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On finite generation of R-subalgebras of $R[X]$

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Abstract

We show that over a complete discrete valuation ring R whose residue field is algebraically closed, any Noetherian R-subalgebra of $R[X]$ is finitely generated and present examples of non-finitely generated Noetherian R-subalgebras of $R[X]$ satisfying various properties. We also give a sufficient codimension-one criterion for a Noetherian R-subalgebra of $R[X]$ to be finitely generated over R when R is locally factorial.

Keywords: Finite generation; Subalgebra of polynomial algebra; Dimension formula; Nagata ring; Discrete valuation ring; Complete local ring; Krull domain

1. Introduction

It is well known that over a field k, a k-subalgebra of the polynomial ring $k[X]$ is finitely generated over k; in particular, it is Noetherian. But even over a discrete valuation ring (R, π), there exist R-subalgebras of $R[X]$ (like $R[\pi X, \pi X^2, \pi X^3, \ldots]$) which are not Noetherian and, therefore, not finitely generated over R. In fact, when $R = k[t]$, Eakin had demonstrated a Noetherian R-subalgebra of $R[X]$ which is not finitely generated over R [6, p. 79].

In this paper we explore conditions under which Noetherian R-subalgebras of $R[X]$ are finitely generated over R. We shall first show that such an algebra is indeed finitely generated

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when R is a complete discrete valuation ring with algebraically closed or real closed residue field. More precisely, we prove (see Theorem 4.2):

Theorem A. *Let (R, π) be a discrete valuation ring (DVR for short) with residue field $k := R/\pi R$, and let A be a Noetherian R-subalgebra of $R[X]$. If R is complete and the algebraic closure $\bar{k}$ of k is a finite extension of k, then A is finitely generated as an R-algebra.*

The result will be illustrated with examples of non-finitely generated Noetherian R-subalgebras of $R[X]$ over DVRs. Example 5.5 will show that even the closed fibres of such algebras need not be finitely generated over the respective residue fields, while Example 5.6 will show that an additional hypothesis of finite generation of fibres does not ensure finite generation of a Noetherian subalgebra. Examples 5.5 and 5.6 will also illustrate the necessity of the hypotheses "$[\bar{k} : k] < \infty$" and "R is complete" in Theorem A. Both these hypotheses can be dropped if the closed fibre is assumed to be integral. More generally, we prove the following codimension-one criterion for finite generation of a Noetherian R-subalgebra of $R[X]$ over a locally factorial Noetherian domain R.

Theorem B. *Let R be a locally factorial Noetherian domain and $A(\neq R)$ a Noetherian R-subalgebra of $R[X]$ such that for every prime ideal p in R of height one, pA is a prime ideal in A. Then A is finitely generated over R and the normalisation of A is isomorphic to the symmetric algebra of an invertible ideal of R.*

Before proving our main theorems, we shall establish a few technical results relating finite generation of certain flat algebra A over a DVR (R, π) with the transcendence of fibres at minimal prime ideals of πA. We state below a special case of one such result (Proposition 3.4):

Proposition. *Let (R, π) be a complete DVR with residue field k and field of fractions K, and let A be a Krull overdomain of R such that the generic fibre $K \otimes_R A \cong A[\pi^{-1}]$ is a finitely generated K-algebra with* $\operatorname{tr.deg}_K A[\pi^{-1}] = 1$. *Then A is finitely generated over R if and only if* $\operatorname{tr.deg}_k A/P > 0$ *for every minimal prime ideal P of πA.*

In Section 2, we compile some known results which will be used in the paper; in Section 3, we prove our auxiliary results on finite generation of certain flat algebras over a DVR; in Section 4, we prove our main theorems; and in Section 5, we describe our examples.

2. Preliminaries

Throughout the paper, rings and algebras will be assumed to be commutative.

Notation. Over a commutative ring R, a polynomial ring in n variables will be denoted by $R^{[n]}$. Over a field k, $k^{(n)}$ will denote a purely transcendental extension field in n variables over k.

Definition. A Noetherian ring R is said to be a Nagata ring (or a pseudo-geometric ring) if, for every prime ideal p of R and for every finite algebraic extension field L of the field of fractions $k(p)$ of R/p, the integral closure of R/p in L is a finite module over R/p.

A Noetherian complete local ring is a Nagata ring [10, Theorem 32.1], and an affine ring (meaning finitely generated ring) over a field is also a Nagata ring [10, Theorem 36.5].

Dimension formula. Let $R \subseteq A$ be integral domains. Let P be a prime ideal of A, $p = P \cap R$ and $k = R_p/pR_p$.

We say that P satisfies the dimension formula relative to R if

$$\operatorname{ht} P + \operatorname{tr.deg}_k A/P = \operatorname{ht} p + \operatorname{tr.deg}_R A. \tag{2.1}$$

When R is a Noetherian domain, one has the dimension inequality [3, Theorem 2]:

$$\operatorname{ht} P + \operatorname{tr.deg}_k A/P \leqslant \operatorname{ht} p + \operatorname{tr.deg}_R A. \tag{2.2}$$

We shall use the following result on dimension formula over Nagata rings [13, Theorem 3.6 and Proposition 3.10].

Proposition 2.1. *Let $D \subseteq B$ be integral domains such that D is a Nagata ring and B is a D-subalgebra of a finitely generated D-algebra. Let P be a prime ideal of B satisfying the dimension formula relative to D. Then B/P is a D-subalgebra of a finitely generated D-algebra. Moreover, if the dimension of D is one and B is algebraic over D, then B itself is a finitely generated D-algebra.*

Corollary 2.2. *Let K be a field and let B be an integral domain such that B is a K-subalgebra of a finitely generated K-algebra. If* $\operatorname{tr.deg}_K B = 1$*, then B is finitely generated over K.*

Proof. Let $y \in B$ be a transcendental element over K, and set $D = K[y]$. Then $\dim D = 1$ and $\operatorname{tr.deg}_D B = 0$. Since D is a Nagata ring, the assertion follows from Proposition 2.1. □

Definition. Let $R \subseteq C$ be integral domains. Then C is said to be a locality (or essentially of finite type) over R if there exists a finitely generated R-algebra B and a prime ideal Q of B such that $C = B_Q$.

If R is a Nagata Dedekind domain and C is a normal locality over R, then C is analytically normal [10, Theorem 37.5]; in particular, C is analytically irreducible. Hence, as a special case of [13, Lemma 4.2], we have the following criterion for locality over such rings.

Lemma 2.3. *Let R be a Nagata Dedekind domain, and A a Noetherian normal overdomain of R such that A is an R-subalgebra of a finitely generated R-algebra. If Q is a prime ideal of A satisfying the dimension formula relative to R, then A_Q is a locality over R.*

Since a complete local ring is a Nagata ring, we sometimes make use of "reduction to complete case." For the reduction, we have the following lemma.

Lemma 2.4. *Let (R, π) be a DVR with field of fractions K, and let B be an R-algebra. Let $\widehat{R}$ denote the completion of R and set $B' = \widehat{R} \otimes_R B$. Then $B'/\pi B' \cong B/\pi B$ and $B'[\pi^{-1}] \cong \widehat{K} \otimes_K B[\pi^{-1}]$, where $\widehat{K} = \widehat{R}[\pi^{-1}]$, the field of fractions of $\widehat{R}$. In particular, if P is a prime ideal in B with $\pi \in P$, then $B'/PB' \cong B/P$.*

Proof. We have

$$B'/\pi B' \cong R/\pi R \otimes_R (\widehat{R} \otimes_R B) \cong \widehat{R}/\pi \widehat{R} \otimes_{R/\pi R} B/\pi B \cong B/\pi B$$

and

$$B'[\pi^{-1}] \cong R[\pi^{-1}] \otimes_R (\widehat{R} \otimes_R B) \cong \widehat{R}[\pi^{-1}] \otimes_{R[\pi^{-1}]} B[\pi^{-1}] \cong \widehat{K} \otimes_K B[\pi^{-1}],$$

as claimed. □

We now state a local–global result.

Theorem 2.5. *Let R be a Noetherian domain and A an R-subalgebra of a finitely generated R-algebra.*

(I) *If A_Q is a locality over R for every prime ideal Q of A, then A is finitely generated over R.*
(II) *If A is locally finitely generated over R, then A is finitely generated over R.*
(III) *If $A_m = R_m^{[1]}$ for every maximal ideal m of R, then $A \cong R[IT]$ for an invertible ideal I of R.*

Proof. (I) and (II) follow from [13, Theorem 2.20], while (III) follows from (II) and the well-known result of Eakin and Heinzer [7]. □

For ready reference, we state an elementary result [11, p. 201].

Lemma 2.6. *Let R be a Noetherian domain and A an R-subalgebra of a finitely generated R-algebra B. If B is integral over A, then A is finitely generated over R.*

We now recall a Lüroth-analogue on subalgebras of polynomial algebras due to Abhyankar, Eakin and Heinzer [2, Theorem 4.1].

Theorem 2.7. *Let R be a factorial domain and A an R-subalgebra of $R[X]$. If $A(\neq R)$ is a factorial domain, then $A = R^{[1]}$.*

In Examples 5.5 and 5.6, we shall use the following application of Cohen's criterion for a ring to be Noetherian.

Lemma 2.8. *Let D be an integral domain. Suppose that there exists a non-zero element t in D such that*

(i) *$D[t^{-1}]$ is a Noetherian ring*;
(ii) *tD is a maximal ideal of D*;
(iii) $\mathrm{ht}(tD) = 1$ (*or, equivalently*, $\bigcap_{n \geqslant 1} t^n D = (0)$).

Then D is a Noetherian ring.

Proof. By Cohen's theorem [10, Theorem 3.4], it suffices to show that every prime ideal of D is finitely generated. By (ii), the principal ideal tD is the only prime ideal containing t.

Now let P be any non-zero prime ideal of D not containing t. By (iii), $P \nsubseteq tD$ and hence by (ii), P and tD are comaximal ideals. Thus there exists $s \in P$ such that $(s,t)D = D$, i.e., the image of t is a unit in D/sD. Therefore, $D/sD = D[t^{-1}]/sD[t^{-1}]$ is a Noetherian ring by (i). Thus P/sD is finitely generated and hence P is finitely generated. □

We shall also use the following criterion for normality.

Lemma 2.9. *Suppose that D is an integral domain. If there exists a non-zero element t in D such that $\sqrt{tD} = tD$ and $D[t^{-1}]$ is normal, then D is normal.*

Proof. Let $\overline{D}$ denote the normalisation of D, and let w be an element in $\overline{D}$. Since $D[t^{-1}]$ is normal, it then follows that $t^n w \in D$ for some $n > 0$. Let P be a prime ideal in D with $t \in P$, and let P' be a prime ideal in $\overline{D}$ lying over P. Then $t^n w \in P' \cap D = P$, so that

$$t^n w \in \bigcap_{t \in P} P = \sqrt{tD} = tD,$$

which implies that $t^{n-1} w \in D$. Repeating this argument, we have $w \in D$, and hence $D = \overline{D}$, as claimed. □

Finally, we quote a recent result on the transcendence degree of A/pA when $R \hookrightarrow A \hookrightarrow R^{[m]}$ and p is a prime ideal in R whose extension remains prime in A [5, Proposition 3.7].

Proposition 2.10. *Let R be a Noetherian domain and let $A(\neq R)$ be an R-subalgebra of $R^{[m]}$. Let p be a prime ideal in R such that pA is a prime ideal in A. Then R/p is algebraically closed in A/pA and* $\operatorname{tr.deg}_{R/p} A/pA > 0$.

3. Finite generation over DVR and transcendence of fibres

In this section we prove two results (Propositions 3.4 and 3.6) on Krull domains which are flat algebras over a DVR (R, π). We show that, under certain conditions, the finite generation of such an algebra A is equivalent to the transcendence of the fibres at minimal prime ideals of πA.

Remark 3.1. Let B be an R-algebra over a DVR (R, π) with residue field $k := R/\pi R$ and field of fractions $K := R[\pi^{-1}]$. Then B has only two fibres over R: the generic fibre $K \otimes_R B \cong B[\pi^{-1}]$ and the closed fibre $k \otimes_R B \cong B/\pi B$. Note that B is flat over R if and only if π is a regular element in B, and B is faithfully flat over R if and only if B is flat over R and $\pi B \neq B$. In particular, in case $B[\pi^{-1}]$ is an integral domain, B is flat over R if and only if B itself is an integral domain.

Suppose in addition that B is an integral domain with $\operatorname{tr.deg}_R B = 1$. Then B is an R-subalgebra of a finitely generated R-algebra if and only if $B[\pi^{-1}]$ is an affine domain over K. Indeed, if B is an R-subalgebra of a finitely generated R-algebra, then $B[\pi^{-1}]$ is a K-subalgebra of a finitely generated K-algebra, so that $B[\pi^{-1}]$ is an affine domain over K by Corollary 2.2. The converse is obvious, because $B \subseteq B[\pi^{-1}]$ and $K = R[\pi^{-1}]$ is finitely generated over R.

Lemma 3.2. *Let (R, π) be a DVR with residue field k and B an overdomain of R with* $\operatorname{tr.deg}_R B \leqslant 1$. *Suppose that there exists a prime ideal P in B such that $\pi \in P$ and* $\operatorname{tr.deg}_k B/P > 0$. *Then* $\operatorname{tr.deg}_R B = 1$ *and* $\operatorname{ht} P = \operatorname{tr.deg}_k B/P = 1$. *In particular, P satisfies the dimension formula relative to R.*

Proof. By the dimension inequality (2.2) for P, we have

$$\operatorname{ht} P + \operatorname{tr.deg}_k B/P \leqslant \operatorname{ht} \pi R + \operatorname{tr.deg}_R B \leqslant 2. \tag{3.1}$$

Since $\operatorname{ht} P > 0$ and $\operatorname{tr.deg}_k B/P > 0$, it follows that $\operatorname{tr.deg}_R B = 1$ and $\operatorname{ht} P = \operatorname{tr.deg}_k B/P = 1$, as claimed. Thus the equalities hold in (3.1), and hence P satisfies the dimension formula relative to R. □

In Section 4 we shall show (Lemma 4.4) that B/P is finitely generated over k under the additional hypothesis that $B[\pi^{-1}]$ is an affine domain over K, the field of fractions of R.

The following lemma gives a criterion for a 2-dimensional Krull domain to be Noetherian.

Lemma 3.3. *Let R be a one-dimensional Noetherian domain with field of fractions K, $S = R \setminus \{0\}$, and let A be a Krull domain such that $R \subseteq A$ and $S^{-1}A$ is a finitely generated K-algebra with $\operatorname{tr.deg}_K S^{-1}A = 1$. Then A is Noetherian. In particular, over a one-dimensional Noetherian domain R, any Krull domain which is an R-subalgebra of $R[X]$ is Noetherian.*

Proof. Since R is a one-dimensional Noetherian domain, the normalisation $\overline{R}$ of R is Noetherian [9, p. 85]. Thus, replacing R by $\overline{R}$, we may assume that R is normal.

By the Mori–Nishimura theorem [9, Theorem 12.7], it suffices to show that A/P is Noetherian for every prime ideal P in A of height one. We set $p = P \cap R$ and $k = R/p$.

First suppose that $p \neq (0)$. Let L be the field of fractions of A. Then R_p is a DVR of K and A_P is a DVR of L dominating R_p. If A/P is algebraic over k, then A/P is a field, and hence A/P is Noetherian. Suppose that A/P is transcendental over k. Then we have $\operatorname{tr.deg}_k A/P = 1$ by Lemma 3.2. Let $x \in L$ be a transcendental element over K, and let $V = A_P \cap K(x)$. Then $[L : K(x)]$ is finite, and V is a DVR of $K(x)$. Let M be the maximal ideal of V. Then $[A_P/PA_P : V/M] < \infty$, so that V/M is transcendental over k. Hence, by [1, Corollary 3.6], we have $V/M = k_1(z) = k_1^{(1)}$ for some finite extension field k_1 of k and $z \in V/M$. Let $w \in A/P$ be a transcendental element over k. Then $k[w] \subseteq A/P \subseteq A_P/PA_P$ and

$$\bigl[A_P/PA_P : k(w)\bigr] = \bigl[A_P/PA_P : k_1(w,z)\bigr]\bigl[k_1(w,z) : k(w)\bigr] < \infty.$$

Therefore, by the Krull–Akizuki theorem [9, Theorem 11.7], A/P is Noetherian.

Next suppose that $p = (0)$. Then

$$R \subseteq A/P \subseteq S^{-1}A/PS^{-1}A,$$

where $S^{-1}A/PS^{-1}A$ is a finite extension of K, because $S^{-1}A$ is an affine K-domain of dimension one. Thus A/P is Noetherian again by the Krull–Akizuki theorem.

For the second assertion, note that if $R \subseteq A \subseteq R[X]$, then $K \subseteq S^{-1}A \subseteq K[X]$, so that $S^{-1}A$ is finitely generated over K by Corollary 2.2. Thus the assertion follows from the first assertion. □

We now give a criterion for finite generation of a flat Krull algebra of transcendence degree one over a DVR in terms of the transcendence of some of its fibres: the necessity statement (1) will show that the R-algebras A and D in our main examples (5.5 and 5.6) are not finitely generated (cf. Lemma 5.4); the sufficiency statement (2) is a crucial step in the proof of Theorem 4.2 (i.e., Theorem A).

Proposition 3.4. *Let (R, π) be a DVR with residue field k and field of fractions K, and let A be a Krull domain such that $R \subseteq A$ and $A[\pi^{-1}]$ is a finitely generated K-algebra with* $\operatorname{tr.deg}_K A[\pi^{-1}] = 1$. *Then*:

(1) *If A is finitely generated over R, then* $\operatorname{tr.deg}_k A/P > 0$ *for every minimal prime ideal P of πA.*
(2) *The converse of* (1) *also holds in case R is a Nagata ring.*

Proof. (1) Let P be a minimal prime ideal of πA. Since R, being of dimension one, is universally catenary [9, p. 255, Corollary 2], it then follows from Ratliff's theorem [9, Theorem 15.6] that P satisfies the dimension formula relative to R, namely,

$$\operatorname{ht} P = \operatorname{ht} \pi R + \operatorname{tr.deg}_R A - \operatorname{tr.deg}_k A/P.$$

Since $\operatorname{ht} P = \operatorname{tr.deg}_R A = 1$, from this we have $\operatorname{tr.deg}_k A/P = 1$, as desired.

(2) Note that $K = R[\pi^{-1}]$, and hence $A[\pi^{-1}]$ is finitely generated over R by assumption. Therefore, to show that A is finitely generated, it suffices, by Theorem 2.5, to show that A_Q is a locality over R for every prime ideal Q of A.

If $\pi \notin Q$, then $A_Q = A[\pi^{-1}]_{Q[\pi^{-1}]}$ is a locality over R. Suppose that $\pi \in Q$. Then, by dimension inequality, we have

$$\operatorname{ht} Q \leqslant \operatorname{ht} \pi R + \operatorname{tr.deg}_R A - \operatorname{tr.deg}_k A/Q = 2 - \operatorname{tr.deg}_k A/Q. \tag{3.2}$$

The equality clearly holds in (3.2) when $\operatorname{ht} Q = 2$. The equality also holds in (3.2) when $\operatorname{ht} Q = 1$, because in this case Q is a minimal prime ideal of πA, so that $\operatorname{tr.deg}_k A/Q > 0$ by assumption. Thus Q satisfies the dimension formula relative to R. Moreover A is Noetherian by Lemma 3.3, and hence A_Q is a locality over R by Lemma 2.3. □

Remark 3.5. Part (2) of Proposition 3.4 does not hold if R is not a Nagata ring. Indeed, if R is not a Nagata ring, then there exists a finite algebraic extension field L of K such that the integral closure R' of R in L is not a finite R-module. Let $A = R'[x]$ where x is an indeterminate. Then A is Noetherian and normal, because so is R' by the Krull–Akizuki theorem. Note that $R'[\pi^{-1}] = L$, and hence $A[\pi^{-1}] = L[x]$. Since L is finite over K, it thus follows that $A[\pi^{-1}]$ is a finitely generated K-algebra with $\operatorname{tr.deg}_K A[\pi^{-1}] = 1$.

Now, let P be a prime ideal in A of height one with $\pi \in P$, and let $p = P \cap R'$. Then p is a minimal prime ideal of $\pi R'$ and $P = pR'[x]$. Thus $A/P = (R'/p)[x]$, which implies that $\operatorname{tr.deg}_k A/P > 0$. However A is not finitely generated over R; if A were finitely generated over R, then $R' = A/xA$ would be finitely generated over R, and hence R' would be a finite R-module, a contradiction.

For example of non-Nagata DVR, see [10, p. 205, Example 3].

However, the following result shows that the hypothesis "R is a Nagata ring" is not required in part (2) of Proposition 3.4 if the field of fractions of A is given to be $K(x)$. Note that in such a case, $A[\pi^{-1}]$ is a normal affine domain over K with field of fractions $K(x)$, and hence, by [1, Lemma 5.1],

$$\text{either}\quad A[\pi^{-1}] = K[x, f^{-1}] \quad \text{or} \quad A[\pi^{-1}] = K[x, f^{-1}] \cap K[x^{-1}]_{(x^{-1})} \tag{3.3}$$

for some $f(\neq 0) \in K[x]$.

Proposition 3.6. *Let (R, π) be a DVR with residue field k and field of fractions K, and let A be a Krull domain with field of fractions $K(x)$ $(= K^{(1)})$ such that $R \subseteq A$ and $A[\pi^{-1}]$ is a finitely generated K-algebra. Then A is finitely generated over R if and only if* $\mathrm{tr.deg}_k A/P > 0$ *for every minimal prime ideal P of πA.*

Proof. It suffices to prove the "if" part (part (1) of Proposition 3.4 covers the "only if" part).

First of all we show that if $(V, \xi V)$ is a DVR of $K(x)$ dominating R such that $\mathrm{tr.deg}_k V/\xi V > 0$, then $V' := \widehat{R} \otimes_R V$ is a Krull domain, where $\widehat{R}$ is the completion of R. Indeed, we have

$$V'[\xi^{-1}] \cong \widehat{R} \otimes_R V[\xi^{-1}] \cong \widehat{R} \otimes_R K(x) = S^{-1}\widehat{K}[x],$$

where $\widehat{K}$ is the field of fractions of $\widehat{R}$ and $S = K[x] \setminus \{0\}$. Thus $V'[\xi^{-1}]$ is a Krull domain. In particular V' is an integral domain, because ξ is a regular element in V' by flatness of $\widehat{R}$ over R, so that $V' \subseteq V'[\xi^{-1}]$. On the other hand, since $\pi \in \xi V'$, it follows from Lemma 2.4 that $V'/\xi V' \cong V/\xi V$, and hence ξ is a prime element in V'. Thus

$$V' = V'[\xi^{-1}] \cap V'_{\xi V'}. \tag{3.4}$$

Moreover, since $\mathrm{tr.deg}_k V'/\xi V' = \mathrm{tr.deg}_k V/\xi V > 0$ by assumption, we have $\mathrm{ht}\,\xi V' = 1$ by Lemma 3.2. Hence $V'_{\xi V'}$ is a DVR, and therefore, by (3.4), V' is a Krull domain as claimed.

We now give a proof of the proposition. If $\pi A = A$, then $A = A[\pi^{-1}]$ is finitely generated over R. Suppose that $\pi A \neq A$, and set $A' = \widehat{R} \otimes_R A$. Since $\widehat{R}$ is faithfully flat over R, if A' is finitely generated over $\widehat{R}$, then so is A over R. Hence it suffices to show that A' is finitely generated over $\widehat{R}$. Let $P_1, \ldots, P_n$ be the minimal prime ideals of πA. Since A is a Krull domain, it follows that

$$A = A[\pi^{-1}] \cap V_1 \cap \cdots \cap V_n,$$

where $V_i = A_{P_i}$ for each i. From this we have

$$A' = (\widehat{R} \otimes_R A[\pi^{-1}]) \cap V'_1 \cap \cdots \cap V'_n,$$

where $V'_i = \widehat{R} \otimes_R V_i$ for each i. From (3.3), it follows that $\widehat{R} \otimes_R A[\pi^{-1}]$ is a normal affine domain over $\widehat{K}$. Since each V'_i is a Krull domain as proved above, we know that A' is a Krull domain such that $\widehat{R} \subseteq A'$. On the other hand, it follows from Lemma 2.4 that $A'[\pi^{-1}] \cong \widehat{K} \otimes_K A[\pi^{-1}]$, and hence $A'[\pi^{-1}]$ is a finitely generated $\widehat{K}$-algebra with $\mathrm{tr.deg}_{\widehat{K}} A'[\pi^{-1}] = 1$. Moreover, again by Lemma 2.4, we have $A'/P_iA' = A/P_i$ for each i, $1 \leqslant i \leqslant n$, which implies that $P_1A', \ldots, P_nA'$ are the minimal prime ideals of $\pi A'$. Since $\widehat{R}$ is a Nagata ring, it thus follows from Proposition 3.4 that A' is finitely generated over $\widehat{R}$. This completes the proof. $\square$

Remark 3.7. Let (R, π) be a DVR with residue field k and field of fractions K, and let

$$B = R\bigl[\pi X, X(\pi X - 1)\bigr] \cong R[U, V]/\bigl(\pi V - U(U-1)\bigr).$$

Then $R \subseteq B \subseteq R[X]$ and $B[\pi^{-1}] = K[X]$. Since

$$B/\pi B \cong R[U, V]/\bigl(\pi, U(U-1)\bigr) \cong k[V] \times k[V],$$

we have $\sqrt{\pi B} = \pi B$, so that B is normal by Lemma 2.9. Thus the R-algebra B is an example of non-polynomial R-subalgebra of $R[X]$ satisfying the hypotheses in the above proposition.

In Example 5.6, we shall see a Noetherian normal R-subalgebra A of $R[X]$ such that A is not finitely generated but $A/\pi A$ is finitely generated; and πA has two minimal prime ideals P_1, P_2 for which $A/P_1 = k^{[1]}$ but $A/P_2 = k$ (Lemma 5.4; also see Remark A.2).

Corollary 3.8. *Let (R, π) be a DVR with residue field k and field of fractions K. Let A be a flat R-algebra such that the generic fibre $A[\pi^{-1}]$ is a normal affine domain over K with field of fractions $K^{(1)}$, and the closed fibre $A/\pi A$ is an integral domain with* $\operatorname{tr.deg}_k A/\pi A > 0$. *Then A is a Noetherian normal domain which is finitely generated over R.*

Proof. Since A is an integral domain (cf. Remark 3.1) and π is a prime element in A, it follows that $A = A[\pi^{-1}] \cap A_{\pi A}$, where $A_{\pi A}$ is a DVR, because $\operatorname{ht} \pi A = 1$ by Lemma 3.2. Thus A is a Krull domain, and therefore A is finitely generated over R by Proposition 3.6. $\square$

Remark 3.9.

(1) Proposition 2.10 shows that the hypothesis $\operatorname{tr.deg}_k A/\pi A > 0$ in Corollary 3.8 is automatically satisfied when $A \hookrightarrow R^{[1]}$. However we need the hypothesis in general (namely, the transcendency does not follow from the other hypotheses in Corollary 3.8): indeed the R-algebra D of Example 5.6 is a Noetherian normal flat algebra over $R = k[t]_{(t)}$ such that $D[t^{-1}] = k(t)^{[1]}$, but $D/tD = k$.
(2) Example 5.3 will show that the hypothesis "$A[\pi^{-1}]$ is normal" is necessary even when $A \hookrightarrow R^{[1]}$.

4. Main results

In this section, we shall prove Theorems A and B mentioned in the introduction (Theorems 4.2, 4.8). In fact, Theorem 4.2 is a slightly generalised version of Theorem A: the hypothesis "$A \subseteq R[X]$" of Theorem A being replaced by the milder condition "A is a flat R-algebra such that $A[\pi^{-1}]$ is a finitely generated algebra over $R[\pi^{-1}]$ with $\dim A[\pi^{-1}] \leqslant 1$."

For convenience, we state below an easy result on reduction.

Lemma 4.1. *Let R be a ring and A an R-algebra. If there exists a finitely generated nilpotent ideal I of A such that A/I is a finitely generated R-algebra, then A is a finitely generated R-algebra. In particular, if A is Noetherian and A/P is a finitely generated R-algebra for every minimal prime ideal P of A, then A is a finitely generated R-algebra.*

Proof. By assumption there exists an R-subalgebra B of A such that B is finitely generated over R and $A = B + I$. Then, writing $I = (a_1, \dots, a_r)A$, we have $A = B[a_1, \dots, a_r] + I^n$ for every $n > 0$. Since $I^n = 0$ for some $n > 0$, it follows that $A = B[a_1, \dots, a_r]$. Thus A is finitely generated over R.

For the second assertion, it suffices to show that $A/\sqrt{0}$ is finitely generated over R. Let $P_1, \dots, P_n$ be the minimal prime ideals of A. Then $\sqrt{0} = P_1 \cap \dots \cap P_n$ and

$$A/\sqrt{0} \hookrightarrow B := A/P_1 \times \dots \times A/P_n,$$

where B is finitely generated over R, because so is each A/P_i by assumption. Note that B is generated by idempotents over $A/\sqrt{0}$, so that B is integral over $A/\sqrt{0}$. Thus $A/\sqrt{0}$ is finitely generated over R by Lemma 2.6. □

We now prove Theorem A in the slightly generalised form below.

Theorem 4.2. *Let (R, π) be a DVR with residue field k and field of fractions K, and let A be a Noetherian flat R-algebra such that $A[\pi^{-1}]$ is a finitely generated K-algebra with* $\dim A[\pi^{-1}] \leqslant 1$. *If R is complete and the algebraic closure $\bar{k}$ of k is a finite extension of k, then A is finitely generated over R.*

Proof. Since the assertion is obvious when $\pi A = A$, we may assume that $\pi A \neq A$.

We first consider the case where A is an integral domain. Replacing A by $A[t](= A^{[1]})$ if necessary, we further assume that $\operatorname{tr.deg}_R A(= \dim A[\pi^{-1}]) = 1$. Let C be the normalisation of A. Then $C[\pi^{-1}]$ is the normalisation of $A[\pi^{-1}]$, so that $C[\pi^{-1}]$ is an affine domain over K, because so is $A[\pi^{-1}]$. Note that C is a Krull domain by the Mori–Nagata theorem [10, Theorem 33.10], and hence C is Noetherian by Lemma 3.3. Therefore, by Lemma 2.6, it suffices to prove the result when A is a Noetherian normal domain.

Let P be a prime ideal in A of height one with $P \cap R = \pi R$. We will show that $\operatorname{tr.deg}_k A/P > 0$; if this is the case, then A is finitely generated over R by Proposition 3.4. Suppose on the contrary that $\operatorname{tr.deg}_k A/P = 0$, and set $V = A_P$, which is a DVR with maximal ideal PV. Then V/PV $(\hookrightarrow \bar{k})$ is a finite extension of k, which implies that $\operatorname{length}_k V/\pi V < \infty$, because $\pi V = P^m V$ for some positive integer m. Hence $V/\pi V$ is a finite k-module. Since R is complete, it then follows that V is a finite R-module [14, p. 259, Corollary 2]. However this contradicts that $\operatorname{tr.deg}_R V = \operatorname{tr.deg}_R A = 1$. Thus A is finitely generated over R.

We now consider the general case (i.e., when A is not necessarily a domain). By Lemma 4.1, it suffices to show that A/P_0 is finitely generated over R for every minimal prime ideal P_0 in A. Note that A/P_0 is R-flat, since A/P_0 is an integral domain. Note also that $\pi \notin P_0$, since A is R-flat. Hence $P_0[\pi^{-1}]$ is a minimal prime ideal of $A[\pi^{-1}]$, so that $(A/P_0)[\pi^{-1}](= A[\pi^{-1}]/P_0[\pi^{-1}])$ is a finitely generated K-algebra of dimension at most one. Therefore A/P_0 is finitely generated over R by what we have proved above. This completes the proof. □

Remark 4.3.

(1) The necessity of the additional hypotheses "R is complete," "$[\bar{k} : k] < \infty$" and "A is Noetherian" in Theorem 4.2 would be shown by the case $k = \boldsymbol{C}$ of Examples 5.6, 5.5 and the case $R = \boldsymbol{C}[\![t]\!]$ of Example 5.3, respectively. In fact, the case $k = \boldsymbol{C}$ of Example 5.6 shows that "R is complete" cannot be weakened to "R is a Nagata ring" even when $A \hookrightarrow R[X]$. Note

that, unlike in Proposition 3.6, we cannot "reduce to the complete case": Example 5.6 will demonstrate that the extended ring $A' = \widehat{R} \otimes_R A$ need not remain Noetherian even if A is Noetherian (see Remark 5.7).

(2) Note that when $\bar{k} \neq k$, then, by the Artin–Schreier characterisation of real closed fields [12, pp. 211–213], the hypothesis "$[\bar{k} : k] < \infty$" is equivalent to each of the following hypotheses: "$[\bar{k} : k] = 2$"; "$\bar{k} = k(i)$ where $i^2 = -1$"; "k is a real closed field." Moreover, in such a case, the characteristic of k is zero.

Let A be a flat algebra over a DVR (R, π). Then the closed fibre $A/\pi A$ could be finitely generated even when A itself is not finitely generated over R. We shall give sufficient conditions for finite generation of the closed fibre. The following result is a step for the purpose.

Lemma 4.4. *Let (R, π) be a DVR with residue field k and field of fractions K, and A a flat R-algebra such that $A[\pi^{-1}]$ is a finitely generated K-algebra of dimension one. Let P be a prime ideal in A such that $\pi \in P$. If* $\operatorname{tr.deg}_k A/P > 0$*, then A/P is finitely generated over k.*

Proof. Let $\widehat{R}$ denote the completion of R, and set $A' = \widehat{R} \otimes_R A$. Then A' is a flat $\widehat{R}$-algebra. By Lemma 2.4, we have $A'/PA' \cong A/P$ and $A'[\pi^{-1}] \cong \widehat{K} \otimes_K A[\pi^{-1}]$, where $\widehat{K}$ is the field of fractions of $\widehat{R}$. Hence $A'[\pi^{-1}]$ is a finitely generated $\widehat{K}$-algebra of dimension one. Thus, replacing A and P by A' and PA', respectively, we may assume that R is complete. Let P_0 be a minimal prime ideal in A such that $P_0 \subseteq P$. Then $\pi \notin P_0$, because π is a regular element in A. Hence $A/P_0 \subseteq A[\pi^{-1}]/P_0[\pi^{-1}]$, which implies that $\operatorname{tr.deg}_R A/P_0 \leqslant 1$. Thus, replacing A by A/P_0, we may also assume that A is an integral domain with $\operatorname{tr.deg}_R A \leqslant 1$. It then follows from Lemma 3.2 that P satisfies the dimension formula relative to R, and hence, by Proposition 2.1, A/P is a subring of a finitely generated ring over k. Since $\operatorname{tr.deg}_k A/P = 1$ by Lemma 3.2, we know from Corollary 2.2 that A/P is finitely generated over k. This completes the proof. □

Proposition 4.5. *Let (R, π) be a DVR with residue field k and field of fractions K; and let A be a flat R-algebra such that $A[\pi^{-1}]$ is a finitely generated K-algebra of dimension one. Suppose that any of the following three conditions is satisfied:*

(i) *$A \hookrightarrow R[X]$ and $\pi A \in \operatorname{Spec} A$;*
(ii) *$A/\pi A$ is Noetherian and $[\bar{k} : k] < \infty$;*
(iii) *$\pi A \in \operatorname{Spec} A$ and $[\bar{k} : k] < \infty$.*

Then $A/\pi A$ is finitely generated over k. Moreover, if the condition (i) *is satisfied, then we have* $\operatorname{tr.deg}_k A/\pi A = 1$.

Proof. First we consider the case (i). In this case we have $\operatorname{tr.deg}_k A/\pi A > 0$ by Proposition 2.10 and hence the assertion follows from Lemmas 4.4 and 3.2.

Next suppose that (ii) is satisfied. Since $A/\pi A$ is Noetherian, by Lemma 4.1, it suffices to show that A/P is finitely generated over k for every minimal prime ideal P of πA. If $\operatorname{tr.deg}_k A/P > 0$, then the assertion follows from Lemma 4.4; while if $\operatorname{tr.deg}_k A/P_i = 0$, then $A/P_i (\hookrightarrow \bar{k})$ is a finite extension of k.

The proof for the final case (iii) is similar to that for (ii): if $\operatorname{tr.deg}_k A/\pi A > 0$, then the assertion again follows from Lemma 4.4; while if $\operatorname{tr.deg}_k A/\pi A = 0$, then $A/\pi A$ is finite over k. This completes the proof. □

Remark 4.6. Let $R = \boldsymbol{C}[\![t]\!]$ where t is transcendental over $\boldsymbol{C}$ and

$$A = R\big[tX, tX^2, \dots, tX^n, \dots\big] = R + (tX)R[X] \subseteq R[X].$$

Then

$$A/tA \cong \boldsymbol{C}[X_1, X_2, \dots, X_n, \dots]/\big(\{X_i X_j \mid i \geqslant 1, j \geqslant 1\}\big)$$

is non-Noetherian and hence not finitely generated as an algebra over $R/tR(=\boldsymbol{C})$. This example illustrates the necessity of the condition "$\pi A \in \operatorname{Spec} A$" in (i) and (iii) and the condition "$A/\pi A$ is Noetherian" in (ii). The ring D of Example 5.5 illustrates the hypothesis "$A \hookrightarrow R[X]$" of (i) and "$[\bar{k} : k] < \infty$" of (ii) and (iii).

The case $R = \boldsymbol{C}[\![t]\!]$ (or $R = \boldsymbol{C}[t]_{(t)}$) of Example 5.3 shows that even all the conditions (i), (ii), (iii) together do not ensure finite generation of A over R; in particular, A need not be finitely generated even when both the generic and closed fibres are affine domains of transcendence degree one (over respective residue fields). However, Theorem 4.8 will show that, under the hypotheses (i), A is finitely generated over R if A is Noetherian. (Also compare with Corollary 3.8.)

The proof of Theorem B will use the following result on primes in normalisation.

Lemma 4.7. *Let A be a Noetherian domain, C the normalisation of A and π a prime element in A. Then π is a prime element in C.*

Proof. Note that C is a Krull domain. If P is a minimal prime ideal of πC, then, by [10, (33.11)], $P \cap A$ is a prime divisor of πA, so that $P \cap A = \pi A$. Hence the local ring $A_{\pi A}$ is dominated by the local ring C_P, which implies that $A_{\pi A} = C_P$, because both $A_{\pi A}$ and C_P are DVRs with the same field of fractions. From this it follows that P is the unique minimal prime ideal of πC, and hence $\pi C = \pi C_P \cap C$. Since $\pi C_P (= \pi A_{\pi A})$ is the maximal ideal of C_P, we know that πC is a prime ideal of C, as claimed. $\square$

In Example 5.3, we shall demonstrate an R-subalgebra A of $R[X]$ over a DVR (R, π) such that π is prime in A but does not remain prime in the normalisation C of A.

We now prove Theorem B.

Theorem 4.8. *Let R be a locally factorial Noetherian domain and $A(\neq R)$ a Noetherian R-subalgebra of $R[X]$ such that for every prime ideal p in R of height one, pA is a prime ideal in A. Then A is finitely generated over R and the normalisation C of A is isomorphic to the symmetric algebra of an invertible ideal of R.*

Proof. We first consider the case where R is a local domain. Let K be the field of fractions of R and let S be the multiplicative set generated by all prime elements in the factorial ring R. Then $K = S^{-1}R$. From $R \subsetneqq C \subseteq R[X]$, we have

$$K \subsetneqq S^{-1}C \subseteq K[X],$$

which implies that $S^{-1}C = K^{[1]}$ since $S^{-1}C$ is normal (cf. [2, (2.6)]). Now, by hypothesis, every prime element p in R remains prime in A and hence, by Lemma 4.7, p remains prime in C. Thus

S is generated by a set of elements which are prime elements in C. Since C is a Krull domain and $S^{-1}C(= K^{[1]})$ is factorial, by Nagata's criterion [8, Corollary 7.3], C is a factorial domain. Now

$$R \subsetneqq C \subseteq R[X]$$

with both R and C being factorial. Therefore $C = R^{[1]}$ by Theorem 2.7. In particular C is finitely generated over R, and hence so is A by Lemma 2.6, because $A \subseteq C$ and C is integral over A. This completes the proof when R is local.

We next consider the general case. Let m be an arbitrary maximal ideal of R. Then $R_m \subseteq A_m \subseteq C_m \subseteq R_m[X]$ and C_m is the normalisation of A_m. Note that $R_m \neq A_m$; if $R_m = A_m$, then $A \subseteq R_m \cap R[X] = R$, a contradiction. Hence, by the local case, A_m is finitely generated over R_m and $C_m = R_m^{[1]}$. The result then follows from Theorem 2.5. $\square$

Remark 4.9. Example 5.3 will show the necessity of the hypothesis that A is Noetherian in Theorem 4.8.

5. Examples

We first record a few properties of two auxiliary rings C and E which will be used in Examples 5.3 and 5.6 (see Remarks 5.7 and A.2).

Lemma 5.1. *Let (R, π) be a DVR with residue field k and field of fractions K, and let $f = aX + b$ be an element of $R[X]$ with $a \in \pi R \setminus \{0\}$ and $b \in R \setminus \pi R$. Define the rings C and E by*

$$C = R + fR[X] \subseteq R[X]$$

and

$$E = R + fK[X] \subseteq K[X].$$

Let P and Q be the prime ideals of C defined by

$$P = \pi R[X] \cap C \quad \text{and} \quad Q = \pi E \cap C.$$

Then the following assertions hold:

(1) *$C = R[X] \cap E$ and $C[\pi^{-1}] = E[\pi^{-1}] = K[X]$.*
(2) *$E/\pi E = k$ and $\operatorname{ht}(\pi E) = 2$.*
(3) *$C/P = k[\bar{f_1}] = k^{[1]}$, where $f_1 = Xf$, and $\operatorname{ht} P = 1$.*
(4) *$C/Q = k$ and $\operatorname{ht} Q = 2$.*
(5) *$\pi C = P \cap Q$ and $P + Q = C$.*
(6) *$C/\pi C \cong C/P \times C/Q \cong k^{[1]} \times k$.*
(7) *E and C are non-Krull normal domains.*

Proof. (1) The inclusion $C \subseteq R[X] \cap E$ is obvious. For the converse inclusion, let g be an element of $R[X] \cap E$, and write $g = c + f\xi(X)$ with $c \in R$ and $\xi(X) \in K[X]$. Then $\xi(X) \in$

$K[X] \cap R[X, f^{-1}]$. Writing $a = \lambda\pi$ for some $\lambda \in R$, we have $f - (\lambda X)\pi = b$, a unit in R. Thus π and f are comaximal in $R[X]$, so that

$$K[X] \cap R\big[X, f^{-1}\big] = R\big[X, \pi^{-1}\big] \cap R\big[X, f^{-1}\big] = R[X].$$

Therefore $\xi(X) \in R[X]$, and hence $g \in C$.

Clearly $C[\pi^{-1}] = K[X]$ and $E[\pi^{-1}] = K[X]$.

(2) Let $J = fK[X]$, an ideal of E. Since $J(=\pi J) \subseteq \pi E$, we have $E/\pi E = k$. In particular, πE is a prime ideal in E.

By the dimension inequality (2.2) for πE, we have $\operatorname{ht}(\pi E) \leqslant 2$. On the other hand, note that $Q := \bigcap_{n>0} \pi^n E$ is a prime ideal in E because π is a prime element. Since $f \in Q$, we have $Q \neq (0)$, and hence $\operatorname{ht}(\pi E) \geqslant 2$. Thus $\operatorname{ht}(\pi E) = 2$, as claimed.

(3) For $g \in C$, let $\bar{g}$ denote the residue class of $g \in C$ in C/P. Set $f_i = X^i f$ for $i \geqslant 0$. Note that $aX \in P$ and hence $af_i = (aX)f_{i-1} \in P$ for every $i \geqslant 1$. We have the relation

$$f_1 f_{n-1} = bf_n + af_{n+1} \tag{5.1}$$

for $n \geqslant 1$, which implies that $\bar{b}\bar{f_n} = \bar{f_1}\bar{f}_{n-1}$. Moreover, we have $\bar{f} = \bar{b}$. Since, C is generated by $\{1, f, f_1, f_2, \ldots\}$ as an R-module, it follows that $C/P = k[\bar{f_1}]$. Note that $\pi R[X] \cap R[f_1] = \pi R[f_1]$, and hence $k[\bar{f_1}] = k^{[1]}$. In particular $\operatorname{tr.deg}_k C/P = 1$, so that $\operatorname{ht} P = 1$ by Lemma 3.2.

(4) Since $k \hookrightarrow C/Q \hookrightarrow E/\pi E$ and $E/\pi E = k$, we have $C/Q = k$.

From the dimension inequality (2.2) for Q, it follows that $\operatorname{ht} Q \leqslant 2$. On the other hand, $I := fR[X](\subsetneq Q)$ is a prime ideal of C such that $C/I = R$, so that $\operatorname{ht} Q \geqslant 2$. Thus $\operatorname{ht} Q = 2$.

(5) Since $C = R[X] \cap E$, we have $\pi C = \pi R[X] \cap \pi E = P \cap Q$. Since $aX \in P$ and $f \in Q$, the relation $b = -aX + f$ shows that P and Q are comaximal.

(6) follows from (5), (3) and (4) by Chinese remainder theorem.

(7) Since π is a prime element in E and $E[\pi^{-1}] = K[X]$ is a normal domain, E is normal by Lemma 2.9. Thus C is normal, because $C = R[X] \cap E$ and both $R[X]$ and E are normal. Since πE and Q are prime ideals of height 2, and Q is a minimal prime ideal of πC, it follows that E and C are not Krull domains. □

Remark 5.2. The explicit descriptions of C and E of Lemma 5.1 as R-algebras are given as follows:

$$C = R\big[aX, X(aX+b), X^2(aX+b), \ldots\big]$$

and

$$E = R\left[aX, \frac{aX+b}{\pi}, \frac{aX+b}{\pi^2}, \ldots\right].$$

Clearly

$$I = fR[X] = \big(f, Xf, X^2 f, \ldots\big)C$$

and

$$J = fK[X] = \big(f, \pi^{-1} f, \pi^{-2} f, \ldots\big)E$$

are ideals of C and E respectively. One can see that I and J are non-finitely generated. Indeed, since C is normal and $R[X]$ is birational to C, $R[X]$ is not integral over C and hence not finitely generated as a C-module. Thus $fR[X](\cong R[X])$ is not finitely generated as a C-module, i.e., I is not finitely generated as an ideal of C. Again, since $\pi^{-1} \notin E$, we have

$$\pi^{-(n+1)} f \notin \pi^{-n} f E = \bigl(f, \pi^{-1} f, \pi^{-2} f, \ldots, \pi^{-n} f\bigr)E$$

and hence J is not finitely generated as an ideal of E.

The prime ideals P and Q of Lemma 5.1 are two-generated:

$$P = \pi R[X] \cap C = (\pi, aX)C$$

and

$$Q = \pi E \cap C = \pi R + fR[X] = \bigl(\pi, f, Xf, X^2 f, \ldots\bigr)C = (\pi, f)C.$$

The last equality follows from the relation $X^i f = (b^{-1}X^i f)f - (b^{-1}a'X^{i+1}f)\pi$ where $a' = \pi^{-1}a \in R$.

Thus the ideal πC of the non-Krull normal domain C has a primary decomposition into finitely generated prime ideals of different heights:

$$\pi C = P \cap Q = (\pi, aX) \cap (\pi, f)$$

where $\mathrm{ht}(\pi, aX) = 1$ and $\mathrm{ht}(\pi, f) = 2$.

Example 5.3. With the same notations as in Lemma 5.1, let

$$A = R + XfR[X] = R\bigl[Xf, X^2 f, X^3 f, \ldots\bigr].$$

Then:

(1) A is a non-Noetherian domain with normalisation C.
(2) $\pi R[X] \cap A = \pi A$; in particular, π is a prime element in A.
(3) $A/\pi A = k[\bar{f_1}] = k^{[1]}$.

Proof. (1) We have $C = A[f]$ and $f^2 = bf + af_1$ with $f_1 = Xf \in A$, so that C is integral over A. Thus C is the normalisation of A, because C is normal and both C and A have the same field of fractions $K(X)$. Thus A is non-Noetherian since so is $C = A[f]$.

(2) We now show that π remains a prime element of A by verifying the equality $\pi R[X] \cap A = \pi A$. Let $g = c + Xfh(X)$ be an element of $\pi R[X] \cap A$ for some $c \in R$ and $h(X) \in R[X]$. Then $c \in \pi R$, and $h(X) \in \pi R[X]$ since $f \notin \pi R[X]$. Thus $g \in \pi A$.

(3) Note that $P \cap A = (\pi R[X] \cap C) \cap A = \pi R[X] \cap A = \pi A$ and $\pi A \cap R[f_1] = \pi R[X] \cap R[f_1] = \pi R[f_1]$, so that $k[\bar{f_1}] \hookrightarrow A/\pi A \hookrightarrow C/P = k[\bar{f_1}]$. Thus $A/\pi A = k[\bar{f_1}] = k^{[1]}$. $\square$

However, as we have seen in Lemma 5.1, π does not remain prime in C, the normalisation of A; in fact, πC decomposes as $\pi C = P \cap Q$ where $\mathrm{ht}\, P = 1$ but $\mathrm{ht}\, Q = 2$.

Note that in this example we have, from relation (5.1), that

$$A\big[\pi^{-1}\big] = K\big[Xf, X^2 f\big] \cong K[U, V]/\big(U^3 - bUV - aV^2\big),$$

and hence the generic fibre $A[\pi^{-1}]$ is a non-normal affine domain with field of fractions $K(X)$.

The example shows that (i) the Noetherian hypothesis on A is needed in Theorem 4.2 ($A/\pi A$ Noetherian does not suffice); (ii) the hypotheses of Proposition 4.5 do not imply that A is finitely generated; (iii) the normality hypothesis on $A[\pi^{-1}]$ is needed in Corollary 3.8; (iv) the Noetherian hypothesis on A is needed in Lemma 4.7; and (v) the Noetherian hypothesis on A is needed in Theorem 4.8.

The next result forms the basis of construction of our main examples (5.5, 5.6) of non-finitely generated Noetherian normal R-subalgebras of $R[X]$ satisfying various properties.

Lemma 5.4. *Let (R, π) be a DVR with residue field k and field of fractions K. Let D be a Noetherian normal ring such that $R \subseteq D \subseteq K[X]$ and $R[X] \nsubseteq D$. Suppose that $D[\pi^{-1}] = K[X]$ and πD is a maximal ideal of D such that $D/\pi D$ is an algebraic extension of k. Set $V_1 = R[X]_{\pi R[X]}$, $V_2 = D_{\pi D}$ and let*

$$A = K[X] \cap V_1 \cap V_2. \tag{5.2}$$

Let P_1 and P_2 be prime ideals of A defined by $P_i = \pi V_i \cap A$ for $i = 1, 2$. Then the following assertions hold:

(1) *$A = R[X] \cap D$; $P_1 = \pi R[X] \cap A$ and $P_2 = \pi D \cap A$.*
(2) *A is a Noetherian normal R-subalgebra of $R[X]$.*
(3) *$\pi A = P_1 \cap P_2$; $P_1 + P_2 = A$ and $A_{P_i} = V_i$ $(i = 1, 2)$.*
(4) *$A/P_1 \cong k^{[1]}$ and $A/P_2 \cong D/\pi D$.*
(5) *$A/\pi A \cong A/P_1 \times A/P_2 \cong k^{[1]} \times D/\pi D$.*
(6) *A is not finitely generated over R.*

Proof. (1) Since $K[X] = R[X][\pi^{-1}] = D[\pi^{-1}]$, we have

$$A = \big(K[X] \cap V_1\big) \cap \big(K[X] \cap V_2\big) = R[X] \cap D.$$

From this it follows that $P_1 = \pi V_1 \cap A = (\pi V_1 \cap R[X]) \cap A = \pi R[X] \cap A$. Similarly we have $P_2 = \pi D \cap A$.

Here we note that $A \neq D$, i.e., $D \nsubseteq R[X]$. Indeed, if $D \subseteq R[X]$, then $\pi R[X] \cap D = \pi D$, because πD is a maximal ideal. Since $R[X][\pi^{-1}] = D[\pi^{-1}]$, it then follows that $D = R[X]$, a contradiction. Also note that $A \neq R[X]$ as $R[X] \nsubseteq D$.

(2) Note that each V_i, $i = 1, 2$, is a DVR of $K(X)$ with uniformising parameter π, and hence A is a Krull domain such that $A[\pi^{-1}] = K[X]$. It thus follows from Lemma 3.3 that A is Noetherian. Since $A \subseteq R[X]$ by (1), we know that A is a Noetherian normal R-subalgebra of $R[X]$.

(3) It follows from (1) that

$$\pi A = \pi R[X] \cap \pi D = \big(\pi R[X] \cap A\big) \cap (\pi D \cap A) = P_1 \cap P_2. \tag{5.3}$$

Since $k \hookrightarrow A/P_2 \hookrightarrow D/\pi D$ and $D/\pi D$ is an algebraic field extension of k, it follows that A/P_2 is an algebraic extension of k, and hence P_2 is a maximal ideal of A.

We check that $P_1 \nsubseteq P_2$. Indeed, if $P_1 \subseteq P_2$, then $\pi A = P_1$ by (5.3), which implies that $\pi A = \pi R[X] \cap A$. Since $A \subseteq R[X]$ and $A[\pi^{-1}] = R[X][\pi^{-1}](= K[X])$, from this it follows that $A = R[X]$, contradicting $R[X] \nsubseteq D$. Therefore $P_1 \nsubseteq P_2$, as desired. Hence $P_1 + P_2 = A$ since P_2 is maximal.

We thus see that (5.3) gives the primary decomposition of πA. Hence $\operatorname{ht} P_i = 1$ for $i = 1, 2$, and therefore each A_{P_i} is a DVR of $K(x)$. Since A_{P_i} is dominated by V_i, we have $A_{P_i} = V_i$ for $i = 1, 2$.

(4) Since $P_1 + P_2 = A$, the assertion follows from [1, Lemma 5.6]. For convenience, we give the proof here.

Let $a \in P_1$ and $b \in P_2$ be elements such that $a + b = 1$. Then $D[b^{-1}] = K[X][b^{-1}]$, because $b \in \pi D$ and $D[\pi^{-1}] = K[X]$. Hence $A[b^{-1}] = R[X][b^{-1}]$ by (1), so that $\pi R[X][b^{-1}] = \pi A[b^{-1}] = P_1 A[b^{-1}]$ by (3). Thus $A \subseteq R[X] \subseteq A[b^{-1}]$ and $P_1 A[b^{-1}] \cap R[X] = \pi R[X]$. Note that $b \equiv 1 \bmod P_1$. Therefore

$$A/P_1 \hookrightarrow R[X]/\pi R[X] \hookrightarrow A\left[b^{-1}\right]/P_1 A\left[b^{-1}\right] = A/P_1,$$

which implies that $A/P_1 \cong R[X]/\pi R[X] \cong k^{[1]}$. Similarly we have $A/P_2 \cong D/\pi D$.

(5) follows from (3) and (4) by Chinese remainder theorem.

(6) Since P_2 is a minimal prime ideal of πA and A/P_2 is algebraic over k, A is not finitely generated over R by Proposition 3.4. □

A further discussion on the R-algebra A and the minimal prime ideals P_1 and P_2 of πA will be made in Remark A.2.

Now if R is an integral domain with field of fractions K and A is an R-subalgebra of $R[X]$, then $K \hookrightarrow K \otimes_R A \hookrightarrow K^{[1]}$. Therefore, by Corollary 2.2, the generic fibre of A is always finitely generated and has transcendence degree one if $A \neq R$. The following example shows that even over a nice equicharacteristic complete DVR like $R = \boldsymbol{Q}[\![t]\!]$, the closed fibre of a Noetherian normal R-subalgebra of $R^{[1]}$ need not be finitely generated; in particular, A itself need not be finitely generated. The example also illustrates the necessity of the hypothesis "$[\bar{k} : k] < \infty$" in Theorem 4.2 and Proposition 4.5((ii) and (iii)), and the hypothesis "$A \hookrightarrow R[X]$" in Proposition 4.5(i).

Example 5.5. Let $R = \boldsymbol{Q}[\![t]\!]$ where t is transcendental over $\boldsymbol{Q}$ and $K = \boldsymbol{Q}(\!(t)\!)$, the field of fractions of R. Let p_n denote the nth prime among the natural numbers and let L be the infinite algebraic field extension of $\boldsymbol{Q}$ generated by all the $\sqrt{p_n}$s, i.e.,

$$L = \boldsymbol{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots, \sqrt{p_n}, \ldots).$$

Let $D = R[x_1, x_2, \ldots]$ be the subring of $\boldsymbol{Q}(\!(t)\!)[X]$ generated by $x_1, x_2, \ldots$, where $x_1 = tX$ and x_n is defined inductively by the relation

$$x_{n+1} = \left(x_n^2 - p_n\right)/t$$

for each $n \geqslant 1$. Thus

$$D = R\left[tX, \frac{(tX)^2 - 2}{t}, \frac{(\frac{(tX)^2-2}{t})^2 - 3}{t}, \frac{(\frac{(\frac{(tX)^2-2}{t})^2-3}{t})^2 - 5}{t}, \ldots\right].$$

Then:

(1) D is a Noetherian normal domain with fibres $D[t^{-1}] = K[X]$ and $D/tD = L$. In particular, D/tD is not finitely generated as a $\boldsymbol{Q}$-algebra.
(2) Let A be the ring defined by (5.2) in Lemma 5.4. Then A is a Noetherian normal R-subalgebra of $R[X]$ such that

$$A/tA \cong \boldsymbol{Q}^{[1]} \times L.$$

In particular, A/tA is not finitely generated as an $R/tR(= \boldsymbol{Q})$-algebra and A is not finitely generated as an R-algebra.

Proof. (1) Clearly $D[t^{-1}] = K[X]$. First we show that $D/tD = L$. Note that

$$L \cong \boldsymbol{Q}[X_1, X_2, X_3, \ldots]/\bigl(X_1^2 - p_1, X_2^2 - p_2, X_3^2 - p_3, \ldots\bigr).$$

Now let I be the kernel of the R-algebra homomorphism $\phi: R[X_1, X_2, \ldots] \to D$ defined by $\phi(X_i) = x_i$ for each i, and let

$$J = \bigl(tX_2 - X_1^2 + p_1, tX_3 - X_2^2 + p_2, tX_4 - X_3^2 + p_3, \ldots\bigr).$$

We have $J \subseteq I$, and hence $D/tD(\cong R[X_1, X_2, \ldots]/(t, I))$ is a surjective image of

$$R[X_1, X_2, X_3, \ldots]/(t, J) \cong L,$$

which is a field. Therefore, $D/tD = L$. In particular, t is a prime element in D and D/tD an algebraic extension field of $\boldsymbol{Q}$ with $[D/tD : \boldsymbol{Q}] = \infty$. Thus D/tD is not finitely generated as a $\boldsymbol{Q}$-algebra.

Next we show that $\operatorname{ht}(tD) = 1$. By the dimension inequality (2.2) for tD, we have

$$\operatorname{ht}(tD) \leqslant \operatorname{ht}(tR) + \operatorname{tr.deg}_R D - \operatorname{tr.deg}_{\boldsymbol{Q}} D/tD = 2.$$

Therefore, if $\operatorname{ht}(tD) \neq 1$, then $\operatorname{ht}(tD) = 2$, so that tD would satisfy the dimension formula (2.1) relative to R. Hence, by Proposition 2.1, the field D/tD would be a subring of a finitely generated ring over $R/tR = \boldsymbol{Q}$, which contradicts $[D/tD : \boldsymbol{Q}] = \infty$. Thus $\operatorname{ht}(tD) = 1$, as claimed. (Remark A.1(1) provides an alternative proof that $\operatorname{ht}(tD) = 1$.)

By Lemmas 2.8 and 2.9, it then follows that D is a Noetherian normal domain.

(2) follows from Lemma 5.4. □

An explicit description of A will be given in Remark A.2.

Note that, in Example 5.5, the closed fibre is not an integral domain. Recall that, by Proposition 4.5, if the closed fibre $A/\pi A$ (of an R-subalgebra A of $R[X]$) is an integral domain, then it must be an affine domain over k of dimension one; and that, by Theorem 4.8, A itself would be finitely generated if it is Noetherian.

We now describe an example to show that the hypothesis "R is complete" is necessary in Theorem 4.2. Unlike Example 5.5, in this example, the closed fibre $A/\pi A$ will be finitely generated over k. The example illustrates that even over a nice DVR like the affine $\boldsymbol{C}$-spot $R = \boldsymbol{C}[t]_{(t)}$, a Noetherian R-subalgebra of $R[X]$ need not be finitely generated over R—not even under the additional hypothesis that all the fibre rings are affine.

Example 5.6. Let $R = k[t]_{(t)}$ where k is an arbitrary field and t is transcendental over k and $K = R[t^{-1}] = k(t)$, the field of fractions of R. Let

$$y = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n + \cdots$$

be an invertible element of $k[\![t]\!]$ (i.e., $a_i \in k$ for all i and $a_0 \neq 0$) which is transcendental over $k(t)$. Let

$$D = R\left[tX, \frac{tX - a_0}{t}, \frac{tX - a_0 - a_1 t}{t^2}, \ldots, \frac{tX - a_0 - a_1 t - \cdots - a_n t^n}{t^{n+1}}, \ldots\right].$$

Then:

(1) D is a Noetherian normal domain such that $D[t^{-1}] = K[X]$ and $D/tD = k$.
(2) Let A be the ring defined by (5.2) in Lemma 5.4. Then A is a Noetherian normal R-subalgebra of $R[X]$ which is not finitely generated over R although $A/tA \cong k^{[1]} \times k$ is finitely generated over k.

Proof. (1) Clearly $D[t^{-1}] = K[X]$ and, by argument as in Example 5.5, one sees that $D/tD = k$. Moreover, since y is transcendental over $k(t)$, D is R-isomorphic to the subring

$$D' := R\left[y, \frac{y - a_0}{t}, \frac{y - a_0 - a_1 t}{t^2}, \ldots\right]$$

of $k[\![t]\!]$, and hence $\bigcap_{n\geqslant 1} t^n D = (0)$ because $\bigcap_{n\geqslant 1} t^n D' \subseteq \bigcap_{n\geqslant 1} t^n k[\![t]\!] = (0)$. Therefore, D is a Noetherian normal domain by Lemmas 2.8 and 2.9.

(2) follows from Lemma 5.4. $\square$

Remark 5.7. Let the notation be as in Example 5.6 and let $\widehat{R}(= k[\![t]\!])$ denote the completion of R. As $\widehat{R}$ is faithfully flat over R and A is not finitely generated over R, it follows that $\widehat{R} \hookrightarrow \widehat{R} \otimes_R A \hookrightarrow \widehat{R}[X]$ and that $\widehat{R} \otimes_R A$ is not finitely generated over $\widehat{R}$. We thus see from Theorem 4.2 that although A is Noetherian, $\widehat{R} \otimes_R A$ does not remain Noetherian when k is real closed or algebraically closed. In fact, the non-Noetherian property of $\widehat{R} \otimes_R A$ holds without the additional assumption, real or algebraic closedness, on k. We show this by giving explicit descriptions of the $\widehat{R}$-algebra $\widehat{R} \otimes_R A$ and of a non-finitely generated ideal of $\widehat{R} \otimes_R A$.

Note that

$$\widehat{R} \otimes_R D = \widehat{R}\left[tX, \frac{tX - a_0}{t}, \frac{tX - a_0 - a_1 t}{t^2}, \ldots\right] = \widehat{R}\left[tX, \frac{tX - y}{t}, \frac{tX - y}{t^2}, \ldots\right],$$

because

$$\frac{tX-(a_0+a_1t+\cdots+a_nt^n)}{t^{n+1}}-\frac{tX-y}{t^{n+1}}=\frac{y-(a_0+a_1t+\cdots+a_nt^n)}{t^{n+1}}\in\widehat{R}.$$

Thus, by Remark 5.2, the ring $\widehat{R}\otimes_R D$ is same as the non-Noetherian ring E of Lemma 5.1 (when $R=k[\![t]\!]$, $a=\pi=t$ and $b=-y$ in Lemma 5.1). Hence, by (1) of Lemma 5.1, the ring

$$\widehat{R}\otimes_R A=\widehat{R}[X]\cap(\widehat{R}\otimes_R D)$$

is same as the non-Noetherian ring C of Lemma 5.1. Thus, by Remark 5.2,

$$\widehat{R}\otimes_R A=\widehat{R}\big[tX, X(tX-y), X^2(tX-y),\ldots\big]$$

and the ideal $I=(tX-y, X(tX-y), X^2(tX-y),\ldots)$ in the above ring is not finitely generated.

Also note that while D is a Noetherian normal domain in which tD is a maximal ideal (necessarily of height one), the extended ring $E=\widehat{R}\otimes_R D$ over the completion $\widehat{R}$ becomes a non-Noetherian normal domain in which the extended ideal tE becomes a maximal ideal of height two ((2) of Lemma 5.1).

Different formulations of the rings A and D of Example 5.6 have appeared earlier in various contexts. When $k=\boldsymbol{C}$, one can see that D of Example 5.6 is isomorphic to $k[\![t]\!]\cap k(t)[y]$ which is a localisation of an example of Bhatwadekar [4, Example 2]. The ring A of Example 5.6 is closely related to the example of Eakin [6, p. 79] and another example of Bhatwadekar [4, Example 1]. In Remark A.2, we shall give explicit generators for A.

Acknowledgment

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Appendix A

After discussions on an earlier draft, S.M. Bhatwadekar formulated the following observation which neatly brings out the underlying principles involved in Examples 5.5 and 5.6.

Remark A.1 *(Bhatwadekar).* Let (R,t) be a DVR with residue field k and field of fractions K. Let $D=R[x_1,x_2,\ldots,x_n,\ldots]$ be an R-subalgebra of $K[X]$ such that $tD\neq D$, $x_1=tX$ and for each $i\geqslant 1$, $tx_{i+1}=f_i(x_i)$ for some polynomial $f_i(X)$ in $R[X]$ whose image $\bar{f_i}(X)$ in $k[X]$ is monic irreducible. Note that $D[t^{-1}]=K[X]$.

Now suppose that P is a prime ideal in D containing tD. By construction, D/P is algebraic over k, so that P is a maximal ideal of D. Also note that, by dimension inequality, $\operatorname{ht} P\leqslant 2$. The following relations hold between the dimension $[D/P:k]$ and $\operatorname{ht} P$:

(1) If $[D/P:k]$ is infinite, then $\operatorname{ht} P=1$.
(2) If R is complete and P is finitely generated, then $[D/P:k]$ is finite if and only if $\operatorname{ht} P=2$.

(1) Suppose that $\operatorname{ht} P \neq 1$ (i.e., $\operatorname{ht} P = 2$). Then there exists a non-zero prime ideal Q in D such that $Q \subsetneqq P$. By construction, D/tD is integral over k, so that $\dim D/tD = 0$. Thus $t \notin Q$ and hence $Q \cap R = 0$. Thus $R \hookrightarrow D/Q$. Now the quotient field of D/Q is $D[t^{-1}]/Q = K[X]/Q$, a finite algebraic extension of K. Hence, by the Krull–Akizuki theorem, $D/P = (D/Q)/(P/Q)$ is a finite extension of k.

(2) Suppose that P is finitely generated and $\operatorname{ht} P = 1$. Then $V := D_P$ is a one-dimensional Noetherian local domain such that $\sqrt{tV} = PV$, the maximal ideal of V. Note that $V/PV = D/P$ since D/P is a field. Thus, if $[D/P : k]$ is finite, then so is $[V/tV : k]$, because $P^nV \subseteq tV$ for some $n > 0$. Now, if further R were complete, then V would be a finite R-module [14, p. 259, Corollary 2] contradicting $\operatorname{tr.deg}_R V = 1$.

Thus, over a complete DVR (R, t) with residue field k, for constructing a transcendental R-algebra D where tD would be a maximal ideal of height one, one is forced to ensure that D/tD is an infinite algebraic extension of k (the starting point of Example 5.5). However, as demonstrated in Example 5.6, for non-complete R, it is indeed possible to construct D such that tD itself is a maximal ideal P satisfying $D/P = R/tR = k$ and yet $\operatorname{ht} P = 1$.

Note that by choosing $f_i(X)$ in $R[X]$ such that

$$L := R[X_1, X_2, \ldots, X_n, \ldots]/\big(t, f_1(X_1), f_2(X_2), \ldots, f_n(X_n), \ldots\big)$$

is a field, one gets D for which tD is a maximal ideal with $D/tD \cong L$, an algebraic field extension of k.

The ring D of Remark A.1 need not be Noetherian in general. However if it satisfies the additional hypothesis that tD is a maximal ideal of height one, then D indeed will be Noetherian. In this case, we give below an explicit description of the R-algebra $A := D \cap R[X]$. In particular, we display explicit generators for the R-algebras A of Examples 5.5 and 5.6. We also make a few observations on the minimal prime ideals of tA.

Remark A.2. Let the notation be as in Remark A.1. Suppose that tD is a maximal ideal of D of height one. In particular, D is Noetherian and normal (by Lemmas 2.8 and 2.9) and D/tD is an algebraic field extension of k. We further assume that $R[X] \nsubseteq D$. Set $A = R[X] \cap D$. Then, by Lemma 5.4, A is a non-finitely generated Noetherian normal R-subalgebra of $R[X]$ satisfying

$$A/tA \cong k^{[1]} \times D/tD.$$

(1) We give an explicit description of the ring A. Let $d_i = \deg f_i(X)$ for $i \geqslant 1$, and define m_n inductively by the relation $m_1 = 0$ and $m_{n+1} = d_n m_n + 1$ for $n \geqslant 1$. Let $y_n = (tX)^{m_n} x_n$ for each $n \geqslant 1$; in particular, $y_1 = x_1$ and $y_2 = (tX)x_2 = Xf_1(x_1)$. Now set

$$B = R[y_1, y_2, \ldots, y_n, \ldots].$$

We show that $B = A$. Note that $B \subseteq D$.

For simplicity, we set $z = tX$ and $w = f_1(tX)$. Thus $z = x_1 = y_1$ and $w = f_1(y_1)$. We also set $F_n(X, Y) = Y^{d_n} f_n(X/Y)$ for every n, so that $F_n(X, Y)$ is a homogeneous polynomial in $R[X, Y]$ of degree d_n. Then, the relations $tx_{n+1} = f_n(x_n)$ and $m_{n+1} = d_n m_n + 1$ imply that

$$ty_{n+1} = y_1 F_n\big(y_n, y_1^{m_n}\big) \tag{A.1}$$

and hence $y_{n+1} = XF_n(y_n, y_1^{m_n})$ for $n \geqslant 1$. Therefore, by induction on n, we have $y_n \in R[X]$ for every n. Thus $y_n \in D \cap R[X](= A)$ for $n \geqslant 1$, and hence $B \subseteq A$.

Since $y_n = z^{m_n} x_n$ and $B \subseteq D$, we have $B[z^{-1}] = D[z^{-1}]$. On the other hand, we have $R[X, z^{-1}] = K[X, z^{-1}]$ because $z = tX$, so that

$$A\left[z^{-1}\right] = D\left[z^{-1}\right] \cap R\left[X, z^{-1}\right] = D\left[z^{-1}\right]. \tag{A.2}$$

Thus $B[z^{-1}] = A[z^{-1}]$.

Since $ty_2 = y_1 w$, the relation (A.1) also implies that

$$wy_{n+1} = y_2 F_n\left(y_n, y_1^{m_n}\right) \tag{A.3}$$

for $n \geqslant 1$. As $y_1 = tX$ and $y_2 = Xf_1(y_1) = Xw$, it follows that

$$B[w^{-1}] = R\left[y_1, y_2, w^{-1}\right] = R\left[X, w^{-1}\right].$$

On the other hand, since $tx_2 = f_1(x_1) = w$ and $D[t^{-1}] = K[X]$, we have

$$D\left[w^{-1}\right] = D\left[1/(tx_2)\right] = K\left[X, w^{-1}\right],$$

so that

$$A\left[w^{-1}\right] = D\left[w^{-1}\right] \cap R\left[X, w^{-1}\right] = R\left[X, w^{-1}\right].$$

Thus $B[w^{-1}] = A[w^{-1}]$.

Note that the hypothesis $R[X] \nsubseteq D$ (i.e., $X \notin D$) implies that $\bar{f}_1(X) \neq X$, where $\bar{f}_1(X)$ denotes the residue class of $f_1(X)$ in $k[X]$. Indeed if not, then $f_1(X) = X + th(X)$ for some $h(X) \in R[X]$, from which it follows that $X = x_2 - h(x_1) \in D$ because $x_2 = t^{-1} f_1(x_1)$ and $x_1 = tX$, a contradiction. Since $\bar{f}_1(X) \neq X$ and $\bar{f}_1(X)$ is monic irreducible in $k[X]$, we have $\bar{f}_1(0) \neq 0$, i.e., $f_1(0)$ is a unit in R. Thus z and w are comaximal both in B and A. It now follows that

$$B = B\left[z^{-1}\right] \cap B\left[w^{-1}\right] = A\left[z^{-1}\right] \cap A\left[w^{-1}\right] = A,$$

as claimed.

The relation $A = R[y_1, y_2, y_3, \ldots]$ gives us a concrete system of generators for the rings A in Examples 5.5 and 5.6. In Example 5.5, we have $f_i(X) = X^2 - p_i$ for each $i \geqslant 1$, so that $d_i = 2$ for all $i \geqslant 1$ and $m_n = 2^{n-1} - 1$ for $n \geqslant 1$. Therefore,

$$A = R\left[tX, X\left((tX)^2 - 2\right), X^3\left(\left((tX)^2 - 2\right)^2 - 3t^2\right), \ldots\right].$$

Similarly, in Example 5.6, we have $f_i(X) = X - a_{i-1}$ for $i \geqslant 1$, so that $d_i = 1$ for each i, $m_n = n - 1$ for every n, and hence the ring A of the example is given by

$$A = R\left[tX, X(tX - a_0), X^2(tX - a_0 - a_1 t), \ldots\right].$$

(2) Next we discuss the minimal prime ideals of tA. Let $P_1 = tR[X] \cap A$ and $P_2 = tD \cap A$. Then $tA = P_1 \cap P_2$ by Lemma 5.4, and hence P_1 and P_2 are the minimal prime ideals of tA. We show that $P_1 = (t, y_1)A$ and $P_2 = (t, w)A$.

Set $I_1 = (t, y_1)A$ and $I_2 = (t, w)A$. Since $y_1 = tX \in tR[X]$ and $w = f_1(x_1) = tx_2 \in tD$, clearly we have $I_i \subseteq P_i$ for $i = 1, 2$. Let $b = f_1(0)$. Then b is a unit in R and $w \equiv b \bmod I_1$, because $w = f_1(y_1)$ and $y_1 \in I_1$. Since $m_n > 0$ for $n \geqslant 2$, it then follows from the relation (A.3) that

$$y_{n+1} \equiv b^{-1} y_2 F_n(y_n, 0) \bmod I_1$$

for each $n \geqslant 2$. Therefore we have $A/I_1 = k[\bar{y}_2]$, where $\bar{y}_2$ denotes the residue class of y_2 in A/I_1. On the other hand, recall that $A/P_1 = k^{[1]}$ by Lemma 5.4. Since A/P_1 is a surjective image of A/I_1, it now follows that $I_1 = P_1$.

For the equality $I_2 = P_2$, note that $f_1(y_1) \equiv 0 \bmod I_2$, and hence the residue class $\bar{y}_1 \in A/I_2$ is a unit in A/I_2, because $b = f_1(0)$ is a unit in R. Note also that $A[y_1^{-1}] = D[x_1^{-1}]$ by the equality (A.2), because $x_1 = y_1 = z$. Since $\bar{y}_1$ is a unit in A/I_2, $t \in I_2$ and tD is a maximal ideal with $x_1 = tX \notin tD$, it follows that

$$A/I_2 = A\big[y_1^{-1}\big]/I_2 A\big[y_1^{-1}\big] = D\big[x_1^{-1}\big]/tD\big[x_1^{-1}\big] = D/tD,$$

and hence I_2 is a maximal ideal of A. Therefore $I_2 = P_2$, as desired.

We have thus obtained $P_1 = (t, tX)A$ and $P_2 = (t, f_1(tX))A$. Therefore, by Lemma 5.4, the primary decomposition of tA is of the form

$$tA = P_1 \cap P_2 = (t, tX)A \cap \big(t, f_1(tX)\big)A.$$

Note that $P_2 = (t, (tX)^2 - 2)A$ in Example 5.5 and $P_2 = (t, tX - a_0)A$ in Example 5.6.

(3) We now examine the extensions of the minimal prime ideals P_1 and P_2 of Example 5.6 to $\widehat{R} \otimes_R A$.

Let the notation be as in Example 5.6, and set $C = \widehat{R} \otimes_R A$ and $E = \widehat{R} \otimes_R D$. Let $P = t\widehat{R}[X] \cap C$ and $Q = tE \cap C$. Then we have $tC = P \cap Q$ by Lemma 5.1 and Remark 5.7. Note that $P = (t, tX)C$ and $Q = (t, tX - y)C$ by Remarks 5.2 and 5.7. Note also that $(t, tX - y)C = (t, tX - a_0)C$ since $y - a_0 \in tC$. It thus follows that $P = P_1 C$ and $Q = P_2 C$. Hence, by Lemmas 2.4, 5.1 and 5.4, we have $C/P \cong A/P_1 = k^{[1]}$, $C/Q \cong A/P_2 = k$ and

$$A/P_1 \times A/P_2 \cong A/tA \cong C/tC \cong C/P \times C/Q.$$

While P_1, P_2 are prime ideals of height one (A being Noetherian), recall that Q is a prime ideal of height two ((4) of Lemma 5.1).

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