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Gröbner–Shirshov basis of quantum group of type E_6^\star

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Professor Jie Xiao on the occasion of his
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ABSTRACT

In this paper we prove that, for type E_6 , the set of the skew-commutator relations of quantum root vectors forms a minimal Gröbner–Shirshov basis.

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1. Introduction

The Gröbner–Shirshov basis theory provides a solution to the reduction problem for various kinds of algebras and gives an algorithm of computing a set of generators for a given ideal which can be used to determine the reduced elements with respect to the relations given by the ideal (see [1,2,4,16]).

In [3], Bokut and Malcolmson developed the theory of Gröbner–Shirshov basis for the quantum enveloping algebras, or the so-called quantum groups and, by using the Jimbo relations given by Yamane in [17], explicitly constructed the basis for the quantum group of type A_n for $(q^8 \neq 1)$. In [13],

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by using the skew commutator relations between quantum root vectors for the quantum group of type \mathbb{G}_2 in [12] and the canonical isomorphism between the positive parts of quantum groups and the Ringel–Hall algebras, the authors give a Gröbner–Shirshov basis for quantum group of type \mathbb{G}_2 . In this paper, by using the Ringel–Hall algebra method, we compute all skew commutator relations between quantum root vectors first, and then prove that the set of these relations forms a minimal Gröbner–Shirshov basis of the quantum group of type E_6 .

2. Some preliminaries

For the convenience of the reader, in this section, we recall some notions about Gröbner–Shirshov basis theory of quantum groups and the Ringel–Hall algebras.

First, we recall some basic notions about Gröbner–Shirshov basis theory from [3]. Let k be a field and X a non-empty set of alphabets. Let $\langle X \rangle$ and $k\langle X \rangle$ be the free semigroup with 1 and a free algebra generated by X , respectively. We choose a monomial ordering $<$ on $\langle X \rangle$ in order to determine a leading term \bar{f} for each element $f \in k\langle X \rangle$. An element $f \in k\langle X \rangle$ will be called monic if the coefficient of the leading term \bar{f} is 1 $\in k$. If f and g are monic elements in $k\langle X \rangle$ with leading terms \bar{f} and \bar{g} , there will be a so-called composition of intersection if there are a and b in $\langle X \rangle$ such that $\bar{f}a = b\bar{g} = \omega$ with total length of \bar{f} is larger than that of b . We write $(f, g)_\omega = fa - bg$ in that case and note that the leading term $\overline{(f, g)_\omega} < \omega$. There will be a composition of inclusion if there are a and b in $\langle X \rangle$ such that $\bar{f} = a\bar{g}b = \omega$. We write $(f, g)_\omega = f - agb$ in that case and again note that the leading term is less than ω .

Let us take some set of relations $S \subseteq k\langle X \rangle$ (which, we will assume, consists of monic elements). Let us denote by (S) the ideal generated by S in $k\langle X \rangle$. Let $p, q \in k\langle X \rangle$ and $\omega \in \langle X \rangle$. We define an *equivalent relation* on $k\langle X \rangle$ as follows: $p \equiv q \pmod{(S; \omega)}$ if and only if $p - q = \sum \alpha_i a_i s_i b_i$, where $\alpha_i \in k$; $a_i, b_i \in \langle X \rangle$; $s_i \in S$; $\overline{a_i s_i b_i} < \omega$. We will say that S is *closed under composition* if for any $f, g \in S$ we have

$$(f, g)_\omega \equiv 0 \pmod{(S; \omega)},$$

whenever the composition $(f, g)_\omega$ is defined and in this case we will say that the composition $(f, g)_\omega$ is trivial with respect to S . If S is not closed under composition, then we will need to expand S by including all nontrivial compositions (inductively) to obtain a completion S^c . If S is complete (i.e. closed under composition) in this sense ($S^c = S$), then Shirshov's Lemma (see [16]) says that any monic element $f \in (S)$ has a reducible leading term $\bar{f} = a\bar{s}b$, where $s \in S$ and $a, b \in \langle X \rangle$. Shirshov's Lemma also says that a linear basis for the factor algebra $k\langle X \rangle / (S)$ (i.e., as a vector space over k) may be obtained by taking the set of irreducible monomials in $\langle X \rangle$.

The set S will then be referred to as a Gröbner–Shirshov basis for the ideal (S) . By abusing the definition we may also refer to S as a Gröbner–Shirshov basis for the factor algebra $k\langle X \rangle / (S)$. The set S will be called a *minimal Gröbner–Shirshov basis* if there is no inclusion composition in S .

Next, we recall the definition of quantum groups from [5] and [10].

Let $A = (a_{ij})$ be an integral symmetrizable $N \times N$ Cartan matrix, so that $a_{ii} = 2$, $a_{ij} \leq 0$ ($i \neq j$), and there exists a diagonal matrix D with nonzero integer diagonal entries d_i such that the product DA is symmetric. Let q be a nonzero element of k so that $q^{4d_i} \neq 1$ for each i . Then the quantum group $U_q(A)$ is the k -algebra generated by $4N$ elements $E_i, K_i^{\pm 1}, F_i$, subject to the following set of relations (for $1 \leq i, j \leq N$):

$$K = \{K_i K_j - K_j K_i, K_i K_i^{-1} - 1, K_i^{-1} K_i - 1, E_j K_i^{\pm 1} - q^{\pm d_i a_{ij}} K_i^{\pm 1} E_j, K_i^{\pm 1} F_j - q^{\pm d_i a_{ij}} F_j K_i^{\pm 1}\},$$

$$T = \left\{ E_i F_j - F_j E_i - \delta_{ij} \frac{K_i^2 - K_i^{-2}}{q^{2d_i} - q^{-2d_i}} \right\},$$

$$S^+ = \left\{ \sum_{\mu=0}^{1-a_{ij}} (-1)^\mu \begin{bmatrix} 1-a_{ij} \\ \mu \end{bmatrix}_t E_i^{1-a_{ij}-\mu} E_j E_i^\mu \mid i \neq j, t = q^{2d_i} \right\},$$

$$S^- = \left\{ \sum_{\mu=0}^{1-a_{ij}} (-1)^\mu \begin{bmatrix} 1-a_{ij} \\ \mu \end{bmatrix}_t F_i^{1-a_{ij}-\mu} F_j F_i^\mu \mid i \neq j, t = q^{2d_i} \right\},$$

where

$$\begin{bmatrix} m \\ n \end{bmatrix}_t = \begin{cases} \prod_{i=1}^n \frac{t^{m-i+1} - t^{i-m-1}}{t^i - t^{-i}} & (\text{for } m > n > 0), \\ 1 & (\text{for } n = 0 \text{ or } n = m). \end{cases}$$

Let $U_q^0(A)$ be the subalgebra of $U_q(A)$ generated by $K_i^{\pm 1}$. Let $U_q^+(A)$ (resp. $U_q^-(A)$) be the subalgebra of $U_q(A)$ generated by E_i (resp. F_i). Then we have following triangular decomposition of $U_q(A)$ (see [15])

$$U_q(A) \cong U_q^+(A) \otimes U_q^0(A) \otimes U_q^-(A).$$

The main result in [3] is following

Theorem 2.1. *If the set S^{+c} (resp. S^{-c}) is a Gröbner–Shirshov basis of $U_q^+(A)$ (resp. $U_q^-(A)$), then the set $S^{+c} \cup K \cup T \cup S^{-c}$ is a Gröbner–Shirshov basis of $U_q(A)$.*

Finally, we recall some basic notions about the twisted generic Ringel–Hall algebras. Because in this paper, we only consider the quantum group of type E_6 , we recall the relevant notions directly for the finite dimensional hereditary algebra of Dynkin type from [6].

Let \mathbb{F} be a finite field, \vec{Q} a (connected) quiver with the underlying graph Q of Dynkin type, that is $Q \in \{\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$, then it is well known that the path algebra $\Lambda(\mathbb{F}, \vec{Q}) = \mathbb{F}\vec{Q}$ is a finite dimensional hereditary \mathbb{F} -algebra of representation finite. By $\Lambda(\mathbb{F}, \vec{Q})$ -mod we denote the category of finite dimensional right $\Lambda(\mathbb{F}, \vec{Q})$ -modules. For $M, N_1, \dots, N_t \in \Lambda(\mathbb{F}, \vec{Q})$ -mod, let F_{N_1, \dots, N_t}^M be the number of filtrations

$$M = M_0 \supseteq M_1 \supseteq \dots \supseteq M_{t-1} \supseteq M_t = 0,$$

such that $M_{i-1}/M_i \cong N_i$ for all $1 \leq i \leq t$.

For each $M \in \Lambda(\mathbb{F}, \vec{Q})$ -mod, we denote by $[M]$ the isomorphism class of M and by $\mathbf{dim} M$ the dimension vector of the $\Lambda(\mathbb{F}, \vec{Q})$ -module M . We have the well-known Euler form $\langle -, - \rangle$ defined by

$$\langle \mathbf{dim} M, \mathbf{dim} N \rangle = \mathbf{dim} \operatorname{Hom}_\Lambda(M, N) - \mathbf{dim} \operatorname{Ext}_\Lambda^1(M, N).$$

Note that $\langle -, - \rangle$ is the symmetrization of $\langle -, - \rangle$.

Let ν be an indeterminate and $\mathbb{Q}(\nu)$ be the rational function field of ν over the field \mathbb{Q} of rational numbers and set $\nu^2 = q$. In order to define the twisted generic Ringel–Hall algebra, we recall the notion of Hall polynomials.

For a Dynkin diagram Q , there is the corresponding semisimple Lie algebra \mathfrak{g} and the Cartan matrix A . Let Φ^+ be the set of positive roots of \mathfrak{g} . According to [7], \mathbf{dim} is a bijection between the set of the isomorphism classes of the indecomposable modules and the set of positive roots Φ^+ of \mathfrak{g} . For each $\alpha \in \Phi^+$, let $M_{\mathbb{F}}(\alpha)$ denote the corresponding indecomposable $\Lambda(\mathbb{F}, \vec{Q})$ -module; thus $\mathbf{dim} M_{\mathbb{F}}(\alpha) = \alpha$. By the Krull–Schmidt theorem, every $\Lambda(\mathbb{F}, \vec{Q})$ -module $M_{\mathbb{F}}$ is isomorphic to

$$M_{\mathbb{F}}(\lambda) = \bigoplus_{\alpha \in \Phi^+} \lambda(\alpha) M_{\mathbb{F}}(\alpha),$$

for some function $\lambda: \Phi^+ \rightarrow \mathbb{N}$. Thus, isoclasses of $\Lambda(\mathbb{F}, \vec{Q})$ -modules are indexed by the set

$$\mathfrak{B} = \mathfrak{B}(\vec{Q}) =: \{\lambda \mid \lambda : \Phi^+ \rightarrow \mathbb{N}\},$$

which is independent of finite field \mathbb{F} . To be consistent, we view each $\alpha \in \Phi^+$ as the function $\Phi^+ \rightarrow \mathbb{N}$, $\beta \mapsto \delta_{\alpha, \beta}$. For later use, we denote by α_i the i 'th simple root in Φ^+ and λ_i the function $\Phi^+ \rightarrow \mathbb{N}$, $\beta \mapsto \delta_{\alpha_i, \beta}$. For any finite field \mathbb{F} and $\lambda, \mu \in \mathfrak{B}(\vec{Q})$, we define

$$\langle \lambda, \mu \rangle = \langle \dim M_{\mathbb{F}}(\lambda), \dim M_{\mathbb{F}}(\mu) \rangle.$$

Then we have

Theorem 2.2 (Ringel). Assume \vec{Q} is a Dynkin quiver. For any $\lambda, \mu, \rho \in \mathfrak{B} = \mathfrak{B}(\vec{Q})$, there exists a polynomial $\varphi_{\mu, \rho}^{\lambda}(T) \in \mathbb{Z}[T]$ such that

$$\varphi_{\mu, \rho}^{\lambda}(|\mathbb{F}|) = F_{M_{\mathbb{F}}(\mu), M_{\mathbb{F}}(\rho)}^{M_{\mathbb{F}}(\lambda)}$$

holds for each finite field \mathbb{F} .

Now, we are ready to define the twisted generic Ringel–Hall algebra.

Definition. The twisted generic Ringel–Hall algebra $\mathcal{H}(\vec{Q})$ of Dynkin quiver \vec{Q} is the free $\mathbb{Q}(\nu)$ -module having basis $\{u_{\lambda} \mid \lambda \in \mathfrak{B}(\vec{Q})\}$ with multiplication defined by

$$u_{\mu} u_{\rho} = \nu^{\langle \mu, \rho \rangle} \sum_{\lambda \in \mathfrak{B}(\vec{Q})} \varphi_{\mu, \rho}^{\lambda}(\nu^2) u_{\lambda}.$$

Then $\mathcal{H}(A)$ is an associative algebra with identity $1 = u_0$, where 0 denotes the zero function in $\mathfrak{B}(\vec{Q})$.

From now on, we fix $k = \mathbb{Q}(\nu)$. Let \vec{Q} be a Dynkin quiver with underlying graph Q and \mathfrak{g} (resp. A) the corresponding semisimple Lie algebra (resp. the Cartan matrix). Then the main result in [11] is

Theorem 2.3 (Ringel). The map $\eta : U_q^+(\mathfrak{g}) = U_q^+(A) \rightarrow \mathcal{H}(\vec{Q})$ given by

$$\eta(E_i) = u_{[\lambda_i]}$$

is a $\mathbb{Q}(\nu)$ -algebra isomorphism.

3. Gröbner–Shirshov basis of quantum group of type E_6

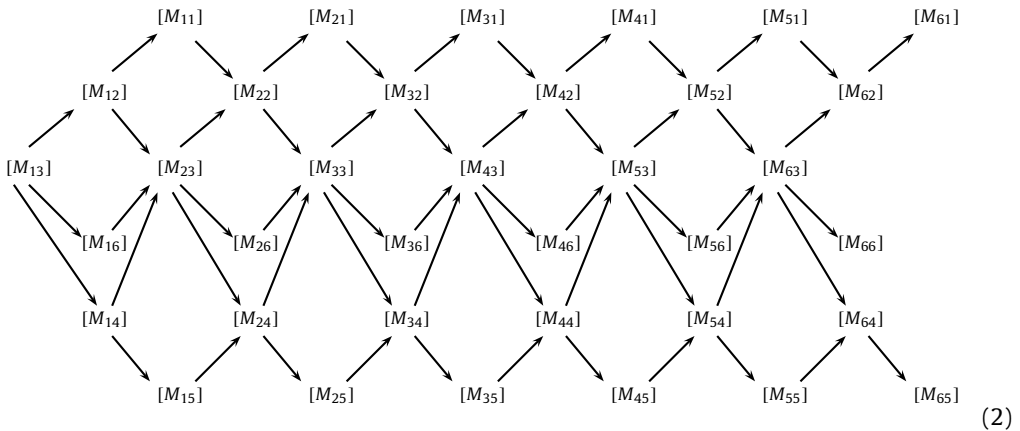
Throughout this section, quantum group $U_q(A)$ means the quantum group U_q of type E_6 :



whose corresponding Cartan matrix A is

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}.$$

The Auslander–Reiten quiver of the algebra given by E_6 is following



where M_{ij} ($1 \leq i, j \leq 6$) are indecomposable representations and $M_{65}, M_{55}, M_{13}, M_{51}, M_{61}, M_{66}$ are the corresponding 6 simple representations.

We let

$$e_{ij} = \nu^{\frac{1}{2}(\dim M_{ij}, \dim M_{ij}) - \dim_F M_{ij}} u_{[M_{ij}]} \quad \text{and} \quad E_{ij} = \eta^{-1}(e_{ij}) \quad (1 \leq i, j \leq 6),$$

where η is the isomorphism in Theorem 2.3 and

$$E_{65} = E_1, \quad E_{55} = E_2, \quad E_{13} = E_3, \quad E_{51} = E_4, \quad E_{61} = E_5, \quad E_{66} = E_6.$$

Then by using the algorithm given in [8] and the Auslander–Reiten translate τ (and its inverse τ^{-1}), we have following relations (for simplicity, we denote these relations by (R1))

1. $E_{i4}E_{i2} = E_{i2}E_{i4},$
2. $E_{i6}E_{i5} = E_{i5}E_{i6},$
3. $E_{i+1,3}E_{i1} = E_{i1}E_{i+1,3},$
4. $E_{i+1,4}E_{i1} = E_{i1}E_{i+1,4},$
5. $E_{i5}E_{i2} = E_{i2}E_{i5},$
6. $E_{i6}E_{i1} = E_{i1}E_{i6},$
7. $E_{i+1,5}E_{i1} = E_{i1}E_{i+1,5},$
8. $E_{i+1,6}E_{i1} = E_{i1}E_{i+1,6},$
9. $E_{i6}E_{i2} = E_{i2}E_{i6},$
10. $E_{i6}E_{i4} = E_{i4}E_{i6},$
11. $E_{i+1,1}E_{i5} = E_{i5}E_{i+1,1},$
12. $E_{i+1,2}E_{i5} = E_{i5}E_{i+1,2},$
13. $E_{i4}E_{i1} = E_{i1}E_{i4},$
14. $E_{65}E_{i5} = E_{i5}E_{65},$
15. $E_{i+1,3}E_{i5} = E_{i5}E_{i+1,3},$
16. $E_{i+1,6}E_{i5} = E_{i5}E_{i+1,6},$
17. $E_{i5}E_{i1} = E_{i1}E_{i5},$
18. $E_{65}E_{i6} = E_{i6}E_{65},$
19. $E_{i+4,5}E_{i1} = E_{i1}E_{i+4,5},$
20. $E_{i+4,1}E_{i5} = E_{i5}E_{i+4,1},$
21. $E_{61}E_{i3} = E_{i3}E_{61},$
22. $E_{65}E_{i3} = E_{i3}E_{65},$
23. $E_{i+2,1}E_{i1} = E_{i1}E_{i+2,1},$
24. $E_{65}E_{i4} = E_{i4}E_{65},$

25. $E_{61}E_{12} = E_{12}E_{61},$
26. $E_{61}E_{11} = E_{11}E_{61},$
27. $E_{i+2,5}E_{i5} = E_{i5}E_{i+2,5},$
28. $E_{61}E_{16} = E_{16}E_{61},$
29. $E_{i2}E_{i3} = \nu E_{i3}E_{i2},$
30. $E_{i+2,3}E_{i1} = \nu E_{i1}E_{i+2,3},$
31. $E_{i+2,5}E_{i4} = \nu E_{i4}E_{i+2,5},$
32. $E_{i6}E_{i3} = \nu E_{i3}E_{i6},$
33. $E_{i+1,2}E_{i1} = \nu E_{i1}E_{i+1,2},$
34. $E_{i+1,3}E_{i2} = \nu E_{i2}E_{i+1,3},$
35. $E_{i4}E_{i3} = \nu E_{i3}E_{i4},$
36. $E_{i+2,6}E_{i1} = \nu E_{i1}E_{i+2,6},$
37. $E_{i+1,4}E_{i6} = \nu E_{i6}E_{i+1,4},$
38. $E_{i5}E_{i3} = \nu E_{i3}E_{i5},$
39. $E_{i+2,1}E_{i2} = \nu E_{i2}E_{i+2,1},$
40. $E_{i+2,5}E_{i1} = \nu E_{i1}E_{i+2,5},$
41. $E_{i1}E_{i3} = \nu E_{i3}E_{i1},$
42. $E_{i+2,3}E_{i5} = \nu E_{i5}E_{i+2,3},$
43. $E_{i+2,6}E_{i5} = \nu E_{i5}E_{i+2,6},$
44. $E_{i1}E_{i2} = \nu E_{i2}E_{i1},$
45. $E_{i+1,1}E_{i6} = \nu E_{i6}E_{i+1,1},$
46. $E_{i+1,1}E_{i4} = \nu E_{i4}E_{i+1,1},$
47. $E_{i5}E_{i4} = \nu E_{i4}E_{i5},$
48. $E_{i+1,2}E_{i4} = \nu E_{i4}E_{i+1,2},$
49. $E_{i+1,4}E_{i2} = \nu E_{i2}E_{i+1,4},$
50. $E_{64}E_{11} = \nu E_{11}E_{64},$
51. $E_{i+1,6}E_{i4} = \nu E_{i4}E_{i+1,6},$
52. $E_{i+1,6}E_{i2} = \nu E_{i2}E_{i+1,6},$
53. $E_{61}E_{15} = \nu E_{15}E_{61},$
54. $E_{i+3,1}E_{i1} = \nu E_{i1}E_{i+3,1},$
55. $E_{i+3,2}E_{i1} = \nu E_{i1}E_{i+3,2},$
56. $E_{62}E_{15} = \nu E_{15}E_{62},$
57. $E_{i+3,1}E_{i6} = \nu E_{i6}E_{i+3,1},$
58. $E_{i+3,5}E_{i6} = \nu E_{i6}E_{i+3,5},$
59. $E_{66}E_{16} = \nu E_{16}E_{66},$
60. $E_{i+4,1}E_{i4} = \nu E_{i4}E_{i+4,1},$
61. $E_{i+4,6}E_{i5} = \nu E_{i5}E_{i+4,6},$
62. $E_{65}E_{11} = \nu E_{11}E_{65},$
63. $E_{i+1,3}E_{i6} = \nu E_{i6}E_{i+1,3},$
64. $E_{i+3,5}E_{i5} = \nu E_{i5}E_{i+3,5},$
65. $E_{i+2,2}E_{i5} = \nu E_{i5}E_{i+2,2},$
66. $E_{i+1,5}E_{i6} = \nu E_{i6}E_{i+1,5},$
67. $E_{i+2,4}E_{i1} = \nu E_{i1}E_{i+2,4},$
68. $E_{i+2,1}E_{i5} = \nu E_{i5}E_{i+2,1},$
69. $E_{i+2,6}E_{i6} = \nu E_{i6}E_{i+2,6},$
70. $E_{i+1,2}E_{i6} = \nu E_{i6}E_{i+1,2},$
71. $E_{i+1,3}E_{i4} = \nu E_{i4}E_{i+1,3},$
72. $E_{i+1,4}E_{i5} = \nu E_{i5}E_{i+1,4},$
73. $E_{i+3,4}E_{i5} = \nu E_{i5}E_{i+3,4},$
74. $E_{i+4,5}E_{i2} = \nu E_{i2}E_{i+4,5},$
75. $E_{i+1,5}E_{i2} = \nu E_{i2}E_{i+1,5},$
76. $E_{i+4,6}E_{i1} = \nu E_{i1}E_{i+4,6},$
77. $E_{i+1,1}E_{i2} = \nu^{-1}E_{i2}E_{i+1,1} + E_{i+1,3},$
78. $E_{i+1,1}E_{i3} = E_{i3}E_{i+1,1} + \nu^{-1}(\nu^2 - 1)E_{i4}E_{i6},$
79. $E_{i+1,1}E_{i1} = \nu^{-1}E_{i1}E_{i+1,1} + E_{i+1,2},$
80. $E_{i+1,5}E_{i3} = E_{i3}E_{i+1,5} + \nu^{-1}(\nu^2 - 1)E_{i2}E_{i6},$
81. $E_{i+1,5}E_{i4} = \nu^{-1}E_{i4}E_{i+1,5} + E_{i+1,3},$
82. $E_{i+1,6}E_{i3} = E_{i3}E_{i+1,6} + \nu^{-1}(\nu^2 - 1)E_{i2}E_{i4},$
83. $E_{i+1,5}E_{i5} = \nu^{-1}E_{i5}E_{i+1,5} + E_{i+1,4},$
84. $E_{i+1,2}E_{i2} = E_{i2}E_{i+1,2} + \nu^{-1}(\nu^2 - 1)E_{i1}E_{i+1,3},$
85. $E_{i+1,6}E_{i6} = \nu^{-1}E_{i6}E_{i+1,6} + E_{i+1,3},$
86. $E_{i+1,4}E_{i4} = E_{i4}E_{i+1,4} + \nu^{-1}(\nu^2 - 1)E_{i5}E_{i+1,3},$
87. $E_{i+3,5}E_{i2} = \nu^{-1}E_{i2}E_{i+3,5} + E_{i+1,4},$
88. $E_{i+2,1}E_{i3} = E_{i3}E_{i+2,1} + \nu^{-1}(\nu^2 - 1)E_{i2}E_{i5},$

89. $E_{i+3,5}E_{i1} = v^{-1}E_{i1}E_{i+3,5} + E_{i+2,6},$
90. $E_{i+2,5}E_{i3} = E_{i3}E_{i+2,5} + v^{-1}(v^2 - 1)E_{i1}E_{i4},$
91. $E_{i+3,6}E_{i1} = v^{-1}E_{i1}E_{i+3,6} + E_{i+2,4},$
92. $E_{i+2,2}E_{i2} = E_{i2}E_{i+2,2} + v^{-1}(v^2 - 1)E_{i+1,4}E_{i+1,6},$
93. $E_{i+3,1}E_{i4} = v^{-1}E_{i4}E_{i+3,1} + E_{i+1,2},$
94. $E_{i+2,5}E_{i2} = E_{i2}E_{i+2,5} + v^{-1}(v^2 - 1)E_{i1}E_{i+1,6},$
95. $E_{i+3,1}E_{i5} = v^{-1}E_{i5}E_{i+3,1} + E_{i+2,6},$
96. $E_{i+2,6}E_{i2} = E_{i2}E_{i+2,6} + v^{-1}(v^2 - 1)E_{i1}E_{i+1,4},$
97. $E_{i+3,6}E_{i5} = v^{-1}E_{i5}E_{i+3,6} + E_{i+2,2},$
98. $E_{i+2,1}E_{i4} = E_{i4}E_{i+2,1} + v^{-1}(v^2 - 1)E_{i5}E_{i+1,6},$
99. $E_{i+2,2}E_{i1} = v^{-1}E_{i1}E_{i+2,2} + E_{i+2,3},$
100. $E_{i+2,4}E_{i4} = E_{i4}E_{i+2,4} + v^{-1}(v^2 - 1)E_{i+1,2}E_{i+1,6},$
101. $E_{i+2,4}E_{i5} = v^{-1}E_{i5}E_{i+2,4} + E_{i+2,3},$
102. $E_{i+2,6}E_{i4} = E_{i4}E_{i+2,6} + v^{-1}(v^2 - 1)E_{i5}E_{i+1,2},$
103. $E_{i+2,1}E_{i6} = v^{-1}E_{i6}E_{i+2,1} + E_{i+1,4},$
104. $E_{i+2,2}E_{i6} = E_{i6}E_{i+2,2} + v^{-1}(v^2 - 1)E_{i+1,1}E_{i+1,4},$
105. $E_{i+2,5}E_{i6} = v^{-1}E_{i6}E_{i+2,5} + E_{i+1,2},$
106. $E_{i+2,3}E_{i6} = E_{i6}E_{i+2,3} + v^{-1}(v^2 - 1)E_{i+1,2}E_{i+1,4},$
107. $E_{62}E_{i3} = v^{-1}E_{i3}E_{62} + E_{i5},$
108. $E_{i+2,4}E_{i6} = E_{i6}E_{i+2,4} + v^{-1}(v^2 - 1)E_{i+1,2}E_{i+1,5},$
109. $E_{i+4,1}E_{i3} = v^{-1}E_{i3}E_{i+4,1} + E_{i4},$
110. $E_{i+4,6}E_{i3} = E_{i3}E_{i+4,6} + v^{-1}(v^2 - 1)E_{i1}E_{i5},$
111. $E_{i+4,5}E_{i3} = v^{-1}E_{i3}E_{i+4,5} + E_{i2},$
112. $E_{i+4,4}E_{i2} = E_{i2}E_{i+4,4} + v^{-1}(v^2 - 1)E_{i+1,5}E_{i+2,1},$
113. $E_{i+4,1}E_{i2} = v^{-1}E_{i2}E_{i+4,1} + E_{i+1,6},$
114. $E_{i+4,6}E_{i2} = E_{i2}E_{i+4,6} + v^{-1}(v^2 - 1)E_{i1}E_{i+2,1},$
115. $E_{i+4,1}E_{i1} = v^{-1}E_{i1}E_{i+4,1} + E_{i+2,5},$
116. $E_{i+4,2}E_{i1} = E_{i1}E_{i+4,2} + v^{-1}(v^2 - 1)E_{i+2,5}E_{i+3,1},$

117. $E_{i+4,4}E_{i1} = v^{-1}E_{i1}E_{i+4,4} + E_{i+3,2},$
118. $E_{i+4,3}E_{i1} = E_{i1}E_{i+4,3} + v^{-1}(v^2 - 1)E_{i+2,5}E_{i+3,2},$
119. $E_{i+4,5}E_{i4} = v^{-1}E_{i4}E_{i+4,5} + E_{i+1,6},$
120. $E_{i+4,2}E_{i4} = E_{i4}E_{i+4,2} + v^{-1}(v^2 - 1)E_{i+1,1}E_{i+2,5},$
121. $E_{i+4,2}E_{i5} = v^{-1}E_{i5}E_{i+4,2} + E_{i+3,4},$
122. $E_{i+4,6}E_{i4} = E_{i4}E_{i+4,6} + v^{-1}(v^2 - 1)E_{i5}E_{i+2,5},$
123. $E_{i+4,5}E_{i5} = v^{-1}E_{i5}E_{i+4,5} + E_{i+2,1},$
124. $E_{i+4,3}E_{i5} = E_{i5}E_{i+4,3} + v^{-1}(v^2 - 1)E_{i+2,1}E_{i+3,4},$
125. $E_{i+4,1}E_{i6} = v^{-1}E_{i6}E_{i+4,1} + E_{i+1,1},$
126. $E_{i+4,4}E_{i5} = E_{i5}E_{i+4,4} + v^{-1}(v^2 - 1)E_{i+2,1}E_{i+3,5},$
127. $E_{i+4,5}E_{i6} = v^{-1}E_{i6}E_{i+4,5} + E_{i+1,5},$
128. $E_{i+4,2}E_{i6} = E_{i6}E_{i+4,2} + v^{-1}(v^2 - 1)E_{i+1,1}E_{i+3,1},$
129. $E_{i+4,6}E_{i6} = v^{-1}E_{i6}E_{i+4,6} + E_{i+2,6},$
130. $E_{i+4,4}E_{i6} = E_{i6}E_{i+4,4} + v^{-1}(v^2 - 1)E_{i+1,5}E_{i+3,5},$
131. $E_{64}E_{13} = v^{-1}E_{13}E_{64} + E_{11},$
132. $E_{i+3,1}E_{i3} = E_{i3}E_{i+3,1} + v^{-1}(v^2 - 1)E_{i1}E_{i6},$
133. $E_{66}E_{13} = v^{-1}E_{13}E_{66} + E_{16},$
134. $E_{i+3,5}E_{i3} = E_{i3}E_{i+3,5} + v^{-1}(v^2 - 1)E_{i5}E_{i6},$
135. $E_{62}E_{12} = v^{-1}E_{12}E_{62} + E_{31},$
136. $E_{i+3,1}E_{i2} = E_{i2}E_{i+3,1} + v^{-1}(v^2 - 1)E_{i1}E_{i+1,5},$
137. $E_{65}E_{12} = v^{-1}E_{12}E_{65} + E_{11},$
138. $E_{i+3,6}E_{i2} = E_{i2}E_{i+3,6} + v^{-1}(v^2 - 1)E_{i+1,5}E_{i+1,6},$
139. $E_{66}E_{12} = v^{-1}E_{12}E_{66} + E_{25},$
140. $E_{i+3,3}E_{i1} = E_{i1}E_{i+3,3} + v^{-1}(v^2 - 1)E_{i+2,4}E_{i+2,6},$
141. $E_{62}E_{11} = v^{-1}E_{11}E_{62} + E_{56},$
142. $E_{i+3,4}E_{i1} = E_{i1}E_{i+3,4} + v^{-1}(v^2 - 1)E_{i+2,5}E_{i+2,6},$
143. $E_{66}E_{11} = v^{-1}E_{11}E_{66} + E_{41},$
144. $E_{i+3,5}E_{i4} = E_{i4}E_{i+3,5} + v^{-1}(v^2 - 1)E_{i5}E_{i+1,1},$
145. $E_{61}E_{14} = v^{-1}E_{14}E_{61} + E_{15},$

146. $E_{i+3,6}E_{i4} = E_{i4}E_{i+3,6} + v^{-1}(v^2 - 1)E_{i+1,1}E_{i+1,6},$
147. $E_{64}E_{14} = v^{-1}E_{14}E_{64} + E_{35},$
148. $E_{i+3,2}E_{i5} = E_{i5}E_{i+3,2} + v^{-1}(v^2 - 1)E_{i+2,1}E_{i+2,6},$
149. $E_{66}E_{14} = v^{-1}E_{14}E_{66} + E_{21},$
150. $E_{i+3,3}E_{i5} = E_{i5}E_{i+3,3} + v^{-1}(v^2 - 1)E_{i+2,2}E_{i+2,6},$
151. $E_{64}E_{15} = v^{-1}E_{15}E_{64} + E_{56},$
152. $E_{i+3,2}E_{i6} = E_{i6}E_{i+3,2} + v^{-1}(v^2 - 1)E_{i+1,5}E_{i+2,6},$
153. $E_{66}E_{15} = v^{-1}E_{15}E_{66} + E_{45},$
154. $E_{i+3,4}E_{i6} = E_{i6}E_{i+3,4} + v^{-1}(v^2 - 1)E_{i+1,1}E_{i+2,6},$
155. $E_{62}E_{16} = v^{-1}E_{16}E_{62} + E_{45},$
156. $E_{i+3,6}E_{i6} = E_{i6}E_{i+3,6} + v^{-1}(v^2 - 1)E_{i+1,1}E_{i+1,5},$
157. $E_{64}E_{16} = v^{-1}E_{16}E_{64} + E_{41},$
158. $E_{64}E_{12} = E_{12}E_{64} + v^{-1}(v^2 - 1)E_{11}E_{55},$
159. $E_{63}E_{11} = E_{11}E_{63} + v^{-1}(v^2 - 1)E_{41}E_{56},$
160. $E_{62}E_{14} = E_{14}E_{62} + v^{-1}(v^2 - 1)E_{15}E_{51},$
161. $E_{63}E_{15} = E_{15}E_{63} + v^{-1}(v^2 - 1)E_{45}E_{56},$
162. $E_{63}E_{16} = E_{16}E_{63} + v^{-1}(v^2 - 1)E_{41}E_{45},$
163. $E_{i+1,2}E_{i3} = vE_{i3}E_{i+1,2} + v^{-2}(v^2 - 1)^2E_{i1}E_{i4}E_{i6},$
164. $E_{i+1,3}E_{i3} = vE_{i3}E_{i+1,3} + v^{-2}(v^2 - 1)^2E_{i2}E_{i4}E_{i6},$
165. $E_{i+1,4}E_{i3} = vE_{i3}E_{i+1,4} + v^{-2}(v^2 - 1)^2E_{i2}E_{i5}E_{i6},$
166. $E_{i+2,6}E_{i3} = vE_{i3}E_{i+2,6} + v^{-2}(v^2 - 1)^2E_{i1}E_{i5}E_{i6},$
167. $E_{i+2,3}E_{i2} = vE_{i2}E_{i+2,3} + v^{-2}(v^2 - 1)^2E_{i1}E_{i+1,4}E_{i+1,6},$
168. $E_{i+2,4}E_{i2} = vE_{i2}E_{i+2,4} + v^{-2}(v^2 - 1)^2E_{i1}E_{i+1,5}E_{i+1,6},$
169. $E_{i+2,2}E_{i4} = vE_{i4}E_{i+2,2} + v^{-2}(v^2 - 1)^2E_{i5}E_{i+1,1}E_{i+1,6},$
170. $E_{i+2,3}E_{i4} = vE_{i4}E_{i+2,3} + v^{-2}(v^2 - 1)^2E_{i5}E_{i+1,2}E_{i+1,6},$
171. $E_{i+3,2}E_{i2} = vE_{i2}E_{i+3,2} + v^{-2}(v^2 - 1)^2E_{i1}E_{i+1,5}E_{i+2,1},$
172. $E_{i+3,4}E_{i4} = vE_{i4}E_{i+3,4} + v^{-2}(v^2 - 1)^2E_{i5}E_{i+1,1}E_{i+2,5},$

173. $E_{i+3,3}E_{i6} = \nu E_{i6}E_{i+3,3} + \nu^{-2}(\nu^2 - 1)^2 E_{i+1,1}E_{i+1,5}E_{i+2,6},$
174. $E_{i+2,2}E_{i3} = E_{i3}E_{i+2,2} + \nu^{-3}(\nu^2 - 1)^2 E_{i2}E_{i5}E_{i+1,1}$
 $+ \nu^{-3}(\nu^2 - 1)^2 E_{i5}E_{i6}E_{i+1,6} + \nu^{-1}(\nu^2 - 1)E_{i4}E_{i+1,4}$
 $+ \nu^{-2}(\nu^2 - 1)(\nu^2 - 2)E_{i5}E_{i+1,3},$
175. $E_{i+2,3}E_{i3} = \nu E_{i3}E_{i+2,3} + \nu^{-2}(\nu^2 - 1)^2 E_{i2}E_{i5}E_{i+1,2}$
 $+ \nu^{-3}(\nu^2 - 1)^2(\nu^2 - 2)E_{i1}E_{i5}E_{i+1,3} + \nu^{-2}(\nu^2 - 1)^2 E_{i1}E_{i4}E_{i+1,4}$
 $+ \nu^{-4}(\nu^2 - 1)^3 E_{i1}E_{i5}E_{i6}E_{i+1,6},$
176. $E_{i+2,4}E_{i3} = E_{i3}E_{i+2,4} + \nu^{-3}(\nu^2 - 1)^2 E_{i1}E_{i4}E_{i+1,5} + \nu^{-3}(\nu^2 - 1)^2 E_{i1}E_{i6}E_{i+1,6}$
 $+ \nu^{-1}(\nu^2 - 1)E_{i2}E_{i+1,2} + \nu^{-2}(\nu^2 - 1)(\nu^2 - 2)E_{i1}E_{i+1,3},$
177. $E_{i+3,2}E_{i3} = E_{i3}E_{i+3,2} + \nu^{-3}(\nu^2 - 1)^2 E_{i1}E_{i5}E_{i+1,5} + \nu^{-3}(\nu^2 - 1)^2 E_{i1}E_{i6}E_{i+2,1}$
 $+ \nu^{-1}(\nu^2 - 1)E_{i2}E_{i+2,6} + \nu^{-2}(\nu^2 - 1)(\nu^2 - 2)E_{i1}E_{i+1,4},$
178. $E_{i+3,4}E_{i3} = E_{i3}E_{i+3,4} + \nu^{-3}(\nu^2 - 1)^2 E_{i1}E_{i5}E_{i+1,1} + \nu^{-3}(\nu^2 - 1)^2 E_{i5}E_{i6}E_{i+2,5}$
 $+ \nu^{-1}(\nu^2 - 1)E_{i4}E_{i+2,6} + \nu^{-2}(\nu^2 - 1)(\nu^2 - 2)E_{i5}E_{i+1,2},$
179. $E_{i+3,6}E_{i3} = \nu^{-1}E_{i3}E_{i+3,6} + \nu^{-1}(\nu^2 - 2)E_{i+1,3} + \nu^{-2}(\nu^2 - 1)E_{i2}E_{i+1,1}$
 $+ \nu^{-2}(\nu^2 - 1)E_{i4}E_{i+1,5} + \nu^{-2}(\nu^2 - 1)E_{i6}E_{i+1,6},$
180. $E_{i+3,3}E_{i2} = E_{i2}E_{i+3,3} + \nu^{-1}(\nu^2 - 1)E_{i+1,4}E_{i+2,4}$
 $+ \nu^{-2}(\nu^2 - 1)(\nu^2 - 2)E_{i+1,5}E_{i+2,3}$
 $+ \nu^{-3}(\nu^2 - 1)^2 E_{i1}E_{i+1,5}E_{i+2,2} + \nu^{-3}(\nu^2 - 1)^2 E_{i+1,5}E_{i+1,6}E_{i+2,6},$
181. $E_{i+3,4}E_{i2} = \nu^{-1}E_{i2}E_{i+3,4} + \nu^{-1}(\nu^2 - 2)E_{i+2,3} + \nu^{-2}(\nu^2 - 1)E_{i1}E_{i+2,2}$
 $+ \nu^{-2}(\nu^2 - 1)E_{i+1,4}E_{i+2,5} + \nu^{-2}(\nu^2 - 1)E_{i+1,6}E_{i+2,6},$
182. $E_{i+3,2}E_{i4} = \nu^{-1}E_{i4}E_{i+3,2} + \nu^{-1}(\nu^2 - 2)E_{i+2,3} + \nu^{-2}(\nu^2 - 1)E_{i5}E_{i+2,4}$
 $+ \nu^{-2}(\nu^2 - 1)E_{i+1,2}E_{i+2,1} + \nu^{-2}(\nu^2 - 1)E_{i+1,6}E_{i+2,6},$
183. $E_{i+3,3}E_{i4} = E_{i4}E_{i+3,3} + \nu^{-1}(\nu^2 - 1)E_{i+1,2}E_{i+2,2} + \nu^{-2}(\nu^2 - 1)(\nu^2 - 2)E_{i+1,1}E_{i+2,3}$
 $+ \nu^{-3}(\nu^2 - 1)^2 E_{i5}E_{i+1,1}E_{i+2,4} + \nu^{-3}(\nu^2 - 1)^2 E_{i+1,1}E_{i+1,6}E_{i+2,6},$
184. $E_{i+3,3}E_{i3} = E_{i3}E_{i+3,3} + \nu^{-2}(\nu^2 - 1)(\nu^2 - 2)E_{i+1,3}E_{i+2,6}$
 $+ \nu^{-3}(\nu^2 - 1)(\nu^4 - 3\nu^2 + 3)E_{i+1,2}E_{i+1,4}$
 $+ \nu^{-2}(\nu^2 - 1)(\nu^2 - 2)E_{i6}E_{i+2,3} + \nu^{-3}(\nu^2 - 1)^2 E_{i1}E_{i6}E_{i+2,2}$
 $+ \nu^{-3}(\nu^2 - 1)^2 E_{i5}E_{i6}E_{i+2,4} + \nu^{-3}(\nu^2 - 1)^2 E_{i2}E_{i+1,1}E_{i+2,6}$
 $+ \nu^{-3}(\nu^2 - 1)^2 E_{i4}E_{i+1,5}E_{i+2,6} + \nu^{-3}(\nu^2 - 1)^2 E_{i6}E_{i+1,6}E_{i+2,6}$
 $+ \nu^{-4}(\nu^2 - 1)^2(\nu^2 - 2)E_{i1}E_{i+1,1}E_{i+1,4}$

- $$\begin{aligned}
& + v^{-4}(\nu^2 - 1)^2(\nu^2 - 2)E_{i5}E_{i+1,2}E_{i+1,5} \\
& + v^{-5}(\nu^2 - 1)^3E_{i1}E_{i5}E_{i+1,1}E_{i+1,5}, \\
185. \quad E_{i+4,2}E_{i3} &= v^{-1}E_{i3}E_{i+4,2} + v^{-1}(\nu^2 - 2)E_{i+1,2} + v^{-2}(\nu^2 - 1)E_{i1}E_{i+1,1} \\
& + v^{-2}(\nu^2 - 1)E_{i6}E_{i+2,5} + v^{-2}(\nu^2 - 1)E_{i4}E_{i+3,1}, \\
186. \quad E_{i+4,4}E_{i3} &= v^{-1}E_{i3}E_{i+4,4} + v^{-1}(\nu^2 - 2)E_{i+1,4} + v^{-2}(\nu^2 - 1)E_{i5}E_{i+1,5} \\
& + v^{-2}(\nu^2 - 1)E_{i6}E_{i+2,1} + v^{-2}(\nu^2 - 1)E_{i2}E_{i+3,5}, \\
187. \quad E_{i+4,2}E_{i2} &= v^{-1}E_{i2}E_{i+4,2} + v^{-1}(\nu^2 - 2)E_{i+2,4} + v^{-2}(\nu^2 - 1)E_{i+1,5}E_{i+2,5} \\
& + v^{-2}(\nu^2 - 1)E_{i1}E_{i+3,6} + v^{-2}(\nu^2 - 1)E_{i+1,6}E_{i+3,1}, \\
188. \quad E_{i+4,4}E_{i4} &= v^{-1}E_{i4}E_{i+4,4} + v^{-1}(\nu^2 - 2)E_{i+2,2} + v^{-2}(\nu^2 - 1)E_{i+1,1}E_{i+2,1} \\
& + v^{-2}(\nu^2 - 1)E_{i5}E_{i+3,6} + v^{-2}(\nu^2 - 1)E_{i+1,6}E_{i+3,5}, \\
189. \quad E_{i+4,3}E_{i6} &= v^{-1}E_{i6}E_{i+4,3} + v^{-1}(\nu^2 - 2)E_{i+3,3} + v^{-2}(\nu^2 - 1)E_{i+2,6}E_{i+3,6} \\
& + v^{-2}(\nu^2 - 1)E_{i+1,1}E_{i+3,2} + v^{-2}(\nu^2 - 1)E_{i+1,5}E_{i+3,4}, \\
190. \quad E_{i+4,3}E_{i2} &= E_{i2}E_{i+4,3} + v^{-1}(\nu^2 - 1)E_{i+1,6}E_{i+3,2} + v^{-2}(\nu^2 - 1)(\nu^2 - 2)E_{i+2,1}E_{i+2,4} \\
& + v^{-3}(\nu^2 - 1)^2E_{i1}E_{i+2,1}E_{i+3,6} + v^{-3}(\nu^2 - 1)^2E_{i+1,5}E_{i+2,1}E_{i+2,5}, \\
191. \quad E_{i+4,3}E_{i4} &= E_{i4}E_{i+4,3} + v^{-1}(\nu^2 - 1)E_{i+1,6}E_{i+3,4} + v^{-2}(\nu^2 - 1)(\nu^2 - 2)E_{i+2,2}E_{i+2,5} \\
& + v^{-3}(\nu^2 - 1)^2E_{i5}E_{i+2,5}E_{i+3,6} + v^{-3}(\nu^2 - 1)^2E_{i+1,1}E_{i+2,1}E_{i+2,5}, \\
192. \quad E_{i+4,3}E_{i3} &= v^{-1}E_{i3}E_{i+4,3} + v^{-2}(\nu^2 - 2)^2E_{i+2,3} + v^{-3}(\nu^2 - 1)(\nu^2 - 2)E_{i1}E_{i+2,2} \\
& + v^{-2}(\nu^2 - 1)E_{i2}E_{i+3,4} + v^{-2}(\nu^2 - 1)E_{i4}E_{i+3,2} \\
& + v^{-3}(\nu^2 - 1)(\nu^2 - 2)E_{i5}E_{i+2,4} + v^{-3}(\nu^2 - 1)(\nu^2 - 2)E_{i+1,2}E_{i+2,1} \\
& + v^{-3}(\nu^2 - 1)(\nu^2 - 2)E_{i+1,4}E_{i+2,5} + v^{-3}(\nu^2 - 1)^2E_{i+1,6}E_{i+2,6} \\
& + v^{-4}(\nu^2 - 1)^2E_{i1}E_{i5}E_{i+3,6} + v^{-4}(\nu^2 - 1)^2E_{i1}E_{i+1,1}E_{i+2,1} \\
& + v^{-4}(\nu^2 - 1)^2E_{i5}E_{i+1,5}E_{i+2,5} + v^{-4}(\nu^2 - 1)^2E_{i6}E_{i+2,1}E_{i+2,5}, \\
193. \quad E_{63}E_{i3} &= v^{-1}E_{i3}E_{63} + v^{-1}(\nu^2 - 2)E_{36} + v^{-2}(\nu^2 - 1)E_{i1}E_{45} \\
& + v^{-2}(\nu^2 - 1)E_{i6}E_{56} + v^{-2}(\nu^2 - 1)E_{i5}E_{41}, \\
194. \quad E_{63}E_{i2} &= v^{-1}E_{i2}E_{63} + v^{-1}(\nu^2 - 2)E_{42} + v^{-2}(\nu^2 - 1)E_{25}E_{56} \\
& + v^{-2}(\nu^2 - 1)E_{31}E_{41} + v^{-2}(\nu^2 - 1)E_{i1}E_{54}, \\
195. \quad E_{63}E_{i4} &= v^{-1}E_{i4}E_{63} + v^{-1}(\nu^2 - 2)E_{44} + v^{-2}(\nu^2 - 1)E_{21}E_{56} \\
& + v^{-2}(\nu^2 - 1)E_{35}E_{45} + v^{-2}(\nu^2 - 1)E_{i5}E_{52}.
\end{aligned}$$

Note that the relations (R1) include the Serre relations S^+ . So $U_q^+(A)$ can be viewed as a factor algebra $\mathbb{Q}(\nu)\langle X \rangle/I$, where $X = \{E_{ij} \mid 1 \leq i, j \leq 6\}$ and I is the ideal generated by the relations (R1).

We define an ordering

$$E_{i+1,6} > E_{i+1,5} > E_{i+1,4} > E_{i+1,1} > E_{i+1,2} > E_{i+1,3} > E_{i6} > E_{i5} > E_{i4} > E_{i1} > E_{i2} > E_{i3}$$

for the elements E_{ij} ($1 \leq i \leq 5$), then this ordering induces a degree-lexicographical ordering on the monomials of these elements. For convenience, we denote by P_i ($1 \leq i \leq 195$) the polynomials obtained from relations in (R1) by subtracting the right-hand side from the left-hand side and let

$$S^{+c} = \{P_i \mid 1 \leq i \leq 195\}.$$

Then, clearly, $S^+ \subset S^{+c}$ and we have following

Theorem 3.1. *The set S^{+c} is a Gröbner–Shirshov basis of the algebra $U_q^+(A)$.*

Proof. Before starting our proof, we do following observations in order to shorten our proof. First, in the oriented graph (1) the vertices 1 and 2 are symmetric to the vertices 5 and 4, respectively, and by observing the relations in (R1), we can see that all relations that involving vertices 1 and 2 are same to those relation that involve the vertices 5 and 4. So we do not need to prove the cases about vertices 5 and 4. Next, because the oriented graph (1) includes the graphs D_4 and A_i ($1 \leq i \leq 5$) as a subgraph for suitable orientation, and the Gröbner–Shirshov basis in these two cases are known (see [3,12,14]), so we also do not need to consider these cases. Hence there are 1440 possible compositions between the elements of S^{+c} that we need to consider. By tedious computation we know that all these compositions are trivial modulo the ideal I . For save space, here we only give the proof of following 4 compositions:

1. We take $P_{29} = E_{i2}E_{i3} - \nu E_{i3}E_{i2}$ ($5 \leq i \leq 6$) or, equivalently,

$$P_{29} = E_{i+4,2}E_{i+4,3} - \nu E_{i+4,3}E_{i+4,2} \quad (1 \leq i \leq 2)$$

and

$$\begin{aligned} P_{190} = & E_{i+4,3}E_{i2} - E_{i2}E_{i+4,3} - \nu^{-1}(\nu^2 - 1)E_{i+1,6}E_{i+3,2} - \nu^{-2}(\nu^2 - 1)(\nu^2 - 2)E_{i+2,1}E_{i+2,4} \\ & - \nu^{-3}(\nu^2 - 1)^2E_{i1}E_{i+2,1}E_{i+3,6} - \nu^{-3}(\nu^2 - 1)^2E_{i+1,5}E_{i+2,1}E_{i+2,5} \quad (1 \leq i \leq 2), \end{aligned}$$

where $\omega = E_{i+4,2}E_{i+4,3}E_{i2}$. So

$$\begin{aligned} (P_{29}, P_{190})_\omega = & P_{29}E_{i2} - E_{i+4,2}P_{190} \\ \equiv & -\nu E_{i+4,3}E_{i+4,2}E_{i2} + E_{i+4,2}E_{i2}E_{i+4,3} + \nu^{-1}(\nu^2 - 1)E_{i+4,2}E_{i+1,6}E_{i+3,2} \\ & + \nu^{-2}(\nu^2 - 1)(\nu^2 - 2)E_{i+4,2}E_{i+2,1}E_{i+2,4} + \nu^{-3}(\nu^2 - 1)^2E_{i+4,2}E_{i1}E_{i+2,1}E_{i+3,6} \\ & + \nu^{-3}(\nu^2 - 1)^2E_{i+4,2}E_{i+1,5}E_{i+2,1}E_{i+2,5} \quad \text{mod}(S^{+c}; \omega) \\ \equiv & -E_{i+4,3}E_{i2}E_{i+4,2} - (\nu^2 - 2)E_{i+4,3}E_{i+2,4} - \nu^{-1}(\nu^2 - 1)E_{i+4,3}E_{i+1,5}E_{i+2,5} \\ & - \nu^{-1}(\nu^2 - 1)E_{i+4,3}E_{i1}E_{i+3,6} - \nu^{-1}(\nu^2 - 1)E_{i+4,3}E_{i+1,6}E_{i+3,1} \\ & + \nu^{-1}E_{i2}E_{i+4,2}E_{i+4,3} + \nu^{-1}(\nu^2 - 2)E_{i+2,4}E_{i+4,3} \\ & + \nu^{-2}(\nu^2 - 1)E_{i+1,5}E_{i+2,5}E_{i+4,3} + \nu^{-2}(\nu^2 - 1)E_{i1}E_{i+3,6}E_{i+4,3} \\ & + \nu^{-2}(\nu^2 - 1)E_{i+1,6}E_{i+3,1}E_{i+4,3} + \nu^{-1}(\nu^2 - 1)E_{i+1,6}E_{i+4,2}E_{i+3,2} \end{aligned}$$

$$\begin{aligned}
& + v^{-2}(v^2 - 1)^2 E_{i+2,5} E_{i+3,6} E_{i+3,2} + v^{-3}(v^2 - 1)(v^2 - 2) E_{i+2,1} E_{i+4,2} E_{i+2,4} \\
& + v^{-2}(v^2 - 1)(v^2 - 2) E_{i+4,3} E_{i+2,4} + v^{-3}(v^2 - 1)^2 E_{i1} E_{i+4,2} E_{i+2,1} E_{i+3,6} \\
& + v^{-4}(v^2 - 1)^3 E_{i+2,5} E_{i+3,1} E_{i+2,1} E_{i+3,6} + v^{-3}(v^2 - 1)^2 E_{i+1,5} E_{i+4,2} E_{i+2,1} E_{i+2,5} \\
& + v^{-4}(v^2 - 1)^3 E_{i+3,1} E_{i+3,6} E_{i+2,1} E_{i+2,5} \pmod{S^{+c}; \omega} \\
\equiv & -E_{i2} E_{i+4,3} E_{i+4,2} - v^{-1}(v^2 - 1) E_{i+1,6} E_{i+3,2} E_{i+4,2} \\
& - v^{-2}(v^2 - 1)(v^2 - 2) E_{i+2,1} E_{i+2,4} E_{i+4,2} - v^{-3}(v^2 - 1)^2 E_{i1} E_{i+2,1} E_{i+3,6} E_{i+4,2} \\
& - v^{-3}(v^2 - 1)^2 E_{i+1,5} E_{i+2,1} E_{i+2,5} E_{i+4,2} - v(v^2 - 2) E_{i+2,4} E_{i+4,3} \\
& - v^{-2}(v^2 - 1)^2 (v^2 - 2) E_{i+2,5} E_{i+3,2} E_{i+3,6} - (v^2 - 1) E_{i+1,5} E_{i+2,5} E_{i+4,3} \\
& - v^{-2}(v^2 - 1)^2 E_{i+2,5} E_{i+3,2} E_{i+3,6} - (v^2 - 1) E_{i1} E_{i+3,6} E_{i+4,3} \\
& - v^{-2}(v^2 - 1)^2 E_{i+2,5} E_{i+3,2} E_{i+3,6} - (v^2 - 1) E_{i+1,6} E_{i+3,1} E_{i+4,3} \\
& - v^{-3}(v^2 - 1)^3 E_{i+2,1} E_{i+2,5} E_{i+3,1} E_{i+3,6} + E_{i2} E_{i+4,3} E_{i+4,2} \\
& + v^{-1}(v^2 - 2) E_{i+2,4} E_{i+4,3} + v^{-2}(v^2 - 1) E_{i+1,5} E_{i+2,5} E_{i+4,3} \\
& + v^{-2}(v^2 - 1) E_{i1} E_{i+3,6} E_{i+4,3} + v^{-2}(v^2 - 1) E_{i+1,6} E_{i+3,1} E_{i+4,3} \\
& + v^{-1}(v^2 - 1) E_{i+1,6} E_{i+3,2} E_{i+4,2} + v^{-2}(v^2 - 1)^2 E_{i+1,6} E_{i+3,1} E_{i+4,3} \\
& + v^{-2}(v^2 - 1)^2 E_{i+2,5} E_{i+3,2} E_{i+3,6} + v^{-2}(v^2 - 1)(v^2 - 2) E_{i+2,1} E_{i+2,4} E_{i+4,2} \\
& + v^{-5}(v^2 - 1)^3 (v^2 - 2) E_{i+2,1} E_{i+2,5} E_{i+3,1} E_{i+3,6} \\
& + v^{-1}(v^2 - 1)(v^2 - 2) E_{i+2,4} E_{i+4,3} \\
& + v^{-4}(v^2 - 1)^3 (v^2 - 2) E_{i+2,5} E_{i+3,2} E_{i+3,6} + v^{-3}(v^2 - 1)^2 E_{i1} E_{i+2,1} E_{i+3,6} E_{i+4,2} \\
& + v^{-2}(v^2 - 1)^2 E_{i1} E_{i+3,6} E_{i+4,3} + v^{-5}(v^2 - 1)^3 E_{i+2,1} E_{i+2,5} E_{i+3,1} E_{i+3,6} \\
& + v^{-4}(v^2 - 1)^3 E_{i+2,5} E_{i+3,2} E_{i+3,6} + v^{-3}(v^2 - 1)^2 E_{i+1,5} E_{i+2,1} E_{i+2,5} E_{i+4,2} \\
& + v^{-2}(v^2 - 1)^2 E_{i+1,5} E_{i+2,5} E_{i+4,3} + v^{-5}(v^2 - 1)^3 E_{i+2,1} E_{i+2,5} E_{i+3,1} E_{i+3,6} \\
& + v^{-4}(v^2 - 1)^3 E_{i+2,5} E_{i+3,2} E_{i+3,6} \pmod{S^{+c}; \omega} \\
\equiv & 0 \pmod{S^{+c}; \omega}.
\end{aligned}$$

2. We take $P_9 = E_{i6} E_{i2} - E_{i2} E_{i6}$ ($3 \leq i \leq 6$) or, equivalently,

$$P_9 = E_{i+3,6} E_{i+3,2} - E_{i+3,2} E_{i+3,6} \quad (1 \leq i \leq 3)$$

and

$$\begin{aligned}
P_{177} = & E_{i+3,2} E_{i3} - E_{i3} E_{i+3,2} - v^{-3}(v^2 - 1)^2 E_{i1} E_{i5} E_{i+1,5} - v^{-3}(v^2 - 1)^2 E_{i1} E_{i6} E_{i+2,1} \\
& - v^{-1}(v^2 - 1) E_{i2} E_{i+2,6} - v^{-2}(v^2 - 1)(v^2 - 2) E_{i1} E_{i+1,4},
\end{aligned}$$

where $\omega = E_{i+3,6} E_{i+3,2} E_{i3}$. So

$$\begin{aligned}
(P_9, P_{177})_\omega &= P_9 E_{i3} - E_{i+3,6} P_{177} \\
&\equiv -E_{i+3,2} E_{i+3,6} E_{i3} + E_{i+3,6} E_{i3} E_{i+3,2} + v^{-3} (v^2 - 1)^2 E_{i+3,6} E_{i1} E_{i5} E_{i+1,5} \\
&\quad + v^{-3} (v^2 - 1)^2 E_{i+3,6} E_{i1} E_{i6} E_{i+2,1} + v^{-1} (v^2 - 1) E_{i+3,6} E_{i2} E_{i+2,6} \\
&\quad + v^{-2} (v^2 - 1) (v^2 - 2) E_{i+3,6} E_{i1} E_{i+1,4} \pmod{S^{+c}; \omega} \\
&\equiv -v^{-1} E_{i+3,2} E_{i3} E_{i+3,6} - v^{-1} (v^2 - 2) E_{i+3,2} E_{i+1,3} \\
&\quad - v^{-2} (v^2 - 1) E_{i+3,2} E_{i2} E_{i+1,1} - v^{-2} (v^2 - 1) E_{i+3,2} E_{i4} E_{i+1,5} \\
&\quad - v^{-2} (v^2 - 1) E_{i+3,2} E_{i6} E_{i+1,6} + E_{i+3,6} E_{i3} E_{i+3,2} \\
&\quad + v^{-4} (v^2 - 1)^2 E_{i1} E_{i+3,6} E_{i5} E_{i+1,5} + v^{-3} (v^2 - 1)^2 E_{i+2,4} E_{i5} E_{i+1,5} \\
&\quad + v^{-4} (v^2 - 1)^2 E_{i1} E_{i+3,6} E_{i6} E_{i+2,1} + v^{-3} (v^2 - 1)^2 E_{i+2,4} E_{i6} E_{i+2,1} \\
&\quad + v^{-1} (v^2 - 1) E_{i2} E_{i+3,6} E_{i+2,6} + v^{-2} (v^2 - 1)^2 E_{i+1,5} E_{i+1,6} E_{i+2,6} \\
&\quad + v^{-3} (v^2 - 1) (v^2 - 2) E_{i1} E_{i+3,6} E_{i+1,4} \\
&\quad + v^{-2} (v^2 - 1) (v^2 - 2) E_{i+2,4} E_{i+1,4} \pmod{S^{+c}; \omega} \\
&\equiv -v^{-1} E_{i3} E_{i+3,2} E_{i+3,6} - v^{-4} (v^2 - 1)^2 E_{i1} E_{i5} E_{i+1,5} E_{i+3,6} \\
&\quad - v^{-4} (v^2 - 1)^2 E_{i1} E_{i6} E_{i+2,1} E_{i+3,6} - v^{-2} (v^2 - 1) E_{i2} E_{i+2,6} E_{i+3,6} \\
&\quad - v^{-3} (v^2 - 1) (v^2 - 2) E_{i1} E_{i+1,4} E_{i+3,6} - v^{-1} (v^2 - 2) E_{i+1,3} E_{i+3,2} \\
&\quad - v^{-4} (v^2 - 1)^2 (v^2 - 2) E_{i+1,2} E_{i+1,5} E_{i+2,1} \\
&\quad - v^{-4} (v^2 - 1)^2 (v^2 - 2) E_{i+1,5} E_{i+1,6} E_{i+2,6} \\
&\quad - v^{-2} (v^2 - 1) (v^2 - 2) E_{i+1,4} E_{i+2,4} - v^{-3} (v^2 - 1) (v^2 - 2)^2 E_{i+1,5} E_{i+2,3} \\
&\quad - v^{-2} (v^2 - 1) E_{i2} E_{i+1,1} E_{i+3,2} - v^{-1} (v^2 - 1) E_{i2} E_{i+3,3} \\
&\quad - v^{-5} (v^2 - 1)^3 E_{i1} E_{i+1,1} E_{i+1,5} E_{i+2,1} - v^{-4} (v^2 - 1)^3 E_{i1} E_{i+1,5} E_{i+2,2} \\
&\quad - v^{-2} (v^2 - 1) E_{i4} E_{i+1,5} E_{i+3,2} - v^{-3} (v^2 - 1) (v^2 - 2) E_{i+1,5} E_{i+2,3} \\
&\quad - v^{-3} (v^2 - 1)^2 E_{i5} E_{i+1,5} E_{i+2,4} - v^{-4} (v^2 - 1)^2 E_{i+1,2} E_{i+1,5} E_{i+2,1} \\
&\quad - v^{-4} (v^2 - 1)^2 E_{i+1,5} E_{i+1,6} E_{i+2,6} - v^{-2} (v^2 - 1) E_{i6} E_{i+1,6} E_{i+3,2} \\
&\quad - v^{-3} (v^2 - 1)^2 E_{i6} E_{i+2,1} E_{i+2,4} - v^{-4} (v^2 - 1)^2 E_{i+1,5} E_{i+1,6} E_{i+2,6} \\
&\quad - v^{-3} (v^2 - 1)^2 E_{i+1,5} E_{i+2,3} + v^{-1} E_{i3} E_{i+3,2} E_{i+3,6} \\
&\quad + v^{-1} (v^2 - 2) E_{i+1,3} E_{i+3,2} + v^{-2} (v^2 - 1) E_{i2} E_{i+1,1} E_{i+3,2} \\
&\quad + v^{-2} (v^2 - 1) E_{i4} E_{i+1,5} E_{i+3,2} + v^{-2} (v^2 - 1) E_{i6} E_{i+1,6} E_{i+3,2} \\
&\quad + v^{-4} (v^2 - 1)^2 E_{i1} E_{i5} E_{i+1,5} E_{i+3,6} + v^{-4} (v^2 - 1)^2 E_{i1} E_{i+1,5} E_{i+2,2} \\
&\quad + v^{-3} (v^2 - 1)^2 E_{i5} E_{i+1,5} E_{i+2,4} + v^{-3} (v^2 - 1)^2 E_{i+1,5} E_{i+2,3} \\
&\quad + v^{-4} (v^2 - 1)^2 E_{i1} E_{i6} E_{i+2,1} E_{i+3,6} + v^{-5} (v^2 - 1)^3 E_{i1} E_{i+1,1} E_{i+1,5} E_{i+2,1}
\end{aligned}$$

$$\begin{aligned}
& + v^{-3}(v^2 - 1)^2 E_{i6} E_{i+2,1} E_{i+2,4} + v^{-4}(v^2 - 1)^3 E_{i+1,2} E_{i+1,5} E_{i+2,1} \\
& + v^{-2}(v^2 - 1) E_{i2} E_{i+2,6} E_{i+3,6} + v^{-1}(v^2 - 1) E_{i2} E_{i+3,3} \\
& + v^{-2}(v^2 - 1)^2 E_{i+1,5} E_{i+1,6} E_{i+2,6} + v^{-3}(v^2 - 1)(v^2 - 2) E_{i1} E_{i+1,4} E_{i+3,6} \\
& + v^{-4}(v^2 - 1)^2 (v^2 - 2) E_{i1} E_{i+1,5} E_{i+2,2} + v^{-2}(v^2 - 1)(v^2 - 2) E_{i+1,4} E_{i+2,4} \\
& + v^{-3}(v^2 - 1)^2 (v^2 - 2) E_{i+1,5} E_{i+2,3} \pmod{S^{+c}; \omega} \\
& \equiv 0 \pmod{S^{+c}; \omega}.
\end{aligned}$$

3. We take $P_{154} = E_{i+3,4} E_{i6} - E_{i6} E_{i+3,4} - v^{-1}(v^2 - 1) E_{i+1,1} E_{i+2,6}$ ($2 \leq i \leq 3$) or, equivalently, $P_{154} = E_{i+4,4} E_{i+1,6} - E_{i+1,6} E_{i+4,4} - v^{-1}(v^2 - 1) E_{i+2,1} E_{i+3,6}$ ($1 \leq i \leq 2$) and

$$P_{82} = E_{i+1,6} E_{i3} - E_{i3} E_{i+1,6} - v^{-1}(v^2 - 1) E_{i2} E_{i4} \quad (1 \leq i \leq 2)$$

where $\omega = E_{i+4,4} E_{i+1,6} E_{i3}$. So

$$\begin{aligned}
(P_{154}, P_{82})_\omega &= P_{154} E_{i3} - E_{i+4,4} P_{82} \\
&\equiv -E_{i+1,6} E_{i+4,4} E_{i3} - v^{-1}(v^2 - 1) E_{i+2,1} E_{i+3,6} E_{i3} \\
&\quad + E_{i+4,4} E_{i3} E_{i+1,6} + v^{-1}(v^2 - 1) E_{i+4,4} E_{i2} E_{i4} \pmod{S^{+c}; \omega} \\
&\equiv -v^{-1} E_{i+1,6} E_{i3} E_{i+4,4} - v^{-1}(v^2 - 2) E_{i+1,6} E_{i+1,4} \\
&\quad - v^{-2}(v^2 - 1) E_{i+1,6} E_{i5} E_{i+1,5} - v^{-2}(v^2 - 1) E_{i+1,6} E_{i6} E_{i+2,1} \\
&\quad - v^{-2}(v^2 - 1) E_{i+1,6} E_{i2} E_{i+3,5} - v^{-2}(v^2 - 1) E_{i+2,1} E_{i3} E_{i+3,6} \\
&\quad - v^{-2}(v^2 - 1)(v^2 - 2) E_{i+2,1} E_{i+1,3} - v^{-3}(v^2 - 1)^2 E_{i+2,1} E_{i2} E_{i+1,1} \\
&\quad - v^{-3}(v^2 - 1)^2 E_{i+2,1} E_{i4} E_{i+1,5} - v^{-3}(v^2 - 1)^2 E_{i+2,1} E_{i6} E_{i+1,6} \\
&\quad + v^{-1} E_{i3} E_{i+4,4} E_{i+1,6} + v^{-1}(v^2 - 2) E_{i+1,4} E_{i+1,6} \\
&\quad + v^{-2}(v^2 - 1) E_{i5} E_{i+1,5} E_{i+1,6} + v^{-2}(v^2 - 1) E_{i6} E_{i+2,1} E_{i+1,6} \\
&\quad + v^{-2}(v^2 - 1) E_{i2} E_{i+3,5} E_{i+1,6} + v^{-1}(v^2 - 1) E_{i2} E_{i+4,4} E_{i4} \\
&\quad + v^{-2}(v^2 - 1)^2 E_{i+1,5} E_{i+2,1} E_{i4} \pmod{S^{+c}; \omega} \\
&\equiv -v^{-1} E_{i3} E_{i+1,6} E_{i+4,4} - v^{-2}(v^2 - 1) E_{i2} E_{i4} E_{i+4,4} \\
&\quad - v^{-1}(v^2 - 2) E_{i+1,4} E_{i+1,6} - v^{-2}(v^2 - 1) E_{i5} E_{i+1,5} E_{i+1,6} \\
&\quad - v^{-3}(v^2 - 1) E_{i6} E_{i+1,6} E_{i+2,1} - v^{-2}(v^2 - 1) E_{i+1,3} E_{i+2,1} \\
&\quad - v^{-1}(v^2 - 1) E_{i2} E_{i+1,6} E_{i+3,5} - v^{-2}(v^2 - 1) E_{i3} E_{i+2,1} E_{i+3,6} \\
&\quad - v^{-3}(v^2 - 1)^2 E_{i2} E_{i5} E_{i+3,6} - v^{-2}(v^2 - 1)(v^2 - 2) E_{i+1,3} E_{i+2,1} \\
&\quad - v^{-3}(v^2 - 1)^2 (v^2 - 2) E_{i+1,4} E_{i+1,6} - v^{-3}(v^2 - 1)^2 E_{i2} E_{i+1,1} E_{i+2,1} \\
&\quad - v^{-2}(v^2 - 1)^2 E_{i2} E_{i+2,2} - v^{-3}(v^2 - 1)^2 E_{i4} E_{i+1,5} E_{i+2,1} \\
&\quad - v^{-4}(v^2 - 1)^3 E_{i5} E_{i+1,5} E_{i+1,6} - v^{-3}(v^2 - 1)^2 E_{i6} E_{i+1,6} E_{i+2,1}
\end{aligned}$$

$$\begin{aligned}
& -v^{-3}(v^2-1)^2 E_{i+1,4} E_{i+1,6} + v^{-1} E_{i3} E_{i+1,6} E_{i+4,4} \\
& + v^{-2}(v^2-1) E_{i3} E_{i+2,1} E_{i+3,6} + v^{-1}(v^2-1) E_{i6} E_{i+1,6} E_{i+2,1} \\
& + v^{-1}(v^2-2) E_{i+1,4} E_{i+1,6} + v^{-2}(v^2-1) E_{i5} E_{i+1,5} E_{i+1,6} \\
& + v^{-3}(v^2-1) E_{i2} E_{i+1,6} E_{i+3,5} + v^{-2}(v^2-1) E_{i2} E_{i+2,2} \\
& + v^{-2}(v^2-1) E_{i2} E_{i4} E_{i+4,4} + v^{-2}(v^2-1)(v^2-2) E_{i2} E_{i+2,2} \\
& + v^{-3}(v^2-1)^2 E_{i2} E_{i+1,1} E_{i+2,1} + v^{-3}(v^2-1)^2 E_{i2} E_{i5} E_{i+3,6} \\
& + v^{-3}(v^2-1)^2 E_{i2} E_{i+1,6} E_{i+3,5} + v^{-3}(v^2-1)^2 E_{i4} E_{i+1,5} E_{i+2,1} \\
& + v^{-2}(v^2-1)^2 E_{i+1,3} E_{i+2,1} + v^{-4}(v^2-1)^3 E_{i5} E_{i+1,5} E_{i+1,6} \\
& + v^{-3}(v^2-1)^3 E_{i+1,4} E_{i+1,6} \pmod{S^{+c}; \omega} \\
& \equiv 0 \pmod{S^{+c}; \omega}.
\end{aligned}$$

4. We take $P_{163} = E_{62} E_{53} - v E_{53} E_{62} - v^{-2}(v^2-1)^2 E_{51} E_{54} E_{56}$ ($i = 5$) and

$$\begin{aligned}
P_{190} &= E_{53} E_{12} - E_{12} E_{53} - v^{-1}(v^2-1) E_{26} E_{42} - v^{-2}(v^2-1)(v^2-2) E_{31} E_{34} \\
&- v^{-3}(v^2-1)^2 E_{11} E_{31} E_{46} - v^{-3}(v^2-1)^2 E_{25} E_{31} E_{35} \quad (i = 1)
\end{aligned}$$

where $\omega = E_{62} E_{53} E_{12}$. So

$$\begin{aligned}
(P_{163}, P_{190})_\omega &= P_{163} E_{12} - E_{62} P_{190} \\
&\equiv -v E_{53} E_{62} E_{12} - v^{-2}(v^2-1)^2 E_{51} E_{54} E_{56} E_{12} + E_{62} E_{12} E_{53} \\
&\quad + v^{-1}(v^2-1) E_{62} E_{26} E_{42} + v^{-2}(v^2-1)(v^2-2) E_{62} E_{31} E_{34} \\
&\quad + v^{-3}(v^2-1)^2 E_{62} E_{11} E_{31} E_{46} + v^{-3}(v^2-1)^2 E_{62} E_{25} E_{31} E_{35} \pmod{S^{+c}; \omega} \\
&\equiv -E_{53} E_{12} E_{62} - v E_{53} E_{31} - v^{-2}(v^2-1)^2 E_{51} E_{54} E_{12} E_{56} \\
&\quad - v^{-3}(v^2-1)^3 E_{51} E_{54} E_{11} E_{31} + v^{-1} E_{12} E_{62} E_{53} \\
&\quad + E_{31} E_{53} + v^{-1}(v^2-1) E_{26} E_{62} E_{42} + v^{-2}(v^2-1)^2 E_{31} E_{51} E_{42} \\
&\quad + v^{-1}(v^2-1)(v^2-2) E_{31} E_{62} E_{34} + v^{-3}(v^2-1)^2 E_{11} E_{31} E_{62} E_{46} \\
&\quad + v^{-3}(v^2-1)^2 E_{56} E_{31} E_{46} + v^{-3}(v^2-1)^2 E_{25} E_{31} E_{62} E_{35} \\
&\quad + v^{-3}(v^2-1)^2 E_{54} E_{31} E_{35} \pmod{S^{+c}; \omega} \\
&\equiv -E_{12} E_{53} E_{62} - v^{-1}(v^2-1) E_{26} E_{42} E_{62} - v^{-2}(v^2-1)(v^2-2) E_{31} E_{34} E_{62} \\
&\quad - v^{-3}(v^2-1)^2 E_{11} E_{31} E_{46} E_{62} - v^{-3}(v^2-1)^2 E_{25} E_{31} E_{35} E_{62} - v^2 E_{31} E_{53} \\
&\quad - v^{-3}(v^2-1)^2 E_{12} E_{51} E_{54} E_{56} - v^{-2}(v^2-1)^2 E_{26} E_{54} E_{56} \\
&\quad - v^{-4}(v^2-1)^3 E_{25} E_{31} E_{51} E_{56} - v^{-3}(v^2-1)^3 E_{31} E_{46} E_{56} \\
&\quad - v^{-4}(v^2-1)^3 E_{11} E_{31} E_{51} E_{54} - v^{-3}(v^2-1)^3 E_{31} E_{35} E_{54}
\end{aligned}$$

$$\begin{aligned}
& -v^{-3}(v^2-1)^3 E_{31}E_{42}E_{51} - v^{-2}(v^2-1)^3 E_{31}E_{53} + E_{12}E_{53}E_{62} \\
& + v^{-3}(v^2-1)^2 E_{12}E_{51}E_{54}E_{56} + E_{31}E_{53} + v^{-1}(v^2-1)E_{26}E_{42}E_{62} \\
& + v^{-2}(v^2-1)^2 E_{26}E_{54}E_{56} + v^{-3}(v^2-1)^2 E_{31}E_{42}E_{51} \\
& + v^{-2}(v^2-1)^2 E_{31}E_{53} + v^{-2}(v^2-1)(v^2-2)E_{31}E_{34}E_{62} \\
& + v^{-2}(v^2-1)(v^2-2)^2 E_{31}E_{53} + v^{-3}(v^2-1)^2(v^2-2)E_{31}E_{35}E_{54} \\
& + v^{-3}(v^2-1)^2(v^2-2)E_{31}E_{42}E_{51} + v^{-3}(v^2-1)^2(v^2-2)E_{31}E_{46}E_{56} \\
& + v^{-3}(v^2-1)^2 E_{11}E_{31}E_{46}E_{62} + v^{-4}(v^2-1)^3 E_{11}E_{31}E_{51}E_{54} \\
& + v^{-3}(v^2-1)^2 E_{31}E_{46}E_{56} + v^{-2}(v^2-1)^2 E_{31}E_{53} \\
& + v^{-3}(v^2-1)^2 E_{25}E_{31}E_{35}E_{62} + v^{-4}(v^2-1)^3 E_{25}E_{31}E_{51}E_{56} \\
& + v^{-3}(v^2-1)^2 E_{31}E_{35}E_{54} + v^{-2}(v^2-1)^2 E_{31}E_{53} \pmod{S^{+c}; \omega} \\
& \equiv 0 \pmod{S^{+c}; \omega}.
\end{aligned}$$

The proof is finished. \square

From [12] we know that each E_{ij} ($1 \leq i, j \leq 6$) can be written as a polynomial of root vectors $E_1, E_2, E_3, E_4, E_5, E_6$ and if we apply the involution ω (see [9]) to these polynomials and denote by F_{ij} the element $\omega(E_{ij})$ in $U_q^-(A)$, then we get a similar relations, say $(R1')$, for the generators in

$$Y = \{F_{ij} \mid 1 \leq i, j \leq 6\},$$

of subalgebra $U_q^-(A)$. It is easy to see that relations $(R1')$ include the Serre relations S^- . So, similarly, if J is the ideal generated by the relations $(R1')$, then the negative part $U_q^-(A)$ of quantum group $U_q(A)$ can be viewed as a factor algebra $\mathbb{Q}(v)\langle Y \rangle / J$ of the free algebra $\mathbb{Q}(v)\langle Y \rangle$ generated by the set Y .

We define an ordering

$$F_{i+1,6} > F_{i+1,5} > F_{i+1,4} > F_{i+1,1} > F_{i+1,2} > F_{i+1,3} > F_{i6} > F_{i5} > F_{i4} > F_{i1} > F_{i2} > F_{i3}$$

for the elements F_{ij} ($1 \leq i \leq 5$), then this ordering induces a degree-lexicographical ordering on the monomials of these elements. Similar to the discussions in the positive part, we denote the polynomials obtained from the relations in $(R1')$ by $Q_{(ij)(kl)}$. Let

$$S^{-c} = \{Q_{(ij)(kl)} \mid 1 \leq i, j, k, l \leq 6\}.$$

Then, clearly, $S^- \subset S^{-c}$ and we have following theorem

Theorem 3.2. *The set S^{-c} is a Gröbner–Shirshov basis of the algebra $U_q^-(A)$.*

If we define an ordering

$$\begin{aligned}
& E_{i+1,6} > E_{i+1,5} > E_{i+1,4} > E_{i+1,1} > E_{i+1,2} > E_{i+1,3} \\
& > E_{i6} > E_{i5} > E_{i4} > E_{i1} > E_{i2} > E_{i3} \\
& > K_1 > K_2 > K_3 > K_4 > K_5 > K_6
\end{aligned}$$

$$\begin{aligned}
 &> F_{i+1,6} > F_{i+1,5} > F_{i+1,4} > F_{i+1,1} > F_{i+1,2} > F_{i+1,3} \\
 &> F_{i6} > F_{i5} > F_{i4} > F_{i1} > F_{i2} > F_{i3}
 \end{aligned}$$

for the elements E_{ij}, F_{ij} ($1 \leq i \leq 5$), $K_1, K_2, K_3, K_4, K_5, K_6$, then this ordering induces a degree-lexicographical ordering on the monomials of these elements.

Now, by Theorem 2.7 in [3], we are able to state our main result

Theorem 3.3. *The set $S^{+c} \cup K \cup T \cup S^{-c}$ is a Gröbner–Shirshov basis of the quantum group $U_q(A)$.*

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