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Vertex-primitive s-arc-transitive digraphs of alternating and symmetric groups [☆]

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A fascinating problem on digraphs is the existence problem of the finite upper bound on s for all vertex-primitive s-arc-transitive digraphs except directed cycles (which is known to be reduced to the almost simple groups case). In this paper, we prove that $s \leq 2$ for all vertex-primitive s-arc-transitive digraphs for almost simple groups with socles alternating groups except one case.

1. Introduction

A *digraph* (directed graph) Γ is a pair $(V\Gamma, \to)$ with vertex set $V\Gamma$ and an antisymmetric irreflexive binary relation $\to$ on $V\Gamma$. All digraphs considered in this paper are finite. For a positive integer s, an s-arc of Γ is a sequence $v_0, v_1, \ldots, v_s$ of vertices such that $v_i \to v_{i+1}$ for each $i = 0, 1, \ldots, s-1$. A 1-arc is also simply called an *arc*. A transi-

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tive permutation group G is *primitive* on a set Ω if G preserves no nontrivial partition of Ω (or equivalently, the point stabilizer of G is maximal in G). For an automorphism group G of Γ, we call that Γ is (G, s)*-arc-transitive* if G is transitive on the set of s-arcs of Γ, and Γ is G*-vertex-primitive* if G is primitive on the vertex set of Γ. It is easy to see that s-arc-transitive digraphs with $s \geq 2$ are necessarily $(s-1)$-arc-transitive.

In sharp contrast with the undirected graphs (where a well known result of Weiss [21] states that finite undirected graphs other than cycles can only be s-arc-transitive for $s \leq 7$), Praeger [19] proved that there are infinite many s-arc-transitive digraphs for unbounded s other than directed cycles. This interesting gap stimulated a series of constructions [6,7,10,17] for such digraphs (which are called *highly transitive digraphs* in the literature). However, finding vertex-primitive s-arc-transitive digraphs with $s \geq 2$ seems to be a very intractable problem; in a survey paper of Praeger [20] in 1990, she said "no such examples have yet been found despite considerable effort by several people". The existence problem of vertex-primitive 2-arc-transitive digraphs besides directed cycles has been open until 2017 by Giudici, Li and Xia [11] by constructing an infinite family of such digraphs with valency 6, and no vertex-primitive 3-arc-transitive digraphs have been found yet. These naturally motivate the following interesting problem (posted by Giudici, Li and Xia [11]).

Question A. Is there an upper bound on s for all vertex-primitive s-arc-transitive digraphs that are not directed cycles?

A group G is said to be *almost simple* if there is a nonabelian simple group T such that $T \lhd G \leq \mathsf{Aut}(T)$. A systematic investigation of the O'Nan-Scott types of primitive permutation groups has reduced Question A to the almost simple case by proving that an upper bound on s for vertex-primitive s-arc-transitive digraphs Γ with $\mathsf{Aut}\Gamma$ almost simple will be an upper bound on s for all vertex-primitive s-arc-transitive digraphs, see [13, Corollary 1.6]. Thus a reasonable strategy for Question A is to investigate the upper bound of s for all almost simple groups (the sporadic simple groups case can generally be done especially with the help of the computer program). In this paper, we will do this for almost simple groups with socle alternating groups. The main result is as follows.

Theorem 1.1. *Let Γ be a vertex-primitive s-arc-transitive digraph, with $\mathsf{Aut}\Gamma$ an almost simple group of socle A_n, and v a vertex of Γ. Then either*

(1) $s \leq 2$*; or*
(2) $(\mathrm{A}_m \wr \mathrm{S}_k) \cap G \leq G_v \leq (\mathrm{S}_m \wr \mathrm{S}_k) \cap G$ *with* $m \geq 8$, $k > 1$ *and* $n = mk, m^k$ *or* $(m!/2)^{k-1}$.

We remark that characterizations for part (2) have been given in Lemmas 5.1–5.4 below, which actually shows $s \leq 2$ except one case.

The layout of this paper is as follows. We give some background results in Section 2. For the digraphs in Theorem 1.1, the vertex stabilizers of the automorphism groups satisfy parts (a)-(f) of Theorem 2.4 below. Parts (a) and (c) are investigated in Section 3, part (d) is considered in Section 4, and the remaining wreath product cases are treated in Section 5. We complete the proof of Theorem 1.1 in Section 6.

2. Background results

We fix the following notations in the subsequent sections, where G is a group, n is a positive integer and p is a prime.

$\pi(G)$: the set of prime divisors of the order of G.
$\pi(n)$: the set of prime divisors of n.
n_p: the maximal power of p dividing n.
$\mathrm{Sym}(\Delta)$: the symmetric group on a set Δ.
$\mathsf{soc}(G)$: the socle of G, namely the product of all minimal normal subgroups of G.
$G^{(\infty)}$: the smallest term of the derived subgroups series of G.

The following result is a consequence (also easy to prove directly) of the so-called Legendre's formula, which will be used repeatedly in this paper.

Lemma 2.1. *For each positive integer n and prime p, we have $(n!)_p < p^{\frac{n}{p-1}}$.*

For positive integers $a, m \geq 2$, a prime r is called a *primitive prime divisor* of $a^m - 1$ if r divides $a^m - 1$ but not divides $a^i - 1$ for each $i = 1, 2, \dots, m-1$. The next is a well-known theorem of Zsigmondy (see [2, Theorem IX.8.3]), where the last statement follows easily by the Fermat's Little Theorem.

Lemma 2.2. *For positive integers $a, m \geq 2$, $a^m - 1$ has a primitive prime divisor r if $(a, m) \neq (2, 6)$ and $(2^e - 1, 2)$ with $e \geq 2$ an integer. Moreover, $r \equiv 1 (\mathsf{mod}\ m)$, and in particular $r > m$.*

The next proposition is obtained by Liebeck, Praeger and Saxl, see [16, P. 296, Corollary 5].

Proposition 2.3. *Let G be an almost simple group with socle T. Suppose that L is a subgroup of G such that $\pi(T) \subseteq \pi(L)$. Then either*

(i) *$T \subseteq L$; or*
(ii) *the possibilities for T and L are given in Table 1.*

The maximal subgroups of alternating and symmetric groups are determined by Liebeck, Praeger and Saxl [14], providing a starting point of this paper.

Table 1
Subgroups L with $\pi(T) \subseteq \pi(L)$ and $T \nsubseteq L$.

Row	T	$L \cap T$	Remark
1	A_m	$\mathrm{A}_l \lhd L \leq \mathrm{S}_l \times \mathrm{S}_{c-l}$	p prime, $p < m \Rightarrow p \leq l$
2	$\mathrm{PSp}_{2m}(q)$ $(m, q$ even$)$	$L \rhd \Omega_{2m}^-(q)$	
3	$\mathrm{P\Omega}_{2m+1}(q)$ $(m$ even$, q$ odd$)$	$L \rhd \Omega_{2m}^-(q)$	
4	$\mathrm{P\Omega}_{2m}^+(q)$ $(m$ even$)$	$L \rhd \Omega_{2m-1}(q)$	
5	$\mathrm{PSp}_4(q)$	$L \rhd \mathrm{PSp}_2(q^2)$	
6	$\mathrm{PSL}_2(p)$ $(p = 2^m - 1)$	$L \leq \mathbb{Z}_p : \mathbb{Z}_{p-1}$	$G = T.2$
7	A_6	$\mathrm{PSL}(2, 5)$	
8	$\mathrm{PSL}_2(8)$	$7.2, P_1$	$G = T.3$
9	$\mathrm{PSL}_3(3)$	$13 : 3$	$G = T.2$
10	$\mathrm{PSL}_6(2)$	$P_1, P_5, \mathrm{PSL}_5(2)$	
11	$\mathrm{PSU}_3(3)$	$\mathrm{PSL}_2(7)$	
12	$\mathrm{PSU}_3(5)$	A_7	
13	$\mathrm{PSU}_4(2)$	$L \leq 2^4.\mathrm{A}_5$ *or* S_6	
14	$\mathrm{PSU}_4(3)$	$\mathrm{PSL}_3(4), \mathrm{A}_7$	
15	$\mathrm{PSU}_5(2)$	$\mathrm{PSL}_2(11)$	
16	$\mathrm{PSU}_6(2)$	M_{22}	
17	$\mathrm{PSp}_4(7)$	A_7	
18	$\mathrm{Sp}_4(8)$	${}^2B_2(8)$	$G = T.3$
19	$\mathrm{Sp}_6(2)$	$\mathrm{S}_8, \mathrm{A}_8, \mathrm{S}_7, \mathrm{A}_7$	
20	$\mathrm{P\Omega}_8^+(2)$	$L \leq P_i$ $(i = 1, 3, 4), \mathrm{A}_9$	
21	$G_2(3)$	$\mathrm{PSL}_2(13)$	
22	${}^2F_4(2)'$	$\mathrm{PSL}_2(25)$	
23	M_{11}	$\mathrm{PSL}_2(11)$	
24	M_{12}	$\mathsf{M}_{11}, \mathrm{PSL}_2(11)$	
25	M_{24}	M_{23}	
26	HS	M_{22}	
27	McL	M_{22}	
28	Co_2	M_{23}	
29	Co_3	M_{23}	

Theorem 2.4. *Let $G = \mathrm{A}_n$ or S_n, and $H \neq \mathrm{A}_n$ a maximal subgroup of G. Then H satisfies one of the following:*

(a) *$H = (\mathrm{S}_m \times \mathrm{S}_k) \cap G$, with $n = m + k$ and $m < k$ (intransitive case);*
(b) *$H = (\mathrm{S}_m \wr \mathrm{S}_k) \cap G$, with $n = mk$, $m > 1$ and $k > 1$ (imprimitive case);*
(c) *$H = \mathrm{AGL}(k, p) \cap G$, with $n = p^k$ and p prime (affine case);*
(d) *$H = (T^k.(\mathsf{Out}(T) \times \mathrm{S}_k)) \cap G$, with T a nonabelian simple group, $k > 1$ and $n = |T|^{k-1}$ (diagonal case);*
(e) *$H = (\mathrm{S}_m \wr \mathrm{S}_k) \cap G$, with $n = m^k$, $m \geq 5$ and $k > 1$ (wreath case);*
(f) *$T \lhd H \leq \mathsf{Aut}(T)$, with T a nonabelian simple group, $T \neq \mathrm{A}_n$ and H acts primitively on Ω (almost simple case).*

The following result presents a necessary and sufficient condition of s-arc-transitivity of digraphs, refer to [13, Lemma 2.2].

Lemma 2.5. *Let Γ be a digraph, and $v_0 \to v_1 \to \cdots \to v_{s-1} \to v_s$ be an s-arc of Γ with $s \geq 2$. Suppose $G \leq \mathsf{Aut}\Gamma$ acts arc-transitively on Γ. Then G acts s-arc-transitively on Γ if and only if*

$$G_{v_1 v_2 \ldots v_i} = G_{v_0 v_1 \ldots v_i} G_{v_1 \ldots v_i v_{i+1}}, \text{ for each } i \in \{1, 2, \ldots, s-1\}.$$

For a group G, an expression $G = HK$ with H and K subgroups of G is called a *factorization* of G, and H and K are called *factors* of G. In particular, $G = HK$ is called a *homogeneous factorization* if H is isomorphic to K, and is called a *maximal factorization* if both H and K are maximal subgroups of G.

Lemma 2.6. ([12, Proposition 3.3]) *Let G be an almost simple group with socle T. Suppose $G = AB$ is a homogeneous factorization. Then one of the following holds.*

(a) *Both A and B contain T.*
(b) *A and B are almost simple groups with socles both isomorphic to S, and (T, S) is listed in the following table, where q is a prime power and $f \geq 2$.*

T	A_6	M_{12}	$\mathrm{Sp}_4(2^f)$	$\mathrm{P\Omega}_8^+(q)$
S	A_5	M_{11}	$\mathrm{Sp}_2(4^f)$	$\Omega_7(q)$

Lemma 2.7. ([12, Lemma 3.5]) *Let $R \wr \mathrm{S}_k$ be a wreath product with base group $M = R_1 \times \cdots \times R_k$, where $R_1 \cong \cdots \cong R_k \cong R$, and $T \wr \mathrm{S}_k \leq G \leq R \wr \mathrm{S}_k$ with $T \leq R$. Suppose $G = AB$ is a homogeneous factorization of G such that A is transitive on $\{R_1, \ldots, R_k\}$. Denote by $\phi_i(A \cap M)$ the projection of $A \cap M$ on R_i for $i = 1, 2, \ldots, k$. Then $\phi_1(A \cap M) \cong \cdots \cong \phi_k(A \cap M)$ and $\pi(T) \subseteq \pi(\phi_1(A \cap M))$.*

Lemma 2.8. *Let G be an almost simple group with socle $T = \mathrm{PSL}_k(q)$, where $k \geq 2$ and $q = p^e$ is a prime power. If $G = HK$ with H and K subgroups of G such that $\pi(H) \cap \pi(K) \supseteq \pi(G) \setminus \pi(p(p-1))$, then either*

(i) *at least one of H and K contains T; or*
(ii) *$k = 2$ and $q = 2^e - 1 \geq 7$ is a Mersenne prime.*

Proof. Let H_0 and K_0 be maximal subgroups of G containing H and K, respectively. Then $G = H_0 K_0$ is a maximal factorization. Such factorizations for G being an almost simple group with socle $\mathrm{PSL}(d, q)$ are classified in [15, TABLE 1]. By checking the list, the lemma follows. $\square$

We give an observation to end this section. Denote by $\mathsf{val}(\Gamma)$ the out-valency of a regular digraph Γ.

Lemma 2.9. *Let Γ be a (G, s)-arc-transitive digraph with $G \leq \mathsf{Aut}\Gamma$ and $s \geq 1$. Then $\mathsf{val}(\Gamma)^s \mid |G_v|$ for each $v \in V\Gamma$.*

Proof. Set $m = \mathsf{val}(\Gamma)$, and let $v = v_0 \to v_1 \to \cdots \to v_s$ be an s-arc of Γ. Since Γ is (G, s)-arc-transitive, $G_{v_0 v_1 \ldots v_{i-1}}$ is transitive on the out-neighbor set $\Gamma^+(v_{i-1}) := \{u \in V\Gamma \mid v_{i-1} \to u\}$ for each $i = 1, 2, \ldots, s$. Then since $|\Gamma^+(v)| = \mathsf{val}(\Gamma) = m$, we deduce $|G_{v_0v_1\dots v_{i-1}} : G_{v_0v_1\dots v_i}| = m$. It follows $|G_v| = |G_{v_0}| = m^s|G_{v_0v_1\dots v_s}|$, and hence $\mathsf{val}(\Gamma)^s \mid |G_v|$. $\square$

3. Subgroups (a) and (c)

For convenience, we make the following hypothesis.

Hypothesis 3.1. Let Γ be a G-vertex-primitive (G, s)-arc-transitive digraph with $\mathsf{val}(\Gamma) \geq 3$, where $s \geq 1$ and $G = \mathrm{A}_n$ or S_n with $n \geq 5$ is an automorphism group of Γ. Take an arc $u \to v$ of Γ, and let $g \in G$ such that $u^g = v$ and set $w = v^g$. Then $u \to v \to w$ is a 2-arc of Γ. Set $\Omega = \{1, 2, \dots, n\}$. Then G acts naturally on Ω.

Under Hypothesis 3.1, $G_{vw} = G_{uv}^g$ and G_v is a maximal subgroup of G. Hence G_v satisfies one of parts $(a) - (f)$ of Theorem 2.4. In this section, we suppose Hypothesis 3.1 holds and investigate the cases where G_v satisfies parts (a) and (c).

Lemma 3.2. *Suppose G_v satisfies part (a) of Theorem 2.4. Then $s = 1$.*

Proof. Suppose for a contradiction that $s \geq 2$. By assumption, $G_v \cong (\mathrm{S}_m \times \mathrm{S}_k) \cap G$ with $n = m + k$ and $m < k$. If $m = 1$, G is 2-transitive on $V\Gamma$, so Γ is an undirected complete graph, a contradiction.

Thus assume $m \geq 2$ in the following. Notice that G has a unique conjugacy class of $(\mathrm{S}_m \times \mathrm{S}_k) \cap G$, the action of G on $V\Gamma$ is permutation equivalent to the natural induced action of G on $\Omega^{\{m\}}$, the set of m-subsets of Ω. We may thus identify $V\Gamma$ with $\Omega^{\{m\}}$ and set $v = \Delta := \{1, 2, \dots, m\}$. Then $G_v = (\mathrm{Sym}\{1, \dots, m\} \times \mathrm{Sym}\{m+1, \dots, n\}) \cap G$. Let ϕ be the projection of G_v on $\mathrm{Sym}\{m+1, \dots, n\}$. It is easy to see that $\phi(G_v) = \mathrm{Sym}\{m+1, \dots, n\}$.

Since $w = \Delta^g \neq \Delta$, we may assume $\Delta^g \cap \{m+1, \dots, n\} = \{h_1, \dots, h_l\}$ and $\Delta^{g^{-1}} \cap \{m+1, \dots, n\} = \{k_1, \dots, k_l\}$, with $1 \leq l \leq m$. Since $s \geq 2$, $G_v = G_{uv}G_{vw}$ by Lemma 2.5, hence $\phi(G_v) = \phi(G_{uv})\phi(G_{vw})$. Consequently, we obtain the following homogeneous factorization

$$\begin{aligned}\mathrm{Sym}\{m+1, \dots, n\} = (&\mathrm{Sym}\{k_1, \dots, k_l\} \times \mathrm{Sym}(\{m+1, \dots, n\} \setminus \{k_1, \dots, k_l\})) \cdot \\ &(\mathrm{Sym}\{h_1, \dots, h_l\} \times \mathrm{Sym}(\{m+1, \dots, n\} \setminus \{h_1, \dots, h_l\})) \quad (1)\end{aligned}$$

If $n - m \leq 4$, as $2 \leq m < k = n - m$, we have $n - m = 3$ or 4, and one easily verifies Equation (1) is impossible in the case, a contradiction. Suppose $n - m \geq 5$, by Lemma 2.6, the only possibility is $n - m = 6$ and $l = 1$ or 5. Then Equation (1) leads to

$$\mathrm{Sym}\{m+1, \dots, m+6\} = \mathrm{Sym}\{i_1, \dots, i_5\}\mathrm{Sym}\{j_1, \cdots, j_5\},$$

where $\{i_1,\dots,i_5\}$ and $\{j_1,\dots,j_5\}$ are distinct subsets of $\{m+1,\dots,m+6\}$. Noting that $\mathrm{Sym}\{i_1,\dots,i_5\}\cap\mathrm{Sym}\{j_1,\cdots,j_5\}\le \mathrm{S}_4$, we derive that $\mathrm{Sym}\{i_1,\dots,i_5\}\mathrm{Sym}\{j_1,\cdots,j_5\}$ is of order divisible by 25, which is a contradiction as $25\nmid|\mathrm{S}_{\{m+1,\dots,m+6\}}|$. $\square$

Lemma 3.3. *Suppose G_v satisfies part (c) of Theorem 2.4. Then $s=1$.*

Proof. Suppose for a contradiction that $s\ge 2$. By assumption, $G_v\cong \mathrm{AGL}(k,p)\cap G$ with $n=p^k$ and p a prime, so $\mathsf{soc}(G_v)\cong\mathbb{Z}_p^k$. If $k=1$, then $\mathbb{Z}_p:\mathbb{Z}_{(p-1)/2}\le G_v\le\mathbb{Z}_p:\mathbb{Z}_{p-1}$, and as $G_v=G_{uv}G_{vw}$ and $G_{vw}=G_{uv}^g$, we have $\mathsf{soc}(G_{uv})=\mathsf{soc}(G_{vw})=\mathsf{soc}(G_v)\cong\mathbb{Z}_p$. It follows

$$(\mathsf{soc}(G_v))^g=(\mathsf{soc}(G_{uv}))^g=\mathsf{soc}(G_{uv}^g)=\mathsf{soc}(G_{vw})=\mathsf{soc}(G_v). \tag{2}$$

Consequently, $\mathsf{soc}(G_v)\lhd G$ as $\langle G_v,g\rangle=G$, hence G acts unfaithfully on $V\Gamma$, a contradiction. If $k\ge 2$, since $n\ge 5$, $(k,p)\ne(2,2)$. If $(k,p)=(2,3)$, then $G_v\cong\mathbb{Z}_3^2:2\mathrm{A}_4$ or $\mathbb{Z}_3^2:2\mathrm{S}_4$, a direct computation by Magma [3] shows that G_v has no homogeneous factorization $G_v=G_{uv}G_{vw}$ with $|G_v:G_{uv}|\ge 3$, a contradiction.

Thus assume in the following that $k\ge 2$, and $(k,p)\ne(2,2)$ and $(2,3)$. Then G_v is insoluble. Let M be a normal subgroup of $\mathrm{AGL}(k,p)$ such that $M\cong\mathbb{Z}_p^k:\mathbb{Z}_{p-1}$, and set $\overline{G_v}=G_vM/M$, $\overline{G_{uv}}=G_{uv}M/M$ and $\overline{G_{vw}}=G_{vw}M/M$. Then $\overline{G_v}$ is almost simple with $\mathsf{soc}(\overline{G_v})\cong\mathrm{PSL}_k(p)$. Since $G_v=G_{uv}G_{vw}$ and $G_{uv}\cong G_{vw}$, we conclude $\pi(G_{uv})=\pi(G_{vw})=\pi(G_v)$ and $\overline{G_v}=\overline{G_{uv}}\,\overline{G_{vw}}$. It follows

$$\pi(\overline{G_{uv}})\cap\pi(\overline{G_{vw}})\supseteq(\pi(G_{uv})\cap\pi(G_{vw}))\setminus\pi(M)\supseteq\pi(\overline{G_v})\setminus\pi(p(p-1)).$$

By Lemma 2.8, either

(i) at least one of $\overline{G_{uv}}$ and $\overline{G_{vw}}$ contains $\mathsf{soc}(\overline{G_v})$; or
(ii) $k=2$ and $p=2^e-1\ge 7$ is a Mersenne prime.

First assume case (i) occurs. Without loss of generality, we may suppose $\overline{G_{uv}}\supseteq\mathsf{soc}(\overline{G_v})$. Since $\overline{G_v}$ is almost simple, $\mathsf{soc}(\overline{G_{uv}})=\mathsf{soc}(\overline{G_v})\cong\mathrm{PSL}_k(p)$. Then as

$$(M\cap G_{vw}).\overline{G_{vw}}\cong G_{vw}\cong G_{uv}\cong(M\cap G_{uv}).\overline{G_{uv}},$$

and M is soluble, both $\overline{G_{uv}}$ and $\overline{G_{vw}}$ have the same unique insoluble composition factor $\mathrm{PSL}_k(p)$. Since

$$\overline{G_{vw}}/(\overline{G_{vw}}\cap\mathsf{soc}(\overline{G_v}))\cong\overline{G_{vw}}\mathsf{soc}(\overline{G_v})/\mathsf{soc}(\overline{G_v})\le\overline{G_v}/\mathsf{soc}(\overline{G_v})$$

is soluble, $\mathrm{PSL}_k(p)$ is a composition factor of $\overline{G_{vw}}\cap\ \mathsf{soc}(\overline{G_v})$, we further conclude $\mathsf{soc}(\overline{G_{vw}})=\mathsf{soc}(\overline{G_v})$ as $\overline{G_v}$ is almost simple.

If both G_{uv} and G_{vw} have nontrivial intersections with $\mathsf{soc}(G_v)$, then $G_{uv} \cap \mathsf{soc}(G_v)$ has a subgroup $\langle a_1 \rangle \cong \mathbb{Z}_p$, and there exist $a_2, \dots, a_k$ such that $\mathsf{soc}(G_v) = \langle a_1 \rangle \times \cdots \times \langle a_k \rangle \cong \mathbb{Z}_p^k$. Set $\Delta = \{\langle a_1 \rangle, \langle a_2 \rangle, \dots, \langle a_k \rangle\}$. Since $\mathsf{soc}(G_v)$ is the unique minimal normal subgroup of $\mathrm{ASL}(k,p)$, $\mathrm{SL}(k,p)$ and so $\mathrm{PSL}_k(p)$ acts transitive on Δ (note that the center $\mathbf{Z}(\mathrm{SL}(k,p))$ acts trivially on Δ), we conclude $\mathsf{soc}(G_{uv}) = \mathsf{soc}(G_v)$ because $\overline{G_{uv}} \supseteq \mathrm{PSL}_k(p)$. Similarly, one has $\mathsf{soc}(G_{vw}) = \mathsf{soc}(G_v)$. It then follows from Equation (2) that $\mathsf{soc}(G_v) \lhd G$, hence G acts unfaithfully on $V\Gamma$, a contradiction.

Suppose one of G_{uv} and G_{vw}, say G_{uv}, has trivial intersection with $\mathsf{soc}(G_v)$. Then $G_{uv} \le \mathrm{GL}(d,p)$, and as $G_v = G_{uv}G_{vw}$ and G_{uv} has a composition factor isomorphic to $\mathrm{PSL}(d,p)$, one easily derives $|G_{uv} : G_{uv} \cap G_{vw}| = rp^d$ and $G_{uv}^{(\infty)} \cong \mathrm{SL}(d,p)$, where $r \mid p-1$. If $d = 2$, then $|G_{uv}|_p = |\mathrm{PSL}(2,p)|_p = p$, so G_{uv} has no subgroup with index rp^2, a contradiction. Suppose $d \geqslant 4$. Let $L = G_{uv}/Z(G_{uv})$ and $R = (G_{uv} \cap G_{vw})Z(G_{uv})/Z(G_{uv})$. Then $\mathsf{soc}(L) \cong \mathrm{PSL}(d,p)$ and $|L{:}R| = r_1 p^d$ with $r_1 \mid r$. Observe that $|\mathrm{PSL}(d,p)|$ is divisible by $(p-1)^2$ and $|\mathrm{PSL}(d,p)|_p = p^{d(d-1)/2} > p^d$, we conclude that $\pi(L) = \pi(R)$. It then follows from Proposition 2.3 that $\mathsf{soc}(L) = \mathrm{PSL}(6,2)$, namely $(d,p) = (6,2)$. Consequently, we have $G_{uv} \cong \mathrm{PSL}(6,2)$, which is a contradiction as $\mathrm{PSL}(6,2)$ has no subgroup of index 2^6.

Therefore, $d = 3$ and $G_{uv}^{(\infty)} \cong \mathrm{SL}(3,p)$. Let M be a maximal subgroup of $G_{uv}^{(\infty)}$ containing $G_{uv}^{(\infty)} \cap G_{vw}$. Since $|G_{uv}^{(\infty)} : G_{uv}^{(\infty)} \cap G_{vw}|$ divides $|G_{uv} : G_{uv} \cap G_{vw}|$, we have $|G_{uv}^{(\infty)}{:}M| = r_2 p^e$, where $r_2 \mid p-1$ and $0 \le e \le 3$. If $e \le 2$, then $|G_{uv}^{(\infty)}/Z(G_{uv}^{(\infty)}) : MZ(G_{uv}^{(\infty)})/Z(G_{uv}^{(\infty)})|$ divides $(p-1)p^2$. Since

$$|G_{uv}^{(\infty)}/Z(G_{uv}^{(\infty)})| = |\mathrm{PSL}(3,p)| = \frac{1}{(3,p-1)} p^3(p-1)^2(p+1)(p^2+p+1),$$

one easily sees that $|G_{uv}^{(\infty)}/Z(G_{uv}^{(\infty)})|$ is divisible by $(p-1)^2$, hence $\pi(G_{uv}^{(\infty)}/Z(G_{uv}^{(\infty)})) = \pi(MZ(G_{uv}^{(\infty)})/Z(G_{uv}^{(\infty)}))$. By Proposition 2.3, we obtain $MZ(G_{uv}^{(\infty)}) = G_{uv}^{(\infty)}$ (notice that $G_{uv}^{(\infty)}/Z(G_{uv}^{(\infty)}) \cong \mathrm{PSL}(3,3)$ is not the case), so the commutator subgroup $M' = (G_{uv}^{(\infty)})' = G_{uv}^{(\infty)}$, a contradiction. Hence $|G_{uv}^{(\infty)} : M| = r_2 p^3$. Checking the maximal subgroups of $\mathrm{SL}(3,p)$, refer to [4, Tables 8.3, 8.4], the only possibility is $M \cong \mathbb{Z}_{p^2+p+1}{:}\mathbb{Z}_3$, so $|G_{uv}^{(\infty)}{:}M| = p^3 \cdot \frac{(p^2-1)(p-1)}{3}$. Then since $|G_{uv}^{(\infty)}{:}M|$ divides $p^3(p-1)$, it follows that $p = 2$, $G_v \cong \mathrm{AGL}(3,2)$ and $G_{uv} \cong G_{vw} \cong \mathrm{GL}(3,2)$. However, in this case, $|V\Gamma| = |G : G_v| = 15$ and $|\mathsf{val}(\Gamma)| = |G_v : G_{uv}| = 8 > |V\Gamma|/2$, so Γ is an undirected graph, again a contradiction.

Now assume case (ii) occurs. Then $\overline{G_v} \cong \mathrm{PSL}_2(p).o$ with $o = 1$ or $\mathbb{Z}_2$, and we may assume none of $\overline{G_{uv}}$ and $\overline{G_{vw}}$ contains $\mathsf{soc}(\overline{G_v})$ by (i). Since $\overline{G_v} = \overline{G_{uv}}\,\overline{G_{vw}}$, we have (interchange $\overline{G_{uv}}$ and $\overline{G_{vw}}$ if necessary) $\overline{G_{uv}} \le \mathbb{Z}_p : \mathbb{Z}_{\frac{p-1}{2}}.o$, hence $|\overline{G_{uv}}|_2 \le |o|$ as $p = 2^e - 1$. Since $\overline{G_{uv}} \cong G_{uv}/(G_{uv} \cap M)$, we have $|G_{uv}|_2 \le |M|_2|o| = 2|o|$. It follows

$$2^{e+1}|o| = 2|\mathrm{PSL}_2(2^e-1).o|_2 = |G_v|_2 \le |G_{uv}|_2^2 \le 4|o|^2 \le 8|o|,$$

implying $e \le 2$ and so $p \le 3$, also a contradiction. $\square$

4. Subgroups (d)

Suppose Hypothesis 3.1 holds. In this section, we consider that case where G_v satisfies part (d) of Theorem 2.4, namely

$$G_v = (T^k.(\mathsf{Out}(T) \times \mathrm{S}_k)) \cap G,$$

with T a nonabelian simple group, $k \geq 2$ and $n = |T|^{k-1}$.

Let M be a normal subgroup of $T^k.(\mathsf{Out}(T) \times \mathrm{S}_k)$ isomorphic to $T^k.\mathsf{Out}(T)$. Set $\overline{G_v} = G_vM/M$, $\overline{G_{uv}} = G_{uv}M/M$ and $\overline{G_{vw}} = G_{vw}M/M$. Notice that $T \wr \mathrm{A}_k = T^k : \mathrm{A}_k \leq G_v \leq \mathsf{Aut}(T) \wr \mathrm{S}_k = \mathsf{Aut}(T)^k : \mathrm{S}_k$, and let ϕ_i be the projection of $G_v \cap M$ on the i-component of $\mathsf{Aut}(T)^k$ for $1 \leq i \leq k$. Clearly, $\mathrm{A}_k \leq \overline{G_v} \leq \mathrm{S}_k$, and $\phi_i(G_v \cap M)$ is almost simple with socle T.

Lemma 4.1. *If Γ is $(G, 2)$-arc-transitive, then (interchange $\overline{G_{uv}}$ and $\overline{G_{vw}}$ if necessary) $\overline{G_{uv}} \leq \mathrm{S}_k$ is transitive and $\phi_1(G_{uv} \cap M) \cong \cdots \cong \phi_k(G_{uv} \cap M)$. Further, either $\phi_i(G_{uv} \cap M) \supseteq T$, or the couple $(T, \phi_i(G_{uv} \cap M))$ (as (T, L) there) satisfies Table 1 of Proposition 2.3.*

Proof. Since Γ is $(G, 2)$-arc-transitive, $G_v = G_{uv}G_{vw}$, hence $\overline{G_v} = \overline{G_{uv}}\,\overline{G_{vw}}$. Since $\overline{G_v} \cong \mathrm{A}_k$ or S_k, by [12, Lemma 2.3], at least one of $\overline{G_{uv}}$ or $\overline{G_{vw}}$, say $\overline{G_{uv}}$, is a transitive subgroup of S_k. It then follows from Lemma 2.7 that $\phi_1(G_{uv} \cap M) \cong \cdots \cong \phi_k(G_{uv} \cap M)$, and $\pi(T) \subseteq \pi(\phi_1(G_{uv} \cap M))$. Now by Proposition 2.3, the lemma follows. $\square$

The following lemma treats the case where $\phi_1(G_{uv} \cap M)$ contains T.

Lemma 4.2. *Assume $T \subseteq \phi_1(G_{uv} \cap M)$. Then $s \leq 2$.*

Proof. Suppose on the contrary $s \geq 3$. By Lemma 4.1, we may assume $\phi_1(G_{uv} \cap M) \cong \ldots \cong \phi_k(G_{uv} \cap M) \cong T.o$ with $o \leq \mathsf{Out}(T)$, and $\overline{G_{uv}} \leq \mathrm{S}_k$ is transitive. It follows that $G_{uv} \cap M$ has a unique insoluble composition factor T with multiplicity (say l) dividing k.

We first prove $l = k$. If not, then $l \leq \frac{k}{2}$ as $l \mid k$. It is known (or see [16, P. 297, Corollary 6]) that there is a prime $r \geq 5$ such that $r \mid |T|$ but $r \nmid |\mathsf{Out}(T)|$. So $|G_{uv} \cap M|_r = |T|_r^l \leq |T|_r^{k/2}$, and hence

$$\mathsf{val}(\Gamma)_r = \frac{|G_v|_r}{|G_{uv}|_r} \geq \frac{|G_v|_r}{|G_{uv} \cap M|_r |\mathrm{S}_k|_r} \geq \frac{|T|_r^k (k!)_r}{|T|_r^{k/2} (k!)_r} = |T|_r^{k/2}.$$

Since $s \geq 3$, by Lemma 2.9, $\mathsf{val}(\Gamma)_r^3 \leq |G_v|_r$, thus $|T|_r^{3k/2} \leq |T|_r^k (k!)_r$. However, as $(k!)_r < r^{\frac{k}{r-1}}$ by Lemma 2.1, we conclude $|T|_r^{k/2} < r^{\frac{k}{r-1}}$, which is a contradiction as $r \geq 5$.

Thus $l = k$. Consequently, $G_{uv} \cap M \supseteq \mathsf{soc}(G_v) \cong T^k$. Since $\overline{G_{uv}}$ is transitive, and the centralizer of $\mathsf{soc}(G_v)$ is G_v is trivial, we further conclude that $\mathsf{soc}(G_v)$ is the unique minimal normal subgroup of G_{uv}, namely $\mathsf{soc}(G_{uv}) = \mathsf{soc}(G_v)$. Set $N = \mathsf{soc}(G_{vw})$. Then $N \cong T^k$ is the unique minimal normal subgroup of G_{vw} as $G_{vw} \cong G_{uv}$. Clearly, $N \cap \mathsf{soc}(G_v) \lhd G_{vw}$, by the minimality of N, either $N \cap \mathsf{soc}(G_v) = 1$ or $N = \mathsf{soc}(G_v)$. For the former case, we have

$$N = N/(N \cap \mathsf{soc}(G_v)) \cong \mathsf{soc}(G_v)N/\mathsf{soc}(G_v) \le G_v/\mathsf{soc}(G_v) \le \mathsf{Out}(T) \times \mathrm{S}_k.$$

By Lemma 2.1, we obtain $|T|_r^k = |N|_r < (k!)_r < r^{\frac{k}{r-1}}$, a contradiction. Therefore, $\mathsf{soc}(G_{vw}) = \mathsf{soc}(G_v) = \mathsf{soc}(G_{uv})$. It then follows from Equation (2) that $\mathsf{soc}(G_v)$ is normal in $\langle G_v, g\rangle = G$, thus G acts unfaithfully on $V\varGamma$, yielding a contradiction. □

To treat the candidates in Table 1, we first prove two lemmas.

Lemma 4.3. *Suppose $\varGamma$ is $(G,2)$-arc-transitive. Then for each prime r, we have*

$$|T|_r < r|\phi_1(G_{uv} \cap M)|_r^2.$$

Proof. Suppose $|\phi_1(G_{uv} \cap M)|_r = r^l$ and $G \cong \mathrm{A}_n.o$ with $o \le \mathbb{Z}_2$. Then

$$|G_{uv}|_r \le |G_{uv} \cap M|_r|\mathrm{A}_k.o|_r \le |\phi_1(G_{uv} \cap M)|_r^k|\mathrm{A}_k.o|_r = r^{kl}(\frac{k!}{2})_r|o|_r.$$

Since $\varGamma$ is $(G,2)$-arc-transitive, $G_v = G_{uv}G_{vw}$, we obtain

$$|T|_r^k|\mathsf{Out}(T)|_r(\frac{k!}{2})_r|o|_r \le |G_v|_r \le |G_{uv}|_r^2 \le r^{2kl}(\frac{k!}{2})_r^2|o|_r^2.$$

This together with Lemma 2.1 implies $|T|_r^k \le r^{2kl}(k!)_r < r^{2kl+\frac{k}{r-1}}$, hence $|T|_r < r^{2l+\frac{1}{r-1}} < r^{2l+1}$, the lemma follows. □

Lemma 4.4. *Suppose $\varGamma$ is $(G,3)$-arc-transitive. Then for each prime r, we have*

$$|T|_r^{2k} < r^{\frac{k}{r-1}}|\phi_1(G_{uv} \cap M)|_r^{3k}|\mathsf{Out}(T)|_r.$$

Proof. Suppose $|\phi_1(G_{uv} \cap M)|_r = r^l$ and $G \cong \mathrm{A}_n.o$ with $o \le \mathbb{Z}_2$. Then $|G_{uv}|_r \le r^{kl}(\frac{k!}{2})_r|o|_r$, and since $\mathsf{val}(\varGamma) = |G_v : G_{uv}|$, we obtain

$$\mathsf{val}(\varGamma)_r = \frac{|G_v|_r}{|G_{uv}|_r} \ge \frac{|T|_r^k|\mathsf{Out}(T)|_r(\frac{k!}{2})_r|o|_r}{r^{kl}(\frac{k!}{2})_r|o|_r} \ge \frac{|T|_r^k}{r^{kl}}.$$

Since $\varGamma$ is $(G,3)$-arc-transitive, by Lemma 2.9, $\mathsf{val}(\varGamma)^3 \mid |G_v|$. It follows that $|T|_r^{3k}$ divides $r^{3kl}|T|_r^k|\mathsf{Out}(T)|_r(k!)_r$, then the lemma follows by Lemma 2.1. □

We now analyse the candidates in Table 1. The discussions need information of the orders and the outer automorphism groups of certain simple groups, for those we refer to [15, P. 18–20].

Table 2
Triples $(|T|_r, |\mathsf{Out}(T)|, |\phi_1(G_{uv} \cap M)|_r)$ of 'sporadic' cases in Lemma 4.5.

l	T	$\lvert T\rvert_r$	$\lvert\mathsf{Out}(T)\rvert$	$\lvert\phi_1(G_{uv} \cap M)\rvert_r$	r
9	$\mathrm{PSL}_3(3)$	3^3	2	3	3
11	$\mathrm{PSU}_3(3)$	3^3	2	3	3
12	$\mathrm{PSU}_3(5)$	5^3	6	5	5
14	$\mathrm{PSU}_4(3)$	3^6	8	3^2	3
15	$\mathrm{PSU}_5(2)$	3^5	2	3	3
16	$\mathrm{PSU}_6(2)$	3^6	6	3^2	3
17	$\mathrm{PSp}_4(7)$	7^4	2	7	7
18	$\mathrm{Sp}_4(8)$	3^4	6	3	3
21	$G_2(3)$	3^6	3	$\leq 3^2$	3
22	${}^2F_4(2)'$	3^3	2	3	3
26	HS	5^3	2	5	5
27	McL	3^6	2	3^2	3
28	Co_2	3^6	1	3^2	3
29	Co_3	3^7	1	3^2	3

Lemma 4.5. *Suppose G_v satisfies Row l of Table 1, where $l \in \{6, 9, 11, 12, 14-18, 21, 22, 26-29\}$. Then $s = 1$.*

Proof. Suppose on the contrary, $s \geq 2$. We divide the proof into two cases.

Row 6. Then $T = \mathrm{PSL}_2(p)$ with $p = 2^m - 1$ a Mesenna prime, and $\phi_1(G_{uv} \cap M) \leq \mathbb{Z}_p : \mathbb{Z}_{p-1}$. It follows $|T|_2 = 2^m$ and $|\phi_1(G_{uv} \cap M)|_2 \leq 2$. Hence Lemma 4.3 implies $2^m < 2^3$, thus $m \leq 2$ and T is soluble, a contradiction.

Remaining rows. Then the simple groups T are specific with no parameter, and either $\phi_1(G_{uv} \cap M) \cap T$ or $\phi_1(G_{uv} \cap M)$ is given in Table 1. Since $\phi_1(G_{uv} \cap M)/(\phi_1(G_{uv} \cap M) \cap T) \leq \mathsf{Out}(T)$, we have $|\phi_1(G_{uv} \cap M)|_r \leq |\mathsf{Out}(T)|_r |\phi_1(G_{uv} \cap M) \cap T|_r$. Then a direct computation shows that the triple $(|T|_r, |\mathsf{Out}(T)|, |\phi_1(G_{uv} \cap M)|_r)$ for some prime r lies in Table 2. For each row there, we always have $|T|_r \geq r|\phi_1(G_{uv} \cap M)|_r^2$, contradicting Lemma 4.3. $\square$

Lemma 4.6. *Suppose G_v satisfies Row l of Table 1 with $1 \in \{2-5, 7, 8, 10, 13, 19, 20, 23-25\}$. Then $s \leq 2$.*

Proof. Suppose on the contrary, $s \geq 3$. We consider each row in the following.

Row 2. Then

$$|T| = |\mathrm{PSp}_{2m}(q)| = q^{m^2} \prod_{i=1}^{m} (q^{2i} - 1)$$

with m, q even. Set $q = 2^e$. Assume $em \neq 6$. By Lemma 2.2, $2^{em} - 1 = q^m - 1$ has a primitive prime divisor $r > em$. Set $(q^m - 1)_r = r^l$. Since r does not divide $q^i - 1$ and $q^{m+i} - 1$ (as $q^{m+i} - 1 = q^m(q^i - 1) + (q^m - 1)$) for each $1 \leq i \leq m - 1$, we have $|T|_r = (q^m - 1)_r (q^{2m} - 1)_r = r^{2l}$. Since $\Omega_{2m}^-(q) \lhd \phi_1(G_{uv} \cap M)$, $r > em$ and

$$|\Omega_{2m}^-(q)| = \frac{1}{2}q^{m(m-1)}(q^m+1)\prod_{i=1}^{m-1}(q^{2i}-1),$$

we conclude $|\phi_1(G_{uv}\cap M)|_r = (q^m-1)_r = r^l$, and as q is even, $|\mathsf{Out}(T)| = e$ if $m \geq 3$, and $|\mathsf{Out}(T)| = 2e$ if $m = 2$, so $r \nmid |\mathsf{Out}(T)|$. Then Lemma 4.4 implies $r^{4kl} < r^{3kl+\frac{k}{r-1}}$, a contradiction.

Assume now $em = 6$. Since m is even, $(q, m) = (2, 6)$ or $(2^3, 2)$, and $T = \mathrm{PSp}_{12}(2)$ or $\mathrm{PSp}_4(8)$ respectively. For both cases, we have $|T|_7 = 7^2$, $|\phi_1(G_{uv}\cap M)|_7 = 7$, and $|\mathsf{Out}(T)| = 1$ if $m = 6$ and $|\mathsf{Out}(T)| = 6$ if $m = 2$. By Lemma 4.4, we obtain $7^{4k} < 7^{3k+\frac{k}{6}}$, also a contradiction.

Row 3. Then

$$|T| = |\mathrm{P}\Omega_{2m+1}(q)| = \frac{1}{2}q^{m^2}\prod_{i=1}^{m}(q^{2i}-1)$$

with m even and $q = p^e$ an odd prime power.

If $(p, e, m) = (2^t-1, 1, 2)$ for some t, one easily deduces $|T|_p = p^4$ and $|\phi_1(G_{uv}\cap M)|_p = p^2$. Since $\mathsf{Out}(T) = \mathbb{Z}_2$, Lemma 4.4 implies $p^{8k} < p^{6k+\frac{k}{p-1}}$, a contradiction.

Assume $(p, e, m) \neq (2^t-1, 1, 2)$. By Lemma 2.2, $p^{em}-1 = q^m-1$ has a primitive prime divisor $r > em$. With similarly discussion as in Row 1, we have $|T|_r = (q^m-1)_r^2$, $|\phi_1(G_{uv}\cap M)|_r = (q^m-1)_r$ and $(r, |\mathsf{Out}(T)|) = (r, 2e) = 1$. It then follows from Lemma 4.4 that $(q^m-1)_r^{2k} < (q^m-1)_r^k r^{\frac{k}{r-1}}$, also a contradiction.

Row 4. Then

$$|T| = |\mathrm{P}\Omega_{2m}^+(q)| = \frac{1}{(4, q^m-1)}q^{m(m-1)}(q^m-1)\prod_{i=1}^{m-1}(q^{2i}-1)$$

with m even. Set $q = p^e$ with p a prime.

Assume first $(p, em) \neq (2, 6)$. Then $p^{em}-1 = q^m-1$ has a primitive prime divisor $r > em$. Set $(q^m-1)_r = r^l$. A similar discussion as in the proof of Row 1 implies $|T|_r = (q^m-1)_r^2 = r^{2l}$, and since

$$|\varOmega_{2m-1}(q)| = \frac{1}{2}q^{(m-1)^2}\prod_{i=1}^{m-1}(q^{2i}-1),$$

we have $|\phi_1(G_{uv}\cap M)|_r = (q^m-1)_r = r^l$. Notice that $|\mathsf{Out}(T)|$ divides $24e$ and $(r, |\mathsf{Out}(T)|) = 1$, Lemma 4.4 implies $r^{4kl} < r^{3kl+\frac{k}{r-1}}$, a contradiction.

Now consider the case $(p, em) = (2, 6)$. Since m is even, we have $(m, q) = (2, 8)$ or $(6, 2)$, and so $T = \mathrm{P}\Omega_4^+(8)$ or $\mathrm{P}\Omega_{12}^+(2)$ respectively. In particular, $|T|_7 = 7^2$ and $|\mathsf{Out}(T)|_7 = 1$ for both cases. Since $\Omega_{2m-1}(q) \lhd \phi_1(G_{uv}\cap M)$, one has $|\phi_1(G_{uv}\cap M)|_7 = 7$. Then Lemma 4.4 leads to $7^{4k} < 7^{3k+\frac{k}{6}}$, also a contradiction.

Table 3
Triples $(|T|_r, |\mathsf{Out}(T)|, |\phi_1(G_{uv} \cap M)|_r)$ in Lemma 4.6.

l	T	$\|T\|_r$	$\|\mathsf{Out}(T)\|$	$\|\phi_1(G_{uv} \cap M)\|_r$	r
7	A_6	3^2	2	3	3
8	$\mathrm{PSL}_2(8)$	3^2	3	3	3
10	$\mathrm{PSL}_6(2)$	3^4	2	3^2	3
13	$\mathrm{PSU}_4(2)$	3^4	2	3^2	3
19	$\mathrm{Sp}_6(2)$	3^4	1	3^2	3
20	$\mathrm{P}\Omega_8^+(2)$	2^{12}	6	$\leq 2^7$	2
23	M_{11}	3^2	1	3	3
24	M_{12}	3^3	2	$\leq 3^2$	3
25	M_{24}	3^3	1	3^2	3

Row 5. Then

$$|T| = |\mathrm{PSp}_4(q)| = \frac{1}{(2, q-1)} q^4(q^2-1)(q^4-1)$$

and $|\mathsf{Out}(T)| = 2e$, where $q = p^e$ is a prime power. Since $\mathrm{PSp}_2(q^2) \lhd \phi_1(G_{uv} \cap M)$, we have $\phi_1(G_{uv} \cap M) \cap T \leq \mathrm{PSp}_2(q^2).\mathbb{Z}_2$. It follows $|\phi_1(G_{uv} \cap M)|$ divides $2|\mathrm{PSp}_2(q^2)||\mathsf{Out}(T)|$ and hence divides $4eq^2(q^4-1)$.

If $(p, e) = (2^t - 1, 1)$, then $|T|_p = p^4$, $\mathsf{Out}(T) = \mathbb{Z}_2$ and $|\phi_1(G_{uv} \cap M)|_p$ divides p^2, by Lemma 4.4, we have $p^{8k} < p^{6k+\frac{k}{p-1}}$, a contradiction. Similarly, if $(p, e) = (2, 3)$, then $|T|_7 = 7^2$, $|\mathsf{Out}(T)| = 6$ and $|\phi_1(G_{uv} \cap M)|_7 = 7$, hence Lemma 4.4 implies $7^{4k} < 7^{3k+\frac{k}{6}}$, a contradiction.

Assume now $(p, e) \neq (2^s - 1, 1)$ and $(2, 3)$. Then $p^{2e} - 1$ has a primitive prime divisor $r > 2e$. Set $(p^{2e} - 1)_r = r^l$. Then $|T|_r = r^{2l}$, $(r, |\mathsf{Out}(T)|) = 1$ and $|\phi_1(G_{uv} \cap M)|_r \leq r^l$. It then follows from Lemma 4.4 that $r^{4kl} < r^{3kl+\frac{k}{r-1}}$, also a contradiction.

Row 7. Since $\mathrm{PSL}(2, 5) \cong A_5$, the above discussion with $c = 6$ draws a contradiction.

Remaining rows. With similar discussions as in the proof of Lemma 4.5, it is routine to compute out that the triple $(|T|_r, |\mathsf{Out}(T)|, |\phi_1(G_{uv} \cap M)|_r)$ with r a prime lies in Table 3. For each row there, we always have $|T|_r^{2k} > r^{\frac{k}{r-1}} |\phi_1(G_{uv} \cap M)|_r^{3k} |\mathsf{Out}(T)|_r$, by Lemma 4.4, it is a contradiction. □

Summarize Lemmas 4.1, 4.2, 4.5 and 4.6, we have the following.

Lemma 4.7. *Suppose G_v satisfies part (d) of Theorem 2.4. Then either*

(i) $s \leq 2$; *or*
(ii) $G_v = (A_m^k.(\mathsf{Out}(A_m) \times S_k) \cap G$ *with* $m \geq 5$.

5. Wreath product subgroups

Suppose Hypothesis 3.1 holds. In this section, we assume that G_v satisfies part (b), (e) or part (ii) of Lemma 4.7. Then we always have

$$(\mathrm{A}_m \wr \mathrm{S}_k) \cap G \leq G_v \leq (\mathrm{S}_m \wr \mathrm{S}_k) \cap G,$$

where $m > 1$ and $k > 1$. Let $M \cong \mathrm{S}_m^k$ be the base group of $\mathrm{S}_m \wr \mathrm{S}_k$, and let $\phi_i(G_v \cap M)$ denote the projections of $\mathrm{G}_v \cap M$ on the i-component of M with $1 \leq i \leq k$. Set $\overline{G_v} = G_vM/M$, $\overline{G_{uv}} = G_{uv}M/M$ and $\overline{G_{vw}} = G_{vw}M/M$. Then $G_v = (G_v \cap M).\overline{G_v}$, $G_{uv} = (G_{uv} \cap M).\overline{G_{uv}}$, and $G_{vw} = (G_{vw} \cap M).\overline{G_{vw}}$.

Lemma 5.1. *With above assumption and suppose $s \geq 2$. Then (interchange $\overline{G_{uv}}$ and $\overline{G_{vw}}$ if necessary) $\overline{G_{uv}} \leq \mathrm{S}_k$ is transitive and $\phi_1(G_{uv} \cap M) \cong \cdots \cong \phi_k(G_{uv} \cap M)$. Further, either*

(i) $\mathrm{A}_m \lhd \phi_i(G_{uv} \cap M)$; *or*
(ii) $\mathrm{A}_l \leq \phi_i(G_{uv} \cap M) \leq \mathrm{S}_l \times \mathrm{S}_{m-l}$, *where m is not a prime and $p \leq l < m$ with p the largest prime less than m.*

Proof. Since $s \geq 2$, $G_v = G_{uv}G_{vw}$, so $\overline{G_v} = \overline{G_{uv}}\,\overline{G_{vw}}$. Since $\overline{G_v} \cong \mathrm{A}_k$ or S_k, by [12, Lemma 2.3], one of $\overline{G_{uv}}$ and $\overline{G_{vw}}$, say $\overline{G_{uv}}$, is a transitive subgroup of S_k, it then follows from Lemma 2.7 that $\phi_1(G_{uv} \cap M) \cong \cdots \cong \phi_k(G_{uv} \cap M)$, and $\pi(\mathrm{A}_m) \subseteq \pi(\phi_i(G_{uv} \cap M))$ for $i = 1, 2, \ldots, k$.

If $m \geq 5$, by Proposition 2.3, $\phi_i(G_{uv} \cap M)$ satisfies part (i) or (ii). If $2 \leq m \leq 4$, since $\pi(\mathrm{A}_m) \subseteq \pi(\phi_i(G_{uv} \cap M))$ and $\phi_i(G_{uv} \cap M) \leq \mathrm{S}_m$, we have $\phi_i(G_{uv} \cap M) = \mathrm{A}_m$ for $m = 2$ and 3, and $\phi_i(G_{uv} \cap M) \rhd \mathrm{A}_4$ or equals to S_3 for $m = 4$, namely $\phi_i(G_{uv} \cap M)$ also satisfies part (i) or (ii). $\square$

If $\Delta_1, \ldots, \Delta_k$ are subsets of Ω with equal size m, such that Ω is the disjoint union of them, then we call $(\Delta_1, \ldots, \Delta_k)$ is a *m-homogeneous partition* of Ω. We first treat part (i) of Lemma 5.1.

Lemma 5.2. *Suppose $\mathrm{A}_m \lhd \phi_i(G_{uv} \cap M)$. Then $s \leq 2$.*

Proof. Suppose for a contradiction that $s \geq 3$. If $m \geq 5$, then the same arguments (view A_m as T there) as in the proof of Lemma 4.2 imply that $\mathsf{soc}(G_v) \cong \mathrm{A}_m^k$ is normal in G, hence G acts unfaithfully on $V\Gamma$, a contradiction.

Thus assume $m \leq 5$. Then G_v satisfies part (b) with $n = mk$, and the action of G on $V\Gamma$ is permutation equivalent to the action of G on the set of all m-homogeneous partitions of Ω, hence we may set $v = (\Delta_1, \ldots, \Delta_k)$, a m-homogeneous partition of Ω. Since $u = v^{g^{-1}} = (\Delta_1^{g^{-1}}, \ldots, \Delta_k^{g^{-1}}) \neq v$, without loss of generality, we may suppose $\Delta_1 \neq \Delta_1^{g^{-1}}$, so $q := |\Delta_1 \cap \Delta_1^{g^{-1}}| < m$. Since $\pi_1(G_{uv} \cap M) \leq \mathrm{S}_m$ fixes both $\Delta_1^{g^{-1}}$ and Δ_1, we have $\pi_1(G_{uv} \cap M) \leq \mathrm{S}_q \times \mathrm{S}_{m-q}$, which contradicts $\mathrm{A}_m \lhd \pi_1(G_{uv} \cap M)$. $\square$

The following two lemmas are regarding part (ii) of Lemma 5.1.

Lemma 5.3. *Suppose that G_v satisfies part (ii) of Lemma 5.1 with $m \leq 7$. Then $s \leq 2$.*

Proof. Set $G = \mathrm{A}_n.o$ with $o \le \mathbb{Z}_2$. Since $m \le 7$ is not a prime, we have $m = 4$ or 6.

Assume $m = 4$. Then $l = 3$ and $\mathrm{A}_3 \lhd \phi_i(G_{uv} \cap M) \le \mathrm{S}_3$, hence $|G_v|_2 = 2^{3k-1}(k!)_2|o|$, and $|G_{uv}|_2 \le |\phi_i(G_{uv} \cap M)|_2^k |\mathrm{A}_k|_2|o| = 2^{k-1}(k!)_2|o|$. It follows $|\mathsf{val}(\Gamma)|_2 \ge |G_v|_2/|G_{uv}|_2 \ge 2^{2k}$. If $s \ge 3$, by Lemma 2.9, we have 2^{6k} divides $|G_v|_2 = 2^{3k-1}(k!)_2|o|$. Consequently, $2^{3k} \mid (k!)_2$, contradicting Lemma 2.1.

Assume $m = 6$. Then $l = 5$ and $\mathrm{A}_5 \lhd \phi_i(G_{uv} \cap M) \le \mathrm{S}_5$, thus $|G_v|_3 = 3^{2k}(k!)_3$, and $|G_{uv}|_3 \le |\phi_i(G_{uv} \cap M|_3^k |\mathrm{A}_k|_3 = 3^k(k!)_3$. Consequently, $|\mathsf{val}(\Gamma)|_3 \ge |G_v|_3/|G_{uv}|_3 \ge 3^k$. If $s \ge 3$, Lemma 2.9 implies that 3^{3k} divides $|G_v|_3 = 3^{2k}(k!)_3$, or equivalently $3^k \mid (k!)_3$, also contradicting Lemma 2.1. □

Lemma 5.4. *Suppose that G_v satisfies part (ii) of Lemma 5.1 with $m \ge 8$. Assume that $G_{uv} = (G_{uv} \cap M).\overline{G_{uv}}$ and $G_{vw} = (G_{vw} \cap M).\overline{G_{vw}}$ are split extension. Then $s = 1$.*

Proof. Suppose for a contradiction that $s \ge 2$.

As $m \ge 8$, we have $l \ge 7$, and as l is equal or greater than the largest prime p less than m, by a well known theorem of Chebyshev, we have $m < 2l$ and so $m - l < l$. By Lemma 5.1, $\overline{G_{uv}} \le \mathrm{S}_k$ is transitive, so $G_{uv} \cap M$ has a insoluble composition factor A_l with multiplicity (say s) dividing k. If $s \le k/2$, then with quite similar arguments (view s, p as l, r there) as in the proof of Lemma 4.2, we have $p^{k/2} = |T|_p^{k/2} < p^{\frac{k}{p-1}}$, a contradiction. Thus $s = k$ and A_l^k is a minimal normal subgroup of G_{uv}, hence G_{vw} has a minimal normal subgroup $N \cong \mathrm{A}_l^k$ as $G_{uv} \cong G_{vw}$. If $N \not\subseteq G_{vw} \cap M$, then $N \le \overline{G_{vw}} \le \mathrm{S}_k$, which leads to $p^k = |N|_p \le (k!)_p < p^{\frac{k}{p-1}}$, contradicting Lemma 2.1. Consequently, $N \lhd G_{vw} \cap M$ and $\phi_i(G_{vw} \cap M)$ has a normal subgroup isomorphic to A_l.

Since $G_v = G_{vw}G_{uv}$, $G_{uv} = (G_{uv} \cap M).\overline{G_{uv}}$ and $G_{vw} = (G_{vw} \cap M).\overline{G_{vw}}$ are split extension, for each $(x_1, \ldots, x_k) \in G_v \cap M$, we have

$$(x_1, \ldots, x_k) = (a_1, \ldots, a_k)\sigma(b_1, \ldots, b_k)\tau,$$

where $(a_1, \ldots, a_k) \in G_{vw} \cap M$, $(b_1, \ldots, b_k) \in G_{uv} \cap M$, and $\sigma, \tau \in \overline{G_v} \le \mathrm{S}_k$. It follows that $(x_1, \ldots, x_k) = (a_1 b_{1^\sigma}, \ldots, a_k b_{k^\sigma})\sigma\tau$. Hence $\sigma\tau = 1$, and $x_i = a_i b_{i^\sigma} \in \phi_i(G_{uv} \cap M)\phi_i(G_{vw} \cap M)$ as $\overline{G_{uv}} \le \mathrm{S}_k$ is transitive. Consequently, we obtain $\phi_i(G_v \cap M) = \phi_i(G_{uv} \cap M)\phi_i(G_{vw} \cap M)$. Now by [15, P. 9, Theorem D and Remark 2], we further conclude that $1 \le m - l \le 5$ and $\phi_i(G_{vw} \cap M)$ is $(m-l)$-homogeneous on m points.

If $m - l = 1$, then $\phi_i(G_{uv} \cap M)$ and $\phi_i(G_{vw} \cap M)$ are almost simple with socle A_{m-1}. Notice that $\pi(\mathrm{A}_m) = \pi(\mathrm{A}_{m-1})$, by [1, Theorem 1.1], we have $m = 6$, a contradiction.

Thus assume $m - l \ge 2$ in the following. Since a 2-homogeneous group is either almost simple or an affine group, and $\mathrm{A}_l \lhd \phi_i(G_{vw} \cap M)$, we obtain that $\phi_i(G_{vw} \cap M)$ is almost simple with socle A_l. Notice that almost simple m-homogeneous groups that are not m-transitive with $m \ge 2$ are with socle $\mathrm{PSL}(2,8), \mathrm{PSL}(2,32)$ or $\mathrm{PSL}(2,q)$ with $q \equiv 3(\mathsf{mod}\ 4)$, see [9, Ch. 7], we further conclude that $\phi_i(G_{vw} \cap M)$ is 2-transitive. However, by [5, Theorem 5.3(S)], an almost simple group with socle A_l and $l \ge 7$ has no 2-transitive permutation representation on m points with $l < m < 2l$, also a contradiction. □

6. Proof of Theorem 1.1

Since the full automorphism groups of the directed cycles are soluble, $\mathsf{val}(\Gamma) \neq 1$. If $\mathsf{val}(\Gamma) = 2$, by [18, Theorem 5], $\mathsf{Aut}\Gamma$ is a dihedral group, a contradiction. Thus $\mathsf{val}(\Gamma) \geq 3$. Since $\mathsf{soc}(G) = \mathrm{A}_n$ with $n \geq 5$, either $G = \mathrm{A}_n$ or S_n, or $n = 6$ and $G = \mathrm{A}_6.2_2, \mathrm{A}_6.2_3$ or $\mathrm{A}_6.2^2$ as in Atlas [8].

Suppose $G = \mathrm{A}_n$ or S_n. Then Γ and G satisfy Hypothesis 3.1, and G_v satisfies one of parts (a)-(f) of Theorem 2.4. If G_v satisfies parts (a) and (c), by Lemmas 3.2 and 3.3, we have $s = 1$. If G_v satisfies parts (b), (d) and (e), by Lemmas 4.7 and 5.1–5.4, either $s \leq 2$ or part (2) of Theorem 1.1 holds. If G_v satisfies part (f), by [12, Corollary 3.4], we have $s \leq 2$.

Suppose now $G \neq \mathrm{A}_n$ or S_n. Then $n = 6$, $G = \mathrm{A}_6.2_2, \mathrm{A}_6.2_3$ or $\mathrm{A}_6.2^2$, and G_v is listed in Atlas [8], and direct computation in [3] shows that no digraph Γ exists in these cases. This completes the proof of Theorem 1.1. $\square$

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