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Representations of principal W-algebra for the superalgebra $Q(n)$ and the super Yangian $YQ(1)$

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ABSTRACT

We classify irreducible representations of finite W-algebra for the queer Lie superalgebra $Q(n)$ associated with the principal nilpotent coadjoint orbits. We use this classification and our previous results to obtain a classification of irreducible finite-dimensional representations of the super Yangian $YQ(1)$.

1. Introduction

The main result of this paper is a classification of simple finite-dimensional modules over the super Yangian $YQ(1)$ associated with the Lie superalgebra $Q(1)$. The Yangians of type Q were introduced by Nazarov in [13] and [14]. In [15] these super Yangians were realized as limits of certain centralizers in the universal enveloping algebras of type Q. Our approach is via finite W-algebras as in [1,2].

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In the classical case a finite W_e-algebra is a quantization of the Slodowy slice to the adjoint orbit of a nilpotent element e of a semisimple Lie algebra $\mathfrak{g}$. Finite-dimensional simple W_e-modules are used for classification of primitive ideals of $U(\mathfrak{g})$. Losev's work gives a new point of view on this classification, [8–10].

In the supercase the theory of the primitive ideals is even more complicated, [3]. It is interesting to generalize Losev's result to the supercase. One step in this direction is to study representations of finite W-algebras for a Lie superalgebra $\mathfrak{g}$. In the case when $\mathfrak{g} = \mathfrak{gl}(m|n)$ and e is the even principal nilpotent, Brown, Brundan and Goodwin classified irreducible representations of W_e and explored the connection with the category $\mathcal{O}$ for $\mathfrak{g}$ using coinvariants functor, [1,2].

First, we study representations of finite W-algebra for the Lie superalgebra $Q(n)$ associated with the principal even nilpotent coadjoint orbit. Note that the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g} = Q(n)$ is not abelian and contains a non-trivial odd part. By our previous results ([17]), we realize W as a subalgebra of the universal enveloping algebra $U(\mathfrak{h})$. One of the main results of the paper is a classification of simple W-modules given in Theorem 4.7 (they are all finite-dimensional by [17]). The technique we use is completely different from one used in [1,2] due to the lack of triangular decomposition of W in our case. Instead, we can describe the restriction of simple $U(\mathfrak{h})$-modules to W and prove that any simple W-module occurs as a constituent of this restriction.

We have shown previously in [17] that a principal W-algebra (for any n) is a quotient of $YQ(1)$. Hence a simple module over a W-algebra can be lifted to a simple $YQ(1)$-module. However, not every simple $YQ(1)$-module can be obtained in this way. We prove that any simple finite-dimensional $YQ(1)$-module is isomorphic to the tensor product of a module lifted from a W-algebra and some one-dimensional module (Theorem 5.13). We also obtain a formula for a generating function for the central character of a simple module. This generating function is rational and probably should be considered as an analogue of the Drinfeld polynomial, see [11] chapters 3, 4.

We plan in a subsequent paper to study the coinvariants functor from the category $\mathcal{O}$ for $Q(n)$ to the category of W-modules.

As M. L. Nazarov pointed to us, it is interesting to generalize the results of [7] to the case of $YQ(1)$ using the centralizer construction of $YQ(n)$ given in [15].

2. Notations and preliminary results

We work in the category of super vector spaces over $\mathbb{C}$. All tensor products are over $\mathbb{C}$ unless specified otherwise. By Π we denote the functor of parity switch $\Pi(X) = X \otimes \mathbb{C}^{0|1}$.

Recall that if X is a simple finite-dimensional $\mathcal{A}$-module for some associative superalgebra $\mathcal{A}$, then $\mathrm{End}_{\mathcal{A}}(X) = \mathbb{C}$ or $\mathrm{End}_{\mathcal{A}}(X) = \mathbb{C}[\epsilon]/(\epsilon^2 - 1)$, where the odd element ϵ provides an $\mathcal{A}$ isomorphism $X \to \Pi(X)$. We say that X is of M-type in the former case and of Q-type in the latter (see [6,4]).

If X and Y are two simple modules over associative superalgebras $\mathcal{A}$ and $\mathcal{B}$, we define the $\mathcal{A}\otimes\mathcal{B}$-module $X\boxtimes Y$ as the usual tensor product if at least one of X, Y is of M-type and the tensor product over $\mathbb{C}[\epsilon]$ if both X and Y are of Q-type.

In this paper we consider the Lie superalgebra $\mathfrak{g} = Q(n)$ defined as follows (see [5]). Equip $\mathbb{C}^{n|n}$ with the odd operator ζ such that $\zeta^2 = -\mathrm{Id}$. Then $Q(n)$ is the centralizer of ζ in the Lie superalgebra $\mathfrak{gl}(n|n)$. It is easy to see that $Q(n)$ consists of matrices of the form $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ where A, B are $n\times n$-matrices. We fix the Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ to be the set of matrices with diagonal A and B. By $\mathfrak{n}^+$ (respectively, $\mathfrak{n}^-$) we denote the nilpotent subalgebras consisting of matrices with strictly upper triangular (respectively, low triangular) A and B. The Lie superalgebra $\mathfrak{g}$ has the triangular decomposition $\mathfrak{g} = \mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$ and we set $\mathfrak{b} = \mathfrak{n}^+\oplus\mathfrak{h}$.

2.1. Finite W-algebra for $Q(n)$

Denote by W^n the finite W-algebra associated with a principal[1] even nilpotent element φ in the coadjoint representation of $Q(n)$. Let us recall the definition (see [19]). Let $\{e_{i,j}, f_{i,j}\,|\,i,j = 1,\dots,n\}$ denote the basis consisting of elementary even and odd matrices. Choose $\varphi\in\mathfrak{g}^*$ such that

$$\varphi(f_{i,j}) = 0, \quad \varphi(e_{i,j}) = \delta_{i,j+1}.$$

Let I_φ be the left ideal in $U(\mathfrak{g})$ generated by $x-\varphi(x)$ for all $x\in\mathfrak{n}^-$. Let $\pi: U(\mathfrak{g})\to U(\mathfrak{g})/I_\varphi$ be the natural projection. Then

$$W^n = \{\pi(y)\in U(\mathfrak{g})/I_\varphi \mid \mathrm{ad}(x)y\in I_\varphi \text{ for all } x\in\mathfrak{n}^-\}.$$

Using identification of $U(\mathfrak{g})/I_\varphi$ with the Whittaker module $U(\mathfrak{g})\otimes_{U(\mathfrak{n}^-)}\mathbb{C}_\varphi \simeq U(\mathfrak{b})\otimes\mathbb{C}$ we can consider W^n as a subalgebra of $U(\mathfrak{b})$. The natural projection $\vartheta: U(\mathfrak{b})\to U(\mathfrak{h})$ with the kernel $\mathfrak{n}^+U(\mathfrak{b})$ is called the *Harish-Chandra homomorphism*. It is proven in [17] that the restriction of ϑ to W^n is injective.

The center of $U(\mathfrak{g})$ is described in [21]. Set

$$\xi_i := (-1)^{i+1}f_{i,i},\ x_i := \xi_i^2 = e_{i,i},$$

then

$$U(\mathfrak{h}) \simeq \mathbb{C}[\xi_1,\dots,\xi_n]/(\xi_i\xi_j+\xi_j\xi_i)_{i<j\le n}.$$

The center of $U(\mathfrak{h})$ coincides with $\mathbb{C}[x_1,\dots,x_n]$ and the image of the center of $U(\mathfrak{g})$ under the Harish-Chandra homomorphism is generated by the polynomials $p_k = x_1^{2k+1}+\cdots+$

[1] There is a unique open orbit in the nilpotent cone of the coadjoint representation, elements of this orbit are called principal.

x_n^{2k+1} for all $k \in \mathbb{N}$, where we denote by $\mathbb{N}$ the set of all non-negative integers. These polynomials are called Q-symmetric polynomials.

In [17] we proved that the center Z of W^n coincides with the image of the center of $U(\mathfrak{g})$ and hence can be also identified with the ring of Q-symmetric polynomials.

2.2. Super Yangians of type Q

Recall that in [13] the Yangians $YQ(n)$ associated with Lie superalgebras $Q(n)$ were defined. In [17] and [18] (Corollary 5.16) we have shown the existence of the surjective homomorphism $\varphi_n : YQ(1) \to W^n$.

Recall that $YQ(1)$ is the associative unital superalgebra over $\mathbb{C}$ with the countable set of generators

$$T_{i,j}^{(m)} \text{ where } m = 1, 2, \dots \text{ and } i, j = \pm 1.$$

The $\mathbb{Z}_2$-grading of the algebra $YQ(1)$ is defined as follows:

$$p(T_{i,j}^{(m)}) = p(i) + p(j), \text{ where } p(1) = 0 \text{ and } p(-1) = 1.$$

To write down defining relations for these generators we employ the formal series in $YQ(1)[[u^{-1}]]$:

$$T_{i,j}(u) = \delta_{ij} \cdot 1 + T_{i,j}^{(1)} u^{-1} + T_{i,j}^{(2)} u^{-2} + \dots . \tag{2.1}$$

Then for all possible indices i, j, k, l we have the relations

$$\begin{aligned}(u^2 - v^2)&[T_{i,j}(u), T_{k,l}(v)] \cdot (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)} \\ &= (u + v)(T_{k,j}(u)T_{i,l}(v) - T_{k,j}(v)T_{i,l}(u)) \\ &- (u - v)(T_{-k,j}(u)T_{-i,l}(v) - T_{k,-j}(v)T_{i,-l}(u)) \cdot (-1)^{p(k)+p(l)},\end{aligned} \tag{2.2}$$

where v is a formal parameter independent of u, so that (2.2) is an equality in the algebra of formal Laurent series in u^{-1}, v^{-1} with coefficients in $YQ(1)$.
For all indices i, j we also have the relations

$$T_{i,j}(-u) = T_{-i,-j}(u). \tag{2.3}$$

Note that the relations (2.2) and (2.3) are equivalent to the following defining relations:

$$\begin{aligned}&([T_{i,j}^{(m+1)}, T_{k,l}^{(r-1)}] - [T_{i,j}^{(m-1)}, T_{k,l}^{(r+1)}]) \cdot (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)} = \\ &T_{k,j}^{(m)} T_{i,l}^{(r-1)} + T_{k,j}^{(m-1)} T_{i,l}^{(r)} - T_{k,j}^{(r-1)} T_{i,l}^{(m)} - T_{k,j}^{(r)} T_{i,l}^{(m-1)} \\ &+ (-1)^{p(k)+p(l)} (-T_{-k,j}^{(m)} T_{-i,l}^{(r-1)} + T_{-k,j}^{(m-1)} T_{-i,l}^{(r)} + T_{k,-j}^{(r-1)} T_{i,-l}^{(m)} - T_{k,-j}^{(r)} T_{i,-l}^{(m-1)}),\end{aligned} \tag{2.4}$$

$$T^{(m)}_{-i,-j} = (-1)^m T^{(m)}_{i,j}, \tag{2.5}$$

where $m, r = 1, \dots$ and $T^{(0)}_{i,j} = \delta_{ij}$.

Recall that $YQ(1)$ is a Hopf superalgebra, see [14], with comultiplication given by the formula

$$\Delta(T^{(r)}_{i,j}) = \sum_{s=0}^{r} \sum_{k} (-1)^{(p(i)+p(k))(p(j)+p(k))} T^{(s)}_{i,k} \otimes T^{(r-s)}_{k,j}.$$

The surjective homomorphism $\varphi_n : YQ(1) \to W^n$ is defined as follows:

$$\varphi_n(T^{(k)}_{1,1}) = (-1)^k [\sum_{1 \le i_1 < i_2 < \dots < i_k \le n} (x_{i_1} + (-1)^{k+1}\xi_{i_1}) \dots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k})]_{even}, \tag{2.6}$$

$$\varphi_n(T^{(k)}_{-1,1}) = (-1)^k [\sum_{1 \le i_1 < i_2 < \dots < i_k \le n} (x_{i_1} + (-1)^{k+1}\xi_{i_1}) \dots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k})]_{odd}.$$

Note that $\varphi_n(T^{(k)}_{1,1}) = \varphi_n(T^{(k)}_{-1,1}) = 0$ if $k > n$.

3. The structure of W-algebra

Using Harish-Chandra homomorphism we realize W^n as a subalgebra in $U(\mathfrak{h})$. It is shown in [18] that W^n has n even generators $z_0, \dots, z_{n-1}$ and n odd generators $\phi_0, \dots \phi_{n-1}$ defined as follows. For $k \ge 0$ we set

$$\phi_0 := \sum_{i=1}^{n} \xi_i, \quad \phi_k := T^k(\phi_0), \tag{3.1}$$

where the matrix of T in the standard basis $\xi_1, \dots, \xi_n$ has 0 on the diagonal and

$$t_{ij} := \begin{cases} x_j & \text{if} \quad i < j, \\ -x_j & \text{if} \quad i > j. \end{cases} \tag{3.2}$$

For odd $k \le n-1$ we define

$$z_k := [\sum_{i_1 < i_2 \dots < i_{k+1}} (x_{i_1} + (-1)^k \xi_{i_1}) \cdots (x_{i_k} - \xi_{i_k})(x_{i_{k+1}} + \xi_{i_{k+1}})]_{even}, \tag{3.3}$$

and for even $k \ge 0$ we set

$$z_k := \frac{1}{2}[\phi_0, \phi_k]. \tag{3.4}$$

Let $W_0^n \subset W^n$ be the subalgebra generated by $z_0, \dots, z_{n-1}$. By [17], Proposition 6.4, W_0^n is isomorphic to the polynomial algebra $\mathbb{C}[z_0, \dots, z_{n-1}]$. Furthermore there are the following relations

$$[\phi_i, \phi_j] = \begin{cases} (-1)^i 2z_{i+j} & \text{if } i+j \text{ is even} \\ 0 & \text{if } i+j \text{ is odd} \end{cases} \tag{3.5}$$

Define the $\mathbb{Z}$-grading on $U(\mathfrak{h})$ by setting the degree of ξ_i to be 1. It induces the filtration on W^n, for every $y \in W^n$ we denote by $\bar{y}$ the term of the highest degree.

Note that for even k, we have $z_k = \bar{z}_k$. Moreover, z_k is in the image under the Harish-Chandra map of the center of the universal enveloping algebra $U(Q(n))$. Therefore by [21] z_{2p} is a Q-symmetric polynomial in $\mathbb{C}[x_1, \dots, x_n]$ of degree $2p+1$. For example,

$$z_0 = x_1 + \cdots + x_n, \quad z_2 = \frac{1}{3}\left((x_1^3 + \cdots + x_n^3) - (x_1 + \cdots + x_n)^3\right).$$

For odd k the leading term is given by the elementary symmetric polynomial

$$\bar{z}_k = \sum_{i_1 < i_2 < \ldots < i_{k+1}} x_{i_1} \cdots x_{i_{k+1}}.$$

Lemma 3.1.

(1) $\operatorname{gr} W_0^n$ *is isomorphic to the algebra of symmetric polynomials* $\mathbb{C}[x_1, \dots, x_n]^{S_n} = \mathbb{C}[\bar{z}_0, \dots, \bar{z}_{n-1}]$ *and the degree of* $\bar{z}_k$ *is* $2k+2$;

(2) $U(\mathfrak{h})$ *is a free right* W_0^n*-module of rank* $2^n n!$.

Proof. Since $\bar{z}_0, \dots, \bar{z}_{n-1}$ are algebraically independent generators of $\mathbb{C}[x_1, \dots, x_n]^{S_n}$ we obtain (1).

It is well-known fact that $\mathbb{C}[x_1, \dots, x_n]$ is a free $\mathbb{C}[x_1, \dots, x_n]^{S_n}$-module of rank $n!$, see, for example, [22] Chapter 4. Since $U(\mathfrak{h})$ is a free $\mathbb{C}[x_1, \dots, x_n]$-module of rank 2^n we get that $U(\mathfrak{h})$ is a free $\mathbb{C}[x_1, \dots, x_n]^{S_n}$-module of rank $m = 2^n n!$. Let us choose a homogeneous basis $b_1, \dots, b_m$ of $U(\mathfrak{h})$ over $\mathbb{C}[x_1, \dots, x_n]^{S_n}$. We claim that it is a basis of $U(\mathfrak{h})$ as a right module over W_0^n. Indeed, let us prove first the linear independence. Suppose

$$\sum_{j=1}^{m} b_j y_j = 0$$

for some $y_j \in W_0^n$. Let $k = \max\{\deg y_j + \deg b_j \,|\, j = 1, \dots, m\}$. If $J = \{j \,|\, \deg y_j + \deg b_j = k\}$ we have $\sum_{j \in J} b_j \bar{y}_j = 0$. By above this implies $\bar{y}_j = 0$ for all $j \in J$ and we obtain all $y_j = 0$. On the other hand, it follows easily by induction on degree that $U(\mathfrak{h}) = \sum_{j=1}^{m} b_j W_0^n$. The proof of (2) is complete. □

Consider $U(\mathfrak{h})$ as a free $U(\mathfrak{h}_{\bar{0}})$-module and let W_1^n denote the free $U(\mathfrak{h}_{\bar{0}})$-submodule generated by $\xi_1, \dots, \xi_n$. Then W_1^n is equipped with $U(\mathfrak{h}_{\bar{0}})$-valued symmetric bilinear form $B(x, y) = [x, y]$.

Lemma 3.2. *Let $p(x_1, \dots, x_n) := \prod_{i<j}(x_i + x_j)$ and Γ denotes the Gram matrix $B(\phi_i, \phi_j)$. Then $\det \Gamma = cp^2 x_1 \cdots x_n$, where c is a non-zero constant.*

Proof. Recall that $\phi_k = T^k \phi_0$. Since the matrix of the form B in the basis $\xi_1, \dots, \xi_n$ is the diagonal matrix $C = \mathrm{diag}(x_1, \dots, x_n)$, then $\Gamma = Y^t C Y$, where Y is the square matrix such that $\phi_i = \sum_{j=1}^n y_{ij} \xi_j$. Hence $\det \Gamma = x_1 \cdots x_n \det Y^2$. Since $B(\phi_i, \phi_j)$ is a symmetric polynomial in $x_1, \dots, x_n$, the determinant of Γ is also a symmetric polynomial. The degree of this polynomial is n^2. Therefore it suffices to prove that $(x_1 + x_2)^2$ divides $\det \Gamma$, or equivalently $x_1 + x_2$ divides $\det Y$. In other words, we have to show that if $x_1 = -x_2$, then $\phi_0, \dots, \phi_{n-1}$ are linearly dependent. Indeed, one can easily see from the form of T that the first and the second coordinates of $T^k \phi_0$ coincide, hence $\phi_0, T\phi_0, \dots, T^{n-1}\phi_0$ are linearly dependent. $\square$

We also will use another generators in W^n introduced in [18], Corollary 5.15:

$$u_k(0) := [\sum_{1 \le i_1 < i_2 < \dots < i_k \le n} (x_{i_1} + (-1)^{k+1} \xi_{i_1}) \cdots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k})]_{even}, \tag{3.6}$$

$$u_k(1) := [\sum_{1 \le i_1 < i_2 < \dots < i_k \le n} (x_{i_1} + (-1)^{k+1} \xi_{i_1}) \cdots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k})]_{odd}.$$

For convenience we assume $u_k(0) = u_k(1) = 0$ for $k > n$.

Let $i + j = n$. We have the natural embedding of the Lie superalgebras $Q(i) \oplus Q(j) \hookrightarrow Q(n)$. If $\mathfrak{h}_r$ denotes the Cartan subalgebra of $Q(r)$, the above embedding induces the isomorphism

$$U(\mathfrak{h}) \simeq U(\mathfrak{h}_i) \otimes U(\mathfrak{h}_j). \tag{3.7}$$

The following lemma implies that we have also the embedding of W-algebras.

Lemma 3.3. *Let $i + j = n$. Then W^n is a subalgebra in the tensor product $W^i \otimes W^j$.*

Proof. Introduce generators in W^i and W^j:

$$u_k^+(0) := [\sum_{1 \le i_1 < i_2 < \dots < i_k \le i} (x_{i_1} + (-1)^{k+1} \xi_{i_1}) \cdots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k})]_{even}, \tag{3.8}$$

$$u_k^+(1) := [\sum_{1 \le i_1 < i_2 < \dots < i_k \le i} (x_{i_1} + (-1)^{k+1} \xi_{i_1}) \cdots (x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k})]_{odd}.$$

$$u_k^-(0) := [\sum_{i+1\le i_1<i_2<\ldots<i_k\le n} (x_{i_1}+(-1)^{k+1}\xi_{i_1})\cdots(x_{i_{k-1}}-\xi_{i_{k-1}})(x_{i_k}+\xi_{i_k})]_{even}, \tag{3.9}$$

$$u_k^-(1) := [\sum_{i+1\le i_1<i_2<\ldots<i_k\le n} (x_{i_1}+(-1)^{k+1}\xi_{i_1})\cdots(x_{i_{k-1}}-\xi_{i_{k-1}})(x_{i_k}+\xi_{i_k})]_{odd}.$$

Then for $d, e, f \in \mathbb{Z}/2\mathbb{Z}$ we have

$$u_k(d) = \sum_{e+f=d}\sum_{a+b=k} (-1)^{eb} u_a^+(e)u_b^-(f). \tag{3.10}$$

Here we assume $u_0^{\pm}(0) = 1$ and $u_0^{\pm}(1) = 0$. $\square$

Corollary 3.4. *If $i_1 + \cdots + i_p = n$, then W^n is a subalgebra in $W^{i_1} \otimes \cdots \otimes W^{i_p}$.*

It is easy to see the following commutative diagram:

$$\begin{tikzcd} YQ(1) \arrow[r, "\Delta"] \arrow[d, "\varphi_{m+n}"'] & YQ(1)\otimes YQ(1) \arrow[d, "\varphi_m\otimes\varphi_n"'] \\ W^{m+n} \arrow[r] & W^m \otimes W^n \end{tikzcd} \tag{3.11}$$

where the bottom horizontal arrow is the composition of the flip $W^n \otimes W^m \to W^m \otimes W^n$ with the map $W^{m+n} \to W^n \otimes W^m$ defined in Lemma 3.3. The appearance of the flip is due to the fact that the flip is used in the identification of $U(\mathfrak{h}) \subset U(Q(l))$ with $U(Q(1))^{\otimes l}$, see the formula before Theorem 5.8 and Theorem 5.14 in [18].

4. Irreducible representations of W^n

4.1. Representations of $U(\mathfrak{h})$

Let $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{C}^n$. We call $\mathbf{s}$ *regular* if $s_i \neq 0$ for all $i \le n$ and *typical* if $s_i + s_j \neq 0$ for all $1 \le i < j \le n$.

It follows from the representation theory of Clifford algebras that all irreducible representations of $U(\mathfrak{h})$ up to change of parity can be parameterized by $\mathbf{s} \in \mathbb{C}^n$. Indeed, let M be an irreducible representation of $U(\mathfrak{h})$. By Schur's lemma every x_i acts on M as a scalar operator $s_i\,\mathrm{Id}$. Let $I_{\mathbf{s}}$ denote the ideal in $U(\mathfrak{h})$ generated by $x_i - s_i$, then the quotient algebra $U(\mathfrak{h})/I_{\mathbf{s}}$ is isomorphic to the Clifford superalgebra $C_{\mathbf{s}}$[2] associated with the quadratic form:

$$B_{\mathbf{s}}(\xi_i, \xi_j) = \delta_{ij} 2s_i.$$

[2] We consider Clifford algebras as superalgebras with the natural $\mathbb{Z}_2$-grading.

Then M is a simple $C_{\mathbf{s}}$-module.

The radical $R_{\mathbf{s}}$ of $C_{\mathbf{s}}$ is generated by the kernel of the form $B_{\mathbf{s}}$. Let $m(\mathbf{s})$ be the number of non-zero coordinates of $\mathbf{s}$, then $C_{\mathbf{s}}/R_{\mathbf{s}}$ is isomorphic to the matrix superalgebra $M(2^{\frac{m}{2}-1}|2^{\frac{m}{2}-1})$ for even m and to the superalgebra $M(2^{\frac{m-1}{2}})\otimes\mathbb{C}[\epsilon]/(\epsilon^2-1)$ for odd m.

Therefore $C_{\mathbf{s}}$ has one (up to isomorphism) simple $\mathbb{Z}_2$-graded module $V(\mathbf{s})$ of type Q for odd $m(\mathbf{s})$, and two simple modules $V(\mathbf{s})$ and $\Pi V(\mathbf{s})$ of type M for even $m(\mathbf{s})$ (see [12]). In the case when $\mathbf{s}$ is regular, the form $B_{\mathbf{s}}$ is non-degenerate and the dimension of $V(\mathbf{s})$ equals 2^k, where $k=\lceil n/2\rceil$. In general, $\dim V(\mathbf{s})=2^{\lceil m(\mathbf{s})/2\rceil}$.

Consider the embedding $Q(p)\oplus Q(q)\hookrightarrow Q(n)$ for $p+q=n$ and the isomorphism (3.7). It induces an isomorphism of $U(\mathfrak{h})$-modules

$$V(\mathbf{s})\simeq V(s_1,\ldots,s_p)\boxtimes V(s_{p+1},\ldots,s_n). \tag{4.1}$$

4.2. Restriction from $U(\mathfrak{h})$ to W^n

We denote by the same symbol $V(\mathbf{s})$ the restriction to W^n of the $U(\mathfrak{h})$-module $V(\mathbf{s})$.

Proposition 4.1. *Let S be a simple W^n-module. Then S is a simple constituent of $V(\mathbf{s})$ for some $\mathbf{s}\in\mathbb{C}^n$.*

Proof. Since W_0^n is commutative and S is finite-dimensional (see [17]), there exists one dimensional W_0^n-submodule $\mathbb{C}_\nu\subset S$ with character ν. Therefore S is a quotient of $\operatorname{Ind}_{W_0^n}^{W^n}\mathbb{C}_\nu$. On the other hand, the embedding $W^n\hookrightarrow U(\mathfrak{h})$ induces the embedding $\operatorname{Ind}_{W_0^n}^{W^n}\mathbb{C}_\nu\hookrightarrow\operatorname{Ind}_{W_0^n}^{U(\mathfrak{h})}\mathbb{C}_\nu$. Thus, S is a simple constituent of $\operatorname{Res}_{W^n}\operatorname{Ind}_{W_0^n}^{U(\mathfrak{h})}\mathbb{C}_\nu$. By Lemma 3.1, $\operatorname{Ind}_{W_0^n}^{U(\mathfrak{h})}\mathbb{C}_\nu$ is finite-dimensional, and hence has simple $U(\mathfrak{h})$-constituents isomorphic to $V(\mathbf{s})$ for some $\mathbf{s}$. Hence S must appear as a simple W^n-constituent of some $V(\mathbf{s})$. $\square$

4.3. Typical representations

Theorem 4.2. *If $\mathbf{s}$ is typical, then $V(\mathbf{s})$ is a simple W^n-module.*

Proof. First, we assume that $\mathbf{s}$ is regular, i.e. $s_i\neq 0$ for all $i=1,\ldots,n$. The specialization $x_i\mapsto s_i$ induces an injective homomorphism $\theta_{\mathbf{s}}: W^n/(I_{\mathbf{s}}\cap W^n)\hookrightarrow C_{\mathbf{s}}$ and a specialization of the quadratic form $B\mapsto B_{\mathbf{s}}$. By Lemma 3.2 $\det\Gamma(\mathbf{s})\neq 0$. Therefore $B_{\mathbf{s}}$ is non-degenerate and $\theta_{\mathbf{s}}$ is an isomorphism. Thus, $V(\mathbf{s})$ remains irreducible when restricted to W^n.

If $\mathbf{s}$ is typical non-regular, there is exactly one i such that $s_i=0$. Let $\mathbf{s}'=(s_1,\ldots,s_{i-1},s_{i+1},\ldots s_n)$. Note that $(\theta_{\mathbf{s}}(\xi_i))$ is a nilpotent ideal of $C_{\mathbf{s}}$ and hence ξ_i acts by zero on $V(\mathbf{s})$. Then $V(\mathbf{s})$ is a simple module over the quotient $C_{\mathbf{s}'}\simeq C_{\mathbf{s}}/(\theta_{\mathbf{s}}(\xi_i))$. Recall Y from the proof of Lemma 3.2 and let Y' denote the minor of Y obtained by removing the i-th column and the i-th row. Then

$$\phi_k = \sum_{j \neq i} y'_{kj} \xi_j \mathrm{mod}\,(\xi_i).$$

Hence $\theta_{\mathbf{s}}(\phi_0), \ldots, \theta_{\mathbf{s}}(\phi_{n-1})$ generate $C_{\mathbf{s}'} \simeq C_{\mathbf{s}}/(\theta_{\mathbf{s}}(\xi_i))$ and the statement follows from the regular case for $n-1$. $\square$

4.4. Simple W^n-modules for $n = 2$

Let $n = 2$, then by Theorem 4.2 $V(\mathbf{s})$ is simple as W^n-module if $s_1 \neq -s_2$. The action of $U(\mathfrak{h})$ in $V(s_1, s_2)$ is given by the following formulas in a suitable basis:

$$\xi_1 \mapsto \begin{pmatrix} 0 & \sqrt{s_1} \\ \sqrt{s_1} & 0 \end{pmatrix}, \quad \xi_2 \mapsto \begin{pmatrix} 0 & \sqrt{s_2}\mathbf{i} \\ -\sqrt{s_2}\mathbf{i} & 0 \end{pmatrix}.$$

Note that W^n is generated by ϕ_0, ϕ_1, z_0 and z_1. Using

$$\phi_0 = \xi_1 + \xi_2, \;\; \phi_1 = x_2\xi_1 - x_1\xi_2, \;\; z_0 = x_1 + x_2, \;\; z_1 = x_1x_2 - \xi_1\xi_2$$

we obtain the following formulas for the generators of W^n:

$$\phi_0 \mapsto \begin{pmatrix} 0 & \sqrt{s_1} + \sqrt{s_2}\mathbf{i} \\ \sqrt{s_1} - \sqrt{s_2}\mathbf{i} & 0 \end{pmatrix}, \quad \phi_1 \mapsto \sqrt{s_1 s_2} \begin{pmatrix} 0 & \sqrt{s_2} - \sqrt{s_1}\mathbf{i} \\ \sqrt{s_2} + \sqrt{s_1}\mathbf{i} & 0 \end{pmatrix}, \tag{4.2}$$

$$z_0 \mapsto (s_1 + s_2)\mathrm{Id}, \quad z_1 \mapsto \begin{pmatrix} s_1s_2 + \sqrt{s_1s_2}\mathbf{i} & 0 \\ 0 & s_1s_2 - \sqrt{s_1s_2}\mathbf{i} \end{pmatrix}. \tag{4.3}$$

Assume that $s_1 = -s_2$. If $s_1, s_2 = 0$ then $V(\mathbf{s})$ is isomorphic to $\mathbb{C} \oplus \Pi\mathbb{C}$, where $\mathbb{C}$ is the trivial module. If $s_1 \neq 0$, we choose $\sqrt{s_1}, \sqrt{s_2}$ so that $\sqrt{s_2} = \sqrt{s_1}\mathbf{i}$. Note that the choice of sign controls the choice of the parity of $V(\mathbf{s})$. The following exact sequence easily follows from (4.2) and (4.3):

$$0 \to \Pi\Gamma_{-s_1^2+s_1} \to V(\mathbf{s}) \to \Gamma_{-s_1^2-s_1} \to 0, \tag{4.4}$$

where Γ_t is the simple module of dimension $(1|0)$ on which ϕ_0, ϕ_1 and z_0 act by zero and z_1 acts by the scalar t. The sequence splits only in the case $s_1 = 0$, when $\Gamma_0 \simeq \mathbb{C}$ is trivial. Thus, using Proposition 4.1, Theorem 4.2, and (4.4) we obtain

Lemma 4.3. *If $n = 2$, then every simple W^n-module is isomorphic to one of the following*

(1) $V(s_1, s_2)$ *or* $\Pi V(s_1, s_2)$ *for* $s_1 \neq -s_2$, $s_1, s_2 \neq 0$;
(2) $V(s, 0)$ *if* $s \neq 0$;
(3) Γ_t *or* $\Pi\Gamma_t$.

4.5. Invariance under permutations

Theorem 4.4. *Let* $\mathbf{s}' = \sigma(\mathbf{s})$ *for some permutation of coordinates.*
(1) If $\mathbf{s}$ *is typical, then* $V(\mathbf{s})$ *is isomorphic to* $V(\mathbf{s}')$ *as a* W^n*-module.*
(2) If $\mathbf{s}$ *is arbitrary, then* $[V(\mathbf{s})] = [V(\mathbf{s}')]$ *or* $[\Pi V(\mathbf{s}')]$*, where* $[X]$ *denotes the class of* X *in the Grothendieck group.*

Proof. First, we will prove the statement for $n = 2$. Assume first that $s_2 \neq -s_1$. In this case $V(s_1, s_2)$ is a $(1|1)$-dimensional simple W^n-module.

Let

$$D = \begin{pmatrix} \sqrt{s_2} + \sqrt{s_1}\mathbf{i} & 0 \\ 0 & \sqrt{s_1} + \sqrt{s_2}\mathbf{i} \end{pmatrix}.$$

Then by direct computation we have

$$D\phi_0 D^{-1} = \begin{pmatrix} 0 & \sqrt{s_2} + \sqrt{s_1}\mathbf{i} \\ \sqrt{s_2} - \sqrt{s_1}\mathbf{i} & 0 \end{pmatrix}$$

and

$$D\phi_1 D^{-1} = \sqrt{s_1 s_2} \begin{pmatrix} 0 & \sqrt{s_1} - \sqrt{s_2}\mathbf{i} \\ \sqrt{s_1} + \sqrt{s_2}\mathbf{i} & 0 \end{pmatrix}.$$

Therefore D defines an isomorphism between $V(s_1, s_2)$ and $V(s_2, s_1)$.

Now consider the case $s_1 = -s_2$. Then the structure of $V(s_1, -s_1)$ is given by the sequence (4.4). Let $V(\mathbf{s}') = V(-s_1, s_1)$, then analogously we have the exact sequence

$$0 \to \Pi\Gamma_{-s_1^2 - s_1} \to V(\mathbf{s}') \to \Gamma_{-s_1^2 + s_1} \to 0. \tag{4.5}$$

The statement (2) now follows directly from comparison of (4.4) and (4.5). Now we will prove the statement for all n. Note that it suffices to consider the case of the adjacent transposition $\sigma = (i, i+1)$.

The embedding of $Q(i-1) \oplus Q(2) \oplus Q(n-i-1)$ into $Q(n)$ provides the isomorphism

$$U(\mathfrak{h}) \simeq U(\mathfrak{h}^-) \otimes U(\mathfrak{h}^0) \otimes U(\mathfrak{h}^+),$$

where $\mathfrak{h}^-$, $\mathfrak{h}^0$ and $\mathfrak{h}^+$ are the Cartan subalgebras of $Q(i-1), Q(2)$ and $Q(n-i-1)$ respectively. Using twice the isomorphism (4.1) we obtain the following isomorphism of $U(\mathfrak{h})$-modules

$$V(\mathbf{s}) \simeq (V(s_1, \dots, s_{i-1}) \boxtimes V(s_i, s_{i+1})) \boxtimes V(s_{i+2}, \dots, s_n).$$

Suppose that $s_i \neq -s_{i+1}$. Let $D_{i,i+1} = 1 \otimes D \otimes 1$. By Corollary 3.4 we have that W^n is a subalgebra in $W^{i-1} \otimes W^2 \otimes W^{n-i-1}$ and hence $D_{i,i+1}$ defines an isomorphism of W^n-modules $V(\mathbf{s})$ and $V(\mathbf{s}')$.

If $s_i = -s_{i+1}$, then the statement follows from (4.4) and (4.5). This completes the proof of the theorem. $\square$

4.6. Construction of simple W^n-modules

Now we give a general construction of a simple W^n-module. Let $r, p, q \in \mathbb{N}$ and $r + 2p + q = n$, $\mathbf{t} = (t_1, \dots, t_p) \in \mathbb{C}^p$, $t_1, \dots, t_p \neq 0$, and $\lambda = (\lambda_1, \dots, \lambda_q) \in \mathbb{C}^q$, $\lambda_1, \dots, \lambda_q \neq 0$, such that $\lambda_i + \lambda_j \neq 0$ for any $1 \leq i \neq j \leq q$. Recall that by Corollary 3.4 we have an embedding $W^n \hookrightarrow W^r \otimes (W^2)^{\otimes p} \otimes W^q$. Set

$$S(\mathbf{t}, \lambda) := \mathbb{C} \boxtimes \Gamma_{t_1} \boxtimes \cdots \boxtimes \Gamma_{t_p} \boxtimes V(\lambda),$$

where the first term $\mathbb{C}$ in the tensor product denotes the trivial W^r-module. For $q = 0$ we use the notation $S(\mathbf{t}, 0)$ and set $V(\lambda) = \mathbb{C}$.

Remark 4.5. The dimension of $S(\mathbf{t}, \lambda)$ equals $2^{\frac{q}{2}}$ for even q and $2^{\frac{q+1}{2}}$ for odd q. Furthermore, $S(\mathbf{t}, \lambda)$ is isomorphic to $\Pi S(\mathbf{t}, \lambda)$ if and only if q is odd.

Lemma 4.6. *All $u_k(1)$ act by zero on $S(\mathbf{t}, 0)$. The action of $u_k(0)$ is given by the formula*

$$u_k(0) = \begin{cases} 0 \textit{ for odd } k, \textit{ and for } k > 2p, \\ \sigma_{\frac{k}{2}}(t_1, \dots, t_p) \textit{ for even } k, \end{cases}$$

where σ_a denote the elementary symmetric polynomials, $0 \leq a \leq p$.

Proof. The first assertion is trivial. We prove the second assertion by induction on p. For $p = 1$ it is a consequence of the definition of Γ_t for $Q(2)$. For $p > 1$ we consider the embedding $Q(n-2) \oplus Q(2) \hookrightarrow Q(n)$. The formula (3.10) degenerates to

$$u_k(0) = u_k^+(0) \otimes 1 + u_{k-1}^+(0) \otimes z_0 + u_{k-2}^+(0) \otimes z_1'.$$

As z_0 acts by zero on Γ_{t_p} the statement now follows from the obvious identity

$$\sigma_{\frac{k}{2}}(t_1, \dots t_p) = \sigma_{\frac{k}{2}}(t_1, \dots t_{p-1}) + t_p \sigma_{\frac{k}{2}-1}(t_1, \dots t_{p-1}). \quad \square$$

Theorem 4.7.

(1) *$S(\mathbf{t}, \lambda)$ is a simple W^n-module;*
(2) *Every simple W^n-module is isomorphic to $S(\mathbf{t}, \lambda)$ up to change of parity.*

Proof. Let $u_k^-(d)$, $d \in \mathbb{Z}/2\mathbb{Z}$, $1 \leq k \leq n$ be as in (3.9) where indices are taken in the interval $[n - q + 1, n]$. If $q = 0$ we set $u_k^-(0) = 1$ and $u_k^-(1) = 0$. Using Lemma 4.6 and formula (3.10) we can easily write the action of $u_k(d)$ in $S(\mathbf{t},\lambda)$ in terms of $u_k^-(d)$ after identifying $S(\mathbf{t},\lambda)$ with $V(\lambda)$:

$$u_k(d) = \sum_{2a+j=k} \sigma_a(t_1,\dots,t_p)u_j^-(d). \tag{4.6}$$

From these formulas we see that $u_k^-(d)$ and $u_k(d)$ generate the same subalgebra in $\mathrm{End}_{\mathbb{C}}(V(\lambda))$. By Theorem 4.2 this proves irreducibility of $S(\mathbf{t},\lambda)$.

To show (2) we use Proposition 4.1. Every simple W^n-module is a subquotient of $V(\mathbf{s})$. By Theorem 4.4 (2) we may assume that $s_1 = \dots = s_r = 0$, $s_i \neq 0$ for $i > r$, $s_{r+1} = -s_{r+2}, \dots, s_{r+2p-1} = -s_{r+2p}$. We can compute $W^r \otimes (W^2)^{\otimes p} \otimes W^q$-simple constituents of $V(\mathbf{s})$. They are $S(\mathbf{t},\lambda)$ (up to change of parity) with $t_j = -s_{r+2j}^2 \pm s_{r+2j}$ and $\lambda_i = s_{r+2p+i}$ (we can assume that all $s_i \neq \pm 1$). By (1) $S(\mathbf{t},\lambda)$ remains simple when restricted to W^n. Hence the statement. $\square$

Remark 4.8. $\Gamma_0 \simeq \mathbb{C} \boxtimes \mathbb{C}$ as W^2-modules ($r = 2$, $p = q = 0$).

4.7. Central characters

Recall that the center of $U(Q(n))$ coincides with the center Z of W^n, see Section 2. Every $\mathbf{s}$ defines the central character $\chi_{\mathbf{s}} : Z \to \mathbb{C}$. Furthermore, Theorem 4.7 (2) implies that every simple W^n-module admits central character $\chi_{\mathbf{s}}$ for some $\mathbf{s}$. For every $\mathbf{s} = (s_1,\dots,s_n)$ we define the *core* $c(\mathbf{s}) = (s_{i_1},\dots,s_{i_m})$ as a subsequence obtained from $\mathbf{s}$ by removing all $s_j = 0$ and all pairs (s_i, s_j) such that $s_i + s_j = 0$. Up to a permutation this result does not depend on the order of removing. Thus, the core is well defined up to permutation. We call m the length of the core. The notion of core is very useful for describing the blocks in the category of finite-dimensional $Q(n)$-modules, see [16] and [20].

Example 4.9. Let $\mathbf{s} = (1, 0, 3, -1, -1)$, then $c(\mathbf{s}) = (3, -1)$.

The following is a reformulation of the central character description in [21].

Lemma 4.10. *Let $\mathbf{s}, \mathbf{s}' \in \mathbb{C}^n$. Then $\chi_{\mathbf{s}} = \chi_{\mathbf{s}'}$ if and only if $\mathbf{s}$ and $\mathbf{s}'$ have the same core (up to permutation).*

It follows from Lemma 4.10 that the core depends only on the central character $\chi_{\mathbf{s}}$, we denote it $c(\chi)$. By Theorem 4.4 we obtain the following.

Corollary 4.11. *Let $\chi : Z \to \mathbb{C}$ be a central character with core $c(\chi)$ of length m. Then W^m-module $V(c(\chi))$ is well-defined. From now on we denote it by $V(\chi)$ and call it the core representation.*

The category $W^n-\text{mod}$ of finite dimensional W^n-modules decomposes into direct sum $\bigoplus (W^n)^\chi - \text{mod}$, where $(W^n)^\chi - \text{mod}$ is the full subcategory of modules admitting generalized central character χ.

Lemma 4.12. *A simple W^n-module S belongs to $(W^n)^\chi - \text{mod}$ if and only if it is isomorphic (up to change of parity) to $S(\mathbf{t}, \lambda)$ with $\lambda = c(\chi)$.*

Proof. We have to compute the central character of $S(\mathbf{t}, \lambda)$. For a Q-symmetric polynomial $p_k = x_1^{2k+1} + \cdots + x_n^{2k+1}$ we have $p_k(\mathbf{t}, \lambda) = \lambda_1^{2k+1} + \cdots + \lambda_q^{2k+1}$. Since p_k generate the center of W^n the statement follows. $\square$

Proposition 4.13. *Two simple modules $S(\mathbf{t}, \lambda)$ and $S(\mathbf{t}', \lambda')$ are isomorphic (up to change of parity) if and only if $p' = p$, $q' = q$, $\mathbf{t}' = \sigma(\mathbf{t})$ and $\lambda' = \tau(\lambda)$ for some $\sigma \in S_p$ and $\tau \in S_q$.*

Proof. First, (4.6) and Theorem 4.4 imply the "if" statement. To prove the "only if" statement, assume that $S(\mathbf{t}, \lambda)$ and $S(\mathbf{t}', \lambda')$ are isomorphic. Then these modules admit the same central character. Therefore by Lemma 4.12 $\lambda' = \tau(\lambda)$ for some $\tau \in S_q$. Hence without loss of generality we may assume that $q' = q$ and $\lambda' = \lambda$.

Denote by $\operatorname{tr} x$ and $\operatorname{tr}' x$ the trace of $x \in W^n$ in $S(\mathbf{t}, \lambda)$ and $S(\mathbf{t}', \lambda)$ respectively. Then we must have

$$\operatorname{tr} u_k(0) = \operatorname{tr}' u_k(0).$$

Using the formula (4.6) we get

$$\operatorname{tr} u_k(0) = \sum_{2a+j=k} \sigma_a(t_1, \ldots, t_p) \operatorname{tr}_{V(\lambda)} u_j^-(0),$$

$$\operatorname{tr}' u_k(0) = \sum_{2a+j=k} \sigma_a(t_1', \ldots, t_{p'}') \operatorname{tr}_{V(\lambda)} u_j^-(0).$$

Let $b_j := \operatorname{tr}_{V(\lambda)} u_j^-(0)$. Without loss of generality we may assume that $p \geq p'$. Then we can rewrite our formula with $p = p'$ assuming $t_i' = 0$ for $p \geq i > p'$. Then the above implies

$$\sigma_a(t_1, \ldots, t_p) b_0 + \sigma_{a-1}(t_1, \ldots, t_p) b_2 + \cdots + \sigma_0(t_1, \ldots, t_p) b_{2a} =$$
$$\sigma_a(t_1', \ldots, t_p') b_0 + \sigma_{a-1}(t_1', \ldots, t_p') b_2 + \cdots + \sigma_0(t_1', \ldots, t_p') b_{2a},$$

where we assume $b_i = 0$ for $i > q$. Since $b_0 = \dim V(\lambda) \neq 0$ the above equations imply $\sigma_a(t_1, \ldots, t_p) = \sigma_a(t_1', \ldots, t_p')$ for all $a = 1, \ldots, p$. Therefore $\mathbf{t}' = \sigma(\mathbf{t})$ for some $\sigma \in S_p$ and in particular, $p' = p$. $\square$

We denote by $\mathcal{P}^l$ the subcategory of W^l-modules which admit trivial generalized central character.

Lemma 4.14. *Let $\chi : Z \to \mathbb{C}$ be a central character with core $c(\chi)$ of length m. Then the functor $W^{n-m} - \mathrm{mod} \to W^n - \mathrm{mod}$ defined by*[3] *$F(M) = \mathrm{Res}_{W^n}(M \otimes V(\chi))$ restricts to the functor $\Phi : \mathcal{P}^{n-m} \to (W^n)^\chi - \mathrm{mod}$. Furthermore, Φ is an exact functor which sends a simple object to a simple object.*

Proof. The first assertion is immediate consequence of Lemma 4.12 and the second follows from the construction of $S(\mathbf{t}, \lambda)$. □

Conjecture 4.15. The functor $\Phi : \mathcal{P}^{n-m} \to (W^n)^\chi - \mathrm{mod}$ defines an equivalence of categories.

5. Representations of the super Yangian of type $Q(1)$

In this section we classify irreducible finite-dimensional representations of $YQ(1)$ and explore their connections with irreducible representations of W^n.

Lemma 5.1. *Let*

$$\eta_i = (-\frac{1}{2})^i \operatorname{ad}^i T_{1,1}^{(2)}(T_{1,-1}^{(1)}), \quad Z_{2i} = \frac{1}{2}[\eta_0, \eta_{2i}]. \tag{5.1}$$

(1) *The following analogue of relations (3.5) holds:*

$$[\eta_i, \eta_j] = \begin{cases} (-1)^i 2Z_{i+j} & \text{if } i+j \text{ is even} \\ 0 & \text{if } i+j \text{ is odd} \end{cases}.$$

(2) *The elements $\{Z_{2i} \mid i \in \mathbb{N}\}$ are algebraically independent generators of the center of $YQ(1)$.*
(3) *The elements η_0 and $\{T_{1,1}^{(2i)} \mid i \in \mathbb{N}\}$ generate $YQ(1)$.*

Proof. The surjective homomorphism $\varphi_n : YQ(1) \to W^n$ defined in (2.6) acts on generators by

$$\varphi_n(\eta_i) = \phi_i, \quad \varphi_n(Z_{2i}) = z_{2i}, \quad 0 \leq i \leq n-1.$$

Moreover, $\bigcap_{n \in \mathbb{N}} \operatorname{Ker} \varphi_n = 0$. Hence all assertions of the lemma follow from the similar statements for W^n. □

[3] We consider here the usual exterior tensor product in contrast with ⊠.

Let M be a simple $YQ(1)$-module. Then M admits a central character χ. We set $\chi_{2k} = \chi(Z_{2k})$ and consider the generating function

$$\chi(u) = \sum_{i=0}^{\infty} \chi_{2i} u^{-2i-1}.$$

Lemma 5.2. *Let M be a finite-dimensional simple $YQ(1)$-module admitting central character χ. Then $\chi(u)$ is a rational function of the form*

$$\frac{a_0 u^{-1} + \cdots + a_{q-1} u^{-2q+1}}{1 + c_1 u^{-2} + \cdots + c_q u^{-2q}}.$$

Proof. Let $\mathbf{C} \subset YQ(1)$ denote the unital subalgebra generated by $\{\eta_i \mid i \in \mathbb{N}\}$. Let $\mathbf{C}_\chi$ denote the quotient of $\mathbf{C}$ by the ideal $(\{Z_{2i} - \chi_{2i} \mid i \in \mathbb{N}\})$. Then the relations (3.5) imply that $\mathbf{C}_\chi$ is isomorphic to the infinite-dimensional Clifford algebra $\mathbf{Cliff}(V, B_\chi)$ on the space V with basis $\{\eta_i \mid i \in \mathbb{N}\}$ and the symmetric form B_χ defined by the formula

$$B_\chi(\eta_i, \eta_j) = \begin{cases} (-1)^i 2\chi_{i+j} & \text{if } i+j \text{ is even} \\ 0 & \text{if } i+j \text{ is odd} \end{cases}. \tag{5.2}$$

Note that M by definition restricts to a certain $\mathbf{C}_\chi$-module. On the other hand, $\mathbf{Cliff}(V, B_\chi)$ admits a finite-dimensional representation if and only if B_χ has a finite rank. Look at the infinite symmetric matrix of B_χ in the basis $\{\eta_i\}$. Then every column of this matrix is a linear combination of the first k columns for some k. The formula (5.2) implies that for some integer $q > 0$ and the coefficients $c_1, \ldots, c_q$ we have a recurrence relation

$$\chi_{2m} = \sum_{i=1}^{q} -c_i \chi_{2m-2i}, \quad \text{for all } m \geq q. \tag{5.3}$$

This condition is equivalent to the rationality of $\chi(u)$. $\square$

Recall the W^n-module $V(\mathbf{s})$ constructed in Section 4. Using the homomorphism φ_n we equip $V(\mathbf{s})$ with a $YQ(1)$-module structure. Our next goal is to compute the central character of $V(\mathbf{s})$. For this we need to compute the $\{z_{2i}\}$ in terms of symmetric polynomials. Recall the notations of Section 3. Note that for any n the elements $\{z_{2i}\}$ of the center can be expressed in terms of symmetric polynomials of $x_1, \ldots, x_n$ and this expression stabilizes as $n \to \infty$. Thus, z_{2i} is a particular element in the ring of symmetric functions of degree $2i + 1$.

Lemma 5.3. *We have the following expression*

$$z_{2k} = -\sum_{i=1}^{k} \sigma_{2i} z_{2k-2i} + \sigma_{2k+1}, \tag{5.4}$$

where $\sigma_p = \sum_{i_1<\cdots<i_p} x_{i_1}\dots x_{i_p}$ *is the elementary symmetric function.*

Proof. We proved in [17], Lemma 5.5 that for W^n the characteristic polynomial $\det(\lambda \operatorname{Id} - T)$ of T equals $\lambda^n + \sum_{i=1}^{\lfloor n/2\rfloor} \sigma_{2i}\lambda^{n-2i}$. As

$$z_{2k}(x_1,\dots,x_n) = [x_1,\dots,x_n]T^{2k}[1,\dots,1]^t,$$

the Hamilton–Cayley identity implies that for $2k \geq n$ we have

$$z_{2k}(x_1,\dots,x_n) = -\sum_{i=1}^{k} \sigma_{2i} z_{2k-2i}(x_1,\dots,x_n).$$

Since the degree of z_{2k} is $2k+1$ it is a polynomial of $\sigma_1,\dots,\sigma_{2k+1}$. Therefore it suffices to prove (5.4) for $n = 2k+1$. We do it by induction on k using the fact that $z_{2k}(x_1,\dots,x_{2k+1})$ is Q-symmetric. Indeed, we already know that

$$z_{2k}(x_1,\dots,x_{2k}) = -\sum_{i=1}^{k} \sigma_{2i} z_{2k-2i}(x_1,\dots,x_{2k}),$$

therefore from substituting $x_{2k+1} = 0$ we get

$$z_{2k}(x_1,\dots,x_{2k+1}) = -\sum_{i=1}^{k} \sigma_{2i} z_{2k-2i}(x_1,\dots,x_{2k+1}) + A\sigma_{2k+1}(x_1,\dots,x_{2k+1}).$$

It remains to find the coefficient A. By Q-symmetry

$$z_{2k}(x_1,\dots,x_{2k-1}) = z_{2k}(x_1,\dots,x_{2k-1},t,-t).$$

This leads to the identity

$$z_{2k}(x_1,\dots,x_{2k-1}) = -\sum_{i=1}^{k} \sigma_{2i} z_{2k-2i}(x_1,\dots,x_{2k-1}) +$$
$$+t^2\sum_{i=1}^{k} \sigma_{2i-2} z_{2k-2i}(x_1,\dots,x_{2k-1}) - At^2\sigma_{2k-1}(x_1,\dots,x_{2k-1}).$$

Furthermore, by induction assumption we have

$$\sum_{i=1}^{k} \sigma_{2i-2} z_{2k-2i}(x_1,\dots,x_{2k-1}) =$$
$$z_{2k-2}(x_1,\dots,x_{2k-1}) + \sum_{i=2}^{k} \sigma_{2i-2} z_{2k-2i}(x_1,\dots,x_{2k-1}) = \sigma_{2k-1}(x_1,\dots,x_{2k-1})$$

Hence $A = 1$. $\square$

Corollary 5.4. *$YQ(1)$-module $V(\mathbf{s})$ admits central character χ where*

$$\chi(u)=\frac{\sum_{i=0}^{\infty}\sigma_{2i+1}(\mathbf{s})u^{-2i-1}}{1+\sum_{i=1}^{\infty}\sigma_{2i}(\mathbf{s})u^{-2i}}.$$

Corollary 5.5. *The elements $\{z_{2k}\mid k=0,\dots,\lfloor\frac{n-1}{2}\rfloor\}$ and $\{\sigma_{2k}\mid k=1,\dots,\lfloor\frac{n}{2}\rfloor\}$ form an algebraically independent set of generators in the ring of symmetric polynomials in n variables.*

Proposition 5.6. *For any rational $\chi(u)$ there exist n and $\mathbf{s}$ such that $V(\mathbf{s})$ admits central character χ.*

Proof. It follows immediately from Corollary 5.4. Indeed, by Lemma 5.2

$$\chi(u)=\frac{a_1u^{-1}+\dots+a_{q-1}u^{-2q+1}}{1+c_1u^{-2}+\dots+c_qu^{-2q}}.$$

Let $n=2q$ and assume that $a_i=0$ for $i\geq q$, $c_j=0$ for $j>q$. One can choose $\mathbf{s}=(s_1,\dots,s_n)$ so that $\sigma_{2k}(s_1,\dots,s_n)=c_k$ and $\sigma_{2k+1}(s_1,\dots,s_n)=a_k$. $\square$

Corollary 5.7. *Any simple finite-dimensional $\mathbf{C}$-module is either trivial or isomorphic to $V(\mathbf{s})$ or $\Pi V(\mathbf{s})$ for some typical regular $\mathbf{s}$.*

Proof. Recall the notations of Section 4. Consider a homomorphism $\mathbf{C}\to C_{\mathbf{s}}$ defined as the composition

$$\mathbf{C}\hookrightarrow YQ(1)\xrightarrow{\varphi_n}W^n\xrightarrow{\theta_{\mathbf{s}}}C_{\mathbf{s}}.$$

This homomorphism is surjective if $\mathbf{s}$ is typical regular, see Theorem 4.2. For any central character χ there exists one up to isomorphism and parity change simple $\mathbf{C}_\chi$-module. By Proposition 5.6 it must be isomorphic to $V(\mathbf{s})$. $\square$

Remark 5.8. If $\mathbf{s}=(s_1,\dots,s_n)$ and $\mathbf{s}'=(s_1,\dots,s_n,s,-s)$ then $V(\mathbf{s})$ and $V(\mathbf{s}')$ admit the same central character. We can see it now from the formula

$$\frac{\sum_{i=0}^{\infty}\sigma_{2i+1}(\mathbf{s}')u^{-2i-1}}{1+\sum_{i=1}^{\infty}\sigma_{2i}(\mathbf{s}')u^{-2i}}=\frac{(1-s^2u^{-2})(\sum_{i=0}^{\infty}\sigma_{2i+1}(\mathbf{s})u^{-2i-1})}{(1-s^2u^{-2})(1+\sum_{i=1}^{\infty}\sigma_{2i}(\mathbf{s})u^{-2i})}.$$

Lemma 5.9. *We have the following expression*

$$[T_{1,1}^{(2k)},\eta_i]=[T_{1,1}^{(2k+2)},\eta_{i-2}]-[T_{1,1}^{(2k)},\eta_{i-1}]+2T_{1,1}^{(2k)}\eta_{i-1},\quad i\geq 2, \tag{5.5}$$
$$[T_{1,1}^{(2k)},\eta_0]=2T_{1,-1}^{(2k)},\quad [T_{1,1}^{(2k)},\eta_1]=-2T_{1,-1}^{(2k+1)}-[T_{1,1}^{(2k)},\eta_0]+2T_{1,1}^{(2k)}\eta_0.$$

Proof. Note that according to (6.4) and (6.5) from [17]

$$[T_{1,1}^{(k)}, T_{1,-1}^{(1)}] = (1 + (-1)^k)T_{1,-1}^{(k)}.$$

Hence $[T_{1,1}^{(2k)}, T_{1,-1}^{(1)}] = 2T_{1,-1}^{(2k)}$. Note also that

$$[T_{1,1}^{(2)}, T_{1,-1}^{(2k+1)}] = 2T_{1,-1}^{(2k+2)}, \tag{5.6}$$

$$[T_{1,1}^{(2)}, T_{1,-1}^{(2k)}] = 2T_{1,-1}^{(2k+1)} + 2T_{1,-1}^{(2k)} - 2T_{1,1}^{(2k)}T_{1,-1}^{(1)}. \tag{5.7}$$

Using (6.9) from [17] we have that $[T_{1,1}^{(2)}, T_{1,1}^{(2k)}] = 0$. Hence

$$[T_{1,1}^{(2k)}, \eta_i] = (-\frac{1}{2})^i \operatorname{ad}^i T_{1,1}^{(2)}([T_{1,1}^{(2k)}, T_{1,-1}^{(1)}]) = \frac{(-1)^i}{2^{i-1}} \operatorname{ad}^i T_{1,1}^{(2)}(T_{1,-1}^{(2k)}).$$

Next,

$$\operatorname{ad}^i T_{1,1}^{(2)}(T_{1,-1}^{(2k)}) = \operatorname{ad}^{i-1} T_{1,1}^{(2)}([T_{1,1}^{(2)}, T_{1,-1}^{(2k)}]) = \operatorname{ad}^{i-1} T_{1,1}^{(2)}(2T_{1,-1}^{(2k+1)} + 2T_{1,-1}^{(2k)} - 2T_{1,1}^{(2k)}T_{1,-1}^{(1)}).$$

Furthermore,

$$\begin{aligned}
&2\operatorname{ad}^{i-1} T_{1,1}^{(2)}(T_{1,-1}^{(2k+1)}) = 2\operatorname{ad}^{i-2} T_{1,1}^{(2)}([T_{1,1}^{(2)}, T_{1,-1}^{(2k+1)}]) = \\
&4\operatorname{ad}^{i-2} T_{1,1}^{(2)}(T_{1,-1}^{(2k+2)}) = (-1)^i 2^{i-1}[T_{1,1}^{(2k+2)}, \eta_{i-2}], \\
&2\operatorname{ad}^{i-1} T_{1,1}^{(2)}(T_{1,-1}^{(2k)}) = (-2)^{i-1}[T_{1,1}^{(2k)}, \eta_{i-1}], \\
-&2\operatorname{ad}^{i-1} T_{1,1}^{(2)} T_{1,1}^{(2k)} T_{1,-1}^{(1)} = -2T_{1,1}^{(2k)} \operatorname{ad}^{i-1} T_{1,1}^{(2)}(T_{1,-1}^{(1)}) = -2T_{1,1}^{(2k)}((-2)^{i-1}\eta_{i-1}) = (-2)^i T_{1,1}^{(2k)}\eta_{i-1}.
\end{aligned}$$

Thus

$$[T_{1,1}^{(2k)}, \eta_i] = \frac{1}{2^{i-1}}(2^{i-1}[T_{1,1}^{(2k+2)}, \eta_{i-2}] - 2^{i-1}[T_{1,1}^{(2k)}, \eta_{i-1}] + 2^i T_{1,1}^{(2k)}\eta_{i-1}),$$

which gives (5.5). □

Corollary 5.10. *Let* $\mathbf{A}$ *be the commutative subalgebra in* $YQ(1)$ *generated by* $T_{1,1}^{(2k)}$ *for* $k \geq 0$. *Then* $YQ(1) = \mathbf{CA} = \mathbf{AC}$.

Proof. We will show that $\mathbf{CA} \subset \mathbf{AC}$. (The proof of the opposite inclusion is similar.) Let D_i denote the span of η_j for $j < i$. By Lemma 5.9 for $i \geq 2$ we have that $\eta_i T_{1,1}^{(2k)} = T_{1,1}^{(2k)}\eta_i$ modulo $D_i\mathbf{A} + \mathbf{A}D_i$. Therefore, it suffices to show that $\eta_i\mathbf{A} \in \mathbf{AC}$ for $i = 0, 1$. Furthermore, the relations in the second line of (5.5) imply that it suffices to show that $T_{1,-1}^{(m)} \in \mathbf{CA} \cap \mathbf{AC}$. This can be done by induction on m. The case $m = 1$ is trivial as $\eta_0 = T_{1,-1}^{(m)}$. For the step of induction if m is even we employ (5.6) and if m is odd (5.7) and the relation

$$[T_{1,1}^{(2)}, \mathbf{C}] \subset \mathbf{C}.$$

Finally, since $\mathbf{A}$ and $\mathbf{C}$ generate $YQ(1)$ we get $YQ(1) = \mathbf{CA} = \mathbf{AC}$. □

Recall that for any Hopf superalgebra R the ideal (R_1) generated by all odd elements is a Hopf ideal and the quotient $R/(R_1)$ is a Hopf algebra.

Lemma 5.11. *The quotient $YQ(1)/(YQ(1)_1)$ is isomorphic to $\mathbf{A} \simeq \mathbb{C}[T_{1,1}^{(2k)}]_{k>0}$ with comultiplication*

$$\Delta T_{1,1}(u^{-2}) = T_{1,1}(u^{-2}) \otimes T_{1,1}(u^{-2}),$$

where $T_{1,1}(u^{-2}) = \sum T_{1,1}^{(2k)} u^{-2k}$.

Proof. Since all η_i generate $(YQ(1)_1)$, Lemma 5.1 implies $YQ(1) = \mathbf{A} + (YQ(1)_1)$. Therefore there exists a surjective homomorphism

$$\mathbf{A} \to YQ(1)/(YQ(1)_1).$$

To prove that it is injective we need to show that $\mathbf{A} \cap (YQ(1)_1) = \{0\}$. It suffices to check that for any $y \in \mathbf{A}$ there exists a one-dimensional $YQ(1)$-module Γ such that $y\Gamma \neq 0$. Let $y = P(T_{1,1}^{(2)} \dots T_{1,1}^{(2k)})$ for some polynomial P and consider the module $\Gamma = S(\mathbf{t}, 0)$ as in Lemma 4.6. Then y acts on Γ by $P(\sigma_1(\mathbf{t}), \dots, \sigma_k(\mathbf{t}))$. By a suitable choice of $\mathbf{t}$ we can get $P(\sigma_1(\mathbf{t}), \dots, \sigma_k(\mathbf{t})) \neq 0$. The comultiplication formula is straightforward as all $T_{1,1}^{(2k+1)} \in (YQ(1)_1)$. □

Let $f(u) = 1 + \sum_{k>0} f_{2k} u^{-2k}$. We denote by Γ_f the one-dimensional $\mathbf{A}$-module, where the action of $T_{1,1}(u^{-2})$ is given by the generating function $f(u)$.

Lemma 5.12. *The isomorphism classes of one-dimensional $YQ(1)$-modules are in bijection with the set $\{\Gamma_f\}$. Furthermore, we have the identity $\Gamma_f \otimes \Gamma_g \simeq \Gamma_{fg}$.*

Proof. Lemma 5.11 reduces the statement to classification of one-dimensional $\mathbf{A}$-modules which is straightforward. □

Theorem 5.13. *Any simple finite-dimensional $YQ(1)$-module is isomorphic to $V(\mathbf{s}) \otimes \Gamma_f$ or $\Pi V(\mathbf{s}) \otimes \Gamma_f$ for some regular typical $\mathbf{s}$ and $f(u) = 1 + \sum_{k>0} f_{2k} u^{-2k}$. Furthermore, $V(\mathbf{s}) \otimes \Gamma_f$ and $V(\mathbf{s}') \otimes \Gamma_g$ are isomorphic up to change of parity if and only if $\mathbf{s}'$ is obtained from $\mathbf{s}$ by permutation of coordinates and $f(u) = g(u)$.*

Proof. We start with regular typical $\mathbf{s}$ and identify $V(\mathbf{s})$ with $V(\mathbf{s}) \otimes \Gamma_{\mathbf{1}}$. Let χ be the central character of $V(\mathbf{s})$ and consider only simple modules with central character χ. We denote by $YQ(1)^\chi$ the quotient of $YQ(1)$ by the ideal generated by $\operatorname{Ker} \chi$. Note that $YQ(1)^\chi = \mathbf{C}_\chi \mathbf{A}$.

Note that the central characters of $V(\mathbf{s})$ and $V(\mathbf{s})\otimes\Gamma_f$ are the same and they are isomorphic as $\mathbf{C}_\chi$-modules. For any finite-dimensional $YQ(1)$-module M and $\theta(u) = 1+\sum\theta_i u^{-2i}$ set

$$M^\theta = \bigcap_{k>0}\Big(\bigcup_{m>0}\operatorname{Ker}(T_{1,1}^{(2k)}-\theta_k)^m\Big).$$

Clearly, we have an isomorphism of $\mathbf{A}$-modules

$$M \simeq \bigoplus_{\theta\in P(M)} M^\theta,$$

for some finite set $P(M)$. Furthermore, we have the following obvious relations

$$P(M\otimes\Gamma_f) = P(M)f, \quad (M\otimes\Gamma_f)^{\theta f} = M^\theta\otimes\Gamma_f. \tag{5.8}$$

This implies that $P(V(\mathbf{s})\otimes\Gamma_f) = P(V(\mathbf{s})\otimes\Gamma_g)$ if and only if $f = g$. Therefore we obtain the second assertion of the theorem.

Consider the natural homomorphism

$$F_\chi : YQ(1)^\chi \to \prod_f \operatorname{End}_{\mathbb{C}}(V(\mathbf{s})\otimes\Gamma_f).$$

Lemma 5.14. *Let $J_\chi = \operatorname{Ker}F_\chi$. Then*

(1) $J_\chi = \mathbf{A}R_\chi = R_\chi\mathbf{A}$, *where $R_\chi = \operatorname{Ann}_{\mathbf{C}_\chi} V(\mathbf{s})$ is the Jacobson radical of $\mathbf{C}_\chi$;*
(2) J_χ *acts by zero on any simple finite-dimensional $YQ(1)^\chi$-module.*

Proof. Let us prove (1). Note that $T_{1,1}^{(2k)}$ acts on $V(\mathbf{s})\otimes\Gamma_f$ as $\sum_{i=0}^k T_{1,1}^{(2i)}\otimes T_{1,1}^{(2k-2i)}$ and η_0 acts as $T_{1,-1}^{(1)}\otimes 1$. Therefore by (5.1) η_i acts as $\eta_i\otimes 1$ for all $i\geq 0$ and hence every $\zeta\in\mathbf{C}$ acts as $\zeta\otimes 1$. This implies $R_\chi\subset J_\chi$. Assume

$$X = \sum_{i=0}^k \zeta_i T_{1,1}^{(2i)} \in J_\chi,\ \zeta_i\in\mathbf{C}_\chi.$$

Set $f = 1+u^{-2k}$. Then since X annihilates both $V(\mathbf{s})$ and $V(\mathbf{s})\otimes\Gamma_f$ and X acts on the latter module as $X\otimes 1+\zeta_k\otimes 1$ we obtain that $\zeta_k\in R_\chi$. Repeating this argument we obtain that all $\zeta_i\in R_\chi$. Thus, $J_\chi = R_\chi\mathbf{A}$. The equality $\mathbf{A}R_\chi = R_\chi\mathbf{A}$ follows from $\mathbf{A}\mathbf{C}_\chi = \mathbf{C}_\chi\mathbf{A}$ by symmetry.

To prove (2) note that $J_\chi = \mathbf{A}R_\chi$ annihilates the induced module $YQ(1)^\chi\otimes_{\mathbf{C}_\chi} V(\mathbf{s})$ and hence any its quotient. On the other hand, up to switch of parity, any simple finite-dimensional $YQ(1)^\chi$-module is a quotient of this induced module. Hence the statement. $\square$

Lemma 5.15. *Let A be an associative subalgebra in the superalgebra $\prod_{i\in I} A_i$ where all A_i are isomorphic to the matrix superalgebra $M(n|n)$. If M is a simple A-module then* $\dim M \leq 2n$.

Proof. We use the fact that A_0 satisfies the Amitsur-Levitzki identity

$$\sum_{\sigma\in S_{2n}} sgn(\sigma)x_{\sigma(1)}\dots x_{\sigma(2n)} = 0, \tag{5.9}$$

for any $x_1,\dots,x_{2n} \in A_0$. Let M be a simple A-module. If $\mathrm{End}_A(M) = \mathbb{C}$ then $A \to \mathrm{End}_{\mathbb{C}}(M)$ is surjective by the Jacobson density theorem. Let $\dim M > 2n$ then $\dim M_0 > n$ or $\dim M_1 > n$, hence one can find $x_1,\dots,x_{2n} \in A_0$ which do not satisfy (5.9). If $\mathrm{End}_A(M) = Q(1)$, then $\dim M_0 = \dim M_1$. The image $A \to \mathrm{End}_{\mathbb{C}}(M)$ coincides with $Q(k) = \mathrm{End}_{Q(1)}(M)$ where $k = \dim M_0$ and the map $A_0 \to \mathrm{End}_{\mathbb{C}}(M_0)$ is surjective. Assume $\dim M > 2n$, then one can find $x_1,\dots,x_{2n} \in A_0$ which do not satisfy (5.9). □

Corollary 5.16. *Let M be a finite-dimensional simple $YQ(1)^\chi$-module. Then M is isomorphic to $V(\mathbf{s})$ or $\Pi V(\mathbf{s})$ for a regular typical $\mathbf{s}$ as a module over $\mathbf{C}_\chi$.*

Proof. The algebra $YQ(1)^\chi/J_\chi$ is a subalgebra in the product of matrix algebras $\mathrm{End}_{\mathbb{C}}(V(\mathbf{s}))$. Hence by Lemma 5.15 $\dim M \leq \dim V(\mathbf{s})$. Since R_χ annihilates M, the module M is isomorphic to a direct sum of several copies of $V(\mathbf{s})$ and $\Pi V(\mathbf{s})$ as a module over $\mathbf{C}_\chi$. This implies the statement. □

Remark 5.17. By Corollary 5.7, $\mathbf{C}_\chi/R_\chi \simeq C_{\mathbf{s}}$. Furthermore, $J_\chi \cap \mathbf{C}_\chi = R_\chi$.

Denote by $\mathbf{1}$ the function $\theta(u) = 1$ and assume that M is a simple finite-dimensional $YQ(1)^\chi$-module such that $M_0^{\mathbf{1}} \neq 0$. Then M is a quotient of the induced module

$$I = (YQ(1)^\chi/J_\chi) \otimes_{\mathbf{A}} \Gamma_{\mathbf{1}}.$$

Note that

$$\dim I \leq \dim(\mathbf{C}_\chi/R_\chi)$$

but we will see later that the equality takes place.

Lemma 5.18. *Let M be a simple $YQ(1)^\chi$-module such that $M_0^{\mathbf{1}} \neq 0$ and M remains simple after restriction to $\mathbf{C}_\chi$. Then there exists a quotient U of I with all simple subquotients isomorphic to M and length equal to* $\dim M_0^{\mathbf{1}}$.

Proof. Let $U = M \otimes (M_0^{\mathbf{1}})^*$. It obviously has a filtration with all quotients isomorphic to M and hence it satisfies the desired property. It remains to construct a surjective map $I \to U$. By Frobenius reciprocity we have a canonical isomorphism

$$\mathrm{Hom}_{YQ(1)}(I,U) \simeq \mathrm{Hom}_{\mathbf{A}}(\Gamma_{\mathbf{1}},U) \simeq \mathrm{Hom}_{\mathbf{A}}(\Gamma_{\mathbf{1}},M^{\mathbf{1}} \otimes (M_0^{\mathbf{1}})^*).$$

Consider the identity map in $\mathrm{Hom}_{\mathbf{A}}(\Gamma_{\mathbf{1}},M^{\mathbf{1}} \otimes (M_0^{\mathbf{1}})^*)$ and denote by γ the corresponding map in $\mathrm{Hom}_{YQ(1)}(I,U)$. Let us prove that γ is surjective. First, observe that any $y \in \mathbf{C}$ acts on $M \otimes (M_0^{\mathbf{1}})^*$ as $y \otimes 1$ by the same argument as in the proof of Lemma 5.14. Choose a basis $\{v_1,\dots,v_r\}$ in $M_0^{\mathbf{1}}$ and let $\{w_1,\dots,w_r\}$ be the corresponding dual basis in $(M_0^{\mathbf{1}})^*$. By construction $\sum v_i \otimes w_i \in \mathrm{Im}\gamma$. Since M is a simple $\mathbf{C}_\chi$-module, by the Jacobson density theorem for every $i = 1,\dots r$ there exists $y_i \in \mathbf{C}_\chi$ such that $y_i v_j = \delta_{i,j} v_1$. This implies $v_1 \otimes w_i \in \mathrm{Im}\gamma$ for all i and hence $M \otimes w_i \in \mathrm{Im}\gamma$ for all i. The surjectivity of γ follows immediately. $\square$

Now let us prove the first assertion of the theorem. Consider first the case $\mathbf{s} = (s_1,\dots,s_n)$ when n is even. Then $\dim V(\mathbf{s}) = 2^{n/2}$, $V(\mathbf{s})$ is not isomorphic to $\Pi V(\mathbf{s})$ and $\dim(\mathbf{C}_\chi/R_\chi) = 2^n$. By Lemma 5.18 and (5.8) for every $\theta \in P(V(\mathbf{s}))$ we have

$$[I : V(\mathbf{s}) \otimes \Gamma_{\theta^{-1}}] \geq \dim V(\mathbf{s})_0^\theta, \quad [I : \Pi V(\mathbf{s}) \otimes \Gamma_{\theta^{-1}}] \geq \dim V(\mathbf{s})_1^\theta.$$

On the other hand, $\dim I \leq \dim(\mathbf{C}_\chi/R_\chi)$. Hence any simple subquotient of I is isomorphic to $V(\mathbf{s}) \otimes \Gamma_{\theta^{-1}}$ or $\Pi V(\mathbf{s}) \otimes \Gamma_{\theta^{-1}}$ and $\dim I = \dim(\mathbf{C}_\chi/R_\chi)$. Therefore every simple $YQ(1)^\chi$-module M with $\mathbf{1} \in P(M)$ is isomorphic to $V(\mathbf{s}) \otimes \Gamma_{\theta^{-1}}$ or $\Pi V(\mathbf{s}) \otimes \Gamma_{\theta^{-1}}$. If $f \in P(M)$ then M is isomorphic to $V(\mathbf{s}) \otimes \Gamma_{f\theta^{-1}}$ or $\Pi V(\mathbf{s}) \otimes \Gamma_{f\theta^{-1}}$. This implies the statement.

Let us consider the case of odd n. Then $\dim V(\mathbf{s}) = 2^{(n+1)/2}$, $V(\mathbf{s})$ is isomorphic to $\Pi V(\mathbf{s})$ and $\dim(\mathbf{C}_\chi/R_\chi) = 2^n$. By Lemma 5.18 and (5.8) for every $\theta \in P(V(\mathbf{s}))$ we have

$$[I : V(\mathbf{s}) \otimes \Gamma_{\theta^{-1}}] \geq \dim V(\mathbf{s})_0^\theta = \dim V(\mathbf{s})_1^\theta.$$

By counting dimensions we again obtain that every simple subquotient of I is isomorphic to $V(\mathbf{s}) \otimes \Gamma_{\theta^{-1}}$. The end of the proof is the same as in the previous case. $\square$

Let us conclude by stating the relation between W^n-modules and $YQ(1)$-modules.

Proposition 5.19. *The simple $YQ(1)$-module $V(\mathbf{s}) \otimes \Gamma_f$ is lifted from some W^{m+n}-module if and only if $f \in \mathbb{C}[u^{-2}]$. Moreover, the smallest m is equal to the degree of the polynomial f.*

Remark 5.20. Note that $m = 2p$ is even. Then Theorem 4.7 and the diagram (3.11) imply $S(t_1,\dots,t_p,\lambda) \simeq V(\lambda) \otimes \Gamma_f$ where

$$f = \prod_{i=1}^{p}(1 + t_i u^{-2}).$$

Proof. Immediately follows from Theorem 4.7. $\square$

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