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Algebraic Frobenius splitting of cotangent bundles of flag varieties

Chuck Hague

Department of Mathematical Sciences, University of Delaware, 501 Ewing Hall, Newark, DE 19716, United States

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ABSTRACT

Following the program of algebraic Frobenius splitting begun by Kumar and Littelmann, we use representation-theoretic techniques to construct a Frobenius splitting of the cotangent bundle of the flag variety of a semisimple algebraic group over an algebraically closed field of positive characteristic. We also show that this splitting is the same as one of the splittings constructed by Kumar, Lauritzen, and Thomsen.

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E-mail address: hague@math.udel.edu.

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1. Introduction

1.1. Background

Let G_k be a semisimple, simply-connected algebraic group over an algebraically closed field k of positive characteristic p and let $B_k \subseteq G_k$ be a Borel subgroup. We assume that p is a good prime for G (cf. Definition 2.1). One of the fundamental results of the theory of Frobenius splitting [12] is that the flag variety G_k/B_k is Frobenius split. In the papers [9] and [10], Kumar and Littelmann use the quantum Frobenius morphism and a variant of its splitting, both due to Lusztig [11], to construct an alternate proof of the splitting of G_k/B_k using purely representation-theoretic constructions; they call this an algebraization of Frobenius splitting.

More precisely, Kumar and Littelmann construct morphisms between induced representations for hyperalgebra and quantum group representations. Upon base-change, these morphisms can be identified with morphisms on the structure sheaf $\mathcal{O}_C$ of an affine cone C over G_k/B_k. In particular, the quantum Frobenius morphism induces the p-th power on $\mathcal{O}_C$ and the quantum splitting morphism induces a splitting of the p-th power morphism on $\mathcal{O}_C$. This implies that C is Frobenius split and hence by a process of sheafification that G_k/B_k is Frobenius split as well.

Gros and Kaneda [7] then showed the argument of Kumar–Littelmann can be simplified; in particular, one does not have to go to the level of quantum groups. Instead, all of the constructions of [9] and [10] can be done purely on the level of hyperalgebras. In particular, they construct a morphism φ which is the hyperalgebra version of the quantum splitting morphism. In this paper, we use the constructions in [7] to continue the Kumar–Littelmann program of algebraic Frobenius splitting and give a purely representation-theoretic proof that the cotangent bundle $\mathcal{T}^*$ of G_k/B_k is Frobenius split, a fact which was first proved by geometric means in [8].

One main advantage of using algebraic Frobenius splitting techniques is that one can concretely write down the splitting. In particular, the hope is that using the algebraic method will make it easier to check that certain subvarieties are compatibly split.

1.2. Algebraic Frobenius splitting

Let X be a projective k-variety and let $\mathcal{L}$ be an ample line bundle on X. Set

$$R_{\mathcal{L}} := \bigoplus_{n\geqslant 0} H^0\big(X, \mathcal{L}^n\big), \tag{1.2.1}$$

the affine cone over X corresponding to $\mathcal{L}$. The main fact in algebraic Frobenius splitting (Lemma 1.1.14 in [2]) is that X is Frobenius split if and only if $\operatorname{Spec}(R_{\mathcal{L}})$ is. In turn, $\operatorname{Spec}(R_{\mathcal{L}})$ is Frobenius split if and only if $R_{\mathcal{L}}$ is a Frobenius split k-algebra: i.e., there exists an $\mathbb{F}_p$-linear endomorphism s of $R_{\mathcal{L}}$ such that (1) $s(f^p g) = f \cdot s(g)$ for all $f, g \in R_{\mathcal{L}}$ (this is called **Frobenius-linearity** of s) and (2) $s(f^p) = f$ for all $f \in R_{\mathcal{L}}$.

We now apply these ideas to the case $X = \mathbb{P}(\mathcal{T}^*)$, the projectivization of the cotangent bundle $\mathcal{T}^*$. Let $U_k \subseteq B_k$ be the unipotent radical of B_k and let U_k^- be the opposite unipotent radical. Let $pr : \mathcal{T}^* \to G_k/B_k$ be the projection and set $F_k := pr^{-1}(U_k^- B_k) \subseteq \mathcal{T}^*$, the fiber over the big cell $U_k^- B_k \subseteq G_k/B_k$. Then F_k is an affine subvariety of $\mathcal{T}^*$ isomorphic to $U_k^- \times U_k$.

Let G be a split form of G_k over $\mathbb{F}_p$. We first construct, for any weight λ of G, a polynomial ring $R_\lambda^{\mathfrak{h}}$ over $\mathbb{F}_p$ such that $R_\lambda^{\mathfrak{h}} \otimes_{\mathbb{F}_p} k \cong k[F_k]$. This ring carries an action of the hyperalgebra of G; taking the locally finite part gives a ring R_λ. When λ is a regular dominant weight, $R_\lambda \otimes_{\mathbb{F}_p} k$ is isomorphic to $R_{\mathcal{L}}$ for a very ample bundle $\mathcal{L}$ on $\mathbb{P}(\mathcal{T}^*)$. Further, upon base-change to k the natural inclusion $R_\lambda \hookrightarrow R_\lambda^{\mathfrak{h}}$ corresponds to the inclusion $R_{\mathcal{L}} \hookrightarrow k[F_k]$.

Now, since $\mathbb{P}(\mathcal{T}^*)$ is split if and only if $\mathcal{T}^*$ is, it suffices to construct a splitting of the k-algebra $R_{\mathcal{L}}$. To this end, we first work over $\mathbb{F}_p$ and construct a splitting $\widetilde{S}$ of $R_\lambda^{\mathfrak{h}}$ that restricts to a splitting of the subalgebra R_λ. Upon base-change, this induces a splitting of $R_{\mathcal{L}}$. Geometrically, this corresponds to a splitting of the ring $k[F_k]$ (or, equivalently, a splitting of the affine scheme F_k) that restricts to a splitting of the subring $R_{\mathcal{L}}$.

1.3. Details

We now give more details on the construction of the rings $R_\lambda^{\mathfrak{h}}$ and R_λ and the splitting morphism $\widetilde{S}$. As above let G be a split form of the group G_k over $\mathbb{F}_p$ and let $T \subseteq G$ be a split maximal torus. Let $B \subseteq G$ be a Borel subgroup of G containing T. Let B^- denote the opposite Borel subgroup. Let $U \subseteq B$ and $U^- \subseteq B^-$ be the respective unipotent radicals. We consider the root spaces of B to correspond to the positive roots. Let Λ denote the weight lattice of T.

Let $\bar{\mathrm{U}}(\mathfrak{n})$ denote the hyperalgebra of U. The torus-locally finite part $\bar{\mathrm{U}}(\mathfrak{n})^\vee$ of the full linear dual of $\bar{\mathrm{U}}(\mathfrak{n})$ is naturally isomorphic to $\mathbb{F}_p[U]$, the coordinate ring of U. Set $\mathfrak{n} := \mathrm{Lie}(U)$; then a Springer isomorphism $U \xrightarrow{\sim} \mathfrak{n}$ induces a B-equivariant isomorphism $\mathbb{F}_p[U] \xrightarrow{\sim} \mathbb{F}_p[\mathfrak{n}]$ and hence a B-equivariant isomorphism $\bar{\mathrm{U}}(\mathfrak{n})^\vee \xrightarrow{\sim} \mathbb{F}_p[\mathfrak{n}]$. Since $\mathbb{F}_p[\mathfrak{n}]$ has a natural B-equivariant grading by polynomial degree, we obtain a B-equivariant grading $\bar{\mathrm{U}}_n(\mathfrak{n})^\vee$ on $\bar{\mathrm{U}}(\mathfrak{n})^\vee$.

In Section 2.2 we construct, for each $\lambda \in \Lambda$, the $\mathbb{F}_p$-algebras $R_\lambda^{\mathfrak{h}}$ and R_λ. These rings are defined by inducing (twists of) the B-modules $\bar{\mathrm{U}}_n(\mathfrak{n})^\vee$ to $\bar{\mathrm{U}}(\mathfrak{g})$-modules. We can interpret this construction in the following way. The rings $R_\lambda^{\mathfrak{h}}$ are all isomorphic to polynomial rings (cf. the proof of Proposition 3.4 below). In particular they are all naturally isomorphic to the ring of functions on $U^- \times U$. Base-changing to k, $R_\lambda^{\mathfrak{h}} \otimes_{\mathbb{F}_p} k$ is isomorphic to the ring of functions on the affine space F_k defined above. Different choices of $\lambda \in \Lambda$ give rise to different $\bar{\mathrm{U}}(\mathfrak{g})$-algebra structures on this polynomial ring, so the rings $R_\lambda^{\mathfrak{h}}$ give a family of $\bar{U}_k(\mathfrak{g})$-module structures on $k[F_k] \cong k[U_k^-] \otimes k[U_k]$, where $\bar{U}_k(\mathfrak{g})$ is the hyperalgebra of G_k. Taking the $\bar{\mathrm{U}}(\mathfrak{g})$-locally finite part of $R_\lambda^{\mathfrak{h}}$ gives the ring R_λ. Remark that the rings R_λ are *not* all isomorphic for various choices of $\lambda \in \Lambda$.

Motivated by [8], the splitting $\widetilde{S}$ of $R_\lambda^{\mathfrak{h}}$ is constructed via the trace methodology described as follows. Given a polynomial ring P and a choice of algebra generators of P there is a Frobenius-linear trace morphism Tr on P, and every Frobenius-linear endomorphism of P is of the form

$$f \mapsto \mathrm{Tr}(f \cdot g) \tag{1.3.1}$$

for some fixed $g \in P$. If $Q \subseteq P$ is a subring we can look for $q \in Q$ such that (1) $\mathrm{Tr}(f \cdot q) \in Q$ for all $f \in Q$ and (2) $\mathrm{Tr}(- \cdot q)$ is a Frobenius splitting of P. This will give a Frobenius splitting of the ring Q.

In particular, since $R_\lambda^{\mathfrak{h}}$ is a polynomial ring we have a Frobenius-linear trace map Tr on $R_\lambda^{\mathfrak{h}}$ corresponding to an appropriate choice of $\mathbb{F}_p$-algebra generators of $R_\lambda^{\mathfrak{h}}$ (cf. Section 2.7). We apply the trace methodology to the subring $R_\lambda \subseteq R_\lambda^{\mathfrak{h}}$. In these constructions we first work over $\mathbb{F}_p$ and then base-change to k later.

In Section 2.4 we construct, using representation-theoretic techniques, a Frobenius-linear endomorphism S of $R_\lambda^{\mathfrak{h}}$ which turns out (Section 2.7) to be the same as the trace morphism Tr. In Section 2.5 we construct an element $\psi_{f_+ \otimes f_-} \in R_\lambda$ for $\lambda = 0$ and in Section 2.6 we show that the Frobenius-linear endomorphism

$$\widetilde{S} : f \mapsto S(\psi_{f_+ \otimes f_-} \cdot f) \tag{1.3.2}$$

of $R_\lambda^{\mathfrak{h}}$ is a Frobenius splitting that preserves R_λ. In particular, $\widetilde{S}$ restricts to a Frobenius splitting of R_λ as desired. (Remark that below we write $M_{f_+ \otimes f_-}$ for multiplication by $\psi_{f_+ \otimes f_-}$ and hence, concisely, $\widetilde{S} = S \circ M_{f_+ \otimes f_-}$.)

In Section 3 we base-change to k and construct the desired splitting of $\mathbb{P}(\mathcal{T}^*)$ and hence obtain a splitting of $\mathcal{T}^*$. We also show that this splitting is the same as one of the homogeneous splittings of $\mathcal{T}^*$ in [8].

2. Algebraic splitting

2.1. Setup

Throughout Section 2 we assume all algebraic groups, algebras, schemes, vector spaces, etc. are over $\mathbb{F}_p$. Recall the groups G, B, U, T, etc. from above.

2.1.1.

Definition 2.1. We say that a prime p is **bad** for a simple algebraic group G in the following cases. If G is of type A_ℓ then no prime is bad; if G is of type B_ℓ, C_ℓ, or D_ℓ then $p=2$ is bad; if G is of type E_6, E_7, F_4, or G_2 then $p=2,3$ are bad; and if G is of type E_8 then $p=2,3,5$ are bad. We say that p is a bad prime for a semisimple algebraic group G if it is bad for any of its simple components, and we say that p is a **good** prime for G if it is not bad.

From here on we assume that p is a good prime for G.

For an algebraic group H over $\mathbb{F}_p$ let $I \subseteq \mathbb{F}_p[H]$ denote the ideal of the identity element. The subspace of the linear dual of $\mathbb{F}_p[H]$ consisting of elements that vanish on some power of I is called the hyperalgebra of H; it has a natural Hopf algebra structure obtained from the Hopf algebra structure on $\mathbb{F}_p[H]$. Let $\bar{\mathrm{U}}(\mathfrak{g})$, $\bar{\mathrm{U}}(\mathfrak{b})$, $\bar{\mathrm{U}}(\mathfrak{b}^-)$, $\bar{\mathrm{U}}(\mathfrak{n})$, $\bar{\mathrm{U}}(\mathfrak{n}^-)$, and $\bar{\mathrm{U}}^0$ denote the hyperalgebras of G, B, B^-, U, U^-, and T, respectively.

The Frobenius morphism $\mathbb{F}_p[G] \to \mathbb{F}_p[G]$, $f \mapsto f^p$ induces a morphism $\mathrm{Fr} : \bar{\mathrm{U}}(\mathfrak{g}) \to \bar{\mathrm{U}}(\mathfrak{g})$ of $\mathbb{F}_p$-algebras. We will denote the restriction of Fr to $\bar{\mathrm{U}}(\mathfrak{b})$, $\bar{\mathrm{U}}(\mathfrak{n})$, etc. by Fr as well. Let ℓ denote the rank of G. $\bar{\mathrm{U}}(\mathfrak{g})$ is generated by elements $E_i^{(n)} \in \bar{\mathrm{U}}(\mathfrak{n})$, $F_i^{(n)} \in \bar{\mathrm{U}}(\mathfrak{n}^-)$, and $\binom{H_i}{n} \in \bar{\mathrm{U}}^0$ for $n \geqslant 0$ and $1 \leqslant i \leqslant \ell$. On these generators, we have

$$\mathrm{Fr}\big(E_i^{(n)}\big) = \begin{cases} E_i^{(n/p)} & \text{if } p \mid n, \\ 0 & \text{if } p \nmid n, \end{cases} \tag{2.1.1a}$$

$$\mathrm{Fr}\big(F_i^{(n)}\big) = \begin{cases} F_i^{(n/p)} & \text{if } p \mid n, \\ 0 & \text{if } p \nmid n \end{cases} \tag{2.1.1b}$$

and

$$\mathrm{Fr}\binom{H_i}{n} = \begin{cases} \binom{H_i}{n/p} & \text{if } p \mid n, \\ 0 & \text{if } p \nmid n. \end{cases} \tag{2.1.1c}$$

2.1.2. By [9] and [11] we have $\mathbb{F}_p$-algebra morphisms $\mathrm{Fr}' : \bar{\mathrm{U}}(\mathfrak{n}) \to \bar{\mathrm{U}}(\mathfrak{n})$, $\mathrm{Fr}'^- : \bar{\mathrm{U}}(\mathfrak{n}^-) \to \bar{\mathrm{U}}(\mathfrak{n}^-)$, and $\mathrm{Fr}'_0 : \bar{\mathrm{U}}^0 \to \bar{\mathrm{U}}^0$ given by

$$\mathrm{Fr}'\big(E_i^{(n)}\big) = E_i^{(pn)}, \tag{2.1.2a}$$

$$\mathrm{Fr}'^-\big(F_i^{(n)}\big) = F_i^{(pn)}, \tag{2.1.2b}$$

and

$$\mathrm{Fr}'_0\binom{H_i}{n} = \binom{H_i}{pn} \tag{2.1.2c}$$

for all $1 \leqslant i \leqslant \ell$ and $n \geqslant 0$.

Set

$$\mu_0 := \prod_{i=1}^{\ell} \binom{H_i - 1}{p-1} = \prod_{i=1}^{\ell} \bigl(1 - H_i^{p-1}\bigr), \tag{2.1.3}$$

an idempotent in $\bar{\mathrm{U}}^0$. By [7, Theorem 1.4], there is a multiplicative morphism

$$\varphi : \bar{\mathrm{U}}(\mathfrak{g}) \to \bar{\mathrm{U}}(\mathfrak{g}) \tag{2.1.4a}$$

given by

$$\varphi(YHX) = \mathrm{Fr}'^{-} Y \cdot \mathrm{Fr}'_0 H \cdot \mathrm{Fr}' X \cdot \mu_0 \tag{2.1.4b}$$

for all $Y \in \bar{\mathrm{U}}(\mathfrak{n}^-)$, $H \in \bar{\mathrm{U}}^0$, and $X \in \bar{\mathrm{U}}(\mathfrak{n})$. Further, μ_0 commutes with all elements in the image of φ, so if we consider $\operatorname{im}\varphi$ as an $\mathbb{F}_p$-algebra with unit μ_0, then φ is an $\mathbb{F}_p$-algebra morphism.

Note that

$$\mathrm{Fr}(H_i) = \mathrm{Fr}\binom{H_i}{1} = 0.$$

Hence $\mathrm{Fr}(H_i^{p-1}) = 0$ which implies $\mathrm{Fr}(\mu_0) = 1$, and we have the following important fact:

$$\mathrm{Fr} \circ \varphi = \mathrm{Id}_{\bar{\mathrm{U}}(\mathfrak{g})}. \tag{2.1.5}$$

Let Λ denote the weight lattice of G. For $\lambda \in \Lambda$ let $c_\lambda : \bar{\mathrm{U}}^0 \to \mathbb{F}_p$ be the character associated to λ. We have the following result from [7].

Lemma 2.2. *(See Lemme 2.1 in [7].) For all $\lambda \in \Lambda$ we have*

$$c_\lambda \circ \varphi|_{\bar{\mathrm{U}}^0} = \begin{cases} c_{\lambda/p} & \text{if } \lambda \in p\Lambda, \\ 0 & \text{if } \lambda \notin p\lambda. \end{cases} \tag{2.1.6a}$$

In particular,

$$c_\lambda(\mu_0) = \begin{cases} 1 & \text{if } \lambda \in p\Lambda, \\ 0 & \text{if } \lambda \notin p\lambda. \end{cases} \tag{2.1.6b}$$

2.2. Algebraic constructions and preliminaries

2.2.1. For a Hopf algebra with comultiplication Δ we use the Sweedler notation

$$\Delta X = \sum X_{(1)} \otimes X_{(2)},$$
$$\bigl((\Delta \otimes \mathrm{Id}) \circ \Delta\bigr)(X) = \sum X_{(1)} \otimes X_{(2)} \otimes X_{(3)},$$

etc. Let ϵ and σ denote the augmentation and coinverse of $\bar{\mathrm{U}}(\mathfrak{g})$, respectively. By a slight abuse of notation we will also use the same notation for the various sub-Hopf algebras $\bar{\mathrm{U}}(\mathfrak{n})$, $\bar{\mathrm{U}}(\mathfrak{n}^-)$, etc. of $\bar{\mathrm{U}}(\mathfrak{g})$.

For any $\bar{\mathrm{U}}^0$-module V (resp. $\bar{\mathrm{U}}(\mathfrak{g})$-module W) let $F_{\mathfrak{h}} V$ (resp. $F_{\mathfrak{g}} W$) denote the $\bar{\mathrm{U}}^0$ (resp. $\bar{\mathrm{U}}(\mathfrak{g})$) locally finite part of V (resp. W). Also set $V^\vee := F_{\mathfrak{h}} V^*$. If V is a module for $\bar{\mathrm{U}}(\mathfrak{g})$, $\bar{\mathrm{U}}(\mathfrak{b})$, or $\bar{\mathrm{U}}(\mathfrak{b}^-)$ then so is $V^\vee$.

Recall that for a Hopf algebra H and algebra A, we say that A is an H**-module algebra** if A is an H-module and

$$h.(ab) = \sum (h_{(1)}.a) \cdot (h_{(2)}.b) \tag{2.2.1}$$

for all $h \in H$ and $a, b \in A$.

We have the conjugation (or adjoint) $\bar{\mathrm{U}}(\mathfrak{b})$-action on $\bar{\mathrm{U}}(\mathfrak{n})$ given by

$$X * Y = \sum X_{(1)} Y \sigma(X_{(2)}), \tag{2.2.2}$$

where σ is the coinverse. This action induces a dual action of $\bar{\mathrm{U}}(\mathfrak{b})$ on $\bar{\mathrm{U}}(\mathfrak{n})^\vee$, also denoted by $*$. Under the adjoint action, $\bar{\mathrm{U}}(\mathfrak{n})$ and $\bar{\mathrm{U}}(\mathfrak{n})^\vee$ become $\bar{\mathrm{U}}(\mathfrak{b})$-module algebras. From here on, we consider $\bar{\mathrm{U}}(\mathfrak{n})$ as a $\bar{\mathrm{U}}(\mathfrak{b})$-module under the $*$-action.

There is a duality pairing between $\mathbb{F}_p[U]$ and $\bar{\mathrm{U}}(\mathfrak{n})$ which defines the Hopf algebra structure on $\bar{\mathrm{U}}(\mathfrak{n})$ (cf. Section I.7 in [6]). There is a natural Hopf algebra structure on $\bar{\mathrm{U}}(\mathfrak{n})^\vee$ obtained from duality with $\bar{\mathrm{U}}(\mathfrak{n})$ and hence a Hopf algebra isomorphism $\mathbb{F}_p[U] \cong \bar{\mathrm{U}}(\mathfrak{n})^\vee$. This is also an isomorphism of $\bar{\mathrm{U}}(\mathfrak{b})$-module algebras, where we take the $\bar{\mathrm{U}}(\mathfrak{b})$-action on $\mathbb{F}_p[U]$ induced by the conjugation action of B on U.

2.2.2. Recall that we are assuming that p is a good prime for G. By [14, Proposition 3.5], there is a B-equivariant Springer isomorphism $U \cong \mathfrak{n}$ which intertwines the conjugation B-action on U with the standard B-action on $\mathfrak{n}$. (There are in fact infinitely many Springer isomorphisms, so let us fix any one of them.) Thus we obtain isomorphisms of $\bar{\mathrm{U}}(\mathfrak{b})$-module algebras

$$\bar{\mathrm{U}}(\mathfrak{n})^\vee \cong \mathbb{F}_p[U] \cong \mathbb{F}_p[\mathfrak{n}] \cong S\big(\mathfrak{n}^*\big). \tag{2.2.3}$$

As $S(\mathfrak{n}^*)$ has a natural $\bar{\mathrm{U}}(\mathfrak{b})$-equivariant algebra grading, this induces a $\bar{\mathrm{U}}(\mathfrak{b})$-equivariant multiplicative grading $\bar{\mathrm{U}}_n(\mathfrak{n})^\vee$ on $\bar{\mathrm{U}}(\mathfrak{n})^\vee$. Dually, we obtain a $\bar{\mathrm{U}}(\mathfrak{b})$-equivariant grading $\bar{\mathrm{U}}_n(\mathfrak{n})$ on $\bar{\mathrm{U}}(\mathfrak{n})$ such that the comultiplication $\Delta : \bar{\mathrm{U}}(\mathfrak{n}) \to \bar{\mathrm{U}}(\mathfrak{n}) \otimes \bar{\mathrm{U}}(\mathfrak{n})$ is gradation-preserving under the induced grading on $\bar{\mathrm{U}}(\mathfrak{n}) \otimes \bar{\mathrm{U}}(\mathfrak{n})$.

Remark 2.3. For all of the proofs below, we only use the fact that there is a $\bar{\mathrm{U}}(\mathfrak{b})$-module algebra isomorphism $\bar{\mathrm{U}}(\mathfrak{n})^\vee \cong S(\mathfrak{n}^*)$; hence we could use any such isomorphism. In particular, instead of a Springer isomorphism, we could use the isomorphism constructed in [4]. Different choices of isomorphisms may, however, result in different splittings.

2.2.3. Induction functors and duality

Let M be a B-module. Then $\mathrm{Hom}_{\bar{\mathrm{U}}(\mathfrak{b})}(\bar{\mathrm{U}}(\mathfrak{g}), M)$ has a $\bar{\mathrm{U}}(\mathfrak{g})$-module structure given by

$$(Y.f)(X) = f(XY) \quad \text{for all } X, Y \in \bar{\mathrm{U}}(\mathfrak{g}) \text{ and } f \in \mathrm{Hom}_{\bar{\mathrm{U}}(\mathfrak{b})}\big(\bar{\mathrm{U}}(\mathfrak{g}), M\big). \tag{2.2.4}$$

For any B-module M set

$$H^0(\bar{X}, M) := F_{\mathfrak{g}} \,\mathrm{Hom}_{\bar{\mathrm{U}}(\mathfrak{b})}\big(\bar{\mathrm{U}}(\mathfrak{g}), M\big) \tag{2.2.5a}$$

and

$$H^0_{\mathfrak{h}}(\bar{X}, M) := F_{\mathfrak{h}} \,\mathrm{Hom}_{\bar{\mathrm{U}}(\mathfrak{b})}\big(\bar{\mathrm{U}}(\mathfrak{g}), M\big). \tag{2.2.5b}$$

Note that we have inclusions of $\bar{\mathrm{U}}(\mathfrak{g})$-modules

$$H^0(\bar{X}, M) \subseteq H^0_{\mathfrak{h}}(\bar{X}, M) \subseteq \mathrm{Hom}_{\bar{\mathrm{U}}(\mathfrak{b})}\big(\bar{\mathrm{U}}(\mathfrak{g}), M\big).$$

We will frequently use the following fact. For any $\bar{\mathrm{U}}^0$-locally finite $\bar{\mathrm{U}}(\mathfrak{b})$-module M we have $\bar{\mathrm{U}}^0$-module isomorphisms

$$H^0_{\mathfrak{h}}(\bar{X}, M) \cong F_{\mathfrak{h}} \operatorname{Hom}_{\mathbb{F}_p}\big(\bar{\mathrm{U}}\big(\mathfrak{n}^-\big), M\big) \cong \bar{\mathrm{U}}\big(\mathfrak{n}^-\big)^\vee \otimes M. \tag{2.2.6}$$

2.2.4. Consider the group algebra $\mathbb{F}_p[\Lambda]$ of the lattice Λ; then $\mathbb{F}_p[\Lambda]$ is naturally a $\bar{\mathrm{U}}^0$-module algebra. We make it into a $\bar{\mathrm{U}}(\mathfrak{b})$-module algebra by giving it a trivial $\bar{\mathrm{U}}(\mathfrak{n})$-action. For each $\lambda \in \Lambda$ let $v_\lambda \in \mathbb{F}_p[\Lambda]$ denote the element corresponding to λ. Then, in particular, we have

$$v_\lambda \cdot v_\mu = v_{\lambda+\mu} \tag{2.2.7}$$

for all $\lambda, \mu \in \Lambda$. We also identify $\mathbb{F}_p.v_0$ with $\mathbb{F}_p$ via the basis element v_0. This induces a bilinear pairing

$$\mathbb{F}_p.v_\lambda \otimes \mathbb{F}_p.v_{-\lambda} \to \mathbb{F}_p.v_0 \to \mathbb{F}_p \tag{2.2.8}$$

for all $\lambda \in \Lambda$.

For $\lambda \in \Lambda$ let χ_λ denote the 1-dimensional $\bar{\mathrm{U}}(\mathfrak{b})$-module corresponding to the character λ of $\bar{\mathrm{U}}^0$ and set

$$H^0(\lambda) := H^0(\bar{X}, \chi_{-\lambda}), \tag{2.2.9}$$

the induced G-module with lowest weight $-\lambda$. In the sequel we will freely identify χ_λ with $\mathbb{F}_p.v_\lambda \subseteq \mathbb{F}_p[\Lambda]$.

Lemma 2.4. *Choose $\lambda \in \Lambda$. There is a natural $\bar{\mathrm{U}}(\mathfrak{g})$-equivariant inclusion*

$$H^0_{\mathfrak{h}}\big(\bar{X}, \bar{\mathrm{U}}(\mathfrak{n})^\vee \otimes \chi_{-\lambda}\big) \hookrightarrow \big(\bar{\mathrm{U}}(\mathfrak{g}) \otimes \bar{\mathrm{U}}(\mathfrak{n}) \otimes \chi_\lambda\big)^*, \tag{2.2.10}$$

where the $\bar{\mathrm{U}}(\mathfrak{g})$-action on $(\bar{\mathrm{U}}(\mathfrak{g}) \otimes \bar{\mathrm{U}}(\mathfrak{n}) \otimes \chi_\lambda)^*$ is given by

$$(Z.f)(X \otimes Y \otimes v_\lambda) = f(XZ \otimes Y \otimes v_\lambda) \tag{2.2.11}$$

for all $X, Z \in \bar{\mathrm{U}}(\mathfrak{g})$ and $Y \in \bar{\mathrm{U}}(\mathfrak{n})$.

Further, the image of the inclusion (2.2.10) consists of the $\bar{\mathrm{U}}^0$-locally finite $f \in (\bar{\mathrm{U}}(\mathfrak{g}) \otimes \bar{\mathrm{U}}(\mathfrak{n}) \otimes \chi_\lambda)^*$ such that

$$f(AX \otimes Y \otimes v_\lambda) = f\big(X \otimes \sigma A * (Y \otimes v_\lambda)\big) \tag{2.2.12}$$

for all $A \in \bar{\mathrm{U}}(\mathfrak{b})$.

Proof. From (2.2.8) we can naturally identify $\chi_{-\lambda}$ with χ_λ^*. Hence for $f \in H^0_{\mathfrak{h}}(\bar{X}, \bar{\mathrm{U}}(\mathfrak{n})^\vee \otimes \chi_{-\lambda})$ and $X \in \bar{\mathrm{U}}(\mathfrak{g})$ we can consider $f(X)$ as an element of $(\bar{\mathrm{U}}(\mathfrak{n}) \otimes \chi_\lambda)^*$. We define the inclusion (2.2.10), denoted by θ, as follows: for $f \in H^0_{\mathfrak{h}}(\bar{X}, \bar{\mathrm{U}}(\mathfrak{n})^\vee \otimes \chi_{-\lambda})$, $X \in \bar{\mathrm{U}}(\mathfrak{g})$, and $Y \in \bar{\mathrm{U}}(\mathfrak{n})$ set

$$\theta(f)(X \otimes Y \otimes v_\lambda) = f(X)(Y \otimes v_\lambda). \tag{2.2.13}$$

The rest of the statements in the lemma are now straightforward to verify. $\square$

In the sequel, for ease of computation we will frequently use this lemma to identify $H^0_{\mathfrak{h}}(\bar{X}, \bar{\mathrm{U}}(\mathfrak{n})^\vee \otimes \chi_{-\lambda})$ with its image under the inclusion (2.2.10). Remark that (2.2.12) is just the statement that f is $\bar{\mathrm{U}}(\mathfrak{b})$-linear.

2.2.5. The algebras $R_\lambda^{\mathfrak{h}}$ and R_λ

For any $\mu, \lambda \in \Lambda$ we have (using the identification (2.2.10) above) a $\bar{\mathrm{U}}(\mathfrak{g})$-equivariant multiplication map

$$H^0_{\mathfrak{h}}\big(\bar{X}, \bar{\mathrm{U}}(\mathfrak{n})^\vee \otimes \chi_{-\mu}\big) \otimes H^0_{\mathfrak{h}}\big(\bar{X}, \bar{\mathrm{U}}(\mathfrak{n})^\vee \otimes \chi_{-\lambda}\big) \to H^0_{\mathfrak{h}}\big(\bar{X}, \bar{\mathrm{U}}(\mathfrak{n})^\vee \otimes \chi_{-\mu-\lambda}\big) \tag{2.2.14a}$$

given by

$$(f \cdot g)(X \otimes Y \otimes v_{\mu+\lambda}) = \sum f(X_{(1)} \otimes Y_{(1)} \otimes v_\mu) \cdot g(X_{(2)} \otimes Y_{(2)} \otimes v_\lambda). \tag{2.2.14b}$$

Since comultiplication in $\bar{\mathrm{U}}(\mathfrak{n})$ preserves the gradation, the multiplication map (2.2.14a) restricts to a degree-preserving map

$$H^0_{\mathfrak{h}}\big(\bar{X}, \bar{\mathrm{U}}_n(\mathfrak{n})^\vee \otimes \chi_{\mu}\big) \otimes H^0_{\mathfrak{h}}\big(\bar{X}, \bar{\mathrm{U}}_m(\mathfrak{n})^\vee \otimes \chi_{\lambda}\big) \to H^0_{\mathfrak{h}}\big(\bar{X}, \bar{\mathrm{U}}_{n+m}(\mathfrak{n})^\vee \otimes \chi_{\mu+\lambda}\big) \tag{2.2.14c}$$

for all $n, m \geqslant 0$.

For $\lambda \in \Lambda$ set

$$R_\lambda^{\mathfrak{h}} := \bigoplus_{n \geqslant 0} H^0_{\mathfrak{h}}\big(\bar{X}, \bar{\mathrm{U}}_n(\mathfrak{n})^\vee \otimes \chi_{-n\lambda}\big). \tag{2.2.15a}$$

By the above, $R_\lambda^{\mathfrak{h}}$ is a $\bar{\mathrm{U}}(\mathfrak{g})$-module algebra. Also set

$$R_\lambda := F_{\mathfrak{g}} R_\lambda^{\mathfrak{h}} = \bigoplus_{n \geqslant 0} H^0\big(\bar{X}, \bar{\mathrm{U}}_n(\mathfrak{n})^\vee \otimes \chi_{-n\lambda}\big). \tag{2.2.15b}$$

Since multiplication is $\bar{\mathrm{U}}(\mathfrak{g})$-equivariant, R_λ is a $\bar{\mathrm{U}}(\mathfrak{g})$-module subalgebra of $R_\lambda^{\mathfrak{h}}$.

Remark 2.5. Note that by (2.2.6) we have a natural $\mathbb{F}_p$-algebra inclusion

$$R_\lambda^{\mathfrak{h}} \hookrightarrow \bar{\mathrm{U}}\big(\mathfrak{n}^-\big)^\vee \otimes \bar{\mathrm{U}}(\mathfrak{n})^\vee \otimes \mathbb{F}_p[\Lambda] \tag{2.2.16}$$

for all $\lambda \in \Lambda$.

2.3. The p-th power morphism $\widetilde{\mathrm{Fr}}^*$

2.3.1. Recall the morphism Fr from Section 2.1.1. Let Fr^* (resp. Fr^{*-}) be the endomorphism of $\bar{\mathrm{U}}(\mathfrak{n})^\vee$ (resp. $\bar{\mathrm{U}}(\mathfrak{n}^-)^\vee$) dual to the endomorphism Fr of $\bar{\mathrm{U}}(\mathfrak{n})$ (resp. $\bar{\mathrm{U}}(\mathfrak{n}^-)$). Note that since Fr is a Hopf algebra morphism, so are Fr^* and Fr^{*-}.

Lemma 2.6. Fr^* (*resp.* Fr^{*-}) *is the p-th power morphism on* $\bar{\mathrm{U}}(\mathfrak{n})^\vee$ (*resp.* $\bar{\mathrm{U}}(\mathfrak{n}^-)^\vee$).

Proof. By definition, Fr is dual to the p-th power morphism on $\mathbb{F}_p[U]$. Since $\bar{\mathrm{U}}(\mathfrak{n})^\vee \cong \mathbb{F}_p[U]$ as $\mathbb{F}_p$-algebras (cf. (2.2.3) above), we have that Fr^* is the p-th power map on $\bar{\mathrm{U}}(\mathfrak{n})^\vee$. The statement about Fr^{*-} is proved similarly. □

2.3.2. Choose $\lambda \in \Lambda$. Since Fr^* is the p-th power morphism on $\bar{\mathrm{U}}(\mathfrak{n})^\vee$ it sends $\bar{\mathrm{U}}_n(\mathfrak{n})^\vee$ to $\bar{\mathrm{U}}_{pn}(\mathfrak{n})^\vee$ and we have an endomorphism $\widetilde{\mathrm{Fr}}^*$ of $R_\lambda^{\mathfrak{h}}$ given by the direct sum of the morphisms

$$H^0_{\mathfrak{h}}\big(\bar{X}, \bar{\mathrm{U}}_n(\mathfrak{n})^\vee \otimes \chi_{-n\lambda}\big) \to H^0_{\mathfrak{h}}\big(\bar{X}, \bar{\mathrm{U}}_{pn}(\mathfrak{n})^\vee \otimes \chi_{-pn\lambda}\big),$$
$$\big(\widetilde{\mathrm{Fr}}^* f\big)(X \otimes Y \otimes v_{pn\lambda}) = f(\mathrm{Fr}\, X \otimes \mathrm{Fr}\, Y \otimes v_{n\lambda}) \tag{2.3.1}$$

for all $X \in \bar{\mathrm{U}}(\mathfrak{g})$ and $Y \in \bar{\mathrm{U}}(\mathfrak{n})$.

Proposition 2.7. *$\widetilde{\mathrm{Fr}}^*$ is the p-th power morphism on $R^{\mathfrak{h}}_\lambda$ (and hence restricts to the p-th power morphism on R_λ).*

Proof. There are natural algebra isomorphisms

$$R^{\mathfrak{h}}_\lambda \cong \bigoplus_{n\geqslant 0} \bar{\mathrm{U}}(\mathfrak{g})^\vee \otimes_{\bar{\mathrm{U}}(\mathfrak{b})} \big(\bar{\mathrm{U}}_n(\mathfrak{n})^\vee \otimes \chi_{-n\lambda}\big) \cong \bigoplus_{n\geqslant 0} \bar{\mathrm{U}}\big(\mathfrak{n}^-\big)^\vee \otimes \bar{\mathrm{U}}_n(\mathfrak{n})^\vee \otimes \chi_{-n\lambda}. \tag{2.3.2}$$

The algebra structure on the ring on the right-hand side of (2.3.2) is induced from the algebra structure on $\bar{\mathrm{U}}(\mathfrak{n}^-)^\vee \otimes \bar{\mathrm{U}}(\mathfrak{n})^\vee$, so it suffices to verify that the endomorphism $\mathrm{Fr}^{*-} \otimes \mathrm{Fr}^*$ of $\bar{\mathrm{U}}(\mathfrak{n}^-)^\vee \otimes \bar{\mathrm{U}}(\mathfrak{n})^\vee$ is the p-th power morphism. But this is clear by Lemma 2.6. □

2.4. The morphism S

2.4.1. The small hyperalgebras

Set $E_0 := \prod_{\beta\in\Delta^+} E_\beta^{(p-1)}$ and $F_0 := \prod_{\beta\in\Delta^+} F_\beta^{(p-1)}$. By [5, Proposition 6.7], E_0 and F_0 are independent of the ordering of the roots. Let ρ denote the half-sum of the positive roots; then E_0 (resp. F_0) has weight $2(p-1)\rho$ (resp. $-2(p-1)\rho$).

Let $\bar{\mathrm{u}}(\mathfrak{n})$ denote the "small" hyperalgebra associated to U, i.e. the sub-Hopf algebra of $\bar{\mathrm{U}}(\mathfrak{n})$ generated by $\prod_{\beta\in\Delta^+} E_\beta^{(m_\beta)}$ for $0 \leqslant m_\beta < p$ (where we take any fixed ordering of Δ^+). Similarly, we have the sub-Hopf algebra $\bar{\mathrm{u}}(\mathfrak{n}^-)$ of $\bar{\mathrm{U}}(\mathfrak{n}^-)$.

Also let $\bar{\mathrm{u}}^0$ denote the sub-Hopf algebra of $\bar{\mathrm{U}}^0$ generated by the elements $\prod_{i=1}^{\ell} \binom{H_i}{n_i}$ for $0 \leqslant n_i < p$. The equality

$$\binom{pn}{m} = 0 \quad \text{for all } n \in \mathbb{Z} \text{ and } 0 \leqslant m < p \tag{2.4.1a}$$

in $\mathbb{F}_p$ implies

$$c_{p\lambda+\mu}(z) = c_\mu(z) \quad \text{for all } \mu, \lambda \in \Lambda \text{ and } z \in \bar{\mathrm{u}}^0. \tag{2.4.1b}$$

For any Hopf algebra H let H^+ denote the augmentation ideal. We have the following useful result.

Lemma 2.8. *(See [5, Lemmas 6.5 and 6.6 and Proposition 6.7].) E_0 (resp. F_0) is central in $\bar{\mathrm{U}}(\mathfrak{n})$ (resp. $\bar{\mathrm{U}}(\mathfrak{n}^-)$). In particular, $E * E_0 = 0$ and $F * F_0 = 0$ for all $E \in \bar{\mathrm{U}}(\mathfrak{n})^+$ and $F \in \bar{\mathrm{U}}(\mathfrak{n}^-)^+$. Further, $E_0 \cdot \bar{\mathrm{u}}(\mathfrak{n})^+ = 0$ and $F_0 \cdot \bar{\mathrm{u}}(\mathfrak{n}^-)^+ = 0$.*

We also need the following technical lemma.

Lemma 2.9.

(1) *$E_0 \cdot \mathrm{Fr}'(Z * Y) = E_0 \cdot (\mathrm{Fr}'\, Z * \mathrm{Fr}'\, Y)$ for all $Y, Z \in \bar{\mathrm{U}}(\mathfrak{n})$.*
(2) *$E_0 \cdot (N * X) = 0$ for all $N \in \bar{\mathrm{u}}(\mathfrak{n})^+$ and $X \in \bar{\mathrm{U}}(\mathfrak{n})$.*

Proof. (1) Since Fr' is an $\mathbb{F}_p$-algebra morphism and since

$$E_0 \cdot (A * B) = A * (E_0 B) \tag{2.4.2}$$

for all $A, B \in \bar{\mathrm{U}}(\mathfrak{n})$ (by the centrality of E_0), it suffices to verify the statement in the case that $Z = E_i^{(m)}$ for some $1 \leqslant i \leqslant \ell$ and $m > 0$. We have

$$\begin{aligned}
E_0 \cdot \big(\big(\mathrm{Fr}' E_i^{(m)}\big) * \mathrm{Fr}' Y\big) &= E_0 \cdot \big(E_i^{(pm)} * \mathrm{Fr}' Y\big) \\
&= \sum_{j=0}^{pm} (-1)^{pm-j} E_0 E_i^{(j)} \mathrm{Fr}'(Y) E_i^{(pm-j)} \\
&= \sum_{j=0}^{m} (-1)^{pm-pj} E_0 E_i^{(pj)} \mathrm{Fr}'(Y) E_i^{(pm-pj)} \quad \text{(by Lemma 2.8)} \\
&= \sum_{j=0}^{m} (-1)^{m-j} E_0 \, \mathrm{Fr}'\big(E_i^{(j)} Y E_i^{(m-j)}\big) \\
&= E_0 \cdot \mathrm{Fr}'\big(E_i^{(m)} * Y\big).
\end{aligned}$$

(2) Since $\bar{\mathrm{u}}(\mathfrak{n})^+$ is generated by $E_i^{(m)}$ for $1 \leqslant i \leqslant \ell$ and $0 < m < p$ it suffices to check that

$$E_0 \cdot \big(E_i^{(m)} * X\big) = 0 \tag{2.4.3}$$

for all $X \in \bar{\mathrm{U}}(\mathfrak{n})$, $1 \leqslant i \leqslant \ell$, and $0 < m < p$. We have (using Lemma 2.8)

$$\begin{aligned}
E_0 \cdot \big(E_i^{(m)} * X\big) &= E_0 \cdot \left(\sum_{j=0}^{m} (-1)^{m-j} E_i^{(j)} X E_i^{(m-j)} \right) \\
&= X E_0 E_i^{(m)} + \sum_{j=1}^{m} (-1)^{m-j} E_0 E_i^{(j)} X E_i^{(m-j)} \quad \big(\text{since } E_0 \text{ is central in } \bar{\mathrm{U}}(\mathfrak{n})\big) \\
&= 0 \quad \big(\text{since } E_0 \cdot \bar{\mathrm{u}}(\mathfrak{n}) = 0\big). \qquad \square
\end{aligned}$$

2.4.2. The morphism S

Set $N := |\Delta^+|$. For $n \geqslant 0$ and $\lambda \in \Lambda$ define a morphism

$$S : H^0_{\mathfrak{h}}\big(\bar{X}, \bar{\mathrm{U}}_{(p-1)N+pn}(\mathfrak{n})^\vee \otimes \chi_{-pn\lambda}\big) \to H^0_{\mathfrak{h}}\big(\bar{X}, \bar{\mathrm{U}}_n(\mathfrak{n})^\vee \otimes \chi_{-n\lambda}\big) \tag{2.4.4a}$$

by

$$(Sf)(X \otimes Y \otimes v_{n\lambda}) = f\big(F_0 \cdot \varphi X \otimes E_0 \cdot \mathrm{Fr}' Y \otimes v_{pn\lambda}\big) \tag{2.4.4b}$$

for all $X \in \bar{\mathrm{U}}(\mathfrak{g})$, $Y \in \bar{\mathrm{U}}_n(\mathfrak{n})$, and $f \in H^0_{\mathfrak{h}}(\bar{X}, \bar{\mathrm{U}}_{(p-1)N+pn}(\mathfrak{n})^\vee \otimes \chi_{-pn\lambda})$. (Here we are considering f as an element of $H^0_{\mathfrak{h}}(\bar{X}, \bar{\mathrm{U}}(\mathfrak{n})^\vee \otimes \chi_{-pn\lambda})$ under the natural inclusion.) Note that S is not a morphism of $\bar{\mathrm{U}}(\mathfrak{g})$-modules.

It is not clear that S is well-defined, so we must prove that. We first have the following technical lemma.

Lemma 2.10. *For all $\mu \in \Lambda$, $m \geqslant 0$, $1 \leqslant i \leqslant \ell$, $X \in \bar{\mathrm{U}}(\mathfrak{g})$, $Y \in \bar{\mathrm{U}}(\mathfrak{n})$, and $f \in H^0_{\mathfrak{h}}(\bar{X}, \bar{\mathrm{U}}(\mathfrak{n})^\vee \otimes \chi_{-p\mu})$, we have*

$$f\big(F_0 E_i^{(pm)} X \otimes E_0 \,\mathrm{Fr}'\, Y \otimes v_{p\mu}\big) = f\big(E_i^{(pm)} F_0 X \otimes E_0 \,\mathrm{Fr}'\, Y \otimes v_{p\mu}\big).$$

Proof. Applying the Cartan involution to Lemme 3.7 in [7] (cf. also the proof of Lemma 4.5 in [10]) we have

$$F_0 E_i^{(pm)} \in E_i^{(pm)} F_0 + \bar{\mathfrak{u}}(\mathfrak{n})^+ \cdot \bar{\mathrm{U}}(\mathfrak{g}) + \sum_{s=0}^{m-1} E_i^{(sp)} z_s \cdot \bar{\mathrm{U}}(\mathfrak{g}), \tag{2.4.5}$$

where $z_s \in \bar{\mathfrak{u}}^0$ are elements such that $\chi_{-2(p-1)\rho}(z_s) = 0$.

Since

$$(X.f)\big(X' \otimes Y' \otimes v_{p\mu}\big) = f\big(X'X \otimes Y' \otimes v_{p\mu}\big)$$

for all $X, X' \in \bar{\mathrm{U}}(\mathfrak{g})$ and $Y' \in \bar{\mathrm{U}}(\mathfrak{n})$, it suffices to show that

$$f\big(F_0 E_i^{(pm)} \otimes E_0 \,\mathrm{Fr}'\, Y \otimes v_{p\mu}\big) = f\big(E_i^{(pm)} F_0 \otimes E_0 \,\mathrm{Fr}'\, Y \otimes v_{p\mu}\big). \tag{2.4.6}$$

By (2.4.5) we have

$$F_0 E_i^{(pm)} \otimes E_0 \,\mathrm{Fr}'\, Y \otimes v_{p\mu} = \left(E_i^{(pm)} F_0 + \sum N_j A_j + \sum_{s=0}^{m-1} E_i^{(sp)} z_s B_s\right) \otimes E_0 \,\mathrm{Fr}'\, Y \otimes v_{p\mu} \tag{2.4.7}$$

for some $A_j, B_s \in \bar{\mathrm{U}}(\mathfrak{g})$, $N_j \in \bar{\mathfrak{u}}(\mathfrak{n})^+$, and $z_s \in \bar{\mathfrak{u}}^0$ such that $\chi_{-2(p-1)\rho}(z_s) = 0$. Now,

$$\begin{aligned}
\sum f\big(N_j A_j \otimes E_0 \,\mathrm{Fr}'\, Y \otimes v_{p\mu}\big) &= \sum f\big(A_j \otimes \big(\sigma(N_j) * \big(E_0 \,\mathrm{Fr}'\, Y \otimes v_{p\mu}\big)\big)\big) \\
&= \sum f\big(A_j \otimes E_0 \cdot \big(\sigma(N_j) * \mathrm{Fr}'\, Y\big) \otimes v_{p\mu}\big) \\
&\quad \big(\text{since } \sigma(N_j) * E_0 = 0 \text{ by Lemma 2.8 and } \sigma(N_j).v_{p\mu} = 0\big) \\
&= 0 \quad \big(\text{by Lemma 2.9 (2)}\big).
\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{s=0}^{m-1} f\big(E_i^{(sp)} z_s B_i \otimes E_0 \,\mathrm{Fr}'\, Y \otimes v_{p\mu}\big) &= \sum_{s=0}^{m-1} f\big(B_i \otimes \sigma\big(E_i^{(sp)} z_s\big) * \big(E_0 \,\mathrm{Fr}'\, Y \otimes v_{p\mu}\big)\big) \\
&= \sum_{s=0}^{m-1} f\big((-1)^s B_i \otimes \sigma(z_s) * \big(\big(E_i^{(sp)} * E_0 \,\mathrm{Fr}'\, Y\big) \otimes v_{p\mu}\big)\big) \\
&= \sum_{s=0}^{m-1} f\big((-1)^s B_i \otimes \big(c_{-2(p-1)\rho}(z_s)\big).\big(E_i^{(sp)} * E_0 \,\mathrm{Fr}'\, Y\big) \otimes v_{p\mu}\big) \\
&\quad \big(\text{by (2.4.1b), since } \big(E_i^{(sp)} * E_0 \,\mathrm{Fr}'\, Y\big) \otimes v_{p\mu} \text{ has weight} \\
&\quad 2(p-1)\rho \bmod p\Lambda\big) \\
&= 0 \quad \big(\text{since } c_{-2(p-1)\rho}(z_s) = 0\big).
\end{aligned}$$

Thus (2.4.6) holds by (2.4.7). $\square$

Proposition 2.11. *The morphism S is well-defined and divides weights by p (i.e., if f is a weight vector of weight μ then $S(f)$ is a weight vector of weight μ/p if $\mu \in p\Lambda$ and $S(f) = 0$ otherwise). Furthermore,*

$$S(\varphi Z.f) = Z.(Sf) \quad \text{for all } Z \in \bar{\mathrm{U}}(\mathfrak{g}) \text{ and } f \in H^0_{\mathfrak{h}}\big(\bar{X}, \bar{\mathrm{U}}_{(p-1)N+pn}(\mathfrak{n})^\vee \otimes \chi_{-pn\lambda}\big). \tag{2.4.8}$$

In particular, S preserves $\bar{\mathrm{U}}(\mathfrak{g})$-locally finite vectors, so that S restricts to a morphism

$$H^0\big(\bar{X}, \bar{\mathrm{U}}_{(p-1)N+pn}(\mathfrak{n})^\vee \otimes \chi_{-pn\lambda}\big) \to H^0\big(\bar{X}, \bar{\mathrm{U}}_n(\mathfrak{n})^\vee \otimes \chi_{-n\lambda}\big). \tag{2.4.9}$$

Proof. To see that S is well-defined, we need to check (cf. (2.2.12)) that for $\lambda \in \Lambda$, $X \in \bar{\mathrm{U}}(\mathfrak{g})$, $Y \in \bar{\mathrm{U}}_n(\mathfrak{n})$, $Z \in \bar{\mathrm{U}}(\mathfrak{b})$, and $f \in H^0_{\mathfrak{h}}(\bar{X}, \bar{\mathrm{U}}_{(p-1)N+pn}(\mathfrak{n})^\vee \otimes \chi_{-pn\lambda})$,

$$(Sf)(ZX \otimes Y \otimes v_{n\lambda}) = (Sf)\big(X \otimes \sigma(Z) * (Y \otimes v_{n\lambda})\big). \tag{2.4.10}$$

(That is, we need to check that S preserves $\bar{\mathrm{U}}(\mathfrak{b})$-linearity.) It suffices to check this for the two cases where $Z = \binom{H_i}{m}$ or $Z = E_i^{(m)}$ for some $1 \leqslant i \leqslant \ell$ and $m \geqslant 0$.

For the first case, set $Z = \binom{H_i}{m}$. For $1 \leqslant i \leqslant \ell$, $m \geqslant 0$, and $n \in \mathbb{Z}$ define

$$\binom{H_i; n}{m} := \frac{(H_i + n)(H_i + n - 1) \cdots (H_i + n - m + 1)}{m!} \in \bar{\mathrm{U}}^0. \tag{2.4.11}$$

We may assume in (2.4.10) that Y is a weight vector of weight μ. Then we have

$$\begin{aligned}
&(Sf)\left(\binom{H_i}{m} X \otimes Y \otimes v_{n\lambda}\right) \\
&= f\left(F_0 \cdot \varphi\left(\binom{H_i}{m} X\right) \otimes E_0 \cdot \mathrm{Fr}' Y \otimes v_{pn\lambda}\right) \\
&= f\left(F_0 \cdot \binom{H_i}{pm} \cdot \varphi(X) \otimes E_0 \cdot \mathrm{Fr}' Y \otimes v_{pn\lambda}\right) \\
&= f\left(\binom{H_i; 2(p-1)}{pm} \cdot F_0 \cdot \varphi(X) \otimes E_0 \cdot \mathrm{Fr}' Y \otimes v_{pn\lambda}\right) \quad \big(\text{by [11, 6.5(a6)]}\big) \\
&= f\left(F_0 \cdot \varphi(X) \otimes \sigma\binom{H_i; 2(p-1)}{pm} * \big(E_0 \cdot \mathrm{Fr}' Y \otimes v_{pn\lambda}\big)\right) \\
&= f\left(F_0 \cdot \varphi(X) \otimes \binom{-H_i; 2(p-1)}{pm} * \big(E_0 \cdot \mathrm{Fr}' Y \otimes v_{pn\lambda}\big)\right) \\
&= f\left(F_0 \cdot \varphi(X) \otimes \binom{-(2(p-1)\rho + p\mu + pn\lambda)(\alpha_i^\vee) + 2(p-1)}{pm} \cdot \big(E_0 \cdot \mathrm{Fr}' Y \otimes v_{pn\lambda}\big)\right) \\
&\qquad \big(\text{since } E_0 \cdot \mathrm{Fr}' Y \otimes v_{pn\lambda} \text{ has weight } 2(p-1)\rho + p\mu + pn\lambda\big) \\
&= f\left(F_0 \cdot \varphi(X) \otimes \binom{-(p\mu + pn\lambda)(\alpha_i^\vee)}{pm} \cdot \big(E_0 \cdot \mathrm{Fr}' Y \otimes v_{pn\lambda}\big)\right) \\
&= f\left(F_0 \cdot \varphi(X) \otimes \binom{-(\mu + n\lambda)(\alpha_i^\vee)}{m} \cdot \big(E_0 \cdot \mathrm{Fr}' Y \otimes v_{pn\lambda}\big)\right)
\end{aligned}$$

$$= (Sf)\left(X \otimes \binom{-(\mu+n\lambda)(\alpha_i^\vee)}{m} \cdot Y \otimes v_{n\lambda}\right)$$
$$= (Sf)\left(X \otimes \sigma\binom{H_i}{m} * (Y \otimes v_{n\lambda})\right) \quad \text{(since } Y \otimes v_{n\lambda} \text{ has weight } \mu + n\lambda\text{)}.$$

For the second case, set $Z = E_i^{(m)}$. Then

$$\begin{aligned}
(Sf)\big(E_i^{(m)} X \otimes Y \otimes v_{n\lambda}\big) &= f\left(F_0 \cdot \varphi\big(E_i^{(m)} X\big) \otimes E_0 \operatorname{Fr}' Y \otimes v_{pn\lambda}\right) \\
&= f\left(F_0 E_i^{(pm)} \varphi(X) \otimes E_0 \operatorname{Fr}' Y \otimes v_{pn\lambda}\right) \\
&= f\left(E_i^{(pm)} F_0 \varphi(X) \otimes E_0 \operatorname{Fr}' Y \otimes v_{pn\lambda}\right) \quad \text{(by Lemma 2.10)} \\
&= f\left(F_0 \varphi(X) \otimes \sigma\left(E_i^{(pm)}\right) * \left(E_0 \operatorname{Fr}' Y \otimes v_{pn\lambda}\right)\right) \\
&= f\left((-1)^m F_0 \varphi(X) \otimes E_0 \cdot \left(E_i^{(pm)} * \operatorname{Fr}' Y\right) \otimes v_{pn\lambda}\right) \quad \text{(by Lemma 2.8)} \\
&= f\left((-1)^m F_0 \varphi(X) \otimes E_0 \cdot \operatorname{Fr}'\big(E_i^{(m)} * Y\big) \otimes v_{pn\lambda}\right) \quad \big(\text{by Lemma 2.9 (1)}\big) \\
&= (Sf)\big(X \otimes \left(\sigma\left(E_i^{(m)}\right) * Y\right) \otimes v_{n\lambda}\big) \\
&= (Sf)\big(X \otimes \sigma\left(E_i^{(m)}\right) * (Y \otimes v_{n\lambda})\big).
\end{aligned}$$

Hence $S^\vee$ is well-defined.

Note that the morphism

$$X \otimes Y \otimes v_{n\lambda} \mapsto F_0 \cdot \varphi X \otimes E_0 \cdot \operatorname{Fr}' Y \otimes v_{pn\lambda}$$

is the morphism dual to S. Since this morphism clearly multiplies weights by p, S divides weights by p. Finally, (2.4.8) follows from (2.2.11) and an easy computation. □

2.4.3. Frobenius-linearity of S

Note that by the formulas in Section 2.1.1 we have

$$\operatorname{Fr}(X) = \epsilon(X) \quad \text{for all } X \in \bar{\mathrm{u}}(\mathfrak{g}). \tag{2.4.12}$$

Lemma 2.12. *The following diagrams commute*:

$$\begin{tikzcd}
\bar{\mathrm{U}}(\mathfrak{g}) \otimes \bar{\mathrm{U}}(\mathfrak{g}) & \bar{\mathrm{U}}(\mathfrak{g}) \otimes \bar{\mathrm{U}}(\mathfrak{g}) \arrow[l, "\mathrm{Id}\otimes\varphi"'] & \\
& & \bar{\mathrm{U}}(\mathfrak{g}) \arrow[ul, "\Delta"'] \arrow[dl, "\varphi"] \\
\bar{\mathrm{U}}(\mathfrak{g}) \otimes \bar{\mathrm{U}}(\mathfrak{g}) \arrow[uu, "\mathrm{Fr}\otimes\mathrm{Id}"] & \bar{\mathrm{U}}(\mathfrak{g})\mu_0 \arrow[l, "\Delta"] &
\end{tikzcd} \tag{2.4.13a}$$

and

$$\begin{tikzcd}
\bar{\mathrm{U}}(\mathfrak{n}) \otimes \bar{\mathrm{U}}(\mathfrak{n}) & \bar{\mathrm{U}}(\mathfrak{n}) \otimes \bar{\mathrm{U}}(\mathfrak{n}) \arrow[l, "\mathrm{Id}\otimes\mathrm{Fr}'"'] & \\
 & & \bar{\mathrm{U}}(\mathfrak{n}) \arrow[ul, "\Delta"'] \arrow[dl, "\mathrm{Fr}'"] \\
\bar{\mathrm{U}}(\mathfrak{n}) \otimes \bar{\mathrm{U}}(\mathfrak{n}) \arrow[uu, "\mathrm{Fr}\otimes\mathrm{Id}"] & \bar{\mathrm{U}}(\mathfrak{n}) \arrow[l, "\Delta"] &
\end{tikzcd} \tag{2.4.13b}$$

Proof. This is implicit in [7] and [10], but we verify it directly for completeness. We first verify (2.4.13a). Since all morphisms in the diagram are multiplicative, it suffices to verify that the diagram commutes for the algebra generators $\{E_i^{(m)}\}_{m\geqslant 0}$, $\{F_i^{(m)}\}_{m\geqslant 0}$, and $\{\binom{H_i}{m}\}_{m\geqslant 0}$ of $\bar{\mathrm{U}}(\mathfrak{g})$. We verify this for $E_i^{(m)}$:

$$\begin{aligned}
\big((\mathrm{Fr}\otimes\mathrm{Id})\circ\Delta\circ\varphi\big)\big(E_i^{(m)}\big) &= \big((\mathrm{Fr}\otimes\mathrm{Id})\circ\Delta\big)\big(E_i^{(pm)}\mu_0\big) \\
&= (\mathrm{Fr}\otimes\mathrm{Id})\left[\sum_{j=0}^{pm}\big(E_i^{(j)}\otimes E_i^{(pm-j)}\big)\cdot\sum(\mu_0)_{(1)}\otimes(\mu_0)_{(2)}\right] \\
&= \sum_{j=0}^{m} E_i^{(j)}\otimes E_i^{(pm-pj)}\cdot\sum\mathrm{Fr}\big((\mu_0)_{(1)}\big)\otimes(\mu_0)_{(2)} \quad \big(\text{by (2.4.12)}\big) \\
&= \left(\sum_{j=0}^{m} E_i^{(j)}\otimes E_i^{(pm-pj)}\right)\cdot(1\otimes\mu_0) \\
&= \sum_{j=0}^{m} E_i^{(j)}\otimes\varphi\big(E_i^{(m-j)}\big) \\
&= \big((\mathrm{Id}\otimes\varphi)\circ\Delta\big)\big(E_i^{(m)}\big).
\end{aligned}$$

The computations for $F_i^{(m)}$ and $\binom{H_i}{m}$ are similar, as is the computation for (2.4.13b). $\square$

Proposition 2.13. *$S(f^p g) = f\cdot S(g)$ for all $n,m\geqslant 0$, $f\in H^0(\bar{X},\bar{\mathrm{U}}_n(\mathfrak{n})^\vee\otimes\chi_{-n\lambda})$, and $g\in H^0(\bar{X},\bar{\mathrm{U}}_{(p-1)N+pm}(\mathfrak{n})^\vee\otimes\chi_{-pm\lambda})$.*

Proof. Choose $X\in\bar{\mathrm{U}}(\mathfrak{g})$ and $Y\in\bar{\mathrm{U}}_{n+m}(\mathfrak{n})$. Then

$$\begin{aligned}
S\big(f^p g\big)(X\otimes Y\otimes v_{(n+m)\lambda}) &= \big(f^p g\big)\big(F_0\cdot\varphi X\otimes E_0\cdot\mathrm{Fr}'\,Y\otimes v_{p(n+m)\lambda}\big) \\
&= \big(\widetilde{\mathrm{Fr}}^*f\cdot g\big)\big(F_0\cdot\varphi X\otimes E_0\cdot\mathrm{Fr}'\,Y\otimes v_{p(n+m)\lambda}\big) \quad (\text{by Proposition 2.7}) \\
&= \sum f\big[\mathrm{Fr}\big((F_0)_{(1)}(\varphi X)_{(1)}\big)\otimes\mathrm{Fr}\big((E_0)_{(1)}\big(\mathrm{Fr}'\,Y\big)_{(1)}\big)\otimes v_{n\lambda}\big] \\
&\quad\cdot g\big[(F_0)_{(2)}(\varphi X)_{(2)}\otimes(E_0)_{(2)}\big(\mathrm{Fr}'\,Y\big)_{(2)}\otimes v_{pm\lambda}\big] \quad \big(\text{by (2.2.14b)}\big) \\
&= \sum f\big[\mathrm{Fr}\big((\varphi X)_{(1)}\big)\otimes\mathrm{Fr}\big(\big(\mathrm{Fr}'\,Y\big)_{(1)}\big)\otimes v_{n\lambda}\big] \\
&\quad\cdot g\big[F_0\cdot(\varphi X)_{(2)}\otimes E_0\cdot\big(\mathrm{Fr}'\,Y\big)_{(2)}\otimes v_{pm\lambda}\big] \quad \big(\text{by (2.4.12)}\big)
\end{aligned}$$

$$= \sum f(X_{(1)} \otimes Y_{(1)} \otimes v_{n\lambda}) \cdot g\bigl(F_0 \cdot \varphi X \otimes E_0 \cdot \mathrm{Fr}' Y \otimes v_{pm\lambda}\bigr)$$

(by Lemma 2.12)

$$= \bigl(f \cdot S(g)\bigr)(X \otimes Y \otimes v_{(n+m)\lambda}). \qquad \square$$

2.5. The section $\psi_{f_+ \otimes f_-}$ and the multiplication $M_{f_+ \otimes f_-}$

In this section we construct a particular section $\psi_{f_+ \otimes f_-} \in H^0(\bar{X}, \bar{\mathrm{U}}(\mathfrak{n})^\vee)$ and define the multiplication morphism $M_{f_+ \otimes f_-} : f \mapsto \psi_{f_+ \otimes f_-} \cdot f$.

2.5.1. The morphism $\bar{\psi}$

Set $\delta := (p-1)\rho$. Recall that the **Steinberg module** for G, denoted St, is the irreducible module of highest weight δ. It is also a Weyl module for G and is self-dual. Let

$$\eta : \mathrm{St} \otimes \mathrm{St} \to \mathbb{F}_p \tag{2.5.1}$$

be the G-equivariant pairing.

Recall that we are taking the conjugation action $*$ of $\bar{\mathrm{U}}(\mathfrak{b})$ on $\bar{\mathrm{U}}(\mathfrak{n})^\vee$. Following [8], define a morphism

$$\bar{\psi} : \mathrm{St} \otimes \mathrm{St} \to \bar{\mathrm{U}}(\mathfrak{n})^\vee, \quad v \otimes w \mapsto \bar{\psi}_{v \otimes w} \tag{2.5.2a}$$

by

$$\bar{\psi}_{v \otimes w}(X) = \eta(v \otimes X.w) \tag{2.5.2b}$$

for $v \otimes w \in \mathrm{St} \otimes \mathrm{St}$ and $X \in \bar{\mathrm{U}}(\mathfrak{n})$. Since

$$\eta(Y.v \otimes w) = \eta(v \otimes \sigma Y.w) \quad \text{for all } v, w \in \mathrm{St} \text{ and } Y \in \bar{\mathrm{U}}(\mathfrak{g})$$

it is easy to check that $\bar{\psi}$ is a $\bar{\mathrm{U}}(\mathfrak{b})$-equivariant morphism.

Let

$$q_{(p-1)N} : H^0\bigl(\bar{X}, \bar{\mathrm{U}}(\mathfrak{n})^\vee\bigr) \twoheadrightarrow H^0\bigl(\bar{X}, \bar{\mathrm{U}}_{(p-1)N}(\mathfrak{n})^\vee\bigr)$$

be the $\bar{\mathrm{U}}(\mathfrak{g})$-equivariant projection. We now define a $\bar{\mathrm{U}}(\mathfrak{g})$-equivariant morphism

$$\psi : \mathrm{St} \otimes \mathrm{St} \to H^0\bigl(\bar{X}, \bar{\mathrm{U}}_{(p-1)N}(\mathfrak{n})^\vee\bigr), \quad v \otimes w \mapsto \psi_{v \otimes w} \tag{2.5.3}$$

by the following composition:

$$\mathrm{St} \otimes \mathrm{St} \xrightarrow{H^0(\bar{\psi})} H^0\bigl(\bar{X}, \bar{\mathrm{U}}(\mathfrak{n})^\vee\bigr) \overset{q_{(p-1)N}}{\twoheadrightarrow} H^0\bigl(\bar{X}, \bar{\mathrm{U}}_{(p-1)N}(\mathfrak{n})^\vee\bigr). \tag{2.5.4}$$

Let $\pi_{(p-1)N} : \bar{\mathrm{U}}(\mathfrak{n}) \twoheadrightarrow \bar{\mathrm{U}}_{(p-1)N}(\mathfrak{n})$ be the $\bar{\mathrm{U}}(\mathfrak{b})$-equivariant projection. Then, considering $H^0(\bar{X}, \bar{\mathrm{U}}_{(p-1)N}(\mathfrak{n})^\vee)$ as a subspace of $H^0(\bar{X}, \bar{\mathrm{U}}(\mathfrak{n})^\vee)$, ψ is given explicitly by

$$\psi_{v \otimes w}(X \otimes Y) = \sum \eta\bigl(X_{(1)}.v \otimes \pi_{(p-1)N}(Y).X_{(2)}.w\bigr) \tag{2.5.5}$$

for $X \in \bar{\mathrm{U}}(\mathfrak{g})$ and $Y \in \bar{\mathrm{U}}(\mathfrak{n})$. (Remark that the projections $q_{(p-1)N}$ and $\pi_{(p-1)N}$ are necessary here because in general $H^0(\bar{\psi})(v \otimes w)$ will not be a homogeneous element of $H^0(\bar{X}, \bar{\mathrm{U}}(\mathfrak{n})^\vee)$.)

Lemma 2.14. $E_0 \in \bar{\mathrm{U}}_{(p-1)N}(\mathfrak{n})$.

Proof. Let $\{y_\beta\}_{\beta\in\Delta^+} \subseteq \bar{\mathrm{U}}_1(\mathfrak{n})^\vee$ be a set of weight elements of $\bar{\mathrm{U}}(\mathfrak{n})^\vee$ that generate $\bar{\mathrm{U}}(\mathfrak{n})^\vee$ as an $\mathbb{F}_p$-algebra such that the weight of y_β is $-\beta$. The ideal $I^{(p)} := \langle y_\beta^p\rangle_{\beta\in\Delta^+}$ is $\bar{\mathrm{U}}(\mathfrak{b})$-stable and the quotient algebra $\bar{\mathrm{U}}(\mathfrak{n})^\vee/I^{(p)} \cong \bar{\mathrm{u}}(\mathfrak{n})^\vee$ is a $\bar{\mathrm{U}}(\mathfrak{b})$-module algebra isomorphic to the coordinate algebra of the first Frobenius kernel of U.

Set

$$y_0 := \prod_{\beta\in\Delta^+} y_\beta^{p-1} \in \bar{\mathrm{U}}_{(p-1)N}(\mathfrak{n})^\vee \tag{2.5.6}$$

and let

$$r : \bar{\mathrm{U}}(\mathfrak{n})^\vee \twoheadrightarrow \bar{\mathrm{u}}(\mathfrak{n})^\vee \twoheadrightarrow \chi_{-2\delta} \tag{2.5.7a}$$

be the $\bar{\mathrm{U}}(\mathfrak{b})$-equivariant projection dual to the morphism

$$\chi_{2\delta} \hookrightarrow \bar{\mathrm{u}}(\mathfrak{n}) \hookrightarrow \bar{\mathrm{U}}(\mathfrak{n}), \quad v_{2\delta} \mapsto E_0. \tag{2.5.7b}$$

Since $r(y_0) \neq 0$ we have $y_0(E_0) \neq 0$. Hence $\pi_{(p-1)N}(E_0) \neq 0$ since $y_0 \in \bar{\mathrm{U}}_{(p-1)N}(\mathfrak{n})^\vee$.

Choose nonnegative integers $\{m_\beta\}_{\beta\in\Delta^+}$ such that not all m_β are equal to $p-1$ and set $y := \prod_{\beta\in\Delta^+} y_\beta^{m_\beta}$. To show that $E_0 \in \bar{\mathrm{U}}_{(p-1)N}(\mathfrak{n})$ it suffices to show that $y(E_0) = 0$, since this would imply that E_0 is dual to the element y_0 with respect to a basis of $\bar{\mathrm{U}}(\mathfrak{n})^\vee$ consisting of homogeneous elements.

If y is not of weight -2δ then $y(E_0) = 0$ by weight considerations, so we can assume that y is of weight -2δ. Thus we have $\sum_{\beta\in\Delta^+} m_\beta\beta = 2\delta$. Since not all m_β are equal to $p-1$, at least one of the m_β must be $\geqslant p$. (Indeed, otherwise there would be an element of $\bar{\mathrm{u}}(\mathfrak{n})$ of weight 2δ that is not in the subspace spanned by E_0, which is false.) Thus we can write $y = y_\gamma^p \cdot y'$ for some $\gamma \in \Delta^+$ and we have

$$\begin{aligned} y(E_0) &= \big(y_\gamma^p \cdot y'\big)(E_0) \\ &= \big(\mathrm{Fr}^* y_\gamma \cdot y'\big)(E_0) \\ &= \sum y_\gamma\big(\mathrm{Fr}\big((E_0)_{(1)}\big)\big) \cdot y'\big((E_0)_{(2)}\big) \\ &= y_\gamma(1) \cdot y'(E_0) \\ &= 0 \quad \big(\text{since } y_\gamma(1) = 0\big). \end{aligned}$$

Hence $E_0 \in \bar{\mathrm{U}}_{(p-1)N}(\mathfrak{n})$. □

In particular, we have

$$\pi_{(p-1)N}(E_0) = E_0. \tag{2.5.8}$$

2.5.2. The section $\psi_{f_+\otimes f_-}$ and the multiplication $M_{f_+\otimes f_-}$

Let $f_+, f_- \in \mathrm{St}$ be nonzero highest and lowest weight vectors, respectively. Then $F_0.f_+$ is a nonzero multiple of f_- and $E_0.f_-$ is a nonzero multiple of f_+ (cf. Exercise 2.3.E(2) in [2]).

By (2.5.5), for $X \in \bar{\mathrm{U}}(\mathfrak{n}^-)$ and $Y \in \bar{\mathrm{U}}(\mathfrak{n})$ we have

$$\psi_{f_+\otimes f_-}(X \otimes Y) = \eta\big(X.f_+ \otimes \pi_{(p-1)N}(Y).f_-\big). \tag{2.5.9}$$

Thus, by rescaling f_+ and f_- if necessary, by Lemma 2.14 we have

$$\psi_{f_+\otimes f_-}(F_0 \otimes E_0) = \eta\big(F_0.f_+ \otimes \pi_{(p-1)N}(E_0).f_-\big) = \eta(F_0.f_+ \otimes E_0.f_-) = 1. \tag{2.5.10}$$

For all $\lambda \in \Lambda$ and $n \geqslant 0$ define a morphism

$$M_{f_+\otimes f_-} : H^0\big(\bar{X}, \bar{\mathrm{U}}_n(\mathfrak{n})^\vee \otimes \chi_{-n\lambda}\big) \to H^0\big(\bar{X}, \bar{\mathrm{U}}_{n+(p-1)N}(\mathfrak{n})^\vee \otimes \chi_{-n\lambda}\big) \tag{2.5.11}$$

given by multiplication by the section $\psi_{f_+\otimes f_-}$. Note that $M_{f_+\otimes f_-}$ is $\bar{\mathrm{U}}^0$-equivariant since $f_+ \otimes f_- \in \mathrm{St} \otimes \mathrm{St}$ is an element of weight 0.

2.6. The splitting $\widetilde{S}$

2.6.1. Define an endomorphism $\widetilde{S}$ of $R_\lambda^{\mathfrak{h}}$ as follows. Set

$$\widetilde{S}\big(H^0_{\mathfrak{h}}\big(\bar{X}, \bar{\mathrm{U}}_m(\mathfrak{n})^\vee \otimes \chi_{-m\lambda}\big)\big) = 0 \quad \text{if } p \nmid m \tag{2.6.1a}$$

and for $n \geqslant 0$ let $\widetilde{S}$ be defined on $H^0_{\mathfrak{h}}(\bar{X}, \bar{\mathrm{U}}_{pn}(\mathfrak{n})^\vee \otimes \chi_{-pn\lambda})$ by the composition

$$\begin{aligned} H^0_{\mathfrak{h}}\big(\bar{X}, \bar{\mathrm{U}}_{pn}(\mathfrak{n})^\vee \otimes \chi_{-pn\lambda}\big) &\xrightarrow{M_{f_+\otimes f_-}} H^0_{\mathfrak{h}}\big(\bar{X}, \bar{\mathrm{U}}_{(p-1)N+pn}(\mathfrak{n})^\vee \otimes \chi_{-pn\lambda}\big) \\ &\xrightarrow{S} \quad H^0_{\mathfrak{h}}\big(\bar{X}, \bar{\mathrm{U}}_n(\mathfrak{n})^\vee \otimes \chi_{-n\lambda}\big). \end{aligned} \tag{2.6.1b}$$

By Proposition 2.11, $\widetilde{S}$ descends to a morphism $R_\lambda \to R_\lambda$.

Definition 2.15. Let A be an $\mathbb{F}_p$-algebra and s an $\mathbb{F}_p$-linear endomorphism of A. We say that s is **Frobenius-linear** if $s(a^p b) = a \cdot s(b)$ for all $a, b \in A$. If s is a Frobenius-linear endomorphism of A such that $s(a^p) = a$ for all $a \in A$ we say that s is a **Frobenius splitting** of A.

Theorem 2.16. *$\widetilde{S}$ is a Frobenius splitting of $R_\lambda^{\mathfrak{h}}$ for all $\lambda \in \Lambda$. In particular, $\widetilde{S}$ descends to a Frobenius splitting of R_λ.*

Proof. Since $\widetilde{S}$ preserves R_λ it suffices to check that $\widetilde{S}$ is a Frobenius splitting of $R_\lambda^{\mathfrak{h}}$. We first check that $\widetilde{S}$ is Frobenius-linear. Choose $n \geqslant 0$ and $f \in H^0_{\mathfrak{h}}(\bar{X}, \bar{\mathrm{U}}_n(\mathfrak{n})^\vee \otimes \chi_{-n\lambda})$. For m with $p \nmid m$ and $h \in H^0_{\mathfrak{h}}(\bar{X}, \bar{\mathrm{U}}_m(\mathfrak{n})^\vee \otimes \chi_{-m\lambda})$ we have

$$f^p h \in H^0_{\mathfrak{h}}\big(\bar{X}, \bar{\mathrm{U}}_{pn+m}(\mathfrak{n})^\vee \otimes \chi_{-(pn+m)\lambda}\big).$$

Thus, since $p \nmid pn + m$, we have

$$\widetilde{S}\big(f^p \cdot h\big) = 0 = f \cdot \widetilde{S}(h). \tag{2.6.2}$$

Now choose $m \geqslant 0$ and $g \in H^0_{\mathfrak{h}}(\bar{X}, \bar{\mathrm{U}}_{pm}(\mathfrak{n})^\vee \otimes \chi_{-pm\lambda})$. Since $M_{f_+\otimes f_-}$ is given by section multiplication we have

$$M_{f_+\otimes f_-}\big(f^p \cdot g\big) = f^p \cdot M_{f_+\otimes f_-}(g).$$

Thus, by Proposition 2.13,

$$\widetilde{S}\big(f^p \cdot g\big) = S\big(f^p \cdot M_{f_+\otimes f_-}(g)\big) = f \cdot \widetilde{S}(g). \tag{2.6.3}$$

Hence $\widetilde{S}$ is Frobenius-linear.

We next verify that $\widetilde{S}$ is a Frobenius splitting. Since $\widetilde{S}$ is Frobenius-linear it suffices to show that $\widetilde{S}(e) = e$, where $e \in R_\lambda$ is the unit. Now, $e \in H^0_{\mathfrak{h}}(\bar{X}, \bar{\mathrm{U}}_0(\mathfrak{n})^\vee)$ is the element such that

$$e(X \otimes Y) = \epsilon(X)\epsilon(Y) \tag{2.6.4}$$

for all $X \in \bar{\mathrm{U}}(\mathfrak{g})$, $Y \in \bar{\mathrm{U}}(\mathfrak{n})$. Since

$$f(ZX \otimes Y) = f(X \otimes \sigma Z * Y)$$

for all $Z \in \bar{\mathrm{U}}(\mathfrak{b})$, $X \in \bar{\mathrm{U}}(\mathfrak{g})$, $Y \in \bar{\mathrm{U}}(\mathfrak{n})$, and $f \in H^0_{\mathfrak{h}}(\bar{X}, \bar{\mathrm{U}}(\mathfrak{n})^\vee)$, by the triangular decomposition of $\bar{\mathrm{U}}(\mathfrak{g})$ we can assume in the following that $X \in \bar{\mathrm{U}}(\mathfrak{n}^-)$. We have

$$\begin{aligned}
\big(\widetilde{S}(e)\big)(X \otimes Y) &= \big((S \circ M_{f_+ \otimes f_-})(e)\big)(X \otimes Y) \\
&= \big(M_{f_+ \otimes f_-}(e)\big)\big(F_0 \cdot \varphi X \otimes E_0 \cdot \mathrm{Fr}' Y\big) \\
&= \big(\mu_0.\big(M_{f_+ \otimes f_-}(e)\big)\big)\big(F_0 \cdot \mathrm{Fr}'^{-} X \otimes E_0 \cdot \mathrm{Fr}' Y\big) \\
&= \big(M_{f_+ \otimes f_-}(e)\big)\big(F_0 \cdot \mathrm{Fr}'^{-} X \otimes E_0 \cdot \mathrm{Fr}' Y\big) \quad \big(\text{since } M_{f_+ \otimes f_-}(e) \text{ has weight } 0\big) \\
&= \sum \eta\big((F_0)_{(1)}.\big(\mathrm{Fr}'^{-} X\big)_{(1)}.f_+ \otimes \pi_{(p-1)N}\big((E_0)_{(1)} \cdot \big(\mathrm{Fr}' Y\big)_{(1)}\big).f_-\big) \\
&\quad\ \cdot e\big((F_0)_{(2)} \cdot \big(\mathrm{Fr}'^{-} X\big)_{(2)} \otimes (E_0)_{(2)} \cdot \big(\mathrm{Fr}' Y\big)_{(2)}\big) \quad \big(\text{by (2.2.14b) and (2.5.9)}\big) \\
&= \eta\big(F_0.\mathrm{Fr}'^{-} X.f_+ \otimes \pi_{(p-1)N}\big(E_0 \cdot \mathrm{Fr}' Y\big).f_-\big) \quad \big(\text{by (2.6.4)}\big) \\
&= \eta\big(F_0.f_+ \otimes \pi_{(p-1)N}(E_0).f_-\big) \cdot \epsilon(X) \cdot \epsilon(Y) \quad (\text{by weight considerations}) \\
&= \epsilon(X) \cdot \epsilon(Y) \quad \big(\text{by (2.5.10)}\big) \\
&= e(X \otimes Y).
\end{aligned}$$

Hence $\widetilde{S}$ is a Frobenius splitting of $R^{\mathfrak{h}}_\lambda$. $\square$

2.7. S and the trace map

In this section we compare S to the local trace map. The results of this section are also crucial in the proof of Proposition 3.4 below. The main result in this section is Proposition 2.20.

Definition 2.17. For any polynomial ring $P := \mathbb{F}_p[z_1, \dots, z_n]$ we have the Frobenius-linear **trace map** $\mathrm{Tr} : P \to P$ which is given on monomials as follows. Set $z_0 := z_1^{p-1} \cdots z_n^{p-1}$. Then

$$\mathrm{Tr}\big(z_0 f^p\big) = f \tag{2.7.1}$$

for all $f \in P$, and if g is a monomial that is not of the form $z_0 f^p$ for some $f \in P$ we set $\mathrm{Tr}(g) = 0$. Up to a nonzero constant, Tr is independent of the choice of generators $z_1, \dots, z_n$ of P.

Remark 2.18. Consider the polynomial ring P as above. For any $h \in P$ we have a Frobenius-linear endomorphism f_h of P given by

$$f_h(g) = \mathrm{Tr}(hg) \quad \text{for all } g \in P. \tag{2.7.2}$$

By Example 1.3.1 in [2], every Frobenius-linear endomorphism of P is of the form f_h for some $h \in P$.

Let $\{x_\beta\}_{\beta\in\Delta^+}$ (resp. $\{y_\beta\}_{\beta\in\Delta^+}$) be eigenfunctions in degree 1 which generate $\mathbb{F}_p[\mathfrak{n}^-]$ (resp. $\mathbb{F}_p[\mathfrak{n}]$) as polynomial rings. By (2.2.3) we may also consider these as elements of $\bar{\mathrm{U}}(\mathfrak{n}^-)^\vee$ (resp. $\bar{\mathrm{U}}(\mathfrak{n})^\vee$). Set

$$y_0 := \prod_{\beta\in\Delta^+} y_\beta^{p-1} \quad \text{and} \quad x_0 := \prod_{\beta\in\Delta^+} x_\beta^{p-1}. \tag{2.7.3}$$

By the proof of Lemma 2.14, after rescaling the x_β, y_β if necessary we have that

$$x_0(F_0) = y_0(E_0) = 1. \tag{2.7.4}$$

These choices of polynomial generators now give trace maps Tr_+ and Tr_- on $\bar{\mathrm{U}}(\mathfrak{n})^\vee$ and $\bar{\mathrm{U}}(\mathfrak{n}^-)^\vee$ respectively as in Definition 2.17.

In the case that $\lambda = 0$ we set $R^\mathfrak{h} := R^\mathfrak{h}_\lambda$. In particular, identifying $R^\mathfrak{h}$ with the polynomial ring $\bar{\mathrm{U}}(\mathfrak{n}^-)^\vee \otimes \bar{\mathrm{U}}(\mathfrak{n})^\vee$, we obtain a trace map

$$\mathrm{Tr}_- \otimes \mathrm{Tr}_+ : R^\mathfrak{h} \to R^\mathfrak{h}. \tag{2.7.5}$$

Define an endomorphism S_- of $\bar{\mathrm{U}}(\mathfrak{n}^-)^\vee$ by

$$(S_- f)(X) = f\big(F_0 \cdot \mathrm{Fr}'^- X\big) \tag{2.7.6a}$$

for all $f \in \bar{\mathrm{U}}(\mathfrak{n}^-)^\vee$ and $X \in \bar{\mathrm{U}}(\mathfrak{n}^-)$. Similarly, define an endomorphism S_+ of $\bar{\mathrm{U}}(\mathfrak{n})^\vee$ by

$$(S_+ g)(Y) = g\big(E_0 \cdot \mathrm{Fr}' Y\big) \tag{2.7.6b}$$

for all $g \in \bar{\mathrm{U}}(\mathfrak{n}^-)$ and $Y \in \bar{\mathrm{U}}(\mathfrak{n})$.

Lemma 2.19. *$S = S_- \otimes S_+$ as endomorphisms of $R^\mathfrak{h}$.*

Proof. Choose $X \in \bar{\mathrm{U}}(\mathfrak{n}^-)$, $Y \in \bar{\mathrm{U}}(\mathfrak{n})$, and $f \in R^\mathfrak{h}$. We need to show that

$$(Sf)(X \otimes Y) = f\big(F_0 \cdot \mathrm{Fr}'^- X \otimes E_0 \cdot \mathrm{Fr}' Y\big). \tag{2.7.7}$$

Now,

$$\begin{aligned}(Sf)(X \otimes Y) &= f\big(F_0 \cdot \varphi X \otimes E_0 \cdot \mathrm{Fr}' Y\big)\\ &= (\mu_0.f)\big(F_0 \cdot \mathrm{Fr}'^- X \otimes E_0 \cdot \mathrm{Fr}' Y\big).\end{aligned} \tag{2.7.8}$$

Without loss of generality we may assume that f is a weight vector of weight $\mu \in \Lambda$ and that X, Y are weight vectors of weight μ_X and μ_Y. Since $F_0 \cdot \mathrm{Fr}'^- X \otimes E_0 \cdot \mathrm{Fr}' Y$ is a weight vector of weight $p(\mu_X + \mu_Y) \in p\Lambda$ we have

$$f\big(F_0 \cdot \mathrm{Fr}'^- X \otimes E_0 \cdot \mathrm{Fr}' Y\big) = 0 \quad \text{unless} \quad \mu = -p(\mu_X + \mu_Y) \in p\Lambda. \tag{2.7.9}$$

In particular, if $\mu \notin p\Lambda$ then $\mu_0.f = 0$ and (2.7.7) follows from (2.7.8) and (2.7.9). On the other hand, if $\mu \in p\Lambda$ then $\mu_0.f = f$ and (2.7.7) follows from (2.7.8). $\square$

Proposition 2.20. *$S_- = \mathrm{Tr}_-$ and $S_+ = \mathrm{Tr}_+$ as endomorphisms of $\bar{\mathrm{U}}(\mathfrak{n}^-)^\vee$ and $\bar{\mathrm{U}}(\mathfrak{n})^\vee$, respectively. In particular, $S = \mathrm{Tr}_- \otimes \mathrm{Tr}_+$ as Frobenius-linear endomorphisms of $R^\mathfrak{h}$.*

Proof. We check that $S_+ = \mathrm{Tr}_+$; the fact that $S_- = \mathrm{Tr}_-$ follows from a similar argument. Since S_+ and Tr_+ are Frobenius-linear endomorphisms, they are completely determined by their values on the monomials $\prod_{\beta\in\Delta^+} y_\beta^{n_\beta}$ for $0 \leqslant n_\beta < p$, so it suffices to check that the values of S_+ and Tr_+ on those monomials are the same.

First consider a monomial $y := \prod_{\beta\in\Delta^+} y_\beta^{n_\beta}$ where $0 \leqslant n_\beta < p$ for all $\beta \in \Delta^+$ and $n_\beta < p-1$ for some β. Then $\mathrm{Tr}_+(y) = 0$ by definition. On the other hand, for all $X \in \bar{\mathrm{U}}(\mathfrak{n})$ we have

$$\big(S_+(y)\big)(X) = y\big(E_0 \cdot \mathrm{Fr}' X\big).$$

We may assume that X is a weight vector. Then $E_0 \cdot \mathrm{Fr}' X$ is a weight vector of weight $\geqslant (p-1)\rho$ and y is a weight vector of weight μ_y with $-(p-1)\rho < \mu_y \leqslant 0$. Hence $y(E_0 \cdot \mathrm{Fr}' X) = 0$ so that $\mathrm{Tr}_+(y) = S_+(y)$.

Next, we have $\mathrm{Tr}_+(y_0) = 1$ by definition. On the other hand, for all $X \in \bar{\mathrm{U}}(\mathfrak{n})$ we have

$$\begin{aligned}\big(S_+(y_0)\big)(X) &= y_0\big(E_0 \cdot \mathrm{Fr}' X\big)\\ &= y_0(E_0)\cdot \epsilon(X) \quad \text{(by weight considerations)}\\ &= \epsilon(X) \quad \big(\text{by (2.7.4)}\big)\\ &= 1(X).\end{aligned}$$

Thus $S_+ = \mathrm{Tr}_+$. $\square$

3. Base-change to *k* and main results

Recall that $k = \bar{\mathbb{F}}_p$. We no longer assume that all schemes are over $\mathbb{F}_p$. Recall that G_k, B_k, T_k, etc. are the groups obtained by base-changing G, B, T, etc. to k. In this section we base-change the above constructions to k and prove that $\mathcal{T}^* = T^*(G_k/B_k)$ is Frobenius split.

3.1. Review of Frobenius splitting facts

In this section we review the theory of Frobenius splitting. The main references are [2] and the seminal paper [12].

Let X be a scheme over k. We define a morphism $F : X \to X$ as follows: let F be the identity map on points and define $F^\# : \mathcal{O}_X \to F_*\mathcal{O}_X$ to be the p-th power map $f \mapsto f^p$. Note that although F is a morphism of $\mathbb{F}_p$-schemes, it is not a morphism of k-schemes. F is called the **absolute Frobenius morphism**.

Definition 3.1. We say that X is **Frobenius split** if there is an $\mathcal{O}_X$-linear map $\varphi : F_*\mathcal{O}_X \to \mathcal{O}_X$ such that $\varphi \circ F^\#$ is the identity map on $\mathcal{O}_X$.

For any invertible sheaf $\mathcal{L}$ on X we set

$$R_{\mathcal{L}} := \bigoplus_{n\geqslant 0} H^0\big(X, \mathcal{L}^n\big). \tag{3.1.1}$$

Recall the definition of a Frobenius split algebra from Definition 2.15. The following fact from [2] is the starting point for algebraic Frobenius splitting.

Proposition 3.2. *(See [2, Lemma 1.1.14].) Let $\mathcal{L}$ be an ample invertible sheaf on a complete k-scheme X. Then X is Frobenius split if and only if the k-algebra $R_{\mathcal{L}}$ is Frobenius split.*

3.2. Splitting of $\mathcal{T}^*$

3.2.1. Base-change

Set $\bar{U}_k(\mathfrak{g}) := \bar{\mathrm{U}}(\mathfrak{g}) \otimes_{\mathbb{F}_p} k$; we have similar definitions for $\bar{U}_k(\mathfrak{b})$, $\bar{U}_k(\mathfrak{b}^-)$, $\bar{U}_k(\mathfrak{n})$, $\bar{U}_k(\mathfrak{n}^-)$, and $\bar{U}_k^0$. Note that $\bar{U}_k(\mathfrak{g})$, $\bar{U}_k(\mathfrak{b})$, $\bar{U}_k(\mathfrak{b}^-)$, etc. are the hyperalgebras of G_k, B_k, B_k^-, etc. For any $\mathbb{F}_p$-module M set $M_k := M \otimes_{\mathbb{F}_p} k$. For $n \geqslant 0$ set

$$\bar{U}_n(\mathfrak{n})_k^\vee := \bar{\mathrm{U}}_n(\mathfrak{n})^\vee \otimes_{\mathbb{F}_p} k, \tag{3.2.1}$$

the degree-n component of $\bar{U}_k(\mathfrak{n})$.

Note that if M is a $\bar{\mathrm{U}}(\mathfrak{g})$, $\bar{\mathrm{U}}(\mathfrak{b})$, etc. module then M_k is a $\bar{U}_k(\mathfrak{g})$, $\bar{U}_k(\mathfrak{b})$, etc. module. For $\lambda \in \Lambda$ let χ_λ^k denote the 1-dimensional $\bar{U}_k(\mathfrak{b})$-module corresponding to the weight λ (equivalently, $\chi_\lambda^k = \chi_\lambda \otimes_{\mathbb{F}_p} k$).

For any $\bar{U}_k^0$ (resp. $\bar{U}_k(\mathfrak{g})$) module V we let, by a slight abuse of notation, $F_\mathfrak{h} V$ (resp. $F_\mathfrak{g} V$) denote the $\bar{U}_k^0$ (resp. $\bar{U}_k(\mathfrak{g})$) locally finite part of V, and we set $V^\vee := F_\mathfrak{h} V^*$.

For any $\bar{U}_k(\mathfrak{b})$-module N set

$$H_k^0(\bar{X}, N) := F_\mathfrak{g} \operatorname{Hom}_{\bar{U}_k(\mathfrak{b})}\big(\bar{U}_k(\mathfrak{g}), N\big). \tag{3.2.2}$$

Note that for any $\bar{\mathrm{U}}(\mathfrak{b})$-module M we have a $\bar{U}_k(\mathfrak{g})$-module isomorphism

$$H_k^0(\bar{X}, M_k) \cong H^0(\bar{X}, M) \otimes_{\mathbb{F}_p} k. \tag{3.2.3}$$

3.2.2. The splitting $\widetilde{S}_k$ of $\mathcal{T}^*$

Fix a regular dominant weight $\lambda \in \Lambda$ and set

$$R^k := R_\lambda \otimes_{\mathbb{F}_p} k = \bigoplus_{n \geqslant 0} H_k^0\big(\bar{X}, \bar{U}_n(\mathfrak{n})_k^\vee \otimes \chi_{-n\lambda}^k\big). \tag{3.2.4}$$

For any B_k-module M let $\mathcal{L}(M)$ denote the G_k-equivariant bundle on G_k/B_k with fiber M. By Proposition 3.7 in [1] we have

$$H^0\big(G_k/B_k, \mathcal{L}(M)\big) \cong H_k^0(\bar{X}, M). \tag{3.2.5}$$

Let $\mathbb{P}(\mathcal{T}^*)$ denote the projectivization of the bundle $\mathcal{T}^*$ and let $\mathcal{L}(\lambda)$ be the line bundle on G_k/B_k corresponding to the B_k-module $\chi_{-\lambda}^k$. Let

$$\mathcal{P}r : \mathbb{P}\big(\mathcal{T}^*\big) \to G_k/B_k \tag{3.2.6}$$

be the projection and set

$$\mathcal{M} := \mathcal{P}r^* \mathcal{L}(\lambda) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{T}^*)}(1). \tag{3.2.7}$$

Recall the ring

$$R_\mathcal{M} = \bigoplus_{n \geqslant 0} H^0\big(\mathbb{P}\big(\mathcal{T}^*\big), \mathcal{M}^n\big) \tag{3.2.8}$$

as in (3.1.1). By the projection formula and (3.2.5) we have

$$R_\mathcal{M} \cong R^k. \tag{3.2.9}$$

Also note that $\mathcal{M}$ is very ample on $\mathbb{P}(\mathcal{T}^*)$ because it is the pullback of the very ample bundle $\mathcal{L}(\lambda) \boxtimes \mathcal{O}_{\mathbb{P}(\mathfrak{g})}(1)$ under the inclusion

$$\mathbb{P}(\mathcal{T}^*) = G_k \times^{B_k} \mathbb{P}(\mathfrak{n}) \hookrightarrow G_k \times^{B_k} \mathbb{P}(\mathfrak{g}) \cong (G_k/B_k) \times \mathbb{P}(\mathfrak{g}). \tag{3.2.10}$$

By Lemma 1.1.11 in [2], if $\mathbb{P}(\mathcal{T}^*)$ is split then so is $\mathcal{T}^*$. Thus, to see that $\mathcal{T}^*$ is split, it suffices by Proposition 3.2 and (3.2.9) to show that R^k is a Frobenius split algebra.

Let $\theta : k \to k$ be the p-th power map and let $\theta' : k \to k$ be the p-th root map. Set

$$\widetilde{\mathrm{Fr}}_k^* := \widetilde{\mathrm{Fr}}^* \otimes_{\mathbb{F}_p} \theta : R^k \to R^k \tag{3.2.11}$$

and set

$$\widetilde{S}_k := \widetilde{S} \otimes_{\mathbb{F}_p} \theta' : R^k \to R^k. \tag{3.2.12}$$

Then, since $\widetilde{\mathrm{Fr}}^*$ is the p-th power morphism on R_λ, $\widetilde{\mathrm{Fr}}_k^*$ is the p-th power morphism on R^k. Also, since $\widetilde{S}$ is Frobenius-linear, so is $\widetilde{S}_k$. Finally, it follows from Theorem 2.16 that $\widetilde{S}_k \circ \widetilde{\mathrm{Fr}}_k^* = \mathrm{Id}$. We summarize this discussion as follows.

Theorem 3.3. *$\widetilde{S}_k$ is a Frobenius splitting of R^k. In particular, $\mathcal{T}^*$ is Frobenius split.*

3.2.3. Comparison with [8]

Set $\mathrm{St}_k := \mathrm{St} \otimes_{\mathbb{F}_p} k$ and let $\eta_k : \mathrm{St}_k \otimes \mathrm{St}_k \to k$ be the duality pairing. In [8] the authors construct, for any element $v \in \mathrm{St}_k \otimes \mathrm{St}_k$ such that $\eta_k(v) \neq 0$, a Frobenius splitting f_v of $\mathcal{T}^*$. Their construction also requires them to fix a Springer isomorphism $U \xrightarrow{\sim} \mathfrak{n}$ so let us assume that the isomorphism used in their construction is the same one we fixed in Section 2.2.2 above. In Section 7 of [8] they then construct, for any splitting f_v, a homogeneous splitting $\pi_{(p-1)N}(f_v)$ of $\mathcal{T}^*$. (In this context "homogeneous" means that the splitting divides degrees by p.)

Recall the highest and lowest weight elements $f_+, f_- \in \mathrm{St}$ as in Section 2.5. Set $f_+^k := f_+ \otimes 1 \in \mathrm{St}_k$ and $f_-^k := f_- \otimes 1 \in \mathrm{St}_k$.

Proposition 3.4. *The splitting of $\mathcal{T}^*$ induced by the splitting $\widetilde{S}_k$ of R^k is the same as the splitting $\pi_{(p-1)N}(f_{f_+^k \otimes f_-^k})$.*

Proof. Let $pr : \mathcal{T}^* \to G_k/B_k$ be the projection. Set

$$F_k := pr^{-1}(U_k^- B_k) \subseteq \mathcal{T}^*,$$

the fiber over the big cell. Then $F_k \cong U_k^- \times \mathfrak{n}_k$. Set $\mathcal{T}_{\mathbb{F}_p}^* := G \times^B \mathfrak{n}$ and set

$$F := U^- B \times \mathfrak{n} \subseteq \mathcal{T}_{\mathbb{F}_p}^*.$$

Then $\mathcal{T}^* = \mathcal{T}_{\mathbb{F}_p}^* \times^{\mathbb{F}_p} k$ and $F_k = F \times^{\mathbb{F}_p} k$. It suffices to check that the two splittings coincide on the open set $F_k \subseteq \mathcal{T}^*$.

Denote by Ψ_k the restriction of the splitting $\pi_{(p-1)N}(f_{f_+^k \otimes f_-^k})$ to F_k. We now define a splitting Ψ of F such that Ψ_k is the base-change to k (along with a twist by the p-th root map θ') of Ψ. Using our chosen Springer isomorphism we have

$$\mathbb{F}_p[F] \cong \mathbb{F}_p[U^-] \otimes \mathbb{F}_p[U]. \tag{3.2.13}$$

For each $m \geqslant 0$ let $\mathbb{F}_p[U]_m$ denote the degree-m component via the identification $\mathbb{F}_p[U] \cong \mathbb{F}_p[\mathfrak{n}]$. Also recall from (2.5.2a) the definition of the morphism $\bar{\psi} : \mathrm{St} \otimes \mathrm{St} \to \bar{\mathrm{U}}(\mathfrak{n})^\vee$. Using the identification

$$H^0\big(\bar{X}, \bar{\mathrm{U}}(\mathfrak{n})^\vee\big) \cong H^0\big(\mathcal{T}^*_{\mathbb{F}_p}, \mathcal{O}_{\mathcal{T}^*_{\mathbb{F}_p}}\big),$$

we obtain a morphism

$$\hat{\psi} := H^0(\bar{\psi}) : \mathrm{St} \otimes \mathrm{St} \to H^0\big(\mathcal{T}^*_{\mathbb{F}_p}, \mathcal{O}_{\mathcal{T}^*_{\mathbb{F}_p}}\big), \quad v \otimes w \mapsto \hat{\psi}_{v \otimes w}. \tag{3.2.14}$$

Now, following [8], Ψ is defined by the direct sum of the following compositions for $n \geqslant 0$:

$$\begin{aligned} \mathbb{F}_p\big[U^-\big] \otimes \mathbb{F}_p[U]_{pn} &\xrightarrow{\cdot \hat{\psi}_{f_+ \otimes f_-}} \mathbb{F}_p\big[U^-\big] \otimes \mathbb{F}_p[U] \\ &\xrightarrow{\mathrm{Tr}_- \otimes \mathrm{Tr}_+} \mathbb{F}_p\big[U^-\big] \otimes \mathbb{F}_p[U] \xrightarrow{q_n} \mathbb{F}_p\big[U^-\big] \otimes \mathbb{F}_p[U]_n, \end{aligned} \tag{3.2.15}$$

where $\cdot \hat{\psi}_{f_+ \otimes f_-}$ denotes multiplication by the function $\hat{\psi}_{f_+ \otimes f_-} \in H^0(\mathcal{T}^*_{\mathbb{F}_p}, \mathcal{O}_{\mathcal{T}^*_{\mathbb{F}_p}})$, $\mathrm{Tr}_- \otimes \mathrm{Tr}_+$ is the trace morphism as in (2.7.5), and q_n is projection onto the n-th homogeneous component $\mathbb{F}_p[U^-] \otimes \mathbb{F}_p[U]_n$. We also set

$$\Psi\big(\mathbb{F}_p[U] \otimes \mathbb{F}_p\big[U^-\big]_m\big) = 0 \quad \text{if } p \nmid m. \tag{3.2.16}$$

It now suffices to verify that the splitting of $\mathbb{F}_p[F]$ induced by $\widetilde{S}$ is the same as Ψ.

Now, the splitting of $\mathbb{F}_p[F]$ induced by the splitting $\widetilde{S}$ of the ring $R^{\mathfrak{h}}_\lambda$ comes from the $\mathbb{F}_p$-algebra isomorphism

$$R^{\mathfrak{h}}_\lambda \cong \mathbb{F}_p[F] \tag{3.2.17}$$

constructed as follows. First, recall that when $\lambda = 0$ we set $R^{\mathfrak{h}} = R^{\mathfrak{h}}_\lambda$. As in Section 2.7, we have isomorphisms

$$\mathbb{F}_p[F] \cong \mathbb{F}_p\big[U^-\big] \otimes \mathbb{F}_p[U] \cong \mathbb{F}_p\big[\mathfrak{n}^-\big] \otimes \mathbb{F}_p[\mathfrak{n}] \cong \bar{\mathrm{U}}\big(\mathfrak{n}^-\big)^\vee \otimes \bar{\mathrm{U}}(\mathfrak{n})^\vee \cong R^{\mathfrak{h}}. \tag{3.2.18}$$

Note that for each $\lambda \in \Lambda$ there is a natural $\mathbb{F}_p$-algebra isomorphism

$$\bigoplus_{n \geqslant 0} \bar{\mathrm{U}}_n(\mathfrak{n})^\vee \otimes \chi_{-n\lambda} \cong \bar{\mathrm{U}}(\mathfrak{n})^\vee \tag{3.2.19a}$$

which is *not*, however, even $\bar{\mathrm{U}}^0$-equivariant. Thus, via the identification (2.3.2), we get a natural $\mathbb{F}_p$-algebra isomorphism

$$r_\lambda : R^{\mathfrak{h}}_\lambda = \bigoplus_{n \geqslant 0} \bar{\mathrm{U}}\big(\mathfrak{n}^-\big)^\vee \otimes \bar{\mathrm{U}}_n(\mathfrak{n})^\vee \otimes \chi_{-n\lambda} \xrightarrow{\sim} \bigoplus_{n \geqslant 0} \bar{\mathrm{U}}\big(\mathfrak{n}^-\big)^\vee \otimes \bar{\mathrm{U}}_n(\mathfrak{n})^\vee = R^{\mathfrak{h}} \tag{3.2.19b}$$

given explicitly by

$$(r_\lambda f)(X \otimes Y) = f(X \otimes Y \otimes v_{n\lambda}) \tag{3.2.19c}$$

for all $n \geqslant 0$, $X \in \bar{\mathrm{U}}(\mathfrak{n}^-)$, and $Y \in \bar{\mathrm{U}}_n(\mathfrak{n})$. (As above, though, this is not $\bar{\mathrm{U}}^0$-equivariant.) Combining (3.2.18) and (3.2.19b) we get the desired isomorphism (3.2.17).

Now, it is easy to see that the following diagram commutes for all λ:

$$\begin{tikzcd} R^{\mathfrak{h}}_{\lambda} \arrow[r, "r_\lambda"] \arrow[d, "\widetilde{S}"'] & R^{\mathfrak{h}} \arrow[d, "\widetilde{S}"] \\ R^{\mathfrak{h}}_{\lambda} \arrow[r, "r_\lambda"'] & R^{\mathfrak{h}}. \end{tikzcd} \tag{3.2.20}$$

Also, by (3.2.18) we can consider Ψ as a splitting of $R^{\mathfrak{h}}$. Hence it suffices to check that Ψ and $\widetilde{S}$ are equal, considered as splittings of $R^{\mathfrak{h}}$.

First, we have that Ψ and $\widetilde{S}$ are both zero on homogeneous elements of $R^{\mathfrak{h}}$ of degree $m \nmid p$. Next, considering $\mathrm{Tr}_- \otimes \mathrm{Tr}_+$ as an endomorphism of $R^{\mathfrak{h}}$ as in Section 2.7, by (3.2.15) we have that Ψ is given on the pn-th homogeneous component of $R^{\mathfrak{h}}$ by the following composition:

$$\bar{\mathrm{U}}(\mathfrak{n}^-)^{\vee} \otimes \bar{\mathrm{U}}_{pn}(\mathfrak{n})^{\vee} \xrightarrow{\cdot\hat{\psi}_{f_+ \otimes f_-}} R^{\mathfrak{h}} \xrightarrow{\mathrm{Tr}_- \otimes \mathrm{Tr}_+} R^{\mathfrak{h}} \xrightarrow{q_n} \bar{\mathrm{U}}(\mathfrak{n}^-)^{\vee} \otimes \bar{\mathrm{U}}_{n}(\mathfrak{n})^{\vee}. \tag{3.2.21a}$$

Here we denote, as above, the projection onto the n-th homogeneous component of $R^{\mathfrak{h}}$ by q_n. Since $\mathrm{Tr}_- \otimes \mathrm{Tr}_+$ sends elements of degree $pn + (p-1)N$ to elements of degree n, this is the same as the composition

$$\bar{\mathrm{U}}(\mathfrak{n}^-)^{\vee} \otimes \bar{\mathrm{U}}_{pn}(\mathfrak{n})^{\vee} \xrightarrow{\cdot\hat{\psi}_{f_+ \otimes f_-}} R^{\mathfrak{h}} \xrightarrow{q_{pn+(p-1)N}} R^{\mathfrak{h}} \xrightarrow{\mathrm{Tr}_- \otimes \mathrm{Tr}_+} \bar{\mathrm{U}}(\mathfrak{n}^-)^{\vee} \otimes \bar{\mathrm{U}}_{n}(\mathfrak{n})^{\vee}. \tag{3.2.21b}$$

On the other hand, recall that $\widetilde{S}$ is given on the pn-th homogeneous component of $R^{\mathfrak{h}}$ by

$$\bar{\mathrm{U}}(\mathfrak{n}^-)^{\vee} \otimes \bar{\mathrm{U}}_{pn}(\mathfrak{n})^{\vee} \xrightarrow{M_{f_+ \otimes f_-}} R^{\mathfrak{h}} \xrightarrow{S} \bar{\mathrm{U}}(\mathfrak{n}^-)^{\vee} \otimes \bar{\mathrm{U}}_{n}(\mathfrak{n})^{\vee}. \tag{3.2.22}$$

Now, by the definition of ψ in (2.5.4), we have that $\psi = q_{(p-1)N} \circ \hat{\psi}$. Hence for all $f \in \bar{\mathrm{U}}(\mathfrak{n}^-)^{\vee} \otimes \bar{\mathrm{U}}_{pn}(\mathfrak{n})^{\vee}$ we have

$$\begin{aligned} M_{f_+ \otimes f_-}(f) &= f \cdot \psi_{f_+ \otimes f_-} \\ &= f \cdot \big(q_{(p-1)N}(\hat{\psi}_{f_+ \otimes f_-})\big) \\ &= q_{pn+(p-1)N}(f \cdot \hat{\psi}_{f_+ \otimes f_-}). \end{aligned} \tag{3.2.23}$$

Also, by Proposition 2.20, $S = \mathrm{Tr}_- \otimes \mathrm{Tr}_+$. Thus (3.2.21b) and (3.2.22) are the same morphism, so we have that Ψ and $\widetilde{S}$ give the same splitting of $R^{\mathfrak{h}}$ as desired. □

Remarks 3.5.

(1) Although the rings R_λ are nonisomorphic for various choices of λ, the splitting of $\mathcal{T}^*$ induced by $\widetilde{S}_k$ does not depend on the choice of regular dominant $\lambda \in \Lambda$. Indeed, $\widetilde{S}_k$ restricts to the same splitting (3.2.22) of the open set $F_k \subseteq \mathcal{T}^*$ regardless of the choice of λ.

(2) For a parabolic subalgebra $\mathfrak{p} \supseteq \mathfrak{b}$ let $\mathfrak{n}_{\mathfrak{p}}$ denote its nilradical. In [13] and [15] it is shown that in type A the splitting $\pi_{(p-1)N}(f_{f_+^k \otimes f_-^k})$ compatibly splits the subbundles $G_k \times^{B_k} (\mathfrak{n}_{\mathfrak{p}})_k$ for every parabolic subalgebra $\mathfrak{p} \supseteq \mathfrak{b}$. A main hope of algebraic Frobenius splitting is to extend this result to other types.

(3) Since the splitting $\pi_{(p-1)N}(f_{f_+\otimes f_-})$ is B-canonical we have that the splitting $\widetilde{S}_k$ is also B-canonical. In the algebraic context B-canonicity is equivalent to the fact that

$$\widetilde{S}(\varphi Z.f) = Z.(\widetilde{S}f) \quad \text{for all } f \in R_\lambda \text{ and } Z \in \bar{\mathrm{U}}(\mathfrak{b}). \tag{3.2.24}$$

However, I do not know how to show this directly.

(4) By Proposition 4.1.17 in [2], if $\mathfrak{n}$ were B-canonically split then one would immediately obtain a B-canonical splitting of $\mathcal{T}^*$ as well. Since $\mathcal{T}^*$ is B-canonically split, it is tempting to try to use algebraic techniques to construct a B-canonical splitting of $\mathfrak{n}$. However, by the following argument due to Kumar, it is known that $\mathfrak{n}$ is *not* B-canonically split.
Indeed, if $\mathfrak{n}$ were B-canonically split, then by Exercise 4.1.E(4) in [2] $\mathcal{T}^*$ would be split compatibly with the divisor $D := (p-1)\pi^*\partial(G_k/B_k)$. Here, $\pi : \mathcal{T}^* \to G_k/B_k$ is the projection and $\partial(G_k/B_k) \subseteq G_k/B_k$ is the divisor $\bigcup_{i=1}^{\ell} X_{w_0 s_i}$, where the $s_i \in W$ are the simple reflections, w_0 is the longest element of the Weyl group, and for any element w of the Weyl group, $X_w := \overline{BwB} \subseteq G_k/B_k$ is the associated Schubert variety. Now,

$$\mathcal{O}_{G_k/B_k}(D) \cong \pi^*\mathcal{L}\big((p-1)\rho\big),$$

so by Lemma 1.4.7 (i) of [2] we would have the following consequence: If $\lambda \in \Lambda$ is such that $\pi^*\mathcal{L}(p\lambda + (p-1)\rho)$ has higher cohomology vanishing on $\mathcal{T}^*$ then so does $\pi^*\mathcal{L}(\lambda)$. By base-change this would also be true in characteristic 0; but this is known to be false (cf. [3]).

(5) Replacing the $*$-action of $\bar{\mathrm{U}}(\mathfrak{b})$ on $\bar{\mathrm{U}}(\mathfrak{n})$ by the multiplication action, one can construct an algebraic splitting of the affine variety $G_k/T_k \cong G_k \times^{B_k} U_k$. Note that here one does not need to use a Springer isomorphism.

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References

[1] Henning Haahr Andersen, Patrick Polo, Ke Xin Wen, Representations of quantum algebras, Invent. Math. 104 (1) (1991) 1–59.
[2] M. Brion, S. Kumar, Frobenius Splitting Methods in Geometry and Representation Theory, Progr. Math., vol. 231, Birkhäuser, Boston, 2005.
[3] B. Broer, Normality of some nilpotent varieties and cohomology of line bundles on the cotangent bundle of the flag variety, in: Lie Theory and Geometry, in: Progr. Math., Birkhäuser, 1994, pp. 1–19.
[4] Eric M. Friedlander, Brian J. Parshall, Rational actions associated to the adjoint representation, Ann. Sci. École Norm. Sup. (4) 20 (2) (1987) 215–226.
[5] W.J. Haboush, Central differential operators on split semisimple groups over fields of positive characteristic, in: Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin, 32ème année, Paris, 1979, in: Lecture Notes in Math., vol. 795, Springer, Berlin, 1980, pp. 35–85.
[6] Jens Carsten Jantzen, Representations of Algebraic Groups, Math. Surveys Monogr., vol. 107, Amer. Math. Soc., 2003.
[7] Masaharu Kaneda, Michel Gros, Contraction par Frobenius de G-modules, arXiv:1004.1939, 2010.
[8] Shrawan Kumar, Niels Lauritzen, Jesper Funch Thomsen, Frobenius splitting of cotangent bundles of flag varieties, Invent. Math. 136 (3) (1999) 603–621.
[9] Shrawan Kumar, Peter Littelmann, Frobenius splitting in characteristic zero and the quantum Frobenius map, J. Pure Appl. Algebra 152 (2000).
[10] Shrawan Kumar, Peter Littelmann, Algebraization of Frobenius splitting via quantum groups, Ann. of Math. 155 (2002) 491–551.
[11] George Lusztig, Quantum groups at roots of 1, Geom. Dedicata 35 (1990) 89–113.
[12] V.B. Mehta, A. Ramanathan, Frobenius splitting and cohomology vanishing for Schubert varieties, Ann. of Math. (2) 122 (1) (1985) 27–40.
[13] V.B. Mehta, Wilberd van der Kallen, A simultaneous Frobenius splitting for closures of conjugacy classes of nilpotent matrices, Compos. Math. 84 (1992) 211–221.
[14] T.A. Springer, The unipotent variety of a semi-simple group, in: Algebraic Geometry, Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968, Oxford Univ. Press, Bombay, 1969, pp. 373–391.
[15] Wilberd van der Kallen, Addendum to: A simultaneous Frobenius splitting for closures of conjugacy classes of nilpotent matrices, arXiv:0803.2960v2.