dots) = h_\mu(x_1, x_2, \dots; tx_1, tx_2, \dots; t^2x_1, t^2x_2, \dots),$$

and compare the coefficient of  $x^\pi$  with the right hand side of (15).

Inserting (15) and (16) into (14), and using the Pieri rule, we have

$$(s_{\lambda/\mu}, h_m)_{t^k, t} = \sum_{\ell \geq 0} \sum_{\beta \in \mathbb{Z}_{\geq 1}^\ell, |\beta|=m} \sum_{0 \leq a_1 < \dots < a_\ell \leq k-1} t^{a_1 \beta_1 + \dots + a_\ell \beta_\ell} |\text{Tab}(\lambda - \mu, \beta)|. \quad (17)$$

Finally, there is a bijection

$$\bigcup_{\ell \geq 0} \bigcup_{\beta \in \mathbb{Z}_{\geq 1}^\ell, |\beta|=m} \{0 \leq a_1 < \dots < a_\ell \leq k-1\} \times \text{Tab}(\lambda - \mu, \beta) \longleftrightarrow \text{Tab}_k(\lambda - \mu, m),$$

in which a box label  $j$  is replaced by  $t^{aj}$ . The statistic  $c(T)$  corresponds to the power of  $t$  in (17), proving the base case.

Now for the induction step, choose any decomposition  $A \sqcup B = \{1, \dots, l\}$ . Let

$$(\nu'_i, \nu''_i) = \begin{cases} (\nu_i, 0) & \text{if } i \in A, \\ (0, \nu_i) & \text{if } i \in B, \end{cases}$$

so that  $g_\nu = g_{\nu'} g_{\nu''}$ . Since

$$(s_\lambda, g_\nu s_\mu) = \sum_{\pi} (s_\lambda, g_{\nu'} s_\pi) (s_\pi, g_{\nu''} s_\mu),$$

it suffices to prove that there is a bijection

$$\text{Tab}_k(\lambda - \mu, \nu) \longleftrightarrow \bigcup_{\pi} \text{Tab}_k(\lambda - \pi, \nu') \times \text{Tab}_k(\pi - \mu, \nu''), \quad (18)$$

such that  $c(T) = c(U)c(V)$  whenever  $T$  maps to  $(U, V)$ .

We may assign to every  $k$ -tableau  $T$  a multiset  $\gamma$  of all the monomials  $at^b$  that appear in  $T$  with multiplicities. This partitions the left side of (18) into disjoint groups

$$\text{Tab}_k(\lambda - \mu, \nu) = \bigcup_{\gamma} \text{Tab}(\lambda - \mu, \gamma).$$

We may write each  $\gamma = \gamma' \sqcup \gamma''$  by assigning the monomial  $at^b$  to  $\gamma'$  if  $a \in A$ , and to  $\gamma''$  if  $a \in B$ . Since all possible pairs  $\gamma', \gamma''$  arise uniquely in this way, it suffices to prove that there exists a bijection

$$\text{Tab}_k(\lambda - \mu, \gamma) \longleftrightarrow \bigcup_{\pi} \text{Tab}_k(\lambda - \pi, \gamma') \times \text{Tab}_k(\pi - \mu, \gamma''). \quad (19)$$

Since (3) is a total ordering. We may identify the possible monomials in an order-preserving way with a subset of the natural numbers. This identifies  $\gamma, \gamma', \gamma''$  with multisets  $c, c', c''$  of integers such that

$$|\text{Tab}_k(\lambda - \mu, \gamma)| = |\text{Tab}(\lambda - \mu, c)|$$

and similarly for  $\gamma', \gamma''$ . This reduces (19) to the case  $k = 1$ , which is true by the usual Littlewood–Richardson rule.  $\square$

We may now prove the theorem.

**Proof.** The existence and uniqueness statement follow from Lemma 1.

We first prove the case  $\kappa = \emptyset$ , by extending the proof of the usual Littlewood–Richardson rule, as explained in [5]. Using the expansion of  $s_\nu$  in the monomial basis, which is well known to be the dual basis to  $h_\nu$  under the standard inner product, we have

$$(s_{\lambda/\mu}, h_\nu)_{t^k, t} = \sum_{\pi} (s_{\lambda/\mu}, s_\pi)_{t^k, t} |\text{Tab}(\pi, \nu)|.$$

Then by Lemma 2, and the invertibility of the triangular matrix  $|\text{Tab}(\pi, \nu)|$ , it suffices to prove that there is a bijection

$$\text{Tab}_k(\lambda - \mu, \nu) \longleftrightarrow \bigcup_{\pi} \text{Tab}_k^0(\lambda - \mu, \pi) \times \text{Tab}(\pi, \nu), \quad (20)$$

producing the correct statistic  $c(T)$ .

Each tableau  $T \in \text{Tab}_k(\lambda - \mu, \nu)$  has an associated filtration

$$F : \mu = \mu^0 \subset \cdots \subset \mu^k = \lambda,$$

so that  $\mu^{i+1} - \mu^i$  is the shape of  $T^i$ . We obtain a decomposition

$$\text{Tab}_k(\lambda - \mu, \nu) \longleftrightarrow \bigcup_F \text{Tab}_k(\lambda - \mu, \nu)_F, \quad (21)$$

into tableaux whose filtration is  $F$ , and similarly for  $\text{Tab}^0(\lambda - \mu, \nu)$  by restriction. It suffices to find a bijection

$$\text{Tab}_k(\lambda - \mu, \nu)_F \longleftrightarrow \bigcup_{\pi} \text{Tab}_k^0(\lambda - \mu, \pi)_F \times \text{Tab}(\pi, \nu), \quad (22)$$

for each  $F$ .

For every such filtration  $F$ , fix a skew shape  $\lambda_F$  which is a disconnected union of the shapes  $\mu_{i+1} - \mu_i$ , positioned in the plane in some way in order from upper right to lower left. For any choice of  $\{\lambda_F\}$ , there is an obvious bijection

$$\text{Tab}_k(\lambda - \mu, \nu)_F \longleftrightarrow \text{Tab}(\lambda_F, \nu)$$

in which the component of  $\lambda_F$  corresponding to  $\mu^{i+1} - \mu^i$  (which may not be connected) is filled with  $T^i$ . It is clear that this bijection respects lattice words, reducing (22) to the case  $k = 1$ , which follows from the usual algorithm of Littlewood–Robinson–Schensted.

For instance, for the filtration

$$F = [1] \subset [3, 2] \subset [5, 2] \subset [5, 3, 1],$$

one choice of  $\lambda_F$  might be

$$\lambda_F = \begin{array}{ccccccc} & & & & & \boxed{0} & \boxed{0} \\ & & & & & \boxed{0} & \boxed{0} \\ & & & \boxed{1} & \boxed{1} & & \\ & & \boxed{2} & & & & \\ \boxed{2} & & & & & & \end{array}.$$

where we have labeled each box with its corresponding index. Under the bijection, the tableau

$$T = \begin{array}{|c|c|c|c|c|} \hline & 1 & 2 & t & t \\ \hline 1 & 2 & t^2 & & \\ \hline t^2 & & & & \\ \hline \end{array} \in \text{Tab}_3([5, 3, 1] - [1], [6, 2]),$$

would map to

$$\begin{array}{c} \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \\ \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \\ \begin{array}{|c|c|} \hline 1 & 1 \\ \hline \end{array} \\ \begin{array}{|c|} \hline 1 \\ \hline \end{array} \end{array}.$$

We now prove the general case. Using the usual Littlewood–Richardson rule to expand  $s_{\nu/\kappa}$ , we have

$$(s_{\lambda/\mu}, s_{\nu/\kappa})_{t^k, t} = \sum_{\pi} (s_{\lambda/\mu}, s_{\pi})_{t^k, t} |\text{Tab}^0(\nu - \kappa, \pi)|.$$

Then applying the special case we just proved, it suffices to find a suitable bijection

$$\text{Tab}_k^0(\lambda - \mu, \nu - \kappa) \longleftrightarrow \bigcup_{\pi} \text{Tab}_k^0(\lambda - \mu, \pi) \times \text{Tab}^0(\nu - \kappa, \pi). \quad (23)$$

Once again, the decomposition (21) reduces this statement to the case  $k = 1$ , which was proved by Zelevinsky [7].  $\square$

Now consider a modification of the bilinear form on  $\Lambda_{t^k, t}$  defined on symmetric functions of homogeneous degree  $d$  by

$$\langle f, g \rangle_k = (f, t^{(1-k)d} \bar{g})_{t^{2k}, t^2}, \quad (24)$$

which is *semi-linear* with respect to the conjugation  $t \mapsto t^{-1}$ , i.e.

$$\langle a(t)f, g \rangle_k = a(t) \langle f, g \rangle_k = \langle f, a(t^{-1})g \rangle_k$$

for any rational function  $a(t)$ . Its expression in the power sum basis takes the form

$$\langle p_{\mu}, p_{\nu} \rangle_k = \delta_{\mu, \nu} \prod_i \left( t^{(1-k)\mu_i} + t^{(3-k)\mu_i} + \dots + t^{(k-1)\mu_i} \right), \quad (25)$$

which is symmetric. With respect to this inner product, we then have

$$s_{\nu}^* s_{\lambda} = \sum_{\mu, \kappa} C_{\mu\nu}^{\kappa\lambda}(t) s_{\mu} s_{\kappa}^*,$$

where

$$C_{\mu\nu}^{\kappa\lambda}(t) = t^{(1-k)(|\lambda| - |\mu|)} c_{\mu\nu}^{\kappa\lambda}(t^2).$$

Call a Laurent polynomial

$$f(t) = a_{-l}t^{-l} + a_{2-l}t^{2-l} + \cdots + a_{l-2}t^{l-2} + a_l t^l$$

symmetric unimodal if  $f(t) = f(t^{-1})$ , and  $a_i \leq a_{i+2}$  for  $i < 0$ .

**Proposition 1.** *The polynomials  $C_{\mu\nu}^{\kappa\lambda}(t)$  are symmetric unimodal.*

**Proof.** First, the symmetry statement follows from Lemma 1, and the obvious symmetry of (25).

Since the sum of unimodal expressions is unimodal, it suffices to check the unimodality of  $\langle s_\mu, s_\nu \rangle_k$ , by Lemma 1, and the Schur-positivity of the skew Schur functions. We have

$$\langle s_\mu, s_\nu \rangle_k = (s_\mu, t^{(1-k)d} \rho_{t^{2k}, t^2} s_\nu) = \sum_{\kappa, \lambda} a_{\kappa\lambda} s_\kappa(t^{1-k}, t^{3-k}, \dots, t^{k-1})(s_\mu, s_\lambda),$$

where  $a_{\kappa\lambda}$  are the multiplicities of the decomposition into irreducibles,

$$\mathbb{S}_\nu(U \boxtimes V) = \bigoplus_{\kappa, \lambda} a_{\kappa\lambda} \mathbb{S}_\kappa(U) \boxtimes \mathbb{S}_\lambda(V)$$

over  $GL(U) \times GL(V)$ , and  $\rho_{q,t}$  is the homomorphism defined in (9). In particular, they are nonnegative integers. The answer now follows from the unimodality of  $s_\kappa(t^{1-k}, t^{3-k}, \dots, t^{k-1})$ , see [5], Chapter I, Section 8, Example 4.  $\square$

A notable feature of Proposition 1 is that neither symmetry nor unimodality is immediately clear from Theorem 1. It would be interesting to give a purely enumerative proof in this way, or a representation-theoretic proof, along the lines of the next section.

### 3. Categorification

#### 3.1. Wreath products of $H^*(\mathbb{P}^{k-1})$

Fix the integer  $k \geq 1$ , and let  $A_k = H^*(\mathbb{P}^{k-1}, \mathbb{C}) \cong \mathbb{C}[x]/x^k$ . Denote by  $A_k^{[n]}$  the smash product of  $A_k$  with the group algebra of the symmetric group  $S_n$ ,

$$A_k^{[n]} = A_k \# \mathbb{C}[S_n].$$

As a vector space,  $A_k^{[n]} = A_k^{\otimes n} \otimes_{\mathbb{C}} \mathbb{C}[S_n]$ , and by convention we take  $A_k^{[0]} = \mathbb{C}$ . We give  $A_k^{[n]}$  a grading by declaring the degree of  $x$  to be 1, and putting  $\mathbb{C}[S_n]$  in degree 0.

Let  $K(A_k^{[n]})$  denote the Grothendieck group of the category of  $\mathbb{Z}$ -graded finitely-generated projective left  $A_k^{[n]}$  modules. The  $\mathbb{Z}$ -grading on  $A_k^{[n]}$  endows  $K(A_k^{[n]})$  with

the structure of a free  $\mathbb{Z}[t, t^{-1}]$  module, where shifting by 1 in the internal grading of a module corresponds to multiplication by  $t$  on the class in the Grothendieck group,

$$[M\{\pm 1\}] = t^{\pm 1}[M] \in K(A_k^{[n]}).$$

If we choose a complete set of minimal idempotents  $e_\lambda \in \mathbb{C}[S_n]$ , so that  $\{\mathbb{C}[S_n]e_\lambda\}_{\lambda \vdash n}$  are representatives of the isomorphism classes of irreducible  $\mathbb{C}[S_n]$  modules, then  $\{A_k^{[n]}e_\lambda\}$  are representatives of the isomorphism classes of indecomposable projective  $A_k^{[n]}$  modules, up to grading shift. It follows that the rank of  $K(A_k^{[n]})$  as a free  $\mathbb{Z}[t, t^{-1}]$  module is the number of partitions of  $n$ . Thus the free  $\mathbb{Z}[t, t^{-1}]$  module  $K(A_k^{[n]})$  comes equipped with a canonical basis, namely, the classes  $\{[A_k^{[n]}e_\lambda]\}_{\lambda \vdash n}$  of the indecomposable projective modules.

Let  $\mathcal{F}_k = \bigoplus_{n=0}^\infty K(A_k^{[n]})$ . Homomorphisms in the module category endow  $\mathcal{F}$  with a bilinear form,

$$\langle [X], [Y] \rangle_{\mathcal{F}_k} = \text{gdim } \text{Hom}(X, Y) \in \mathbb{Z}[t, t^{-1}],$$

where  $\text{gdim } V$  denotes the graded dimension of a  $\mathbb{Z}$ -graded vector space  $V$ . By convention, the individual summands  $K(A_k^{[n]})$  for different  $n$  are orthogonal with respect to this bilinear form. This form is *semi-linear* with respect to  $t$ , meaning

$$\langle t^{\pm 1}[M], [N] \rangle_{\mathcal{F}_k} = t^{\pm 1} \langle [M], [N] \rangle_{\mathcal{F}_k} = \langle [M], t^{\mp 1}[N] \rangle_{\mathcal{F}_k}. \quad (26)$$

The following lemma can be checked easily using the fact that  $A_k^{[n]}$  has a nondegenerate trace  $\text{tr} : A_k^{[n]} \rightarrow \mathbb{C}$ .

**Lemma 3.** *With respect to the basis  $\{[A_k^{[n]}e_\lambda]\}$ , the matrix of the bilinear form  $\langle \cdot, \cdot \rangle_{\mathcal{F}_k}$  is symmetric.*

(The above symmetry together with the semi-linearity can be thought of as defining a Hermitian inner product on  $\mathcal{F}_k$  after specializing  $t$  to a point on the unit circle.)

### 3.2. Bimodules

The standard embeddings  $S_m \subset S_n$ ,  $m \leq n$  of small symmetric groups into larger ones give rise to embeddings of algebras  $A_k^{[m]} \subset A_k^{[n]}$ . Let  $\lambda \vdash m$  a partition, and let  $e_\lambda \in \mathbb{C}[S_m]$  be an associated minimal idempotent in the group algebra. We view  $e_\lambda$  as an element of  $A_k^{[n]}$  for any  $n > m$  via the embeddings  $\mathbb{C}[S_m] \subset A_k^{[m]} \subset A_k^{[n]}$ , and set

$$P^\lambda = A_k^{[n]}e_\lambda \text{ and } Q^\lambda = e_\lambda A_k^{[n]}\{m\}.$$

The internal grading shift  $\{m\}$  in the definition of  $Q^\lambda$  is for convenience; it ensures that various hom spaces occurring later have a grading which is symmetric about the origin.

The space  $P^\lambda$  is naturally an  $(A_k^{[n]}, A_k^{[n-m]})$  bimodule, while the space  $Q^\lambda$  is naturally an  $(A_k^{[n-m]}, A_k^{[n]})$  bimodule. By convention, we set  $P^\lambda = Q^\lambda = 0$  when  $m > n$ , and  $P^\emptyset = Q^\emptyset = A_k^{[n]}$  as an  $(A_k^{[n]}, A_k^{[n]})$  bimodule.

We denote tensor products of these bimodules by concatenation, where the tensor product is understood to be over the algebra  $A_k^{[l]}$  acting on both sides of the tensor product. So, for example, if  $\lambda \vdash m$  and  $\mu \vdash l$ , then for each  $n \geq \max\{m, l\}$ ,

$$P^\lambda Q^\mu = P^\lambda \otimes_{A_k^{[n-m]}} Q^\mu$$

is an  $(A_k^{[n]}, A_k^{[n-m+l]})$  bimodule. On the other hand,

$$Q^\mu P^\lambda = Q^\mu \otimes_{A_k^{[n]}} P^\lambda$$

is an  $(A_k^{[n-l]}, A_k^{[n-m]})$  bimodule.

Denote by  $1_0$  the trivial module over  $A_k^{[0]} = \mathbb{C}$ . The  $\mathbb{Z}[t, t^{-1}]$ -module  $\mathcal{F}_k$  comes equipped with various natural bases indexed by partitions  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \vdash n$ . Examples include

- (1)  $S^\lambda := P^\lambda 1_0$  (this module was denoted  $A_k^{[n]} e_\lambda$  in the previous subsection),
- (2)  $E^\lambda := P^{(1^{\lambda_1})} P^{(1^{\lambda_2})} \dots P^{(1^{\lambda_r})} 1_0$ , and
- (3)  $H^\lambda := P^{(\lambda_1)} P^{(\lambda_2)} \dots P^{(\lambda_r)} 1_0$ .

### 3.3. The character map

For the remainder of this paper, we consider the integral form  $\Lambda_{t^k, t}^{\mathbb{Z}}$  of  $\Lambda_{t^k, t}$ ; by definition  $\Lambda_{t^k, t}^{\mathbb{Z}}$  is the free  $\mathbb{Z}[t, t^{-1}]$ -module spanned by the Schur functions. We define a map of  $\mathbb{Z}[t, t^{-1}]$  modules

$$\Phi : \mathcal{F}_k \longrightarrow \Lambda_{t^k, t}^{\mathbb{Z}}, \quad \Phi([S_\lambda]) = s_\lambda,$$

by sending each canonical basis vector to the corresponding Schur function. It is straightforward to check that  $\Phi([E^\lambda])$  is the elementary symmetric function  $e_\lambda$ , while  $\Phi([H^\lambda])$  is the complete symmetric function  $h_\lambda$ .

The proof of the following theorem will be given at the end of this section.

**Theorem 2.** *The map  $\Phi$  is an isometry with respect to  $\langle \cdot, \cdot \rangle_{\mathcal{F}_k}$ ,  $\langle \cdot, \cdot \rangle_k$ .*

The above theorem translates to the statement that the module categories for the algebras  $A_k^{[n]}$  for all  $n$  together categorify Macdonald's ring of symmetric functions at  $q = t^k$ . When  $k = 1$ , the map  $\Phi$  is just the Frobenius character map, and the above theorem is of course well known. Note that  $\Lambda_{t, t}$  is the ring of symmetric functions (over  $\mathbb{Z}[t, t^{-1}]$ ) endowed with a bilinear form with respect to which the Schur functions  $\{s_\lambda\}$

are orthonormal basis. On the other hand,  $A_1^{[n]} = \mathbb{C}[S_n]$  is a semi-simple algebra, whence the classes of indecomposable projective (irreducible) modules give an orthonormal basis in the Grothendieck group.

### 3.4. Representation-theoretic interpretation of $c_{\mu\nu}^{\kappa\lambda}(t)$

As a consequence of [Theorem 2](#), the generalized Littlewood–Richardson coefficients  $c_{\mu\nu}^{\kappa\lambda}(t)$  inherit an interpretation as the graded dimension of vector spaces arising in the representation theory of the algebras  $A_k^{[n]}$ . To explain this interpretation, we recall the following.

**Proposition 2.** *The bimodules  $P^\lambda Q^\mu$  are indecomposable. Any bimodule of the form  $Q^\alpha P^\beta$  decomposes as a direct sum of (graded shifts of) the indecomposable bimodules  $\{P^\lambda Q^\mu\}_{\lambda, \mu}$ .*

**Proof.** As explained in, for example, [\[1, Proposition 6\]](#), it is straightforward to check that  $\text{End}(P^\lambda Q^\mu)$  is a non-negatively graded algebra whose degree 0 piece is one-dimensional. Thus  $P^\lambda Q^\mu$  is indecomposable. The fact that all indecomposable bimodules are of the form  $P^\lambda Q^\mu$  also follows just as in [\[1, Proposition 6\]](#).  $\square$

For partitions  $\kappa, \lambda, \mu, \nu$ , we may therefore define a  $\mathbb{Z}$  graded vector  $C_{\mu\nu}^{\kappa\lambda}$  as the multiplicity space of  $P^\mu Q^\kappa$  in the decomposition of  $Q^\nu P^\lambda$  into indecomposable bimodules:

$$Q^\nu P^\lambda = \bigoplus_{\mu, \kappa} P^\mu Q^\kappa \otimes_{\mathbb{C}} C_{\mu\nu}^{\kappa\lambda}.$$

**Theorem 3.** *The graded dimension of  $C_{\mu\nu}^{\kappa\lambda}$  is equal to the generalized Littlewood–Richardson coefficient  $C_{\mu\nu}^{\kappa\lambda}(t)$ .*

**Proof.** This follows immediately from [Theorems 1 and 2](#), and the normalization [\(24\)](#).  $\square$

In light of [Proposition 1](#) and [Theorem 3](#) above, it is tempting to speculate that the graded vector space  $C_{\mu\nu}^{\kappa\lambda}$  can be endowed with a linear action of the Lie algebra  $\mathfrak{sl}_2$  in a way that aligns the weight space decomposition with the grading; this would give a more conceptual explanation of the symmetry and unimodality of these coefficients.

### 3.5. Proof of [Theorem 2](#)

We now give the proof of [Theorem 2](#). For each  $n \geq m$  the  $(A_k^{[n]}, A_k^{[n-m]})$  bimodule  $P^\lambda$  is flat, as is the bimodule  $Q^\lambda$ . Summing over  $n$ , we have induced endomorphisms of the Grothendieck group

$$[P^\lambda], [Q^\lambda] : \mathcal{F}_k \longrightarrow \mathcal{F}_k.$$



The proof of the following proposition is given in [1]. That reference was concerned with the particular case  $k = 2$ , although the proof carries over with easy modification to general  $k \geq 1$ .

**Proposition 3.** (See [1, Proposition 2].) *The operators  $[P^\lambda], [Q^\lambda]$  satisfy the following properties.*

- (1)  $[P^\lambda] \circ [P^\mu] = \sum_{\nu} d_{\lambda\mu}^{\nu} [P^{\nu}]$ , where  $d_{\lambda\mu}^{\nu} \geq 0$  is the (ordinary) Littlewood–Richardson coefficient.
- (2)  $[Q^{(n)}] \circ [P^{(m)}] = \sum_{l \geq 0} \binom{k+l-1}{l}_t [P^{(m-l)}] \circ [Q^{(n-l)}]$ , where  $\binom{x}{y}_t$  is the quantum binomial coefficient.

In the above proposition, the quantum binomial coefficients are normalized to be symmetric about the origin, e.g.  $\binom{4}{2}_t = t^{-4} + t^{-2} + 2 + t^2 + t^4$ .

Now, considering symmetric functions instead of Grothendieck groups, we define endomorphisms

$$p^\lambda, q^\lambda : \Lambda_{t^k, t} \longrightarrow \Lambda_{t^k, t}$$

by letting  $p^\lambda$  be multiplication by the Schur function  $s_\lambda$  and letting  $q^\lambda$  be its adjoint with respect to  $\langle \cdot, \cdot \rangle_k$ .

**Proposition 4.** *The operators  $p^\lambda, q^\lambda$  satisfy the following properties.*

- (1)  $p^\lambda \circ p^\mu = \sum_{\nu} d_{\lambda\mu}^{\nu} p^\nu$ , where  $d_{\lambda\mu}^{\nu} \geq 0$  is the (ordinary) Littlewood–Richardson coefficient.
- (2)  $q^{(n)} \circ p^{(m)} = \sum_{l \geq 0} \binom{k+l-1}{l}_t p^{(m-l)} q^{(n-l)}$ .

**Proof.** The first statement is clear. The second follows from Example 2 and (24).  $\square$

Now Propositions 3 and 4 imply the theorem. For, the inner product  $\langle [H^\lambda], [H^\mu] \rangle_{\mathcal{F}_k}$  can be computed as

$$\begin{aligned} & \langle [P^{(\lambda_1)}][P^{(\lambda_2)}] \dots [P^{(\lambda_r)}] 1_0, [P^{(\mu_1)}][P^{(\mu_2)}] \dots [P^{(\mu_s)}] 1_0 \rangle_{\mathcal{F}_k} \\ &= \langle [Q^{(\mu_1)}][P^{(\lambda_1)}][P^{(\lambda_2)}] \dots [(P^{(\lambda_r)})] 1_0, [P^{(\mu_2)}] \dots [P^{(\mu_s)}] 1_0 \rangle_{\mathcal{F}_k} \end{aligned}$$

where the last equality used the adjointness of  $P^{(\mu_1)}$  and  $Q^{(\mu_1)}$ . Now we use the second part of Proposition 3 to write  $Q^{(\mu_1)}[P^{(\lambda_1)}][P^{(\mu_2)}] \dots [P^{(\mu_s)}] 1_0$  as a sum of  $[H^\kappa]$ s for smaller  $\kappa$ , inductively determining the  $\langle [H^\lambda], [H^\mu] \rangle_{\mathcal{F}_k}$  in terms of inner products involving smaller partitions.

Similarly, the inner product

$$\langle h_\lambda, h_\mu \rangle_k = \langle p^{\lambda_1} p^{\lambda_2} \dots p^{\lambda_r} 1_0, p^{\mu_1} p^{\mu_2} \dots p^{\mu_s} 1_0 \rangle_k$$

can be computed using the adjointness of  $p^{(\mu_1)}$  and  $q^{(\mu_1)}$ , together with the second part of [Proposition 4](#). Since the structure constants in [Propositions 3 and 4](#) agree, we conclude by induction that

$$\langle [H_\lambda], [H_\mu] \rangle_{\mathcal{F}_k} = \langle h_\lambda, h_\mu \rangle_k = \langle \Phi([H_\lambda]), \Phi([H_\mu]) \rangle_k,$$

as desired.

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