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On the finitistic dimension conjecture of Artin algebras ☆

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Abstract

Let A be an Artin algebra and e be an idempotent element of A. We prove that if A has representation dimension at most three, then the finitistic dimension of eAe is finite, and deduce that if quasi-hereditary algebras have representation dimensions at most three, then the finitistic dimension conjecture holds.

Keywords: Artin algebras; Representation dimension; Finitistic dimension

1. Introduction

Let A be an Artin algebra, A-mod the category of finitely generated left A-modules, and A-ind the full subcategory of A-mod containing exactly one representative of each isomorphism class of indecomposable A-modules. We denote by $\operatorname{pd} X$ and $\operatorname{id} X$ the projective and injective dimension of an A-module X, respectively. We denote by $\operatorname{gl.dim} A$ the global dimension of A.

In 1960, Bass has introduced in [B] the following definitions of a ring A:

$$\operatorname{fin.dim} A = \sup\{\operatorname{pd}_A M \mid M \in A\text{-mod},\ \operatorname{pd}_A M < \infty\},$$

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and he formulated two dimension conjectures:

(1) $\operatorname{fin.dim} A = \operatorname{Fin.dim} A$;
(2) $\operatorname{fin.dim} A < \infty$.

In [Z1], Zimmermann-Huisgen constructs a class of monomial algebras A such that $\operatorname{fin.dim} A \neq \operatorname{Fin.dim} A$, so (1) does not hold, (2) is still open now, and it is called finitistic dimension conjecture for Artin algebras.

For convenience, we also denote by $\operatorname{fin.dim} A$ the finitistic dimension of an Artin algebra A. Finitistic dimension conjecture is equivalent to that the finitistic dimension of any Artin algebra is finite. So far, it was proved to be true only for a few classes of Artin algebras. For example, monomial algebras [GKK], algebras where the cube of the radical is zero [GZ], and the algebras given in [IT,AR,W] and [X1,X2,X3,X4]. However, the finitistic dimension conjecture is still open in general and it is far from to be proven.

Let A be an Artin algebra, and $0 \longrightarrow {}_AA \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$ be the minimal injective resolution of A. Nakayama conjectured in [N] that A is self-injective whenever all I_j is projective. Up to now, Nakayama conjecture is still open. It is well known that finitistic dimension conjecture implies Nakayama conjecture, and this motivated further research on finitistic dimension conjecture. We refer to [Z2] and [X1,X2,X3,X4] for the background and some new progress about this conjecture.

Igusa and Todorov proved in [IT] that if all Artin algebras have representation dimensions at most three, then the finitistic dimension conjecture holds. Later, Rouquier constructed in [R] the exterior algebras of non-zero finite dimensional vector spaces such that the representation dimensions of which can be arbitrarily large. Recently, Xi provided in [X4] several sufficient conditions to the question of when $\operatorname{gl.dim} A \leqslant 4$ implies $\operatorname{fin.dim} eAe < \infty$, where e is an idempotent element of A.

In this paper, given an idempotent element $e \in A$, we study the pair of Artin algebras A and eAe in another direction. By a modification of Igusa and Todorov's results given in [IT], we prove that $\operatorname{rep.dim} A \leqslant 3$ implies that $\operatorname{fin.dim} eAe < \infty$, and deduce that if the representation dimensions of quasi-hereditary algebras are at most three, then the finitistic dimension conjecture holds. Our results are very different from Xi's in [X4], where Xi proved that $\operatorname{gl.dim} A \leqslant 4$ and $\operatorname{rep.dim} A/AeA \leqslant 3$ imply that $\operatorname{fin.dim} eAe < \infty$. Note that exterior algebras are self-injective algebras, they are not quasi-hereditary algebras, so our results are helpful to the resolution of the finitistic dimension conjecture.

Throughout this paper, we follow the standard terminology and notation used in the representation theory of algebras, see [ARS] and [ASS]. Given an Artin algebra A, we denote by A-mod the category of all finitely generated left A-modules. For an A-module M, we denote by $\operatorname{add} M$ the full subcategory having as objects the direct sums of indecomposable summands of M, by $\Omega^i M$ the ith syzygy of M. Then, $\mathcal{P} = \operatorname{add}{}_AA$ is the full subcategory consisting of all finitely generated projective A-modules, and $\mathcal{I} = \operatorname{add}{}_ADA$ is the full subcategory consisting of all finitely generated injective A-modules, where $D: A\text{-mod} \longrightarrow A^{op}\text{-mod}$ is the standard duality, and A^{op} is the opposite algebra of A.

Let $\mathcal{X}$ be a full subcategory of A-mod. When we say that $\mathcal{X}$ is a full subcategory, we always mean that $\mathcal{X}$ is closed under direct summands. We denote by $\operatorname{gen} \mathcal{X}$ ($\operatorname{cogen} \mathcal{X}$) the full subcategory of A-mod generated (cogenerated) by $\mathcal{X}$, see [AF] and [ASS]. If $\mathcal{X} = \{M\}$, we set $\mathcal{X} = M$ and denote $\operatorname{gen} \mathcal{X}$ ($\operatorname{cogen} \mathcal{X}$) by $\operatorname{gen} M$ ($\operatorname{cogen} M$). If $\mathcal{X}$ contains only finite pairwise non-isomorphic indecomposable A-modules, we call $\mathcal{X}$ is of finite type.

2. Finitistic dimension conjecture

An A-module V is called a *generator–cogenerator* if every indecomposable projective module and every indecomposable injective module is a direct summand of V. We recall from [A] that the number

$$\operatorname{rep.dim} A = \inf\bigl\{\operatorname{gl.dim}\operatorname{End}_A(V) \bigm| V \text{ is a generator–cogenerator}\bigr\}$$

is called the *representation dimension* of an Artin algebra A.

The following two lemmas proved in [A] and [X4] will be used later.

Lemma 2.1. *Let V be a generator–cogenerator of A-mod and $n \geqslant 3$ an integer. The following two statements are equivalent:*

(1) *For any $X \in A$-ind, there is an exact sequence*

$$0 \longrightarrow V_{n-2} \longrightarrow \cdots \longrightarrow V_1 \longrightarrow V_0 \longrightarrow X \longrightarrow 0$$

with $V_i \in \operatorname{add}({}_AV)$ for $j = 0, \ldots, n-2$, such that

$$0 \longrightarrow \operatorname{Hom}_A(V, V_{n-2}) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_A(V, V_1) \longrightarrow \operatorname{Hom}_A(V, V_0) \longrightarrow \operatorname{Hom}_A(V, X) \longrightarrow 0$$

is exact.

(2) $\operatorname{gl.dim}\operatorname{End}_A V \leqslant n$.

Throughout this paper, given an idempotent element $e \in A$ of an Artin algebra A, we set $B = eAe$ and view Ae as an A–B-bimodule.

Lemma 2.2. *Let A be an Artin algebra, e an idempotent element in A, and $B = eAe$. Suppose M is an arbitrary B-module. Then for any $i \geqslant 0$,*

$$\Omega_B^{i+2}(M) \simeq e\Omega_A\bigl(Ae \otimes_B \Omega_B^{i+1}(M)\bigr) \simeq e\Omega_A^2\bigl(Ae \otimes_B \Omega_B^{i}(M)\bigr) \oplus eP,$$

where P is a projective A-module depending on M.

Theorem 2.3. *Let A be an Artin algebra, e an idempotent element in A, and $B = eAe$. If* $\operatorname{rep.dim} A \leqslant 3$*, then* $\operatorname{fin.dim} B$ *is finite.*

Proof. It follows from Lemma 2.1 and the inequality $\operatorname{rep.dim} A \leqslant 3$ that there exists a generator–cogenerator V for A-mod, such that for any A-module X, there is an exact sequence $0 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow X \longrightarrow 0$, with $V_1, V_0 \in \operatorname{add} V$, such that

$$0 \longrightarrow \operatorname{Hom}_A(V, V_1) \longrightarrow \operatorname{Hom}_A(V, V_0) \longrightarrow \operatorname{Hom}_A(V, X) \longrightarrow 0$$

is exact. Suppose M is a B-module with finite projective dimension. Obviously $\Omega_A^2(Ae \otimes_B M)$ is an A-module and there is a short exact sequence

$$0 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow \Omega_A^2(Ae \otimes_B M) \longrightarrow 0, \tag{$*$}$$

with $V_1, V_0 \in \operatorname{add} V$, such that

$$0 \longrightarrow \operatorname{Hom}_A(V, V_1) \longrightarrow \operatorname{Hom}_A(V, V_0) \longrightarrow \operatorname{Hom}_A\big(V, \Omega_A^2(Ae \otimes_B M)\big) \longrightarrow 0$$

is exact. We apply $\operatorname{Hom}_A(Ae, -)$ to the sequence $(*)$ and obtain the following exact sequence

$$0 \longrightarrow eV_1 \longrightarrow eV_0 \longrightarrow e\Omega_A^2(Ae \otimes_B M) \longrightarrow 0.$$

By Lemma 2.2, there is a projective A-module P such that $e\Omega_A^2(Ae \otimes_B M) \oplus eP \simeq \Omega_B^2(M)$, so we may write the sequence as

$$0 \longrightarrow eV_1 \longrightarrow eV_0 \oplus eP \longrightarrow e\Omega_A^2(Ae \otimes_B M) \oplus eP \longrightarrow 0,$$

namely

$$0 \longrightarrow eV_1 \longrightarrow eV_0 \oplus eP \longrightarrow \Omega_B^2(M) \longrightarrow 0.$$

Now it follows that

$$\begin{aligned} \operatorname{pd}_B M &\leqslant \operatorname{pd} \Omega_B^2(M) + 2 \\ &\leqslant \psi_B(eV_1 \oplus eV_0 \oplus eP) + 3 \\ &\leqslant \psi_B(eV \oplus eA) + 3 \end{aligned}$$

thus $\operatorname{fin.dim} B \leqslant \psi_B(eV \oplus eA) + 3$, where $\psi_B : B\text{-mod} \longrightarrow \mathbb{N}$ is the Igusa–Todorov function defined in [IT]. It follows that the finitistic dimension of B is finite. $\square$

Auslander proved in [A] that every Artin algebra B is of the form eAe with A being of finite global dimension. Dlab and Ringel proved in [DR] that the algebra A even can be chosen to be quasi-hereditary. So we have the following corollary.

Corollary 2.4. *If* $\operatorname{rep.dim} A \leqslant 3$ *for every quasi-hereditary algebra A, then the finitistic dimension conjecture holds.*

Theorem 2.5. *Let A be an Artin algebra, e an idempotent element in A, and $B = eAe$. If* $\operatorname{add}\{\Omega_A^3(X) \mid X \in A\text{-mod}\}$ *is of finite type, then* $\operatorname{fin.dim} B$ *is finite.*

Proof. Suppose M is a B-module with finite projective dimension. Let $\pi : P \longrightarrow \Omega_A^2(Ae \otimes_B M)$ be the projective cover of $\Omega_A^2(Ae \otimes_B M)$ in A-mod, then there is an exact sequence

$$0 \longrightarrow \Omega_A^3(Ae \otimes_B M) \longrightarrow P \xrightarrow{\pi} \Omega_A^2(Ae \otimes_B M) \longrightarrow 0. \tag{$*$}$$

By applying the exact functor $\mathrm{Hom}_A(Ae,-)$ to the exact sequence $(*)$, we obtain the following exact sequence

$$0 \longrightarrow e\Omega_A^3(Ae \otimes_B M) \longrightarrow eP \longrightarrow e\Omega_A^2(Ae \otimes_B M) \longrightarrow 0.$$

By Lemma 2.2, there is a projective A-module Q such that $e\Omega_A^2(Ae \otimes_B M) \oplus eQ \simeq \Omega_B^2(M)$, so we may rewrite the above sequence as

$$0 \longrightarrow e\Omega_A^3(Ae \otimes_B M) \longrightarrow eP \oplus eQ \longrightarrow \Omega_B^2(M) \longrightarrow 0.$$

Since $\mathrm{add}\{\Omega_A^3(X) \mid X \in A\text{-mod}\}$ is of finite type, we may assume $X_1, \ldots, X_t$ are a complete list of pairwise non-isomorphic indecomposable A-modules in $\mathrm{add}\{\Omega_A^3(X) \mid X \in A\text{-mod}\}$. Since the module $\Omega_A^3(Ae \otimes_B M)$ lies in $\mathrm{add}\{\Omega_A^3(X) \mid X \in A\text{-mod}\}$, we may write $\Omega_A^3(Ae \otimes M) = \bigoplus_{i=1}^t X_i^{m_i}$. So it follows that

$$\begin{aligned}
\mathrm{pd}_B M &\leqslant \mathrm{pd}\, \Omega_B^2(M) + 2 \\
&\leqslant \psi_B\left(e\left(\bigoplus_{i=1}^{t} X_i^{m_i}\right) \oplus eP \oplus eQ\right) + 3 \\
&\leqslant \psi_B(eX_1 \oplus eX_2 \oplus \cdots \oplus eX_t \oplus eA) + 3,
\end{aligned}$$

$\mathrm{fin.dim}\, B \leqslant \psi_B(eX_1 \oplus eX_2 \oplus \cdots \oplus eX_t \oplus eA) + 3$, where $\psi_B : B\text{-mod} \longrightarrow \mathbb{N}$ is the Igusa–Todorov function defined in [IT]. It follows that the finitistic dimension of B is finite. □

Corollary 2.6. *Let A be an Artin algebra, e an idempotent element in A, and $B = eAe$. If* $\mathrm{gl.dim}\, A \leqslant 3$, *then* $\mathrm{fin.dim}\, B$ *is finite.*

Corollary 2.7. *Let A be an Artin algebra, e an idempotent element in A, and $B = eAe$. If* $\mathrm{cogen}\, A$ *is of finite type, then* $\mathrm{fin.dim}\, B$ *is finite.*

Corollary 2.8. *Let A be an Artin algebra, e an idempotent element in A, and $B = eAe$. Then* $\mathrm{fin.dim}\, B$ *is finite if one of the following conditions holds*:

(1) *A is stably hereditary in the sense that each indecomposable submodule of an indecomposable projective module is either projective or simple and each indecomposable factor module of an indecomposable injective module is either injective or simple* (*see* [X1]).
(2) *A is a special biserial algebra* (*see* [SW]).
(3) *A is a tilted algebra* (*see* [ASS]) *or A is a laura algebra* (*see* [APT]).
(4) *A is the trivial extension of an iterated tilted algebra* (*see* [ASS]).
(5) *A is an algebra such that one of the functors* $\mathrm{Hom}_A(-, A)$ *or* $\mathrm{Hom}_A(D(A), -)$ *is of finite length.*
(6) *A is weakly stable hereditary in the sense that every indecomposable submodule of a projective module is either projective or simple.*

Proof. It was shown in [X1,EHIS,APT] and [CP] that all the algebras displayed in (1)–(5) have representation dimensions at most three, and therefore, the result follows from Theorem 2.3 in these cases.

Let A be a weakly stable hereditary algebra. By Corollary 2.7, we only need to show that cogen A is of finite type. For, let X be an indecomposable module lying in cogen A. Then there exists a positive integer n and an exact sequence $0 \longrightarrow X \xrightarrow{i} A^n \longrightarrow \operatorname{coker} i \longrightarrow 0$. Since A is a weakly stable hereditary algebra, X is projective or simple. So cogen A is of finite type. $\square$

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