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Corners of Leavitt path algebras of finite graphs are Leavitt path algebras

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ABSTRACT

We achieve an extremely useful description (up to isomorphism) of the Leavitt path algebra $L_K(E)$ of a finite graph E with coefficients in a field K as a direct sum of matrix rings over K, direct sum with a corner of the Leavitt path algebra $L_K(F)$ of a graph F for which every regular vertex is the base of a loop. Moreover, in this case one may transform the graph E into the graph F via some step-by-step procedure, using the "source elimination" and "collapsing" processes. We use this to establish the main result of the article, that every nonzero corner of a Leavitt path algebra of a finite graph is isomorphic to a Leavitt path algebra. Indeed, we prove a more general result, to wit, that the endomorphism ring of any nonzero finitely generated projective $L_K(E)$-module is isomorphic to the Leavitt path algebra of a graph explicitly constructed from E. Consequently, this yields in particular that every unital K-algebra which is Morita equivalent to a Leavitt path algebra is indeed isomorphic to a Leavitt path algebra.

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1. Introduction and preliminaries

Given a (row-finite) directed graph E and any field K, the first author and Aranda Pino in [2], and independently Ara, Moreno, and Pardo in [9], introduced the *Leavitt path algebra* $L_K(E)$. Leavitt path algebras generalize the Leavitt algebras $L_K(1, n)$ of [17], and also contain many other interesting classes of algebras. In addition, Leavitt path algebras are intimately related to graph C^*-algebras (see [18]). During the past fifteen years, Leavitt path algebras have become a topic of intense investigation by mathematicians from across the mathematical spectrum. For a detailed history and overview of Leavitt path algebras we refer the reader to the survey article [1].

One of the interesting questions in the theory of Leavitt path algebras is to find relationships between graphs E and F such that their corresponding Leavitt path algebras are Morita equivalent. In [5] the first author, Louly, Pardo, and Smith established some basic transformations of graphs which preserve isomorphism or Morita equivalence of the associated Leavitt path algebras. Motivated by these, along with the "collapsing" process introduced by Sørensen in [19], we present here another sufficient condition for Morita equivalence between Leavitt path algebras (Theorem 2.11).[1] This equivalence result in turn provides the vehicle to establish Theorem 2.18, which allows us to associate (modulo some easily-handled direct summands) the Leavitt path algebra of a finite graph with a (full) corner of the Leavitt path algebra of a graph which is a special type of extension of a "totally looped" graph (a graph for which every non-sink vertex is the base of a loop).

This totally looped property turns out to play an important bridge role in the analysis, as follows. We establish in Corollary 3.11 that any nonzero corner (full or not) of any Leavitt path algebra over a graph which arises as such a "strands of hair" extension of a totally looped graph is isomorphic to a Leavitt path algebra. We then use this Corollary, together with Theorem 2.18, to establish the (perhaps surprising) generalization of the Corollary to corners of Leavitt path algebras of all finite graphs (Theorem 3.15; see also Remark 3.19). As a consequence, this yields the (seemingly more-general) Theorem 3.17, which establishes that the endomorphism ring of any nonzero finitely generated projective module over a Leavitt path algebra is again a Leavitt path algebra. As well, Theorem 3.15 easily yields that every unital K-algebra that is Morita equivalent to a Leavitt path algebra is indeed isomorphic to a Leavitt path algebra.

We now present a streamlined version of the necessary background ideas. We refer the reader to [8] and [16] for information about general ring-theoretic constructions, and to [4] for additional information about Leavitt path algebras.

A (directed) graph $E = (E^0, E^1, s, r)$ consists of two disjoint sets E^0 and E^1, called *vertices* and *edges* respectively, together with two maps $s, r : E^1 \longrightarrow E^0$. The vertices $s(e)$ and $r(e)$ are referred to as the *source* and the *range* of the edge e, respectively.

[1] We thank M. Özaydin for pointing out that a more general version of Theorem 2.11 has been established in [15, Theorem 4.1]; what we here call the "collapsing process" is called the "reduction algorithm" on pp. 919-920 of [15]. The approach taken to establish [15, Theorem 4.1] is different from our approach.

A graph E is *finite* if both sets E^0 and E^1 are finite. To streamline the presentation and help illuminate the key ideas, we will focus on finite graphs throughout this article, although some of the results we establish also hold for more general graphs. A vertex v for which $s^{-1}(v)$ is empty is called a *sink*; a vertex v for which $r^{-1}(v)$ is empty is called a *source*; a vertex v is called an *isolated vertex* if it is both a source and a sink; and a vertex v (in a finite graph) is *regular* if it is not a sink. A graph E is said to be *source-free* if it has no sources. The "trivial" graph with one vertex and no edges is denoted by E_{triv}.

A *path* $p = e_1 \cdots e_n$ in a graph E is a sequence of edges $e_1, \dots, e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, we say that the path p starts at the vertex $s(p) := s(e_1)$ and ends at the vertex $r(p) := r(e_n)$, and has *length* $|p| := n$. We consider the elements of E^0 to be paths of length 0. We denote by $\mathrm{Path}(E)$ the set of all paths in E. A *cycle based at* v is a path $p = e_1 \cdots e_n$ with $s(p) = r(p) = v$, and for which the vertices $s(e_1), s(e_2), \dots, s(e_n)$ are distinct. A cycle c is called a *loop* if $|c| = 1$. A graph E is *acyclic* if it has no cycles.

A subgraph F of a finite graph E is called *complete* in case, for every $v \in F^0$ for which $s_F^{-1}(v) \neq \emptyset$, then $s_F^{-1}(v) = s_E^{-1}(v)$. Less formally: F is complete in case for every vertex v of F, if v emits at least one edge in F, then all edges which v emits in E are included in F.

For vertices $v, w \in E^0$, we write $v \geq w$ if there exists a path in E from v to w, i.e., there exists $p \in \mathrm{Path}(E)$ with $s(p) = v$ and $r(p) = w$. If $v \geq w$ and $v \neq w$, then there necessarily exists a path $q = e_1 e_2 \cdots e_t$ from v to w for which the vertices $v = s(e_1), s(e_2), \dots, s(e_t), w = r(e_t)$ are distinct. (An easy induction argument shows that any path from v to w having minimal length will have this property.)

Let H be a subset of E^0. H is called *hereditary* if for all $v \in H$ and $w \in E^0$, $v \geq w$ implies $w \in H$. For a subset S of E^0, the *hereditary closure* $T(S)$ of S is the (hereditary) subset $\{w \in E^0 \mid s \geq w \text{ for some } s \in S\}$ of E^0. H is called *saturated* if whenever v is a regular vertex in E^0 with the property that $r(s^{-1}(v)) \subseteq H$, then $v \in H$.

Definition 1.1. We call the graph E *totally looped* in case every regular vertex of E is the base of at least one loop.

For an arbitrary graph $E = (E^0, E^1, s, r)$ and any field K, the *Leavitt path algebra* $L_K(E)$ *of the graph* E *with coefficients in* K is the K-algebra generated by the sets E^0 and E^1, together with a set of variables $\{e^* \mid e \in E^1\}$, satisfying the following relations for all $v, w \in E^0$ and $e, f \in E^1$:

(1) $vw = \delta_{v,w} w$;
(2) $s(e)e = e = er(e)$ and $r(e)e^* = e^* = e^* s(e)$;
(3) $e^* f = \delta_{e,f} r(e)$;
(4) $v = \sum_{e \in s^{-1}(v)} ee^*$ for any regular vertex v.

Remark 1.2. We will often use the fact that, by relation (4), if $v \in E^0$ and $s^{-1}(v)$ is a single edge (say $s^{-1}(v) = \{f\}$), then $ff^* = v$.

If the graph E is finite, then $L_K(E)$ is a unital ring having identity $1 = \sum_{v \in E^0} v$ (see, e.g., [2, Lemma 1.6]). For any path $p = e_1 e_2 \cdots e_n$, the element $e_n^* \cdots e_2^* e_1^*$ of $L_K(E)$ is denoted by p^*. It can be shown ([2, Lemma 1.7]) that $L_K(E)$ is spanned as a K-vector space by $\{pq^* \mid p, q \in E^*, r(p) = r(q)\}$. Indeed, $L_K(E)$ is a $\mathbb{Z}$-graded K-algebra: $L_K(E) = \bigoplus_{n \in \mathbb{Z}} L_K(E)_n$, where for each $n \in \mathbb{Z}$, the degree n component $L_K(E)_n$ is the set $\mathrm{span}_K\{pq^* \mid p, q \in \mathrm{Path}(E), r(p) = r(q), |p| - |q| = n\}$.

For any unital ring R, $\mathcal{V}(R)$ denotes the set of isomorphism classes (denoted by $[P]$) of finitely generated projective left R-modules. $\mathcal{V}(R)$ is an abelian monoid with operation

$$[P] + [Q] = [P \oplus Q]$$

for any isomorphism classes $[P]$ and $[Q]$. On the other hand, for any directed graph $E = (E^0, E^1, s, r)$ the monoid M_E is defined as follows. Denote by T the free abelian monoid (written additively) with generators E^0, and define relations on T by setting

$$v = \textstyle\sum_{e \in s^{-1}(v)} r(e)$$

for every regular vertex $v \in E^0$. Let $\sim_E$ be the congruence relation on T generated by these relations. Then M_E is defined to be the quotient monoid $T/_{\sim_E}$; we denote an element of M_E by $[x]$, where $x \in T$. The foundational result about Leavitt path algebras for our work is the following

Theorem 1.3 *([9, Theorem 3.5]). Let E be a finite graph and K any field. Then the map $[v] \longmapsto [L_K(E)v]$ yields an isomorphism of abelian monoids $M_E \cong \mathcal{V}(L_K(E))$. Specifically, these two useful consequences follow immediately.*

(1) For any regular vertex $v \in E^0$, $L_K(E)v \cong \bigoplus_{e \in s^{-1}(v)} L_K(E)r(e)$ as left $L_K(E)$-modules.

(2) For any nonzero finitely generated projective left $L_K(E)$-module Q, there exists a sequence of (not necessarily distinct) vertices $v_1, v_2, \ldots, v_\ell$ in E for which $Q \cong \bigoplus_{i=1}^{\ell} L_K(E)v_i$; restated, there exists a subset of (distinct) vertices X of E^0 and positive integers $\{n_x \mid x \in X\}$ for which $Q \cong \bigoplus_{x \in X} n_x L_K(E)x$.

We emphasize that the direct sums indicated in the above Theorem are external direct sums. Also, throughout, for a positive integer n and left R-module M, the direct sum of n copies of M is denoted nM.

We collect up in the next result some properties of Leavitt path algebras.

Proposition 1.4. *Let K be any field.*

(1) $L_K(E_{triv}) \cong K$ as K-algebras.

(2) Let $E_1, E_2, \ldots, E_n$ be finite graphs. The disjoint union $E_1 \sqcup E_2 \sqcup \cdots \sqcup E_n$ of graphs is defined as expected. Then $L_K(E_1 \sqcup E_2 \sqcup \cdots \sqcup E_n) \cong \bigoplus_{i=1}^n L_K(E_i)$, a ring direct sum as K-algebras.

(3) ([4, Lemma 1.6.6]) Let F be a complete subgraph of E. Then $L_K(F)$ is a subalgebra of $L_K(E)$.

We finish the introductory section by reminding the reader of some well-known, general ring-theoretic results which will be of great importance in this analysis. These can be found in [8, Chapters 1, 2] and [16, Sections 17, 18]. Throughout, "ring" means "unital ring, with $1_R \neq 0$". If f is a nonzero idempotent in the ring R, then the *corner* of R by f is the ring fRf. (However, for notational convenience, we allow the phrase "direct sum with a corner of ..." to include the situation where the direct summand is the zero ring; see e.g. the Abstract.) An idempotent $f \in R$ is called *full* in case $R = RfR$. Rings R and S are *Morita equivalent* in case the category $R - Mod$ of left R-modules and the category $S - Mod$ of left S-modules are equivalent. For a left R-module ${}_RM$, we write R-endomorphisms of M on the right (i.e., the side opposite the scalars); so for $f, g \in \mathrm{End}_R(M)$, $(m)fg$ means "first f, then g".

Proposition 1.5. *Let R be a unital ring.*

(1) Let e, f be idempotents in R. Then $\mathrm{Hom}_R(Re, Rf) \cong eRf$ as abelian groups.

(2) Suppose $P \cong \bigoplus_{i=1}^n P_i$ as left R-modules. Then

$$\mathrm{End}_R(P) \cong \Big(\mathrm{Hom}_R(P_i, P_j)\Big),$$

the ring of $n \times n$ matrices for which the entry in the i-th row, j-th column is an element of $\mathrm{Hom}_R(P_i, P_j)$, for all $1 \leq i, j \leq n$. In particular, if $P \cong \bigoplus_{i=1}^n Re_i$, an external direct sum of the left R-modules Re_i for idempotents e_i, then

$$\mathrm{End}_R(P) \cong \Big(e_iRe_j\Big),$$

the ring of $n \times n$ matrices for which the entry in the i-th row, j-th column is an element of e_iRe_j, for all $1 \leq i, j \leq n$.

Theorem 1.6. *Let R and S be unital rings.*

(1) "Morita's Theorem": R is Morita equivalent to S if and only if there exist a positive integer n and a full idempotent $f \in M_n(R)$ such that $S \cong fM_n(R)f$.

(2) Suppose R decomposes as a ring direct sum $R = \bigoplus_{i=1}^n R_i$. Then S is Morita equivalent to R if and only if S decomposes as a ring direct sum $S = \bigoplus_{i=1}^n S_i$ where R_i is Morita equivalent to S_i for all $1 \leq i \leq n$.

(3) Let Q be a nonzero finitely generated projective left R-module, and suppose that Q is generated by n elements. Then $\mathrm{End}_R(Q) \cong \mathrm{End}_{M_n(R)}(M_n(R)q)$ for some idempotent $q \in M_n(R)$.

2. Collapsing at a regular vertex that is not the base of a loop

In this section we achieve (Theorem 2.18) an extremely useful description (up to isomorphism) of the Leavitt path algebra of any finite graph as a direct sum of matrix rings over the coefficient field, direct sum with a corner of a Leavitt path algebra of a graph which is a special type of extension of a totally looped graph.

Here is the strategy. We begin by establishing a Morita equivalence property which is similar to an analogous property of graph C^*-algebras. The C^*-result, called Move (R), was shown in [19, Proposition 3.2]; it followed as a special case of the result [14, Theorem 3.1]. Move (R) applies only to very restricted configurations of vertices and edges in a graph. By subsequently applying two additional previously-studied graph transformations (first "in-split", then in turn "out-split"), both of which preserve Morita equivalence of the associated Leavitt path algebras, we are able to eliminate the restrictions on the configurations and achieve a significant generalization of Move (R), called "collapsing"; this is the gist of Theorem 2.11. We then use Theorem 2.11 to establish Theorem 2.18.

Definition 2.1. Let $E = (E^0, E^1, r, s)$ be a finite graph. Let $w \in E^0$ be a vertex such that w emits exactly one edge (call it f), and f is not a loop, and such that w receives edges from at most one vertex. That is, $|s^{-1}(w)| = 1$ (but $r(s^{-1}(w)) \neq \{w\}$), and either w is a source vertex or $|s(r^{-1}(w))| = 1$. If w is not a source, then denote by v the only vertex that emits to w. Define the "Move (R) at w" graph $G = (G^0, G^1, r_G, s_G)$ by setting

$$G^0 = E^0 \setminus \{w\}, \qquad G^1 = (E^1 \setminus (r^{-1}(w) \cup \{f\})) \cup \{[ef] \mid e \in r^{-1}(w)\},$$

where range and source maps extend those of E, and satisfy $r_G([ef]) = r(f)$ and $s_G([ef]) = s(e) = v$. (Note that, in case w is a source, then G^1 is simply $E^1 \setminus \{f\}$.)

Remark 2.2. We note that if there is a loop based at w, then the Move (R) construction at w does not yield a well-defined graph, because in that case the range function r_G would be undefined at edges of the form $[ef]$ (since that vertex w has been eliminated).

Proposition 2.3. *Let K be any field. Let E be a finite graph, let $w \in E^0$ be a vertex of the type described in Definition 2.1, and let G be the corresponding Move (R) at w graph. Then $L_K(E)$ is Morita equivalent to $L_K(G)$.*

Proof. We construct a K-algebra homomorphism

$$\psi : L_K(G) \longrightarrow L_K(E)$$

given on the generators of the free K-algebra $K\langle u, g, g^* \mid u \in G^0, g \in G^1\rangle$ as follows:

$$\psi(u) = u, \qquad \psi(g) = \begin{cases} ef & \text{if } g = [ef]\ (e \in r^{-1}(w)), \\ g & \text{otherwise} \end{cases}$$

and

$$\psi(g^*) = \begin{cases} f^*e^* & \text{if } g = [ef] \ (e \in r^{-1}(w)), \\ g^* & \text{otherwise.} \end{cases}$$

To ensure that the map ψ induces a K-algebra homomorphism from $L_K(G)$ to $L_K(E)$, we must verify that all elements of the following forms:

$$\begin{aligned}
&uu' - \delta_{u,u'}u \text{ for all } u, u' \in G^0, \\
&s_G(g)g - g \text{ and } g - gr_G(g) \text{ for all } g \in G^1, \\
&r_G(g)g^* - g^* \text{ and } g^* - g^*s_G(g) \text{ for all } g \in G^1, \\
&g^*h - \delta_{g,h}r_G(g) \text{ for all } g, h \in G^1, \\
&u - \textstyle\sum_{g\in s_G^{-1}(u)} gg^* \text{ for a regular vertex } u \in G^0
\end{aligned}$$

are in the kernel of ψ. Here we verify only the last one, since the first four can be established easily. Let $u \in G^0$ be a regular vertex. If w is a source then $s_G^{-1}(u) = s^{-1}(u)$, and so $\psi(u - \sum_{g\in s_G^{-1}(u)} gg^*) = u - \sum_{g\in s^{-1}(u)} gg^* = 0$, as desired. Suppose w is not a source. Then by hypothesis w receives from only one vertex, v say. If $u \neq v$ then the statement is proved by a similar argument to that given above. So consider the case when $u = v$. We note that $ff^* = w$ by Remark 1.2, and $s_G^{-1}(v) = (s^{-1}(v) \setminus r^{-1}(w)) \sqcup \{[ef] \mid e \in r^{-1}(w)\}$. Hence

$$\begin{aligned}
\psi(v - \textstyle\sum_{g\in s_G^{-1}(v)} gg^*) &= v - \textstyle\sum_{g\in s^{-1}(v)\setminus r^{-1}(w)} gg^* - \sum_{e\in r^{-1}(w)}(ef)(f^*e^*) \\
&= v - \textstyle\sum_{g\in s^{-1}(v)\setminus r^{-1}(w)} gg^* - \sum_{e\in r^{-1}(w)} ee^* \\
&= v - \textstyle\sum_{g\in s^{-1}(v)} gg^* = 0.
\end{aligned}$$

We next prove that $\psi : L_K(G) \to L_K(E)$ is injective. To the contrary, suppose there exists a nonzero element $x \in \ker(\psi)$. Then, by the Reduction Theorem (see, e.g., [4, Theorem 2.2.11]), there exist $a, b \in L_K(G)$ such that either $axb = u \neq 0$ for some $u \in G^0$, or $axb = p(c) \neq 0$, where c is a cycle in G and $p(x)$ is a nonzero polynomial in $K[x, x^{-1}]$.

In the first case, since $axb \in \ker(\psi)$, this would imply that $u = \psi(u) = 0$ in $L_K(E)$; but each vertex is well-known to be a nonzero element inside the Leavitt path algebra, a contradiction.

So we are in the second case: there exists a cycle c in G such that $axb = \sum_{i=-n}^{m} k_i c^i$, where $k_i \in K$ and we interpret c^i as $(c^*)^{-i}$ for negative i, and we interpret c^0 as $u := s(c)$. We then have $\sum_{i=-n}^{m} k_i\psi(c)^i = \psi(axb) = 0$ in $L_K(E)$. We write $c = g_1 \cdots g_m$, where $g_i \in G^1$. If none of the g_i's are in $\{[ef] \mid e \in r^{-1}(w)\}$, then $\psi(c) = \psi(g_1)\cdots\psi(g_m) = g_1 \cdots g_m$, so $\psi(c)$ is also a cycle in E. Otherwise, there exists $1 \le k \le m$ such that $g_k = [ef]$ for some $e \in r^{-1}(w)$. We then have that $g_i \in E^1 \setminus (r^{-1}(w) \cup \{f\})$ for all $i \neq k$, since c is a cycle. It shows that $\psi(c) = \psi(g_1...g_{k-1}[ef]g_{k+1}...g_m) = g_1...g_{k-1}efg_{k+1}...g_m$, so $\psi(c)$ is a cycle in E. Thus, in any case $\psi(c)$ is a cycle in E. But the $\mathbb{Z}$-grading in $L_K(E)$ shows that an equation of the type $\sum_{i=-n}^{m} k_i\psi(c)^i = \psi(axb) = 0$ cannot hold in $L_K(E)$. This shows that ψ is injective, so $L_K(G)$ is isomorphic to $Im(\psi)$.

Let $\epsilon = \psi(1_{L_K(G)}) = \sum_{u\in E^0, u\neq w} u$. We claim that $Im(\psi) = \epsilon L_K(E)\epsilon$. Clearly $Im(\psi) \subseteq \epsilon L_K(E)\epsilon$. To show the other inclusion, we only need to verify that for all nonzero monomials $x \in \epsilon L_K(E)\epsilon$, $\psi^{-1}(x) \neq \emptyset$. We can express such a monomial x of the form $x = pq^*$, where $p = g_1 \cdots g_m$ and $q = f_1 \cdots f_n$ are paths in E such that $r(p) = r(q)$ and $s(p), s(q) \neq w$. Consider the following three cases. (We note in advance that the last two possibilities of Case 1 and Case 3 cannot occur when w is a source.)

Case 1. $r(p) = r(q) = r(f)$. If both g_m and f_n are different from f, then g_i and $f_j \in E^1 \setminus (r^{-1}(w) \cup \{f\})$, so $x = pq^* \in L_K(G)$ and $\psi(x) = x$. If $g_m = f$ then $f_n \neq f$ (otherwise, since $ff^* = w$, we get that $x = g_1...g_{m-1}ff^*f^*_{n-1}...f^*_1 = g_1...g_{m-1}f^*_{n-1}...f^*_1$ and $r(g_1...g_{m-1}) = r(f_1...f_{n-1}) \neq r(f)$, a contradiction). We then immediately obtain that $m \geq 2$, $g_{m-1} \in r^{-1}(w)$ (since $s(p) = s(g_1) \neq w$), and $g_i, f_j \in E^1 \setminus (r^{-1}(w) \cup \{f\})$ for all $1 \leq i \leq m-2$ and $1 \leq j \leq n$. Set $\alpha = g_1...g_{m-2}[g_{m-1}f]f^*_n...f^*_1 \in L_K(G)$. We have that $\psi(\alpha) = g_1...g_{m-2}g_{m-1}ff^*_n...f^*_1 = x$. If $f_n = f$ then g_m is different from f (otherwise, since $ff^* = w$, $x = g_1...g_{m-1}ff^*f^*_{n-1}...f^*_1 = g_1...g_{m-1}f^*_{n-1}...f^*_1$ and $r(g_1...g_{m-1}) = r(f_1...f_{n-1}) \neq r(f)$, a contradiction). We then have that, for $n \geq 2, f_{n-1}$ is in $r^{-1}(w)$ (since $s(q) = s(f_1) \neq w$), and g_i, f_j are not in $r^{-1}(w) \cup \{f\}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n-2$. Define $\beta := g_1...g_m[f_{n-1}f]^*f^*_{n-2}...f^*_1 \in L_K(G)$. Then we have $\varphi(\beta) = g_1...g_mf^*f^*_{n-1}f^*_{n-2}...f^*_1 = x$.

Case 2. $r(p) = r(q) \notin \{r(f), w\}$. We claim that there exists a path $p' \in G$ such that $\psi(p') = p$. Indeed, if $r(g_i) \neq w$ for all $1 \leq i \leq m$, then $g_i \in E^1 \setminus (r^{-1}(w) \cup \{f\})$ for all $1 \leq i \leq m$. Let $p' := p$. We immediately get that p' is a path in G and $\psi(p') = p$. If $r(g_i) = w$ for some $1 \leq i \leq m$, then we denote by $g_{i_1}, g_{i_2}, \cdots, g_{i_k}$, where $1 \leq i_1 < i_2 < \cdots < i_k < m$, all edges having the range w. Since $r(p) = r(g_m) \notin \{r(f), w\}$, we must have that $g_{i_j+1} = f$ for all $1 \leq j \leq k$. Let $p' := g_1 \cdots g_{i_1-1}[g_{i_1}f]g_{i_1+2} \cdots g_{i_2-1}[g_{i_2}f]g_{i_2+2} \cdots g_{i_k-1}[g_{i_k}f]g_{i_k+2} \cdots g_m$, that means, p' is a path in G obtained from p by replacing $g_{i_j}f$ $(1 \leq j \leq k)$ by $[g_{i_j}f]$. We have that $\psi(p') = p$.

Similarly, we obtain that there exists a path q' in G such that $\psi(q') = q$. We then have that $x' := p'(q')^* \in L_K(G)$ and $\psi(x') = \psi(p'(q')^*) = pq^* = x$.

Case 3. $r(p) = r(q) = w$. We have that g_m and f_n are in $r^{-1}(w)$ and g_i $(1 \leq i \leq m-1), f_j$ $(1 \leq j \leq n-1) \in E^1 \setminus (r^{-1}(w) \cup \{f\})$, and so

$$x = g_1...g_mf^*_n...f^*_1 = g_1...g_m.w.f^*_n...f^*_1 = g_1...g_mff^*f^*_n...f^*_1 = \psi(\beta)$$

for $\beta := g_1...g_{m-1}[g_nf][f_nf]^*f^*_{n-1}...f^*_1 \in L_K(G)$.

In any case we always have $\psi^{-1}(x) \neq \emptyset$, so $Im(\psi) = \epsilon L_K(E)\epsilon$, thus showing that $L_K(G)$ is isomorphic to $\epsilon L_K(E)\epsilon$.

To establish the Morita equivalence, we show that $L_K(E) = L_K(E)\epsilon L_K(E)$ (see the $n = 1$ case of Theorem 1.6(1)). It is enough to show that w is in $L_K(E)\epsilon L_K(E)$.

Since $r(f)$ is in the ideal $L_K(E)\epsilon L_K(E)$, the edge f is in $L_K(E)\epsilon L_K(E)$. Then $w = ff^* \in L_K(E)\epsilon L_K(E)$. This proves that $L_K(E)\epsilon L_K(E) = L_K(E)$. Hence $L_K(G)$ is Morita equivalent to $L_K(E)$, finishing the proof. $\square$

The key Morita equivalence result for us (Theorem 2.11) is inspired by [19, Theorem 5.2]. To achieve it, we show that Move (R) may be applied at vertices that are more general than those given in Definition 2.1, and that the corresponding Leavitt path algebras are Morita equivalent. Following [19], we call this generalization of Move (R) a "collapse".

Definition 2.4. [Collapse at a regular vertex which is not the base of a loop] Let $E = (E^0, E^1, r, s)$ be a finite graph, and let $v \in E^0$ be a regular vertex which is not the base of a loop. Define the "collapse at v" graph $G = (G^0, G^1, r_G, s_G)$ by setting

$$G^0 = E^0 \setminus \{v\}, \quad G^1 = (E^1 \setminus (r^{-1}(v) \cup s^{-1}(v))) \cup \{[ef] \mid e \in r^{-1}(v),\ f \in s^{-1}(v)\},$$

where range and source maps extend those of E, and satisfy $r_G([ef]) = r(f)$ and $s_G([ef]) = s(e)$. (We note that, in case v is a source, then G^1 is simply $E^1 \setminus s^{-1}(v)$.)

Remark 2.5. As with Move (R), the requirement that there be no loop based at the collapsing vertex is necessary so that the collapsing process gives a well-defined graph.

Specific examples of this collapsing process are given below in Example 2.17.

We extend Proposition 2.3 in two stages. In the first stage, we show that the requirement that the collapsing vertex receive edges from at most one vertex can be eliminated (Proposition 2.8).

Definition 2.6 *([5, Definitions 1.9]: the "in-split" graph).* Let $E = (E^0, E^1, r, s)$ be a graph and $v \in E^0$ a vertex that is not a source. Partition $r^{-1}(v)$ into a finite number, say n, of disjoint nonempty subsets $\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n$. We form the *in-split graph* $E_{is} = (E^0_{is}, E^1_{is}, r_{is}, s_{is})$ from E using the partition $\{\mathcal{E}_i \mid i = 1, ..., n\}$ as follows: $E^0_{is} = (E^0 \setminus \{v\}) \cup \{v_1, v_2, ..., v_n\}$,

$$E^1_{is} = \{e_1, e_2, ..., e_n \mid e \in E^1, s(e) = v\} \cup \{f \mid f \in E^1 \setminus s^{-1}(v)\},$$

and define r_{is}, $s_{is} : E^1_{is} \longrightarrow E^0_{is}$ by setting $s_{is}(e_j) = v_j$, $s_{is}(f) = s(f)$, and

$$r_{is}(x) = \begin{cases} r(f) & \text{if } x = f \notin r^{-1}(v) \\ v_i & \text{if } x = f \in r^{-1}(v) \text{ and } f \in \mathcal{E}_i \\ r(e) & \text{if } x = e_j \text{ and } e \notin r^{-1}(v) \\ v_i & \text{if } x = e_j,\ e \in r^{-1}(v) \text{ and } e \in \mathcal{E}_i \end{cases}.$$

Proposition 2.7 *(essentially [5, Proposition 1.11 and Corollary 3.9]). Let K be any field. Let E be a finite graph and $v \in E^0$ a vertex that is not a source. Then $L_K(E)$ is Morita equivalent to $L_K(E_{is})$.*

Proof. The quoted result [5, Corollary 3.9] applies to constructions more general than the in-split construction; however, the tools used to prove [5, Corollary 3.9] only allow for the desired conclusion when the field K is infinite. Accordingly, we provide here a short proof of Proposition 2.7 which holds for all fields.

We define the elements $\{Q_u \mid u \in E^0\}$ and $\{T_e, T_{e^*} \mid e \in E^1\}$ of $L_K(E_{is})$ by setting

$$Q_u = \begin{cases} v_1 & \text{if } u = v, \\ u & \text{otherwise,} \end{cases}$$

$$T_e = \begin{cases} \sum_{f \in s^{-1}(v)} e f_i f_1^* & \text{if } e \in (r^{-1}(v) \cap \mathcal{E}_i) \setminus s^{-1}(v),\ s^{-1}(v) \neq \emptyset \\ e & \text{if } e \in (r^{-1}(v) \cap \mathcal{E}_i) \setminus s^{-1}(v),\ s^{-1}(v) = \emptyset \\ \sum_{f \in s^{-1}(v)} e_1 f_i f_1^* & \text{if } e \in r^{-1}(v) \cap \mathcal{E}_i \cap s^{-1}(v) \\ e & \text{if } e \notin r^{-1}(v), \qquad \text{and} \end{cases}$$

$$T_{e^*} = \begin{cases} \sum_{f \in s^{-1}(v)} f_1 f_i^* e^* & \text{if } e \in (r^{-1}(v) \cap \mathcal{E}_i) \setminus s^{-1}(v),\ s^{-1}(v) \neq \emptyset \\ e^* & \text{if } e \in (r^{-1}(v) \cap \mathcal{E}_i) \setminus s^{-1}(v),\ s^{-1}(v) = \emptyset \\ \sum_{f \in s^{-1}(v)} f_1 f_i^* e_1^* & \text{if } e \in r^{-1}(v) \cap \mathcal{E}_i \cap s^{-1}(v) \\ e^* & \text{if } e \notin r^{-1}(v). \end{cases}$$

By repeating verbatim the corresponding argument in the proof of [5, Proposition 1.11], there exists an K-algebra homomorphism $\pi : L_K(E) \longrightarrow L_K(E_{is})$, which maps $u \longmapsto Q_u$, $e \longmapsto T_e$ and $e^* \longmapsto T_{e^*}$, such that $\pi(L_K(E)) = \pi(1_{L_K(E)}) L_K(E_{is}) \pi(1_{L_K(E)})$. Note that $\pi(1_{L_K(E)}) = v_1 + \sum_{u \in E^0 \setminus \{v\}} u =: \epsilon$.

Since Q_u has degree 0, T_e has degree 1, and T_{e^*} has degree -1 for all $u \in E^0$ and $e \in E^1$, π is thus a $\mathbb{Z}$-graded homomorphism, whence the injectivity of π is guaranteed by [20, Theorem 4.8]. So we have $L_K(E) \cong \epsilon L_K(E_{is}) \epsilon$.

To obtain the Morita equivalence, we invoke Theorem 1.6(1) (with $n = 1$); so we need only establish that ϵ is full, i.e., that $L_K(E_{is}) = L_K(E_{is}) \epsilon L_K(E_{is})$. It is enough to show that v_i is in $L_K(E_{is}) \epsilon L_K(E_{is})$ for all $2 \leq i \leq n$. We consider the following cases.

Case 1. $\mathcal{E}_i$ does not contain loops for all i. Then, for each $1 \leq i \leq n$, there exists $f_i \in r^{-1}(v) \cap \mathcal{E}_i$ such that $s(f_i) \neq v$. Therefore, $s_{is}(f_i) = s(f_i) \in E^0 \setminus \{v\}$ and $r_{is}(f_i) = v_i$ for all i. This implies that $s_{is}(f_i)$ is in the ideal $L_K(E_{is}) \epsilon L_K(E_{is})$, and so the edge $f_i = s_{is}(f_i) f_i$ is in $L_K(E_{is}) \epsilon L_K(E_{is})$. Then $v_i = r_{is}(f_i) = f_i^* f_i \in L_K(E_{is}) \epsilon L_K(E_{is})$ for all i. This proves that $L_K(E_{is}) \epsilon L_K(E_{is}) = L_K(E_{is})$ in this case.

Case 2. There exists $1 \leq i \leq n$ such that $\mathcal{E}_i$ contains a loop. We may, without loss of generality, that there exists $1 \leq k \leq n$ such that $\mathcal{E}_i$ $(1 \leq i \leq k)$ contains a loop and $\mathcal{E}_j$ $(k+1 \leq j \leq n)$ does not contain loops. Then, similar to Case 1, we get that v_j is in the ideal $L_K(E_{is}) \epsilon L_K(E_{is})$ for all $k+1 \leq j \leq n$. For each $1 \leq i \leq k$, there exists $f_i \in \mathcal{E}_i$ such that $s(e^i) = r(e^i) = v$. We have that $s_{is}(e_1^i) = v_1$ and $r_{is}(e_1^i) = v_i$ for all $1 \leq i \leq k$.

Since v_1 is in the ideal $L_K(E_{is})\epsilon L_K(E_{is})$, the edge $e_1^i = v_1 e_1^i$ is in $L_K(E_{is})\epsilon L_K(E_{is})$. Then $v_i = r_{is}(e_1^i) = (e_1^i)^* e_1^i$ is in the ideal $L_K(E)\epsilon L_K(E)$ for all $1 \le i \le k$. This implies that $L_K(E_{is})\epsilon L_K(E_{is}) = L_K(E_{is})$ in this case as well.

Thus $L_K(E)$ is Morita equivalent to $L_K(E_{is})$, finishing the proof. □

Here is the first generalization of Proposition 2.3, in which we remove the hypotheses that the collapsing vertex receives edges from at most one other vertex.

Proposition 2.8. *Let $E = (E^0, E^1, r, s)$ be a finite graph, and let $v \in E^0$ be a regular vertex which emits precisely one edge, f_0 say. Assume that $r(f_0) \neq v$ (i.e., that f_0 is not a loop). Let G be the "collapse at v" graph. Let K be any field. Then $L_K(E)$ is Morita equivalent to $L_K(G)$.*

Proof. If v receives from at most one vertex (i.e., $|s(r^{-1}(v))| \le 1$) then the statement follows immediately from Proposition 2.3. We assume then that v receives from the vertices $\{u_1, u_2, ..., u_n\}$, where $n \ge 2$. Partition $r^{-1}(v)$ into disjoint nonempty subsets $\mathcal{E}_i = r^{-1}(v) \cap s^{-1}(u_i)$ for all $i = 1, 2, ..., n$. By using Proposition 2.7 at v according to the partition $\{\mathcal{E}_i\}$, we get that $L_K(E)$ is Morita equivalent to $L_K(E_{is})$. But the graph E_{is} is the graph E, with v replaced by n vertices $v_1, v_2, ..., v_n$, each receiving from exactly one of the vertices v received from, and each emitting one edge to $r(f_0)$. So we may apply Move (R) step by step at v_1 through v_n. At each step, the Morita equivalence of the corresponding Leavitt path algebra is ensured by Proposition 2.3. But this sequence of graph transformations, which first in-splits at v and then performs Move (R) at each of the v_i, yields precisely the collapse at v graph G. Hence $L_K(E)$ is Morita equivalent to $L_K(G)$. □

We now show how to eliminate the restriction in Proposition 2.8 which requires that the collapsing vertex emits just one edge.

Definition 2.9 *([3, Definition 2.6]: the "out-split" graph).* Let $E = (E^0, E^1, r, s)$ be a graph and $v \in E^0$ a vertex that is not a sink. Partition $s^{-1}(v)$ into a finite number, say n, of disjoint nonempty subsets $\mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_n$. We form the *out-split graph* $E_{os} = (E^0_{os}, E^1_{os}, r_{os}, s_{os})$ from E using the partition $\{\mathcal{E}_i \mid i = 1, ..., n\}$ as follows: $E^0_{os} = (E^0 \setminus \{v\}) \cup \{v^1, v^2, ..., v^n\}$,

$$E^1_{os} = \{e^1, e^2, ..., e^n \mid e \in E^1, r(e) = v\} \cup \{f \mid f \in E^1 \setminus r^{-1}(v)\},$$

and define r_{os}, $s_{os} : E^1_{os} \longrightarrow E^0_{os}$ by setting $r_{os}(e^j) = v^j$, $r_{os}(f) = r(f)$, and

$$s_{os}(x) = \begin{cases} s(f) & \text{if } x = f \notin s^{-1}(v) \\ v^i & \text{if } x = f \in s^{-1}(v) \text{ and } f \in \mathcal{E}_i \\ s(e) & \text{if } x = e^j \text{ and } e \notin s^{-1}(v) \\ v^i & \text{if } x = e^j,\ e \in s^{-1}(v) \text{ and } e \in \mathcal{E}_i \end{cases}.$$

Proposition 2.10 *([3, Theorem 2.8]). Let K be any field. Let E be a finite graph and $v \in E^0$ a regular vertex. Then $L_K(E) \cong L_K(E_{os})$ as $\mathbb{Z}$-graded K-algebras. In particular, $L_K(E)$ is Morita equivalent to $L_K(E_{os})$.*

We are now in position to achieve the key Morita equivalence result. (See also [15, Theorem 4.1].)

Theorem 2.11. *Let K be any field. Let E be a finite graph, let $v \in E^0$ be a regular vertex which is not the base of a loop, and let G be the "collapse at v" graph. Then $L_K(E)$ is Morita equivalent to $L_K(G)$.*

Proof. If $|s^{-1}(v)| = 1$ then the result follows immediately from Proposition 2.8. We assume that $e_1, e_2, ..., e_n$ are the edges with source v, where $n \geq 2$. Partition $s^{-1}(v)$ into disjoint nonempty subsets $\mathcal{E}_i = \{e_i\}$ for $i = 1, 2, ..., n$. Applying Proposition 2.10 at v according to the partition $\{\mathcal{E}_i\}$ of $s^{-1}(v)$, we get that $L_K(E)$ is isomorphic to $L_K(E_{os})$. Since v is not the base of a loop, for each $1 \leq i \leq n$, v^i emits exactly one edge e_i, and e_i is not a loop. In particular, each v^i satisfies the hypotheses of Proposition 2.8. So we may apply the collapsing process step by step at v^1 through v^n. At each step, the Morita equivalence of the corresponding Leavitt path algebra is preserved by Proposition 2.8. But this sequence of graph transformations, which first out-splits at v and then collapses at each of the v^i, yields precisely the collapse at v graph G. Hence $L_K(E)$ is Morita equivalent to $L_K(G)$, as desired. □

Definition 2.12 *([5, Definition 1.2] and [10, Notation 2.4]).* Let $E = (E^0, E^1, r, s)$ be a graph, and let $v \in E^0$ be a source. We form the *source elimination* graph $E_{\setminus v}$ of E as follows:

$$(E_{\setminus v})^0 = E^0 \setminus \{v\},\ (E_{\setminus v})^1 = E^1 \setminus s^{-1}(v),\ s_{E_{\setminus v}} = s|_{(E_{\setminus v})^1} \text{ and } r_{E_{\setminus v}} = r|_{(E_{\setminus v})^1}.$$

In other words, $E_{\setminus v}$ denotes the graph constructed from E by deleting v and all of edges in E emitting from v.

We note that the source elimination process is allowed at isolated vertices. If v is a source vertex in a graph E, and v is not an isolated vertex, then clearly the source elimination process at v coincides with the "collapsing at v" move. So Theorem 2.11 immediately gives the following previously-established result.

Corollary 2.13 *([10, Lemma 4.3]). Let E be a finite graph and K any field. If $v \in E^0$ is a source vertex which is not isolated, then $L_K(E)$ is Morita equivalent to $L_K(E_{\setminus v})$.*

We note that Theorem 2.11 yields [10, Lemma 4.4] as well.

Let E be a finite graph. If E is acyclic, then repeated application of the source elimination process to E yields the empty graph. On the other hand, if E contains a cycle, then repeated application of the source elimination process will yield a source-free graph E_{sf} which necessarily contains a cycle.

Consider the sequence of graphs which arises in some step-by-step process of source eliminations

$$E = E_0 \to E_1 \to \cdots \to E_i \to \cdots \to E_t = E_{sf}.$$

To avoid defining a graph to be the empty set, we define E_{sf} to be the graph E_{triv} (consisting of one vertex and no edges) in case $E_{t-1} = E_{triv}$.

Remark 2.14. Although there in general are many different orders in which a step-by-step source elimination process can be carried out, the resulting source-free subgraph E_{sf} is always the same (see, e.g., [6, Lemma 3.13]).

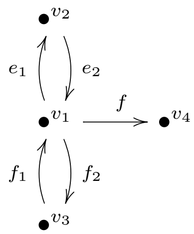
Remark 2.15. For a finite graph E, we perform a sequence of source eliminations $E = E_0 \to E_1 \to \cdots \to E_i \to \cdots \to E_t = E_{sf}$. By Remark 2.14 we may assume that the final k steps in the process involve eliminating any isolated vertices which may arise. The non-negative integer k is precisely the number of vertices of E which are sinks in E, but which are not in E_{sf}^0. As observed in [6], k may also be viewed as the number of sinks u in E for which every path $p \in \mathrm{Path}(E)$ having $r(p) = u$ contains no closed subpath. Let V_k denote the graph with k vertices and no edges. Then repeated application of Lemma 2.13 gives that $L_K(E)$ is Morita equivalent to $L_K(V_k \sqcup E_{sf})$ (when E_{sf} is nontrivial), and to $L_K(V_k)$ (when E_{sf} is the trivial graph V_1).

Remark 2.16. For a finite graph E, we form the (uniquely-determined) graph E_{sf}. In case E_{sf} is not the trivial graph, we may then perform a step-by-step process in which we produce a sequence of graphs

$$E_{sf} = F_0 \to F_1 \to \cdots \to \cdots \to F_\ell := F,$$

where each F_{i+1} is formed from F_i by performing a collapsing at some vertex v of F_i which is not the base of a loop. (We note that by the construction of E_{sf} there will be no isolated vertices in any of the F_i.) In this way, E_{sf} is transformed to a totally looped graph F in which there are no isolated vertices.

Example 2.17. Although the process of source elimination yields a graph E_{sf} which is unique up to graph isomorphism, the process described in Remark 2.16, if carried out in different orders, does not necessarily yield isomorphic graphs. For instance, let E be the graph



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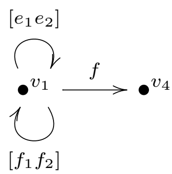
We note that $E = E_{sf}$, i.e., E has no sources.

If we first collapse E at v_2, and then subsequently at v_3, the resulting totally looped graph F_2 is



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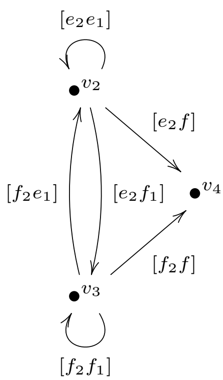
On the other hand, if we instead collapse E at v_1, then the resulting totally looped graph F_1 is



Clearly F_1 and F_2 are not isomorphic as graphs. (We note that both F_1 and F_2 are indeed totally looped; no loop is required at a sink, specifically, at v_4.)

We are now in position to establish the main result of this section.

Theorem 2.18. *Let K be any field, and let E be a finite graph. Let k denote the number of vertices of E which are sinks in E, but which are not in E^0_{sf}. Let F denote any totally looped graph which is constructed from the graph E_{sf} via some step-by-step process, where at each step we collapse at a regular vertex which is not the base of a loop. Then*

$$L_K(E) \textit{ is Morita equivalent to } (\prod_{i=1}^{k} K_i) \oplus L_K(F),$$

where $K_i \cong K$ *for* $1 \le i \le k$. *Consequently, there exist* $k \ge 0$, *and (when* $k \ge 1$*) positive integers* $m_1, \ldots, m_k$, *a positive integer* n, *and a full idempotent* $p \in M_n(L_K(F))$, *for which*

$$L_K(E) \cong (M_{m_1}(K) \oplus M_{m_2}(K) \oplus \cdots \oplus M_{m_k}(K)) \oplus pM_n(L_K(F))p$$

as K*-algebras.*

Proof. By Remark 2.15 we have that $L_K(E)$ is Morita equivalent to $L_K(V_k \sqcup E_{sf})$. Applying k times statements (1) and (2) of Proposition 1.4 gives that $L_K(E)$ is Morita equivalent to $(\prod_{i=1}^{k} K_i) \oplus L_K(E_{sf})$. Then one forms the sequence of graphs $E_{sf} = F_0 \to F_1 \to \cdots \to \cdots \to F_\ell := F$, where each F_{i+1} is produced from F_i by collapsing at some regular vertex of F_i which is not the base of a loop. The Morita equivalence then follows directly from repeated application of Theorem 2.11.

To establish the isomorphism, we use the well-known fact that for a field K, the only rings Morita equivalent to K are of the form $M_n(K)$ for some positive integer n. So the consequence follows immediately, using statements (1) and (2) of Theorem 1.6. □

3. Corners of unital Leavitt path algebras

The main goal of this section (indeed, of this article) is to show that every corner of a Leavitt path algebra of a finite graph is also isomorphic to a Leavitt path algebra (Theorem 3.15). Consequently, we achieve what on the surface seems to be a more general result (Theorem 3.17): for any finite graph E and any nonzero finitely generated projective left $L_K(E)$-module Q, the endomorphism ring $\mathrm{End}_{L_K(E)}(Q)$ is isomorphic to $L_K(F)$ for some finite graph F.

Definition 3.1 *([4, Definition 2.2.21]: "the restriction graph").* Let E be a graph and let H be a hereditary subset of E^0. We denote by E_H the *restriction graph*:

$$E_H^0 := H, \quad E_H^1 := \{e \in E^1 \mid s(e) \in H\},$$

and the source and range maps in E_H are simply the source and range maps in E, restricted to H. (We note that H must be hereditary in order for the construction E_H to actually yield a graph, specifically, so that the restriction of the range function r to edges having $s(e) \in H$ is defined.)

Remark 3.2. By construction, the restriction graph E_H is a complete subgraph of E, so that by Proposition 1.4(3) we may view $L_K(E_H)$ as a K-subalgebra of $L_K(E)$.

Definition 3.3. (The "strands of hair" extension of a graph) Let E be a finite graph, with $E^0 = \{v_1, v_2, \ldots, v_t\}$. Let $n_1, n_2, \ldots, n_t$ be a sequence of positive integers. We define the *strands of hair extension* (concisely: *hair extension*) graph $E^+(n_1, n_2, \ldots, n_t)$ to be the graph E, together with an extension by a "strand of hair" of length $n_i - 1$ at each v_i. Graphically, $E^+(n_1, ..., n_t)$ is formed by adding these vertices and edges to E:

$$\bullet^{v_i^{n_i-1}} \xrightarrow{e_i^{n_i-1}} \cdots \bullet^{v_i^2} \xrightarrow{e_i^2} \bullet^{v_i^1} \xrightarrow{e_i^1}$$

where $r(e_i^1) = v_i$. (So if $n_i = 1$, then we attach no new edges at v_i.) If the sequence $n_1, n_2, \ldots, n_t$ is understood from context, we will denote $E^+(n_1, n_2, \ldots, n_t)$ simply by E^+.

Remark 3.4. By construction, E is a complete subgraph of any hair extension $E^+(n_1, n_2, \ldots, n_t)$, so that by Proposition 1.4(3) we may view $L_K(E)$ as a K-subalgebra of $L_K(E^+(n_1, n_2, \ldots, n_t))$.

For clarification, we consider the following example.

Example 3.5. If E is the graph

$$\begin{tikzcd} \bullet^{v_1} \arrow[loop above] \arrow[r] & \bullet^{v_2} \end{tikzcd}$$

then $E^+(3, 2)$ is the graph

$$\begin{tikzcd} \bullet^{v_1^2} \arrow[r] & \bullet^{v_1^1} \arrow[r] & \bullet^{v_1} \arrow[loop above] \arrow[r] & \bullet^{v_2} & \bullet^{v_2^1} \arrow[l] \end{tikzcd}$$

The following Proposition illuminates exactly why the totally looped property should play such a central role in this analysis. We first need a lemma.

Lemma 3.6. *(cf. [11, Lemma 4.5]) Let E be totally looped. Then every subset of E^0 is saturated.*

Proof. Let H be a subset of E^0 and $v \in E^0$ a regular vertex with $r(s^{-1}(v)) \subseteq H$. By hypothesis v is the base of a loop f, i.e., $r(f) = v = s(f)$. So $v \in r(s^{-1}(v))$, and so $v \in H$. Thus H is saturated. $\square$

Proposition 3.7. *Let $F = \{v_1, v_2, \ldots, v_t\}$ be a totally looped finite graph. Let $n_1, n_2, \ldots, n_t$ be a sequence of positive integers, and let E denote the hair extension graph $F^+(n_1, n_2, \ldots, n_t)$. Let Q be a nonzero finitely generated projective left $L_K(E)$-module. Then there exist positive integers m_i $(1 \le i \le u)$, and a hereditary subset $T = \{v_{j_1}, v_{j_2}, \ldots, v_{j_u}\}$ of F^0 such that*

$$Q \cong \bigoplus_{i=1}^{u} m_i L_K(E) v_{j_i}.$$

Moreover, if Q is a generator for $L_K(E)$-Mod, then $T = F^0$.

Proof. By Theorem 1.3(2) we have that

$$Q \cong \bigoplus_{w \in E^0} m''_w L_K(E) w$$

for some (not necessarily unique) non-negative integers m''_w. For any $w \in E^0$ which is of the form v_i^j for some $j \geq 1$ (i.e., for each "added" vertex w), we have $L_K(E)w = L_K(E)v_i^j \cong L_K(E)v_i$ (by using Theorem 1.3(1) $j-1$ times). So by replacing appropriate summands, we have

$$Q \cong \bigoplus_{w \in F^0} m'_w L_K(E) w \tag{$*$}$$

for some non-negative integers m'_w. Clearly we can eliminate any summand for which $m'_w = 0$. Denote by T_1 the set of remaining vertices (i.e., the set of vertices w in F^0 for which $m'_w \geq 1$ in $(*)$). So

$$Q \cong \bigoplus_{w \in T_1} m'_w L_K(E) w. \tag{$**$}$$

Let T denote the hereditary closure of T_1. (Note: The hereditary closure of T_1 is the same regardless of whether we view $T_1 \subseteq F^0$ or $T_1 \subseteq E^0$; either way, $T \subseteq F^0$.) We claim that

$$Q \cong \bigoplus_{w \in T} m_w L_K(E) w,$$

where each $m_w \geq 1$. For, let $z \in T$, let $v \in T_1$, and suppose that there is a path $p = e_1 e_2 \cdots e_x$ from v to z. By an observation made in the Introduction, we may assume that the sequence of vertices $v = s(e_1), s(e_2), \ldots, s(e_x), r(e_x) = w$ which appear in p contains no repeats. In particular, no e_i is a loop. By Theorem 1.3(1) we have $L_K(E)v \cong \oplus_{e \in s^{-1}(v)} L_K(E) r(e)$. Since $v \in F^0$ we have that $r(e) \in F^0$ for all $e \in s^{-1}(v)$. But because F is totally looped, $v = r(f)$ for at least one loop $f \in s^{-1}(v)$; in addition, $f \neq e_1$ because e_1 is not a loop. So this decomposition yields that

$$L_K(E)v \;\cong\; L_K(E)v \;\oplus L_K(E)r(e_1) \;\oplus \bigoplus_{e \in s^{-1}(v) \setminus \{f, e_1\}} L_K(E) r(e). \tag{$\dagger$}$$

Now replace any one of the summands isomorphic to $L_K(E)v$ which appears in the decomposition $(**)$ of Q by the isomorphic version of $L_K(E)v$ given in $(\dagger)$; note that such a replacement does not decrease the number of copies of the summand $L_K(E)v$ which appear in (∗∗). Continuing this same process now on the summand $L_K(E)r(e_1)$, we see that after x steps we arrive at a direct sum decomposition of Q which includes a summand isomorphic to $L_K(E)z$, and which has not decreased the number of summands isomorphic to any given $L_K(E)w$ which appeared in decomposition (∗∗) of Q. (Note that $L_K(E)z$ will appear as a summand of Q in no more than x steps, because there are no repeats in the sequence of vertices in p.) This completes the proof of the claim, and establishes the displayed isomorphism of the statement.

For the second part, suppose that Q is in addition a generator for $L_K(E)$-Mod. Let $w \in F^0$. Then for some positive integer s there is a split epimorphism $sQ \to L_K(E)w \to 0$; so there are maps $\varphi \in \mathrm{Hom}_{L_K(E)}(sQ, L_K(E)w)$ and $\psi \in \mathrm{Hom}_{L_K(E)}(L_K(E)w, sQ)$ for which $\psi\varphi$ is the identity map on $L_K(E)w$. But using Proposition 1.5(1) and the standard decomposition of maps to and from finite direct sums, the equation $w = (w)\psi\varphi$ yields elements $r_{i,\ell}$ and $r'_{i,\ell}$ in $L_K(E)$, with $1 \le i \le u$ and $1 \le \ell \le s \cdot m_i$, for which

$$w = \sum_{i=1}^{u} \sum_{\ell=1}^{s \cdot m_i} w r_{i,\ell} v_{j_i} r'_{i,\ell} w.$$

Because w and each v_{j_i} is in F, and because there are no paths which start in F^0 and end in any of the added vertices which produce E as a hair extension of F, each expression $w r_{i,\ell} v_{j_i}$ and $v_{j_i} r'_{i,\ell} w$ is an element of $L_K(F)$. Since w and each v_{j_i} an idempotent, we can therefore view each term $w r_{i,\ell} v_{j_i} r'_{i,\ell} w$ in the displayed sum as an element of the ideal of $L_K(F)$ generated by the set of vertices $T \subseteq F^0$. We have thus established that the ideal $I(T)$ of $L_K(F)$ generated by T contains all vertices of F^0, and so $I(T) = L_K(F)$. But T is not only hereditary, it is by default saturated as well (Lemma 3.6). We now apply [4, Theorem 2.5.9] to conclude that $T = F^0$. □

Theorem 3.8. *Let F be a finite totally looped graph, let $E = F^+$ be a hair extension of F, and let Q be a nonzero finitely generated projective left $L_K(E)$-module. Then $\mathrm{End}_{L_K(E)}(Q)$ is isomorphic to a Leavitt path algebra. Specifically, $\mathrm{End}_{L_K(E)}(Q) \cong L_K(G)$, where G is a hair extension of the restriction graph F_T of F by some hereditary subset T of F^0. (In particular, G is a hair extension of a totally looped graph.)*

Moreover, if Q is in addition a generator for $L_K(E)$-Mod, then $\mathrm{End}_{L_K(E)}(Q) \cong L_K(G)$ where G is a hair extension of F.

Proof. By Proposition 3.7 we may decompose Q as

$$Q \cong \bigoplus_{v \in T} m_v L_K(E)v,$$

where T is a hereditary subset of F^0, and each $m_v \ge 1$. Write $T = \{v_1, v_2, \dots, v_u\}$. Note that there are $\sigma = \sum_{v \in T} m_v$ direct summands in the decomposition. By Proposition 1.5 $\mathrm{End}_{L_K(E)}(Q)$ is isomorphic to a $\sigma \times \sigma$ matrix ring, with entries described as follows. The indicated matrices may be viewed as consisting of rectangular blocks of size $m_{v_i} \times m_{v_j}$, where, for $1 \le i \le u$, $1 \le j \le u$, the entries of the (i, j) block are elements of the K-vector space $v_i L_K(E) v_j$.

On the other hand, because T is hereditary, we may construct the restriction graph F_T of F. Furthermore, because $m_i \ge 1$ for all $1 \le i \le u$, we may construct the hair extension $G = F_T^+(m_1, m_2, \dots, m_u)$ of F_T. For each $1 \le i \le u$, and each $1 \le y \le m_i - 1$, let $p_i^y := e_i^y \cdots e_i^1$ denote the (unique) path in $G = F_T^+$ having $s(p_i^y) = v_i^y$, and $r(p_i^y) = v_i$. Note that, because of the specific configuration of the added vertices and edges used to build G as F_T^+, repeated application of Remark 1.2 gives that $p_i^y (p_i^y)^* = v_i^y$ in $L_K(G)$. Note also that $|G^0| = \sum_{1 \le i \le u} m_i$, which is precisely σ. Writing $L_K(G) = \oplus_{v \in G^0} L_K(G) v$ and again using Proposition 1.5, we get that $L_K(G)$ is isomorphic to the $\sigma \times \sigma$ matrix ring with entries described as follows. For $1 \le i, j \le u$, and $0 \le y \le m_i - 1$, $0 \le z \le m_j - 1$, the entries in the row indexed by (m_i, y) and column indexed by (m_j, z) are elements of $v_i^y L_K(G) v_j^z$, where we interpret v_i^0 as v_i.

We now show that these two $\sigma \times \sigma$ matrix rings are isomorphic as K-algebras. To do so, we show first that for each pair (m_i, y), (m_j, z) with $1 \le i, j \le u$, and $1 \le y \le m_i - 1$, $1 \le z \le m_j - 1$, there is a K-vector space isomorphism

$$\varphi = \varphi_{(m_i,y),(m_j,z)} : v_i L_K(E) v_j \to v_i^y L_K(G) v_j^z.$$

For $r \in L_K(E)$ we define

$$\varphi_{(m_i,y),(m_j,z)}(v_i r v_j) = p_i^y v_i r v_j (p_j^z)^*.$$

Writing r as a sum of elements of the form $k\alpha\beta^*$ with $k \in K$ and $\alpha, \beta \in \mathrm{Path}(E)$, we have that $v_i r v_j$ may be viewed as a sum of elements $k\alpha\beta^*$ with $s(\alpha) = v_i$ and $s(\beta) = v_j$. Because any path which starts in T must have all of its vertices in T (because T is hereditary, and there is no path from $T \subseteq F^0$ to any of the added vertices which yield G as F_T^+), we have that the expression $v_i r v_j$ is indeed an element of $L_K(G)$, which in turn yields that $p_i^y v_i r v_j (p_j^z)^* \in v_i^y L_K(G) v_j^z$.

That φ is K-linear is clear. Further, φ is a monomorphism: if $p_i^y v_i r v_j (p_j^z)^* = 0$ then multiplying on the left by $(p_i^y)^*$ and on the right by p_j^z yields $v_i r v_j = 0$. To show φ is surjective: for $v_i^y s v_j^z \in v_i^y L_K(G) v_j^z$ with $s \in L_K(G)$, define $s' = (p_i^y)^* v_i^y s v_j^z p_j^z \in v_i L_K(G) v_j$. But using the fact that there are no paths from elements of T to any of the newly added vertices which yield G as $(F_T)^+$, we have as above that s' may be viewed as an element of $L_K(E)$. Then, using the previous observation that $p_i^y (p_i^y)^* = v_i^y$ in $L_K(G)$, we conclude that $\varphi(s') = p_i^y (p_i^y)^* v_i^y s v_j^z p_j^z (p_j^z)^* = v_i^y s v_j^z$, and thus φ is surjective.

We now define Φ to be the K-space isomorphism between the two matrix rings induced by applying each of the $\varphi_{(m_i,y),(m_j,z)}$ componentwise. We need only show that these componentwise isomorphisms respect the corresponding matrix multiplications. But to do so, it suffices to show that the maps behave correctly in each component. That is, we need only show, for each m_ℓ $(1 \le \ell \le u)$ and each x $(1 \le x \le m_\ell)$, that

$$\varphi_{(m_i,y),(m_\ell,x)}(v_i r v_\ell) \cdot \varphi_{(m_\ell,x),(m_j,z)}(v_\ell r' v_j) = \varphi_{(m_i,y),(m_j,z)}(v_i r v_\ell r' v_j).$$

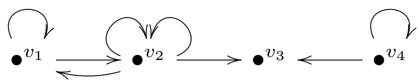
But this is immediate, as $(p_\ell^x)^* p_\ell^x = v_\ell$ for each $v_\ell \in T$ and $1 \le x \le m_\ell$.

The additional statement follows from the final assertion of Proposition 3.7. $\square$

Remark 3.9. In the previous proof, E is an arbitrary hair extension of F, by some sequence of $|F^0|$ integers. As well, G is a hair extension of a subgraph F_T of F, by some sequence of $|F_T^0|$ integers. In general there need be no relationship whatsoever between the two sequences of integers.

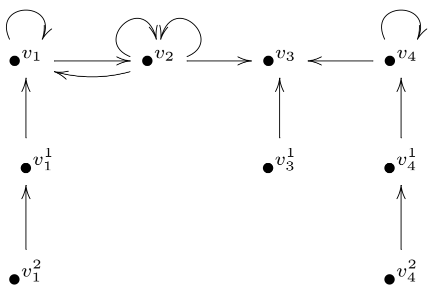
The following example will help illuminate the ideas of Theorem 3.8.

Example 3.10. Let F be graph

$F \quad =$ 

.

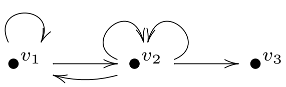
Then F is totally looped (note that v_3 is a sink, so no loop is required at v_3). Let E be the hair extension $F^+(3,1,2,3)$ of F, pictured here:

$E \quad =$ 

Let R denote $L_K(E)$. Consider the (arbitrarily chosen) nonzero finitely generated projective left R-module $Q = Rv_1^2$. We write Q in the form indicated in Proposition 3.7, as follows. Using the isomorphism of Theorem 1.3(1) multiple times, we have

$$Q \cong Rv_1^1 \cong Rv_1 \cong Rv_1 \oplus Rv_2 \cong Rv_1 \oplus (Rv_1 \oplus 2Rv_2 \oplus Rv_3) \cong 2Rv_1 \oplus 2Rv_2 \oplus Rv_3.$$

The hereditary subset of F^0 corresponding to Q is $T = \{v_1, v_2, v_3\}$, so

$F_T \quad =$ 

.

The decomposition

$$Q \cong 2Rv_1 \oplus 2Rv_2 \oplus Rv_3$$

dictates that we construct the hair extension $G = F_T^+(2,2,1)$ of F_T, graphically,

$$G \quad = \quad \begin{tikzcd} \bullet^{v_1} \arrow[loop above] \arrow[r, shift left] & \bullet^{v_2} \arrow[l, shift left] \arrow[loop above] \arrow[loop above] \arrow[r] & \bullet^{v_3} \\ \bullet^{v_1^1} \arrow[u] & \bullet^{v_2^1} \arrow[u] & \end{tikzcd} .$$

By Proposition 3.7 we have $\operatorname{End}_{L_K(E)}(Q) \cong L_K(G)$. For notational simplification, let S denote $L_K(G)$. Then the explicit description of these two algebras as matrix rings as described in the proof of Proposition 3.7 is:

$$\begin{pmatrix} v_1Rv_1 & v_1Rv_1 & v_1Rv_2 & v_1Rv_2 & v_1Rv_3 \\ v_1Rv_1 & v_1Rv_1 & v_1Rv_2 & v_1Rv_2 & v_1Rv_3 \\ v_2Rv_1 & v_2Rv_1 & v_2Rv_2 & v_2Rv_2 & v_2Rv_3 \\ v_2Rv_1 & v_2Rv_1 & v_2Rv_2 & v_2Rv_2 & v_2Rv_3 \\ v_3Rv_1 & v_3Rv_1 & v_3Rv_2 & v_3Rv_2 & v_3Rv_3 \end{pmatrix} \cong \begin{pmatrix} v_1Sv_1 & v_1Sv_1^1 & v_1Sv_2 & v_1Sv_2^1 & v_1Sv_3 \\ v_1^1Sv_1 & v_1^1Sv_1^1 & v_1^1Sv_2 & v_1^1Sv_2^1 & v_1^1Sv_3 \\ v_2Sv_1 & v_2Sv_1^1 & v_2Sv_2 & v_2Sv_2^1 & v_2Sv_3 \\ v_2^1Sv_1 & v_2^1Sv_1^1 & v_2^1Sv_2 & v_2^1Sv_2^1 & v_2^1Sv_3 \\ v_3Sv_1 & v_3Sv_1^1 & v_3Sv_2 & v_3Sv_2^1 & v_3Sv_3 \end{pmatrix}.$$

Corollary 3.11. *Let E be a graph which arises as a hair extension of a totally looped finite graph F. Let ε be a nonzero idempotent in $L_K(E)$. Then the corner algebra $\varepsilon L_K(E)\varepsilon$ is isomorphic to a Leavitt path algebra. More specifically, $\varepsilon L_K(E)\varepsilon$ is isomorphic to the Leavitt path algebra of a hair extension of F_T, where F_T is the (totally looped) restriction graph of F to some hereditary subset T of F^0.*

Moreover, in case ε is a full idempotent in $L_K(E)$, then $\varepsilon L_K(E)\varepsilon$ is isomorphic the Leavitt path algebra of a hair extension of F itself.

Proof. The first statement follows directly from Theorem 3.8, as $L_K(E)\varepsilon$ is a nonzero finitely generated projective left $L_K(E)$-module, and $\operatorname{End}_{L_K(E)}(L_K(E)\varepsilon) \cong \varepsilon L_K(E)\varepsilon$. The statement about full idempotents follows from Proposition 3.7, because an idempotent f in a ring R is full precisely when Rf is a generator for R-Mod. $\square$

There is a specific hair extension construction for arbitrary graphs which is already known, and which will be useful in establishing our main result.

Definition 3.12 *([7, Definition 9.1]).* For any finite graph E and positive integer n, let M_nE denote the hair extension graph

$$M_nE = E^+(n, n, \dots, n).$$

In other words, M_nE is constructed from E by attaching a strand of hair of length $n-1$ of the form

$$\bullet^{v^{n-1}} \longrightarrow \cdots \bullet^{v^2} \longrightarrow \bullet^{v^1} \longrightarrow$$

to each $v \in E^0$.

Proposition 3.13 *([7, Proposition 9.3]). Let K be any field, E a finite graph and n a positive integer. Then there exists a K-algebra isomorphism*

$$\varphi : M_n(L_K(E)) \longrightarrow L_K(M_nE).$$

In particular, any full $n \times n$ matrix ring over a Leavitt path algebra is isomorphic to the Leavitt path algebra of a hair extension of E.

For each positive integer n we denote by A_n the "straight line graph" having n vertices and $n-1$ edges:

$$A_n \quad = \quad \bullet^{v_{n-1}} \longrightarrow \bullet^{v_{n-2}} \longrightarrow \cdots \bullet^{v_2} \longrightarrow \bullet^{v_1} \longrightarrow \bullet^{v_0} \ .$$

Easily (or, applying Proposition 3.13 to E_{triv}) we get

Lemma 3.14. *For any positive integer n, $L_K(A_n) \cong M_n(K)$.*

We are finally in position to achieve the main result of this article.

Theorem 3.15. *Let K be any field. Let E be any finite graph, and let ε be any nonzero idempotent in $L_K(E)$. Then the corner $\varepsilon L_K(E)\varepsilon$ of $L_K(E)$ is isomorphic to a Leavitt path algebra.*

Proof. By Theorem 2.18 we have

$$L_K(E) \cong (M_{m_1}(K) \oplus M_{m_2}(K) \oplus \cdots \oplus M_{m_k}(K)) \oplus pM_n(L_K(F))p$$

for some $k \geq 0$, integers $m_1, \ldots, m_k$, and some full idempotent p in $M_n(L_K(F))$ for some positive integer n and totally looped graph F. By Proposition 3.13 $M_n(L_K(F)) \cong L_K(M_nF)$. Let γ denote the image of p under this isomorphism; so γ is a full idempotent in $L_K(M_nF)$. Then as K-algebras we have

$$L_K(E) \cong (M_{m_1}(K) \oplus M_{m_2}(K) \oplus \cdots \oplus M_{m_k}(K)) \oplus \gamma L_K(M_nF)\gamma.$$

Since $M_nF = F^+(n, n, \ldots, n)$ is a hair extension of the totally looped graph F, Corollary 3.11 yields that $\gamma L_K(M_nF)\gamma \cong L_K(G_1)$ for some finite graph G_1, where G_1 is a hair extension of F. So we get

$$L_K(E) \cong (M_{m_1}(K) \oplus M_{m_2}(K) \oplus \cdots \oplus M_{m_k}(K)) \oplus L_K(G_1);$$

denote this K-algebra isomorphism by Φ. Write $\Phi(\varepsilon) = (\epsilon_1, \epsilon_2, \ldots, \epsilon_k, \epsilon)$; then each of ϵ and ϵ_i $(1 \leq i \leq k)$ is an idempotent. Reordering if necessary, we may eliminate any summand for which $\epsilon_i = 0$, and thereby get

$$\varepsilon L_K(E)\varepsilon \cong (\epsilon_1 M_{m_1}(K)\epsilon_1 \oplus \epsilon_2 M_{m_2}(K)\epsilon_2 \oplus \cdots \oplus \epsilon_\ell M_{m_\ell}(K)\epsilon_\ell) \oplus \epsilon L_K(G_1)\epsilon$$

for some $\ell \leq k$. If $\epsilon = 0$ then we eliminate the summand $\epsilon L_K(G_1)\epsilon$; otherwise, again invoking Corollary 3.11, we have that $\epsilon L_K(G_1)\epsilon \cong L_K(G)$ for some graph G. Thus we have

$$\varepsilon L_K(E)\varepsilon \cong (\epsilon_1 M_{m_1}(K)\epsilon_1 \oplus \epsilon_2 M_{m_2}(K)\epsilon_2 \oplus \cdots \oplus \epsilon_\ell M_{m_\ell}(K)\epsilon_\ell) \oplus L_K(G).$$

It is well-known that any corner of a full matrix ring over a field K is isomorphic to a full matrix ring (possibly of smaller size) over K. So

$$\varepsilon L_K(E)\varepsilon \cong (M_{n_1}(K) \oplus M_{n_2}(K) \oplus \cdots \oplus M_{n_\ell}(K)) \oplus L_K(G)$$

for some integers $1 \leq n_i \leq m_i$ $(1 \leq i \leq \ell)$. Since $M_t(K) \cong L_K(A_t)$ for any positive integer t (Lemma 3.14), this last isomorphism with Proposition 1.4(2) yields

$$\varepsilon L_K(E)\varepsilon \cong L_K(A_{n_1} \sqcup A_{n_2} \sqcup \cdots \sqcup A_{n_\ell} \sqcup G),$$

thus establishing the result. □

Corollary 3.16. *Let K be any field. Let A be a K-algebra which is Morita equivalent to a Leavitt path algebra. Then A is isomorphic to a Leavitt path algebra.*

Proof. If A is Morita equivalent to $L_K(E)$, then (see Theorem 1.6(1)) there exists a positive integer n and a (full) idempotent $p \in M_n(L_K(E))$ for which $A \cong pM_n(L_K(E))p$. But using Proposition 3.13, we have $pM_n(L_K(E))p \cong \varepsilon L_K(M_nE)\varepsilon$ for some idempotent $\varepsilon \in L_K(M_nE)$. Finally, we invoke Theorem 3.15 to get the desired result. □

Although the following result seems on the surface to be a generalization of Theorem 3.15, this result in fact follows as a consequence of Theorem 3.15.

Theorem 3.17. *Let K be any field. Let E be any finite graph, and let Q be a nonzero finitely generated projective left $L_K(E)$-module. Then $\mathrm{End}_{L_K(E)}(Q)$ is isomorphic to a Leavitt path algebra.*

Proof. Suppose Q is generated by n elements as an $L_K(E)$-module. Then by Theorem 1.6(3), under the standard Morita equivalence Ψ between $L_K(E) - Mod$ and $M_n(L_K(E)) - Mod$, $\Psi(Q)$ is isomorphic to a direct summand of $M_n(L_K(E))$, i.e., $\Psi(Q) \cong M_n(L_K(E))q$ for some idempotent $q \in M_n(L_K(E))$. But the equivalence yields that

$$\mathrm{End}_{L_K(E)}(Q) \cong \mathrm{End}_{M_n(L_K(E))}(\Psi(Q)) \cong \mathrm{End}_{M_n(L_K(E))}(M_n(L_K(E))q).$$

This in turn by Proposition 3.13 is isomorphic to $\mathrm{End}_{L_K(M_nE)}(L_K(M_nE)\varepsilon)$ for some idempotent $\varepsilon \in L_K(M_nE)$, which in turn is isomorphic to $\varepsilon L_K(M_nE)\varepsilon$. Now Theorem 3.15 gives the result. □

We close this article with a series of remarks.

Remark 3.18. There is a tight but not fully understood connection between results about Leavitt path algebras and results about their graph C^*-algebra analogs. The connection continues in this context as well. Specifically, Arklint and Ruiz [11], and Arklint, Gabe and Ruiz [12] have established (among many other things) that for a finite graph E, any corner $pC^*(E)p$ of the graph C^*-algebra $C^*(E)$ by a projection p is isomorphic to a graph C^*-algebra.

Remark 3.19. Our ability to establish Theorem 3.15 for *all* nonzero idempotents ε in $L_K(E)$ may seem surprising. Specifically, we need not assume that ε possess any additional properties (e.g., that ε be full, or that $\varepsilon \in L_K(E)_0$, or that $\varepsilon = \varepsilon^*$). The point here is that by the theorem of Ara, Moreno and Pardo (Theorem 1.3(2)), we have a description up to isomorphism of any left $L_K(E)$-module of the form $L_K(E)\varepsilon$ for *all* idempotents ε of $L_K(E)$ in terms of idempotents of the form $L_K(E)v$ where $v \in E^0$. The foundational result which is used to establish Theorem 1.3 is the fact that the Leavitt path algebra $L_K(E)$ may be viewed as the "Bergman algebra" of the monoid M_E, which allows the extremely powerful [13, Theorem 6.2] to be invoked.

Remark 3.20. By combining the germane ideas in the proofs of Theorem 3.15 and Corollary 3.16, we can in fact establish a more precise result about algebras which are Morita equivalent to Leavitt path algebras. Specifically, let K be any field, A a unital K-algebra, E a finite graph, and F a finite graph in which every regular vertex is the base of a loop that is obtained from the graph E_{sf} via some step-by-step process of collapsing at a regular vertex which is not the base of a loop. Then A is Morita equivalent to $L_K(E)$ if and only if

(1) there exists a finite acyclic graph V such that the number of sinks in V is equal to the number of all those sinks of E which are not in E^0_{sf},
(2) there exist a positive integer k, and a hereditary subset H of $(M_kF)^0$ containing F^0, and
(3) A is isomorphic to $L_K(V \sqcup (M_kF)_H)$.

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References

[1] G. Abrams, Leavitt path algebras: the first decade, Bull. Math. Sci. 5 (2015) 59–120.
[2] G. Abrams, G. Aranda Pino, The Leavitt path algebra of a graph, J. Algebra 293 (2005) 319–334.
[3] G. Abrams, P.N. Ánh, A. Louly, E. Pardo, The classification question for Leavitt path algebras, J. Algebra 320 (2008) 1983–2026.
[4] G. Abrams, P. Ara, M. Siles Molina, Leavitt Path Algebras, Lecture Notes in Mathematics Series, vol. 2191, Springer-Verlag Inc., 2017.
[5] G. Abrams, A. Louly, E. Pardo, C. Smith, Flow invariants in the classification of Leavitt path algebras, J. Algebra 333 (2011) 202–231.
[6] G. Abrams, T.G. Nam, N.T. Phuc, Leavitt path algebras having unbounded generating number, J. Pure Appl. Algebra 221 (2017) 1322–1343.
[7] G. Abrams, M. Tomforde, Isomorphism and Morita equivalence of graph algebras, Trans. Amer. Math. Soc. 363 (2011) 3733–3767.
[8] F. Anderson, K. Fuller, Rings and Categories of Modules, second edition, Graduate Texts in Mathematics Series, vol. 13, Springer-Verlag Inc., 1992.
[9] P. Ara, M.A. Moreno, E. Pardo, Nonstable K-theory for graph algebras, Algebr. Represent. Theory 10 (2007) 157–178.
[10] P. Ara, K.M. Rangaswamy, Finitely presented simple modules over Leavitt path algebras, J. Algebra 417 (2014) 333–352.
[11] S. Arklint, E. Ruiz, Corners of Cuntz-Krieger algebras, Trans. Amer. Math. Soc. 367 (2015) 7595–7612.
[12] S. Arklint, J. Gabe, E. Ruiz, Hereditary C^*-subalgebras of graph C^*-algebras, arXiv:1604.03085v2.
[13] G.M. Bergman, Coproducts and some universal ring constructions, Trans. Amer. Math. Soc. 200 (1974) 33–88.
[14] T. Crisp, D. Gow, Contractible subgraphs and Morita equivalence of graph C^*-algebras, Proc. Amer. Math. Soc. 134 (7) (2006) 2003–2013.
[15] A. Koç, M. Özaydin, Finite-dimensional representations of Leavitt path algebras, Forum Math. 30 (4) (2018) 915–928.
[16] T.Y. Lam, Lectures on Modules and Rings, Graduate Texts in Mathematics Series, vol. 189, Springer-Verlag Inc., 1999.
[17] W. Leavitt, The module type of a ring, Trans. Amer. Math. Soc. 42 (1962) 113–130.
[18] I. Raeburn, Graph Algebras, CBMS Regional Conference Series in Mathematics, vol. 103, American Mathematical Society, Providence, RI, 2005, vi+113 pp. Published for the Conference Board of the Mathematical Sciences, Washington, DC.
[19] A.P.W. Sørensen, Geometric classification of simple graph algebras, Ergodic Theory Dynam. Systems 33 (2013) 1199–1220.
[20] M. Tomforde, Uniqueness theorems and ideal structure for Leavitt path algebras, J. Algebra 318 (2007) 270–299.