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Decomposition of torsion pairs on module categories ☆

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A R T I C L E I N F O

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A B S T R A C T

In this article, we generalize the concept of torsion pair and study its structure. As a trial of obtaining all torsion pairs, we decompose torsion pairs by projective modules and injective modules. Then we calculate torsion pairs on the algebra KA_n and tube categories. At last we study the structure of torsion pairs on the module categories of finite-dimensional hereditary algebras.

1. Introduction

The concept of torsion pair on abelian category was introduced by Dickson in 1966 [8]. From that time on, torsion pair has become a useful tool to study the structure of module categories. However, it seems there is no useful way to find all torsion pairs of a given algebra, although there are indeed some ways to construct torsion pairs among which the most well known is the tilting theory. As a trial, we try to find a way to obtain all torsion pairs of hereditary algebras. This topic is also discussed by Assem and Kerner in [1] where they classify and characterize the torsion pairs of hereditary algebras by partial tilting modules.

In Section 2, we study the general theory where we introduce torsion $n+1$-tuple as a generalization of torsion pair and study its structure which would be used in the later sections frequently. The main skill in this section is from [12] and [6] where they study HN-filtration for some categories.

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It is helpful to study this finer structure of module categories. For example perpendicular category [9] can be viewed to be obtained by a torsion 3-tuple.

In [1], Assem and Kerner show a relation between some particular partial tilting modules and torsion pairs. In Section 3, we adopt their ways by restricting to projective modules and injective modules to try to decompose all torsion pairs. We give a method for how to decompose a classic torsion pair to torsion $n+1$-tuple and find their correspondence.

In Section 4, we apply the theory in Section 3 to path algebras. As examples, we obtain all the torsion pairs on path algebra KA_n and tube categories. This topic also has been studied in [3] and [5]. But we think our results would be clearer in some aspects.

Section 5 is devoted to obtain all torsion pairs of hereditary algebras. We define an operation called the translation of torsion pairs. Combining this with the operation developed in Sections 3 and 4, the issue comes down to find all torsion pairs on regular component. For tame hereditary algebras, this problem is equivalent to calculate all torsion pairs on the tube categories in Section 4.

If there is no special instruction, all modules are assumed to be finitely generated left modules. For an artin algebra Λ, we denote by Λ-mod the category of all finitely generated left Λ-modules, by $\mathcal{P}(\Lambda)$ the full subcategory of all projective Λ-modules, by $\mathcal{I}(\Lambda)$ the full subcategory of all injective Λ-modules. If $\mathcal{D}$ is a full subcategory of Λ-mod, then we denote by ${}^{\perp}\mathcal{D}$ the full subcategory of Λ-mod$\{M \in \Lambda\text{-mod} \mid \mathrm{Hom}(M,N)=0,\ \forall N \in \mathcal{D}\}$, by $\mathcal{D}^{\perp}$ the full subcategory of Λ-mod$\{N \in \Lambda\text{-mod} \mid \mathrm{Hom}(M,N)=0,\ \forall M \in \mathcal{D}\}$, and by $\mathrm{Ind}\,\mathcal{D}$ the set of nonzero pairwise non-isomorphic indecomposable modules in $\mathcal{D}$. Subcategories are always assumed to be closed under isomorphism.

2. Torsion $n+1$-tuple

In this section, we assume that Λ is an artin algebra and $\mathcal{C}$ is an extension-closed full subcategory of Λ-mod. If $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ are full subcategories of Λ-mod, then we denote the minimal extension-closed full subcategory containing $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ by $\langle \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n \rangle$. The following definition is well known but slightly different from that in [2].

Definition 2.1. A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\mathcal{C}$ is called a torsion pair on $\mathcal{C}$ if the following conditions are satisfied.

(1) $\mathrm{Hom}(X,Y)=0$ for all $X \in \mathcal{T}$, $Y \in \mathcal{F}$.
(2) $\forall X \in \mathcal{C}$, there exists a short exact sequence on Λ-mod

$$0 \to X_{\mathcal{T}} \to X \to X_{\mathcal{F}} \to 0$$

such that $X_{\mathcal{T}} \in \mathcal{T}$ and $X_{\mathcal{F}} \in \mathcal{F}$.

Remark 2.2.

(1) Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair on $\mathcal{C}$. Then $\mathcal{T} = {}^{\perp}\mathcal{F} \cap \mathcal{C}$; $\mathcal{F} = \mathcal{T}^{\perp} \cap \mathcal{C}$; $\mathcal{T}$ and $\mathcal{F}$ are closed under extensions.
(2) It is well known that the exact sequence is unique up to isomorphism. So we call it a canonical short exact sequence of X induced by $(\mathcal{T}, \mathcal{F})$.

Definition 2.3. Let $n \geqslant 1$. An $n+1$-tuple $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{n+1})$ of full subcategories of $\mathcal{C}$ is called a torsion $n+1$-tuple on $\mathcal{C}$ if the following conditions are satisfied.

(1) $\mathcal{C}_1 = \mathcal{C} \cap {}^{\perp}\langle \mathcal{C}_2, \ldots, \mathcal{C}_{n+1} \rangle$, $\mathcal{C}_i = \mathcal{C} \cap \langle \mathcal{C}_1, \ldots, \mathcal{C}_{i-1} \rangle^{\perp} \cap {}^{\perp}\langle \mathcal{C}_{i+1}, \ldots, \mathcal{C}_{n+1} \rangle$ for $i = 2, \ldots, n$, $\mathcal{C}_{n+1} = \mathcal{C} \cap \langle \mathcal{C}_1, \ldots, \mathcal{C}_n \rangle^{\perp}$.
(2) $(\langle \mathcal{C}_1, \ldots, \mathcal{C}_i \rangle, \langle \mathcal{C}_{i+1}, \ldots, \mathcal{C}_{n+1} \rangle)$ is a torsion pair on $\mathcal{C}$ for $i = 1, 2, \ldots, n$.

Moreover, if (1) is not satisfied, we call it a defect torsion $n+1$-tuple on $\mathcal{C}$.

Remark 2.4.

(1) The torsion pair is a torsion 2-tuple.
(2) We abbreviate the condition (1) as that $\mathcal{C}_i = \mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1}\rangle^{\perp} \cap {}^{\perp}\langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1}\rangle$ for $i = 1, \dots, n+1$. The similar way would be used in the rest of the section.

Lemma 2.5. *Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ be full subcategories of Λ-mod. Then*

(1) $\langle \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\rangle = \langle\langle \mathcal{C}_1, \mathcal{C}_2\rangle, \mathcal{C}_3\rangle = \langle \mathcal{C}_1, \langle \mathcal{C}_2, \mathcal{C}_3\rangle\rangle$;
(2) ${}^{\perp}\langle \mathcal{C}_1, \mathcal{C}_2\rangle = {}^{\perp}\mathcal{C}_1 \cap {}^{\perp}\mathcal{C}_2$, $\langle \mathcal{C}_1, \mathcal{C}_2\rangle^{\perp} = \mathcal{C}_1{}^{\perp} \cap \mathcal{C}_2^{\perp}$;
(3) ${}^{\perp}\langle \mathcal{C}_1\rangle = {}^{\perp}\mathcal{C}_1$, $\langle \mathcal{C}_1\rangle^{\perp} = \mathcal{C}_1{}^{\perp}$.

Proposition 2.6. *Let $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1})$ be a torsion $n+1$-tuple on $\mathcal{C}$ $(n \neq 1)$. And for some $i = 1, 2, \dots, n+1$, $k \neq 1$, let $(\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2, \dots, \tilde{\mathcal{C}}_{k+1})$ be a torsion $k+1$-tuple on $\mathcal{C}_i$. Then $(\mathcal{C}_1, \dots, \mathcal{C}_{i-1}, \tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_{k+1}, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1})$ is a torsion $n+k+1$-tuple on $\mathcal{C}$.*

Proof. *Step* 1. Because $C_i = \langle \tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2, \dots, \tilde{\mathcal{C}}_{k+1}\rangle$, thus if $\mathcal{C}_s \in \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{i-1}\}$, then

$$\mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_{s-1}\rangle^{\perp} \cap {}^{\perp}\langle \mathcal{C}_{s+1}, \dots, \mathcal{C}_{i-1}, \tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_{k+1}, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1}\rangle = \mathcal{C}_s.$$

Similar argument holds for $\mathcal{C}_s \in \{\mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1}\}$.
If $\tilde{\mathcal{C}}_s \in \{\tilde{\mathcal{C}}_1, \tilde{\mathcal{C}}_2, \dots, \tilde{\mathcal{C}}_{k+1}\}$, then

$$\begin{aligned}
&\mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1}, \tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_{s-1}\rangle^{\perp} \cap {}^{\perp}\langle \tilde{\mathcal{C}}_{s+1}, \dots, \tilde{\mathcal{C}}_{k+1}, \dots, \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1}\rangle \\
&\quad = \mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1}\rangle^{\perp} \cap \langle \tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_{s-1}\rangle^{\perp} \cap {}^{\perp}\langle \tilde{\mathcal{C}}_{s+1}, \dots, \tilde{\mathcal{C}}_{k+1}\rangle \cap {}^{\perp}\langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1}\rangle \\
&\quad = \mathcal{C}_i \cap \langle \tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_{s-1}\rangle^{\perp} \cap {}^{\perp}\langle \tilde{\mathcal{C}}_{s+1}, \dots, \tilde{\mathcal{C}}_{k+1}\rangle = \tilde{\mathcal{C}}_s.
\end{aligned}$$

Step 2. Let $1 \leqslant s \leqslant k$. We claim that $(\langle \mathcal{C}_1, \dots, \tilde{\mathcal{C}}_s\rangle, \langle \tilde{\mathcal{C}}_{s+1}, \dots, \mathcal{C}_{n+1}\rangle)$ is a torsion pair on $\mathcal{C}$.

Given $X \in \mathcal{C}$, the torsion pair $(\langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1}\rangle, \langle \mathcal{C}_i, \dots, \mathcal{C}_{n+1}\rangle)$ on $\mathcal{C}$ induces a canonical short exact sequence

$$0 \to X_1 \xrightarrow{i_1} X \xrightarrow{\pi_1} X_2 \to 0.$$

For X_2, consider the canonical short exact sequence induced by the torsion pair $(\langle \mathcal{C}_1, \dots, \mathcal{C}_i\rangle, \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1}\rangle)$

$$0 \to X_3 \xrightarrow{i_2} X_2 \xrightarrow{\pi_2} X_4 \to 0.$$

Because $X_3 \in \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1}\rangle^{\perp}$ since $X_2 \in \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1}\rangle^{\perp}$, so $X_3 \in \mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1}\rangle^{\perp} \cap {}^{\perp}\langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1}\rangle = \mathcal{C}_i$.

For X_3, the torsion pair $(\langle \tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_s\rangle, \langle \tilde{\mathcal{C}}_{s+1}, \dots, \tilde{\mathcal{C}}_{k+1}\rangle)$ on $\mathcal{C}_i$ induces a canonical short exact sequence

$$0 \to X_5 \xrightarrow{i_3} X_3 \xrightarrow{\pi_3} X_6 \to 0.$$

By push-out of i_2 and π_3, we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & X_5 \arrow[r, equal] \arrow[d, "i_3"'] & X_5 \arrow[d, "i_4"] & & \\
0 \arrow[r] & X_3 \arrow[r, "i_2"] \arrow[d, "\pi_3"'] & X_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_4"] & X_4 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X_6 \arrow[r] \arrow[d] & X_{\mathcal{F}} \arrow[r] \arrow[d] & X_4 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Hence we have a short exact sequence

$$0 \to X_5 \xrightarrow{i_4} X_2 \xrightarrow{\pi_4} X_{\mathcal{F}} \to 0$$

such that $X_{\mathcal{F}} \in \langle \tilde{\mathcal{C}}_{s+1}, \ldots, \tilde{\mathcal{C}}_{k+1}, \mathcal{C}_{i+1}, \ldots, \mathcal{C}_{n+1} \rangle$.

By pull-back of i_4 and π_1, we have the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X_1 \arrow[r] \arrow[d, equal] & X_{\mathcal{T}} \arrow[r] \arrow[d] & X_5 \arrow[r] \arrow[d, "i_4"] & 0 \\
0 \arrow[r] & X_1 \arrow[r, "i_1"'] & X \arrow[r, "\pi_1"'] \arrow[d] & X_2 \arrow[r] \arrow[d, "\pi_4"] & 0 \\
 & & X_{\mathcal{F}} \arrow[r, equal] \arrow[d] & X_{\mathcal{F}} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Hence we have the following exact sequence

$$0 \to X_{\mathcal{T}} \to X \to X_{\mathcal{F}} \to 0$$

such that $X_{\mathcal{T}} \in \langle \mathcal{C}_1, \ldots, \tilde{\mathcal{C}}_s \rangle$ and $X_{\mathcal{F}} \in \langle \tilde{\mathcal{C}}_{s+1}, \ldots, \mathcal{C}_{n+1} \rangle$. $\square$

Now we give the following definition which is helpful to learn the structure of torsion $n+1$-tuple.

Definition 2.7. For $n \geqslant 1$, series $\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2), \ldots, (\mathcal{T}_n, \mathcal{F}_n)\}$ of torsion pairs on $\mathcal{C}$ are called a chain of torsion classes of length n if $\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \cdots \subseteq \mathcal{T}_n$ (equivalently, $\mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \cdots \supseteq \mathcal{F}_n$).

Remark 2.8. For convenience, we take the denotation: $\mathcal{F}_0 = \mathcal{T}_{n+1} = \mathcal{C}$, $\mathcal{F}_{n+1} = \mathcal{T}_0 = \{0\}$. This assumption will simplify some descriptions.

Definition 2.9. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair on $\mathcal{C}$, and $\mathcal{D}$ be a subcategory of $\mathcal{C}$. We call $(D^1_{(\mathcal{T},\mathcal{F})}(\mathcal{D}), D^2_{(\mathcal{T},\mathcal{F})}(\mathcal{D}))$ a decomposition of $\mathcal{D}$ along $(\mathcal{T}, \mathcal{F})$, where $D^1_{(\mathcal{T},\mathcal{F})}(\mathcal{D}) = \{X \mid \exists$ exact sequence $0 \to X \to M \to Y \to 0$ with $X \in \mathcal{T},\ Y \in \mathcal{F},\ M \in \mathcal{D}\}$, $D^2_{(\mathcal{T},\mathcal{F})}(\mathcal{D}) = \{Y \mid \exists$ exact sequence $0 \to X \to M \to Y \to 0$ with $X \in \mathcal{T},\ Y \in \mathcal{F},\ M \in \mathcal{D}\}$.

Lemma 2.10. *If $\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2)\}$ is a chain of torsion classes of length 2 on $\mathcal{C}$, then*

$$\mathcal{F}_1 \cap \mathcal{T}_2 = D^2_{(\mathcal{T}_1,\mathcal{F}_1)}(\mathcal{T}_2) = D^1_{(\mathcal{T}_2,\mathcal{F}_2)}(\mathcal{F}_1).$$

Proof. $\mathcal{F}_1 \cap \mathcal{T}_2 \subseteq D^2_{(\mathcal{T}_1,\mathcal{F}_1)}(\mathcal{T}_2)$ is clear. Now given $X \in \mathcal{T}_2$, consider the canonical short exact sequence induced by $(\mathcal{T}_1, \mathcal{F}_1)$

$$0 \to X_{\mathcal{T}_1} \to X \to X_{\mathcal{F}_1} \to 0.$$

However $X_{\mathcal{F}_1} \in \mathcal{T}_2$ since so is X. Thus $X_{\mathcal{F}_1} \in \mathcal{T}_2 \cap \mathcal{F}_1$. Hence $\mathcal{F}_1 \cap \mathcal{T}_2 \supseteq D^2_{(\mathcal{T}_1,\mathcal{F}_1)}(\mathcal{T}_2)$.

The other half is similar. □

Chain of torsion classes will induce filtrations for modules in $\mathcal{C}$ as following.

Proposition 2.11. *Let $\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2), \dots, (\mathcal{T}_n, \mathcal{F}_n)\}$ be a chain of torsion classes of length n on $\mathcal{C}(n \geqslant 1)$. Then for every module X in $\mathcal{C}$, there is a filtration*

$$\begin{tikzcd} 0 \arrow[r, equal] & X_0 \arrow[r] & X_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & X_{n+1} \arrow[r, equal] \arrow[dl] & X \\ & S_1 & & S_{n+1} & & \end{tikzcd}$$

such that $0 \to X_i \to X_{i+1} \to S_{i+1} \to 0$ is an exact sequence, $S_{i+1} \in \mathcal{F}_i \cap \mathcal{T}_{i+1}$ and $X_{i+1} \in \mathcal{T}_{i+1}$ for $i = 0, 1, 2, \dots, n$.

Proof. Using induction on n. For $n = 1$, it's clear. Now suppose the proposition holds for $n = k$ for some $k \geqslant 1$. Let us consider $n = k + 1$. For $X \in \mathcal{C}$, $(\mathcal{T}_{k+1}, \mathcal{F}_{k+1})$ induces the following canonical short exact sequence

$$0 \to X_{k+1} \to X \to S_{k+2} \to 0.$$

Because $\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2), \dots, (\mathcal{T}_k, \mathcal{F}_k)\}$ is a chain of torsion classes of length k on $\mathcal{C}$, by induction, there is a filtration

$$\begin{tikzcd} 0 \arrow[r, equal] & X_0 \arrow[r] & X_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & X_k \arrow[r] \arrow[dl] & X_{k+1} \arrow[r, equal] \arrow[dl] & X_{k+1} \\ & S_1 & & S_k & S_{k+1} & & \end{tikzcd}$$

such that $0 \to X_i \to X_{i+1} \to S_{i+1} \to 0$ is a short exact sequence for $i = 0, 1, 2, \dots, k$, and $S_1 \in \mathcal{T}_1$, $S_i \in \mathcal{F}_{i-1} \cap \mathcal{T}_i$ for $1 < i < k+1$, $S_{k+1} \in \mathcal{F}_k$, $X_i \in \mathcal{T}_i$ for $1 \leqslant i \leqslant k+1$. Thus $S_{k+1} \in \mathcal{F}_k \cap \mathcal{T}_{k+1}$ since $X_{k+1} \in \mathcal{T}_{k+1}$. So we get the desired filtration. $\square$

In fact the above filtration is unique up to isomorphism since all decompositions induced by torsion pairs are canonical. So we say this is a canonical filtration (or decomposition) induced by the chain of torsion classes.

Proposition 2.12. *If $\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2), \dots, (\mathcal{T}_n, \mathcal{F}_n)\}$ is a chain of torsion classes of length n on $\mathcal{C}$ $(n \geqslant 1)$, then $\mathcal{F}_i \cap \mathcal{T}_{i+k} = \langle \mathcal{F}_i \cap \mathcal{T}_{i+1}, \mathcal{F}_{i+1} \cap \mathcal{T}_{i+2}, \dots, \mathcal{F}_{i+k-1} \cap \mathcal{T}_{i+k} \rangle$ for $0 \leqslant i < i+k \leqslant n+1$.*

Proof. For $X \in \mathcal{F}_i \cap \mathcal{T}_{i+k}$, by Proposition 2.11, the chain of torsion classes induces a canonical filtration of X

$$\begin{tikzcd}
0 \arrow[r, equal] & X_0 \cdots \arrow[r] & X_{i+1} \arrow[r] \arrow[dl] & \cdots \arrow[r] & X_{i+k} \arrow[r] \arrow[dl] & \cdots \arrow[r] & X_{n+1} \arrow[r, equal] \arrow[dl] & X \\
& S_{i+1} & & S_{i+k} & & S_{n+1} & &
\end{tikzcd}$$

Since X_{i+1} is a submodule of X, $X_{i+1} \in \mathcal{F}_i$. And because $X_i \in \mathcal{T}_i$, so $\mathrm{Hom}(X_i, X_{i+1}) = 0$, thus $X_i = 0$ since it is a submodule of X_{i+1}. Thus $S_1 = S_2 = \cdots = S_i = 0$.

If $i + k < n + 1$, then $\mathrm{Hom}(X_{n+1}, S_{n+1}) = 0$ since $X_{n+1} = X \in \mathcal{T}_{i+k}$ and $S_{n+1} \in \mathcal{F}_n$. So $S_{n+1} = 0$ and $X_n = X_{n+1} = X$ since S_{n+1} is a quotient module of X. Similarly, we have $S_{i+k+1} = S_{i+k+2} = \cdots = S_n = 0$. Thus $X \in \langle \mathcal{F}_i \cap \mathcal{T}_{i+1}, \mathcal{F}_{i+1} \cap \mathcal{T}_{i+2}, \dots, \mathcal{F}_{i+k-1} \cap \mathcal{T}_{i+k} \rangle$.

The other direction is clear. $\square$

The following demonstrates the relation between torsion $n+1$-tuple and chain of torsion classes of length n.

Theorem 2.13. *There is a one-to-one correspondence between the set of chains of torsion classes of length n on $\mathcal{C}$ and the set of torsion $n+1$-tuples on $\mathcal{C}$ $(n \geqslant 1)$*

$$\left\{ \begin{array}{c} \{(\mathcal{T}_1, \mathcal{F}_1), \dots, (\mathcal{T}_n, \mathcal{F}_n)\}: \\ \textit{chain of torsion classes of length } n \end{array} \right\} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \left\{ \begin{array}{l} (\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1}): \\ \textit{torsion } n+1\textit{-tuple} \end{array} \right\}$$

such that $\alpha(\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2), \dots, (\mathcal{T}_n, \mathcal{F}_n)\}) = (\mathcal{T}_1, \mathcal{F}_1 \cap \mathcal{T}_2, \dots, \mathcal{F}_{n-1} \cap \mathcal{T}_n, \mathcal{F}_n)$ and $\beta((\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1})) = \{(\langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle, \langle \mathcal{C}_i, \dots, \mathcal{C}_{n+1} \rangle) \mid i = 1, 2, \dots, n\}$.

Proof. *Step* 1. Claim: $(\mathcal{T}_1, \mathcal{F}_1 \cap \mathcal{T}_2, \dots, \mathcal{F}_n)$ is a torsion $n+1$-tuple on $\mathcal{C}$.

(1) $\mathcal{F}_{i-1} \cap \mathcal{T}_i = \mathcal{C} \cap \mathcal{T}_{i-1}^{\perp} \cap \mathcal{C} \cap {}^{\perp}\mathcal{F}_i = \mathcal{C} \cap \mathcal{T}_{i-1}^{\perp} \cap {}^{\perp}\mathcal{F}_i = \mathcal{C} \cap \langle \mathcal{T}_1, \mathcal{F}_1 \cap \mathcal{T}_2, \dots, \mathcal{F}_{i-2} \cap \mathcal{T}_{i-1} \rangle^{\perp} \cap {}^{\perp}\langle \mathcal{F}_i \cap \mathcal{T}_{i+1}, \dots, \mathcal{F}_n \rangle$ by Proposition 2.12.
(2) Obviously, $(\langle \mathcal{T}_1, \mathcal{F}_1 \cap \mathcal{T}_2, \dots, \mathcal{F}_{i-1} \cap \mathcal{T}_i \rangle, \langle \mathcal{F}_i \cap \mathcal{T}_{i+1}, \dots, \mathcal{F}_n \rangle) = (\mathcal{T}_i, \mathcal{F}_i)$.

Step 2. It's clear that $\{(\langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle, \langle \mathcal{C}_i, \dots, \mathcal{C}_{n+1} \rangle) \mid i = 1, 2, \dots, n\}$ is a chain of torsion classes of length n on $\mathcal{C}$.

Step 3. $\beta\alpha(\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2), \dots, (\mathcal{T}_n, \mathcal{F}_n)\}) = \beta((\mathcal{T}_1, \mathcal{F}_1 \cap \mathcal{T}_2, \dots, \mathcal{F}_{n-1} \cap \mathcal{T}_n, \mathcal{F}_n)) = \{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2), \dots, (\mathcal{T}_n, \mathcal{F}_n)\}$ by Proposition 2.12.

Step 4. $\alpha\beta((\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1})) = \alpha(\{(\langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle, \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle)\} \mid i = 1, 2, \dots, n) = (\langle \mathcal{C}_i, \dots, \mathcal{C}_{n+1} \rangle \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle \mid i = 1, 2, \dots, n+1) = (\mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1} \rangle^{\perp} \cap {}^{\perp}\langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle \mid i = 1, 2, \dots, n+1) = (\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1})$. $\square$

The following is another characterization of torsion n-tuples.

Proposition 2.14. *Let $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1}$ be full subcategories of $\mathcal{C}$. Then we have $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1})$ is a torsion $n+1$-tuple on $\mathcal{C}$ if and only if:*

(1) $\mathrm{Hom}(X, Y) = 0$ *for* $X \in \mathcal{C}_i$, $Y \in \mathcal{C}_j$, $1 \leqslant i < j \leqslant n+1$.
(2) *For $X \in \mathcal{C}$, there is a filtration*

$$\begin{tikzcd} 0 \arrow[r, equal] & X_0 \arrow[r] & X_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & X_{n+1} \arrow[r, equal] \arrow[dl] & X \\ & S_1 & & S_{n+1} & & \end{tikzcd}$$

such that $0 \to X_i \to X_{i+1} \to S_{i+1} \to 0$ is a short exact sequence with $S_{i+1} \in \mathcal{C}_{i+1}$ for $0 \leqslant i \leqslant n$.

Proof. *Step* 1. Suppose $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1})$ is a torsion $n+1$-tuple on $\mathcal{C}$. Let $\mathcal{T}_i = \langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle$ and $\mathcal{F}_i = \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle$ for $i = 1, 2, \dots, n$. Then by Theorem 2.13 we know $\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2), \dots, (\mathcal{T}_n, \mathcal{F}_n)\}$ is a chain of torsion classes of length n and $\mathcal{C}_i = \mathcal{F}_{i-1} \cap \mathcal{T}_i$ for $1 \leqslant i \leqslant n+1$. Hence, given $X \in \mathcal{C}$, the canonical filtration of X induced by the chain of torsion classes is the desired in (2).

Step 2. Conversely, suppose the tuple $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1})$ satisfies (1) and (2).

Let $1 \leqslant i \leqslant n+1$. For $X \in \mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1} \rangle^{\perp} \cap {}^{\perp}\langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle$, by (2), there is a filtration

$$\begin{tikzcd} 0 \arrow[r, equal] & X_0 \arrow[r] & X_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & X_i \arrow[r] \arrow[dl] & X_{i+1} \arrow[r] \arrow[dl] & \cdots \arrow[r] & X_{n+1} \arrow[r, equal] \arrow[dl] & X \\ & S_1 & & S_i & S_{i+1} & & S_{n+1} & & \end{tikzcd}$$

such that $0 \to X_i \to X_{i+1} \to S_{i+1} \to 0$ is an exact sequence and $S_{i+1} \in \mathcal{C}_{i+1}$ for $0 \leqslant i \leqslant n$. Just like the proof of Proposition 2.12, we have $X_0 = X_1 = \cdots = X_{i-1} = 0$ and $X_i = X_{i+1} = \cdots = X_{n+1} = X$. So $X = X_i = S_i \in \mathcal{C}_i$. Thus it's clear that $\mathcal{C}_i = \mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1} \rangle^{\perp} \cap {}^{\perp}\langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle$ for $i = 1, 2, \dots, n+1$.

Now we show that for $i = 1, 2, \dots, n$, $(\langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle, \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle)$ is a torsion pair on $\mathcal{C}$. It's clear that $\mathrm{Hom}(\langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle, \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle) = 0$. Now given $X \in \mathcal{C}$, then there is a filtration of X as above. And it is clear that $X_i \in \langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle$. We claim that $X/X_i \in \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle$.

In fact, by Snake Lemma we have the following commutative diagram

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & X_i \arrow[r] \arrow[d, equal] & X_{i+1} \arrow[r] \arrow[d] & S_{i+1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X_i \arrow[r] & X_{i+2} \arrow[r] \arrow[d] & X_{i+2}/X_i \arrow[r] \arrow[d] & 0 \\ & & S_{i+2} \arrow[r, equal] \arrow[d] & S_{i+2} \arrow[d] & \\ & & 0 & 0 & \end{tikzcd}$$

Hence $X_{i+2}/X_i \in \langle \mathcal{C}_{i+1}, \mathcal{C}_{i+2} \rangle$.

Inductively, we obtain that $X_{i+k}/X_i \in \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_k \rangle$ for $1 \leqslant k \leqslant n+1-i$, especially, $X_{n+1}/X_i \in \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle$. $0 \to X_i \to X_{n+1} \to X_{n+1}/X_i \to 0$ is the desired short exact sequence. $\square$

The above filtration is also unique up to isomorphism since it is isomorphic to the filtration induced by the corresponding chain of torsion classes. We call it a canonical filtration (or decomposition) induced by the torsion tuple.

The following lemma is well known [8].

Lemma 2.15. *If $\mathcal{B}$ is a subcategory of Λ-mod, then ${}^{\perp}(({}^{\perp}\mathcal{B})^{\perp}) = {}^{\perp}\mathcal{B}$, $({}^{\perp}(\mathcal{B}^{\perp}))^{\perp} = \mathcal{B}^{\perp}$, $({}^{\perp}\mathcal{B}, ({}^{\perp}\mathcal{B})^{\perp})$ and $(\mathcal{B}^{\perp}, {}^{\perp}(\mathcal{B}^{\perp}))$ are both torsion pairs on Λ-mod.*

The following shows that (2) in Definition 2.3 is superfluous if $\mathcal{C} = \Lambda$-mod.

Corollary 2.16. *Let $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1}$ be full subcategories of Λ-mod$(n \geqslant 1)$. If $\mathcal{C}_i = \langle \mathcal{C}_1, \dots, \mathcal{C}_{i-1} \rangle^{\perp} \cap {}^{\perp}\langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{n+1} \rangle$ for $i = 1, 2, \dots, n+1$, then $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1})$ is a torsion $n+1$-tuple on Λ-mod.*

Proof. By Lemma 2.15, there is a fact: $({}^{\perp}\mathcal{C}_{n+1}, \mathcal{C}_{n+1})$ is a torsion pair since $\mathcal{C}_{n+1} = \langle \mathcal{C}_1, \dots, \mathcal{C}_n \rangle^{\perp}$.

We use induction on n to prove the corollary. If $n = 1$, it's clear that $(\mathcal{C}_1, \mathcal{C}_2)$ is a torsion pair by the above fact. Now suppose that the corollary is true for $n = k \geqslant 1$, we consider the case $n = k+1$. It is enough to show the condition (2) in Proposition 2.14 holds since the first condition is clear.

Step 1. Claim: $\langle \mathcal{C}_{k+1}, \mathcal{C}_{k+2} \rangle = \langle \mathcal{C}_1, \dots, \mathcal{C}_k \rangle^{\perp}$.

$\forall X \in \langle \mathcal{C}_1, \dots, \mathcal{C}_k \rangle^{\perp}$, consider the canonical short exact sequence $0 \to X_{k+1} \to X \to T_{k+2} \to 0$ induced by the torsion pair $({}^{\perp}\mathcal{C}_{k+2}, \mathcal{C}_{k+2})$. $X_{k+1} \in \langle \mathcal{C}_1, \dots, \mathcal{C}_k \rangle^{\perp}$ since $X \in \langle \mathcal{C}_1, \dots, \mathcal{C}_k \rangle^{\perp}$. Thus $X_{k+1} \in \mathcal{C}_{k+1}$ and $X \in \langle \mathcal{C}_{k+1}, \mathcal{C}_{k+2} \rangle$.

Step 2. By induction, $(\mathcal{C}_1, \dots, \mathcal{C}_k, \langle \mathcal{C}_{k+1}, \mathcal{C}_{k+2} \rangle)$ is a torsion k-tuple on Λ-mod. So for $X \in \Lambda$-mod, it induces a canonical filtration of X

$$\begin{tikzcd}
0 \arrow[r, equal] & X_0 \arrow[r] & X_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & X_{k-1} \arrow[r] \arrow[dl] & X_k \arrow[r] \arrow[dl] & X_{k+1} \arrow[r, equal] \arrow[dl] & X \\
& S_1 & & S_{k-1} & S_k & S & &
\end{tikzcd}$$

For S, the torsion pair $({}^{\perp}\mathcal{C}_{k+2}, \mathcal{C}_{k+2})$ induces the canonical short exact sequence $0 \to S_{k+1} \to S \to S_{k+2} \to 0$ such that $S_{k+1} \in {}^{\perp}\mathcal{C}_{k+2}$ and $S_{k+2} \in \mathcal{C}_{k+2}$. Because $S \in \langle \mathcal{C}_{k+1}, \mathcal{C}_{k+2} \rangle$, then $S \in \langle \mathcal{C}_1, \dots, \mathcal{C}_k \rangle^{\perp}$, so $S_{k+1} \in \langle \mathcal{C}_1, \dots, \mathcal{C}_k \rangle^{\perp}$, hence $S_{k+1} \in \mathcal{C}_{k+1}$ since $S_{k+1} \in {}^{\perp}\mathcal{C}_{k+2}$. By pull-back of $(X \to S, S_{k+1} \to S)$, we have the following commutative diagram

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X_k \arrow[r] \arrow[d, equal] & X_{k+1} \arrow[r] \arrow[d] & S_{k+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_k \arrow[r] & X \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & 0 \\
& & S_{k+2} \arrow[r, equal] \arrow[d] & S_{k+2} \arrow[d] & \\
& & 0 & 0 &
\end{tikzcd}$$

Adding the exact sequence $0 \to X_{k+1} \to X_{k+2} \to sS_{k+2} \to 0$ to the above filtration, we get the desired filtration. $\square$

Proposition 2.17. *Let $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1})$ be a torsion $n+1$-tuple on $\mathcal{C}(n \geqslant 1)$ and $1 \leqslant i+1 < i+k \leqslant n+1$. Then*

(1) $\langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{i+k} \rangle = \mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle^{\perp} \cap {}^{\perp}\langle \mathcal{C}_{i+k+1}, \dots, \mathcal{C}_{n+1} \rangle$;
(2) $(\mathcal{C}_{i+1}, \dots, \mathcal{C}_{i+k})$ *is a torsion k-tuple on* $\langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{i+k} \rangle$;
(3) $(\mathcal{C}_1, \dots, \mathcal{C}_i, \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{i+k} \rangle, \mathcal{C}_{i+k+1}, \dots, \mathcal{C}_{n+1})$ *is a torsion $(n+2-k)$-tuple.*

Proof. For $X \in \mathcal{C}$, consider the canonical filtration induced by the torsion tuple

$$\begin{tikzcd}
0 \arrow[r, equal] & X_0 \cdots \arrow[r] & X_{i+1} \arrow[r] \arrow[dl] & \cdots \arrow[r] & X_{i+k} \arrow[r] \arrow[dl] & \cdots \arrow[r] & X_{n+1} \arrow[r, equal] \arrow[dl] & X \\
 & S_{i+1} & & S_{i+k} & & S_{n+1} & &
\end{tikzcd}$$

(1) If $X \in \mathcal{C} \cap \langle \mathcal{C}_1, \dots, \mathcal{C}_i \rangle^{\perp} \cap {}^{\perp}\langle \mathcal{C}_{i+k+1}, \dots, \mathcal{C}_{n+1} \rangle$, then $X_0 = X_1 = \cdots = X_{i-1} = 0$ and $X_{i+k+1} = X_{i+k+2} = \cdots = X_{n+1} = X$. Hence $X \in \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{i+k} \rangle$. The other direction is clear.

(2) By (1) and Proposition 2.14, it's clear.

(3) Use the similar techniques in Proposition 2.14, we have the following short exact sequence

$$0 \to X_i \to X_{i+k} \to X_{i+k}/X_i \to 0$$

such that $X_{i+k}/X_i \in \langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{i+k} \rangle$ Let $\hat{S} = X_{i+k}/X_i$, then we have the filtration

$$\begin{tikzcd}
0 \arrow[r, equal] & X_0 \cdots \arrow[r] & X_i \arrow[r] \arrow[dl] & X_{i+k} \arrow[r] \arrow[dl] & \cdots \arrow[r] & X_{n+1} \arrow[r, equal] \arrow[dl] & X \\
 & S_i & \hat{S} & & S_{n+1} & &
\end{tikzcd}$$

Then by Proposition 2.14 the proof is completed. $\square$

Corollary 2.18. *Let $\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2), \dots, (\mathcal{T}_n, \mathcal{F}_n)\}$ be a chain of torsion classes of length n on $\mathcal{C}$. Then $\{(\mathcal{T}_{i+1} \cap \mathcal{F}_i, \mathcal{F}_{i+1} \cap \mathcal{T}_{i+k+1}), \dots, (\mathcal{T}_{i+k} \cap \mathcal{F}_i, \mathcal{F}_{i+k} \cap \mathcal{T}_{i+k+1})\}$ is a chain of torsion classes of length k on $\mathcal{T}_{i+k+1} \cap \mathcal{F}_i$ for $0 \leqslant i < i+k+1 \leqslant n+1$.*

Proof. For $l = 1, 2, \dots, n+1$, let $\mathcal{C}_l = \mathcal{F}_{l-1} \cap \mathcal{T}_l$. So $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1})$ is a torsion $n+1$-tuple on $\mathcal{C}$ by Theorem 2.13, and $(\mathcal{C}_{i+1}, \dots, \mathcal{C}_{i+k+1})$ is a torsion $k+1$-tuple on $\langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{i+k+1} \rangle$ by Proposition 2.17. Thus $\{(\langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{i+l} \rangle, \langle \mathcal{C}_{i+l+1}, \dots, \mathcal{C}_{i+k+l} \rangle) \mid l = 1, 2, \dots, k\}$ is a chain of torsion classes of length k on $\langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{i+k+1} \rangle$. But $\langle \mathcal{C}_{i+1}, \dots, \mathcal{C}_{i+l} \rangle = \mathcal{T}_{i+l} \cap \mathcal{F}_i$, $\langle \mathcal{C}_{i+l+1}, \dots, \mathcal{C}_{i+k+l} \rangle = \mathcal{F}_{i+l} \cap \mathcal{T}_{i+k+l}$ by Proposition 2.12. The corollary is proved. $\square$

Corollary 2.19. *If $(\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_{n+1})$ is a defect torsion $n+1$-tuple on $\mathcal{C}$, then there is a unique torsion $n+1$-tuple $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{n+1})$ on $\mathcal{C}$ such that $\mathcal{D}_i \subseteq \mathcal{C}_i$ for $1 \leqslant i \leqslant n+1$.*

Proof. Let $\mathcal{T}_i = \langle \mathcal{D}_1, \dots, \mathcal{D}_i \rangle$, $\mathcal{F}_i = \langle \mathcal{D}_{i+1}, \dots, \mathcal{D}_n \rangle$. Then $\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2), \dots, (\mathcal{T}_n, \mathcal{F}_n)\}$ is a chain of torsion classes of length n on $\mathcal{C}$.

For $1 \leqslant i \leqslant n+1$, let $\mathcal{C}_i = \mathcal{F}_{i-1} \cap \mathcal{T}_i$, then $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{n+1})$ is a torsion $n+1$-tuple on $\mathcal{C}$ such that $\mathcal{D}_i \subseteq \mathcal{C}_i$ by Theorem 2.13.

Suppose $(\mathcal{C}_1', \mathcal{C}_2', \ldots, \mathcal{C}_{n+1}')$ is another torsion $n+1$-tuple on $\mathcal{C}$ such that $\mathcal{D}_i \subseteq \mathcal{C}_i'$. Then for $1 \leqslant i \leqslant n$, $\mathcal{T}_i = \langle \mathcal{D}_1, \ldots, \mathcal{D}_i \rangle \subseteq \langle \mathcal{C}_1', \ldots, \mathcal{C}_i' \rangle = \mathcal{T}_i'$. Similarly, $\mathcal{F} \subseteq \mathcal{F}_i'$. Hence $\mathcal{T}_i = \mathcal{T}_i'$ and $\mathcal{F}_i = \mathcal{F}_i'$. Therefore, $\mathcal{C}_i = \mathcal{F}_{i-1} \cap \mathcal{T}_i = \mathcal{F}_{i-1}' \cap \mathcal{T}_i' = \mathcal{C}_i'$. $\square$

Proposition 2.20. *Let $\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2)\}$ be a chain of torsion classes of length 2 on $\mathcal{C}$. Then we have the following one-to-one correspondence* (*we denote torsion pair by tp. for convenience here*):

$$\{(\mathcal{T}', \mathcal{F}'):\ tp.\ on\ \mathcal{F}_1 \cap \mathcal{T}_2\} \underset{G}{\overset{F}{\rightleftarrows}} \{(\mathcal{T}_3, \mathcal{F}_3):\ tp.\ on\ \mathcal{C}\ with\ \mathcal{T}_1 \subseteq \mathcal{T}_3 \subseteq \mathcal{T}_2\}$$

where $F((\mathcal{T}', \mathcal{F}')) = (\langle \mathcal{T}_1, \mathcal{T}' \rangle, \langle \mathcal{F}', \mathcal{F}_2 \rangle)$, $G((\mathcal{T}_3, \mathcal{F}_3)) = (\mathcal{T}_3 \cap \mathcal{F}_1, \mathcal{F}_3 \cap \mathcal{T}_2)$.

Proof. Given the torsion pair $(\mathcal{T}_3, \mathcal{F}_3)$ on $\mathcal{C}$ with $\mathcal{T}_1 \subseteq \mathcal{T}_3 \subseteq \mathcal{T}_2$, then by definition $\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_3, \mathcal{F}_3), (\mathcal{T}_2, \mathcal{F}_2)\}$ is a chain of torsion classes of length 3 on $\mathcal{C}$. So by Corollary 2.18, $G((\mathcal{T}_3, \mathcal{F}_3))$ is a torsion pair on $\mathcal{F}_1 \cap \mathcal{T}_2$. By Theorem 2.13, $FG((\mathcal{T}_3, \mathcal{F}_3)) = (\langle \mathcal{T}_1, \mathcal{T}_3 \cap \mathcal{F}_1 \rangle, \langle \mathcal{F}_3 \cap \mathcal{T}_2, \mathcal{F}_2 \rangle) = (\mathcal{T}_3, \mathcal{F}_3)$.

On the other hand, suppose $(\mathcal{T}', \mathcal{F}')$ is a torsion pair on $\mathcal{F}_1 \cap \mathcal{T}_2$. Since $\{(\mathcal{T}_1, \mathcal{F}_1), (\mathcal{T}_2, \mathcal{F}_2)\}$ is a chain of torsion classes of length 2 on $\mathcal{C}$, by Theorem 2.13, $(\mathcal{T}_1, \mathcal{F}_1 \cap \mathcal{T}_2, \mathcal{F}_2)$ is a torsion 3-tuple on $\mathcal{C}$. By Proposition 2.6, $(\mathcal{T}_1, \mathcal{T}', \mathcal{F}', \mathcal{F}_2)$ is a torsion 4-tuple on $\mathcal{C}$. Then $F((\mathcal{T}', \mathcal{F}'))$ is a torsion pair on $\mathcal{C}$ by Theorem 2.13.

By Proposition 2.17, we also learn that $(\mathcal{T}_1, \mathcal{T}')$ is a torsion pair (equivalently, torsion 2-tuple) on $\langle \mathcal{T}_1, \mathcal{T}' \rangle$ and $(\mathcal{F}', \mathcal{F}_2)$ is a torsion pair on $\langle \mathcal{F}', \mathcal{F}_2 \rangle$. Hence we know that $\langle \mathcal{T}_1, \mathcal{T}' \rangle \cap \mathcal{F}_1 = \langle \mathcal{T}_1, \mathcal{T}' \rangle \cap \mathcal{T}_1^{\perp} = \mathcal{T}'$. Similarly $\langle \mathcal{F}', \mathcal{F}_2 \rangle \cap \mathcal{T}_2 = \mathcal{F}'$. Therefore, $GF((\mathcal{T}', \mathcal{F}')) = (\mathcal{T}', \mathcal{F}')$. $\square$

3. Decomposition by projective and injective modules

In this section, we assume Λ is a basic artin algebra. We define $\mathbf{E}(\Lambda) = \{(\mathcal{T}, \mathcal{F})$ is a torsion pair on Λ-mod $\mid \mathcal{T} \cap \mathcal{P}(\Lambda) = \mathcal{F} \cap \mathcal{I}(\Lambda) = \{0\}\}$. For a set Ψ we denote the number of the elements of Ψ by $\#\Psi$. The following definition is slightly different from [2, p. 191, 1.10].

Definition 3.1. Let $\mathcal{C}$ be a full subcategory of Λ-mod and $M \in \Lambda$-mod. Then M is called Ext-projective in $\mathcal{C}$ if $\mathrm{Ext}^1_\Lambda(M, \mathcal{C}) = 0$. Dually, it is called Ext-injective in $\mathcal{C}$ if $\mathrm{Ext}^1_\Lambda(\mathcal{C}, M) = 0$.

If Λ' is a quotient algebra of Λ (i.e. $\exists$ a surjective algebra homomorphism $\pi : \Lambda \twoheadrightarrow \Lambda'$), then there is a canonical way to view Λ'-mod as a full subcategory of Λ-mod as following: for $M \in \Lambda'$-mod, $a \in \Lambda$, $m \in M$, $a \cdot m = \pi(a)m$.

For a Λ-module M, we denote by $Gen(M)$ the minimal additive full subcategory closed under quotients and containing M of Λ-mod, by $Cogen(M)$ the minimal additive full subcategory closed under submodules and containing M of Λ-mod. The following lemma is well known.

Lemma 3.2. *Let e be an idempotent of Λ. Then*

(1) $(\Lambda e)^{\perp} = {}^{\perp}(D(e\Lambda)) = \Lambda/\Lambda e\Lambda$-mod;
(2) $(Gen(\Lambda e), \Lambda/\Lambda e\Lambda$-mod) *and* $(\Lambda/\Lambda e\Lambda$-mod, $Cogen(D(e\Lambda)))$ *are both torsion pairs on* Λ-mod.

The following lemma is part of [2, p. 191, 1.11].

Lemma 3.3. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair on Λ-mod and $0 \neq X \in \Lambda$-mod.*

(1) *If $X \in \mathcal{F}$, then X is* Ext*-projective in $\mathcal{F}$ if and only if there is a projective Λ-module P and a canonical short exact sequence $0 \to K \to P \to X \to 0$ induced by $(\mathcal{T}, \mathcal{F})$.*

(2) *If $X \in \mathcal{T}$, then X is* Ext-*injective in $\mathcal{T}$ if and only if there is an injective Λ-module I and a canonical short exact sequence $0 \to X \to I \to L \to 0$ induced by $(\mathcal{T}, \mathcal{F})$.*

It is obvious that the projective module in (1) can be chosen as the projective cover of X, the injective module in (2) can be chosen as the injective envelope of X. The following is a generalization of the above lemma.

Proposition 3.4. *Let $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{n+1})$ be a torsion $n+1$-tuple on Λ-mod$(n \geqslant 1)$. Then there exists bijections*:

(1) $F : \operatorname{Ind}\mathcal{P}(\Lambda) \to \{X \in \operatorname{Ind}\mathcal{C}_i \mid X$ *is* Ext*-projective in $\langle \mathcal{C}_i, \mathcal{C}_{i+1}, \ldots, \mathcal{C}_{n+1}\rangle$ for some $i = 1, 2, \ldots, n+1$*$\}$;
(2) $G : \operatorname{Ind}\mathcal{I}(\Lambda) \to \{Y \in \operatorname{Ind}\mathcal{C}_j \mid Y$ *is* Ext*-injective in $\langle \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_j\rangle$ for some $j = 1, 2, \ldots, n+1$*$\}$.

Proof. We only prove (1). The proof of (2) is similar.

Step 1. Let $P \in \operatorname{Ind}\mathcal{P}(\Lambda)$. Then the torsion tuple induces the following canonical filtration of P

$$\begin{tikzcd}
0 \arrow[r, equal] & X_0 \arrow[r] & \cdots \arrow[r] & X_{i-1} \arrow[r] \arrow[dl] & X_i \arrow[r] \arrow[dl] & X_{i+1} \arrow[r] \arrow[dl] & \cdots \arrow[r] & X_{n+1} \arrow[r, equal] \arrow[dl] & P \\
 & & S_{i-1} & S_i & S_{i+1} & & S_{n+1} & &
\end{tikzcd}$$

There exists $1 \leqslant i \leqslant n+1$ such that $S_{i+1} = S_{i+2} = \cdots = 0$ and $S_i \neq 0$. Then the torsion pair $(\langle \mathcal{C}_1, \ldots, \mathcal{C}_{i-1}\rangle, \langle \mathcal{C}_i, \ldots, \mathcal{C}_{n+1}\rangle)$ induces the following canonical short exact sequence $0 \to X_{i-1} \to P \to S_i \to 0$. By Lemma 3.3, S_i is Ext-projective in $\langle \mathcal{C}_i, \ldots, \mathcal{C}_{n+1}\rangle$. We define $F(P) = S_i$.

Conversely, let $X \in \operatorname{Ind}\mathcal{C}_i$ such that X is Ext-projective in $\langle \mathcal{C}_i, \mathcal{C}_{i+1}, \ldots, \mathcal{C}_{n+1}\rangle$. Then we define $F^{-1}(X)$ to be the projective cover of X.

Step 2. It is clear that $F^{-1}F(P) = P$ for $P \in \operatorname{Ind}\mathcal{P}(\Lambda)$.

On the other hand, let $1 \leqslant i \leqslant n+1$ and $X \in \operatorname{Ind}\mathcal{C}_i$ which is Ext-projective in $\langle \mathcal{C}_i, \mathcal{C}_{i+1}, \ldots, \mathcal{C}_{n+1}\rangle$. By Lemma 3.3, the torsion pair $(\langle \mathcal{C}_1, \ldots, \mathcal{C}_{i-1}\rangle, \langle \mathcal{C}_i, \ldots, \mathcal{C}_{n+1}\rangle)$ induces the following canonical short exact sequence of $F^{-1}(X)$: $0 \to K \to F^{-1}(X) \to X$. Thus $F(F^{-1}(X)) = X$. $\square$

Corollary 3.5. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair on Λ-*mod*. Then*

(1) *there is an idempotent e such that $\mathcal{T} \cap \mathcal{P}(\Lambda) = \operatorname{add}\Lambda e$, and $\mathcal{T} \cap (\Lambda e)^{\perp}$ has no nonzero* Ext-*projective modules in $(\Lambda e)^{\perp}$;*
(2) *there is an idempotent e such that $\mathcal{F} \cap \mathcal{I}(\Lambda) = \operatorname{add} D(e\Lambda)$, and ${}^{\perp}D(e\Lambda) \cap \mathcal{F}$ has no nonzero* Ext-*injective modules in ${}^{\perp}D(e\Lambda)$.*

Proof. We only prove (1). The proof of (2) is similar.

There is indeed an idempotent e such that $\mathcal{T} \cap \mathcal{P}(\Lambda) = \operatorname{add}\Lambda e$. Suppose that $0 \neq X \in \mathcal{T} \cap (\Lambda e)^{\perp}$ is Ext-projective in $(\Lambda e)^{\perp}$. Then by Lemma 3.3, there exist a projective Λ-module P and the canonical short exact sequence $0 \to K \to P \to X \to 0$ induced by the torsion pair $(Gen(\Lambda e), (\Lambda e)^{\perp})$. Thus $P \in \mathcal{T}$ since $K \in Gen(\Lambda e) \subseteq \mathcal{T}$. Thus $P \in \operatorname{add}\Lambda e$. Hence $X \in Gen(\Lambda e)$. This is a contradiction. $\square$

Let $\mathcal{C}$ be an extension-closed full subcategory of Λ-mod. Then we say $\mathcal{C}$ is a Serre class of Λ-mod if the quotient modules and submodules of the modules in $\mathcal{C}$ still belong to $\mathcal{C}$. The following lemma is obvious.

Lemma 3.6. *Let $\mathcal{C}$ be a Serre class of Λ-module and $0 \neq X \in \mathcal{C}$.*

(1) *If X is* Ext-*projective in $\mathcal{C}$ and not a projective Λ-module, then the projective cover of X doesn't belong to $\mathcal{C}$.*

(2) *If X is* Ext*-injective in $\mathcal{C}$ not an injective Λ-module, then the injective envelope of X doesn't belong to $\mathcal{C}$.*

Corollary 3.7. *Let $(\mathcal{T},\mathcal{F})$ be a torsion pair on Λ-mod and e^0, e^1 be two orthogonal idempotents of Λ.*

(1) *If $\mathcal{F}\cap\mathcal{I}(\Lambda)=\operatorname{add}D(e^0\Lambda)$, $\mathcal{T}\cap\mathcal{P}(\Lambda/\Lambda e^0\Lambda)=\operatorname{add}(\Lambda/\Lambda e^0\Lambda)e^1$, then $\mathcal{T}\cap\mathcal{P}(\Lambda)=\operatorname{add}\Lambda e^1\cap\Lambda/\Lambda e^0\Lambda\text{-mod}$.*
(2) *If $\mathcal{T}\cap\mathcal{P}(\Lambda)=\operatorname{add}\Lambda e^0$, $\mathcal{F}\cap\mathcal{I}(\Lambda/\Lambda e^0\Lambda)=\operatorname{add}D(e^1(\Lambda/\Lambda e^0\Lambda))$, then $\mathcal{F}\cap\mathcal{I}(\Lambda)=\operatorname{add}D(e^1\Lambda)\cap\Lambda/\Lambda e^0\Lambda\text{-mod}$.*

Proof. We only prove (1). The proof of (2) is similar.

Since Λ is a basic artin algebra, $\operatorname{add}\Lambda e^1$ consists of the projective covers of modules in $\operatorname{add}(\Lambda/\Lambda e^0\Lambda)e^1$. Suppose $0\neq P\in\mathcal{T}\cap\mathcal{P}(\Lambda)$. Then $P\in\Lambda/\Lambda e^0\Lambda$-mod since $\mathcal{T}\subseteq\Lambda/\Lambda e^0\Lambda\text{-mod}={}^{\perp}(D(e^0\Lambda))$. On the other hand, P is Ext-projective in $\Lambda/\Lambda e^0\Lambda$-mod. So $P\in\operatorname{add}(\Lambda/\Lambda e^0\Lambda)e^1$. Since P is the projective cover of itself, $P\in\operatorname{add}\Lambda e^1$. Therefore, $P\in\operatorname{add}\Lambda e^1\cap\Lambda/\Lambda e^0\Lambda$-mod.

Conversely, suppose $0\neq P\in\operatorname{add}\Lambda e^1\cap\Lambda/\Lambda e^0\Lambda$-mod. Then P is the projective cover for some $X\in\operatorname{add}(\Lambda/\Lambda e^0\Lambda)e^1$ which means X is Ext-projective in $\Lambda/\Lambda e^0\Lambda$-mod. By Lemma 3.6, $P=X$ (otherwise, $P\notin\Lambda/\Lambda e^0\Lambda$-mod). Thus $P\in\mathcal{T}\cap\mathcal{P}(\Lambda)$. □

Now we start to decompose torsion pairs by projective modules and injective modules. We always assume that $\Delta=\{e_1,e_2,\dots,e_n\}$ is a fixed complete set of primitive orthogonal idempotents of Λ. For $m\geqslant 0$, given $S=\{\Delta_0,\Delta_1,\Delta_2,\dots,\Delta_m\mid\Delta_i\subseteq\Delta\}$ such that $\Delta_1,\Delta_2,\dots,\Delta_m\neq\phi$ and $\Delta_i\cap\Delta_j=\phi$ for $i\neq j$, we define: for $0\leqslant i\leqslant m$, $e_S^i=\sum_{e\in\Delta_i}e$, $\varepsilon_S^i=\sum_{j=0}^i e_S^j$; $\Lambda_S^0=\Lambda$, $\Lambda_S^1=\frac{\Lambda_S^0}{\Lambda_S^0 e_S^0\Lambda_S^0}=\frac{\Lambda}{\Lambda\varepsilon_S^0\Lambda},\dots,\Lambda_S^{m+1}=\frac{\Lambda_S^m}{\Lambda_S^m e_S^m\Lambda_S^m}=\frac{\Lambda}{\Lambda\varepsilon_S^m\Lambda}$; $\mathrm{P}_i(\Lambda_S^i)=\Lambda_S^i e_S^i$, $\mathrm{I}_i(\Lambda_S^i)=D(e_S^i\Lambda_S^i)$.

Definition 3.8. Let S be as above. It is called a 1-type part partition of Λ if: (1) $\forall 2\leqslant 2i\leqslant m$ and $e\in\Delta_{2i}$, $e_S^{2i-1}\Lambda_S^{2i-1}e\neq 0$; (2) $\forall 3\leqslant 2i+1\leqslant m$ and $e\in\Delta_{2i+1}$, $e\Lambda_S^{2i}e_S^{2i}\neq 0$.

Dually, S is called a 2-type part partition if: (1) $\forall 2\leqslant 2i\leqslant m$ and $e\in\Delta_{2i}$, $e\Lambda_S^{2i-1}e_S^{2i-1}\neq 0$; (2) $\forall 3\leqslant 2i+1\leqslant m$ and $e\in\Delta_{2i+1}$, $e_S^{2i}\Lambda_S^{2i}e\neq 0$.

Lemma 3.9. *Let I be an ideal of Λ, and e, e' be two idempotents. Then $\operatorname{Hom}_\Lambda((\Lambda/I)\cdot e, D(e'\cdot\Lambda/I))=0$ if and only if $e'\cdot\Lambda/I\cdot e=0$.*

Proof. It's clear that $\operatorname{Hom}_\Lambda((\Lambda/I)\cdot e, D(e'\cdot\Lambda/I))=\operatorname{Hom}_{\Lambda/I}((\Lambda/I)\cdot e, D(e'\cdot\Lambda/I))=D(e'\cdot\Lambda/I\cdot e)$. □

We give the following notations for describing our theorem easily

$$\begin{aligned}
\mathfrak{M}&=\bigl\{(\mathcal{T},\mathcal{F})\bigm|(\mathcal{T},\mathcal{F})\text{ is a torsion pair on }\Lambda\text{-mod}\bigr\},\\
\mathfrak{N}&=\bigl\{\bigl(S=\{\Delta_0,\Delta_1,\Delta_2,\dots,\Delta_m\},(\mathcal{T}',\mathcal{F}')\bigr)\bigm|S\text{ is a 1-type part partition and}\\
&\qquad(\mathcal{T}',\mathcal{F}')\in\mathbf{E}\bigl(\Lambda_S^{m+1}\bigr)\bigr\},\\
\mathfrak{N}'&=\bigl\{\bigl(S'=\{\Delta_0',\Delta_1',\Delta_2',\dots,\Delta_n'\},(\mathcal{T}'',\mathcal{F}'')\bigr)\bigm|S'\text{ is a 2-type part partition and}\\
&\qquad(\mathcal{T}'',\mathcal{F}'')\in\mathbf{E}\bigl(\Lambda_{S'}^{n+1}\bigr)\bigr\}.
\end{aligned}$$

Now we are in a position to give a demonstration of how to decompose a torsion pair into a torsion tuple by projective modules and injective modules.

Let $(\mathcal{T},\mathcal{F})$ be a torsion pair on Λ-mod.

Operation 1. Let $\mathcal{T}^0=\mathcal{T}$, $\mathcal{F}^0=\mathcal{F}$, $\Lambda^0=\Lambda$, there exists a set $\Delta_0\subseteq\Delta$ such that $\mathcal{T}^0\cap\mathcal{P}(\Lambda^0)=\operatorname{add}\bigoplus_{e\in\Delta_0}\Lambda^0 e$. Let $e^0=\sum_{e\in\Delta_0}e$, $\mathrm{P}_0(\Lambda^0)=\Lambda^0e^0$, $\mathcal{T}^1=\mathcal{T}^0\cap(\mathrm{P}_0(\Lambda^0))^{\perp}$, $\mathcal{F}^1=\mathcal{F}^0$, $\Lambda^1=\Lambda/\Lambda e^0\Lambda$.

Then $(\mathcal{T}^1, \mathcal{F}^1)$ is a torsion pair on Λ^1-mod and $\mathcal{T}^1 \cap \mathcal{P}(\Lambda^1) = \{0\}$ by Corollary 3.5. Hence we have a torsion 3-tuple $(Gen\,\mathrm{P}_0(\Lambda^0), \mathcal{T}^1, \mathcal{F}^1)$ on Λ-mod.

Operation 2. There exists $\Delta_1 \subseteq \Delta - \Delta_0$ such that $\mathcal{F}^1 \cap \mathcal{I}(\Lambda^1) = \operatorname{add} \bigoplus_{e \in \Delta_1} D(e\Lambda^1)$. Let $e^1 = \sum_{e \in \Delta_1} e$, $\varepsilon = e^0 + e^1$, $\mathrm{I}_1(\Lambda^1) = D(e^1\Lambda^1)$, $\mathcal{T}^2 = \mathcal{T}^1$, $\mathcal{F}^2 = \mathcal{F}^1 \cap {}^{\perp}\mathrm{I}_1(\Lambda^1)$, $\Lambda^2 = \Lambda/\Lambda\varepsilon^1\Lambda$. Then $(\mathcal{T}^2, \mathcal{F}^2)$ is a torsion pair on Λ^2-mod and $\mathcal{F}^2 \cap \mathcal{I}(\Lambda^2) = \{0\}$ by Corollary 3.5. Hence we have a torsion 4-tuple $(Gen\,\mathrm{P}_0(\Lambda^0), \mathcal{T}^2, \mathcal{F}^2, Cogen\,\mathrm{I}_1(\Lambda^1))$ on Λ-mod.

The above operations go on alternatively until we get $m \geqslant 0$ such that the torsion pair $(\mathcal{T}^{m+1}, \mathcal{F}^{m+1}) \in \mathbf{E}(\Lambda^{m+1})$. Finally, we obtain the following.

(1) $\{\Delta_0, \Delta_1, \Delta_2, \ldots, \Delta_m \mid \Delta_i \subseteq \Delta\}$ such that $\Delta_1, \Delta_2, \ldots, \Delta_m \neq \phi$ and $\Delta_i \cap \Delta_j = \phi$ for $i \neq j$.
(2) $\Lambda = \Lambda^0 \to \Lambda^1 \to \cdots \to \Lambda^{m+1}$ is a series of quotient algebras.
(3) For $1 \leqslant 2i-1 \leqslant m-1$, $\mathcal{T}^{2i} = \mathcal{T}^{2i-1}$, for $0 \leqslant 2i \leqslant m-1$, $\mathcal{F}^{2i+1} = \mathcal{F}^{2i}$.
(4) For $1 \leqslant 2i-1 \leqslant m$, $(\mathcal{T}^{2i-1}, \mathcal{F}^{2i-1})$ is a torsion pair on Λ^{2i-1}-mod such that $\mathcal{T}^{2i-1} \cap \mathcal{P}(\Lambda^{2i-1}) = \{0\}$. For $2 \leqslant 2j \leqslant m$, $(\mathcal{T}^{2j}, \mathcal{F}^{2j})$ is a torsion pair on Λ^{2j}-mod such that $\mathcal{F}^{2j} \cap \mathcal{I}(\Lambda^{2j}) = \{0\}$.
(5) $(Gen\,\mathrm{P}_0(\Lambda^0), Gen\,\mathrm{P}_2(\Lambda^2), \ldots, \mathcal{T}^{m+1}, \mathcal{F}^{m+1}, \ldots, Cogen\,\mathrm{I}_3(\Lambda^3), Cogen\,\mathrm{I}_1(\Lambda^1))$ is a torsion $(m+3)$-tuple on Λ-mod.

Theorem 3.10. *There is a one-to-one correspondence between $\mathfrak{M}$ and $\mathfrak{N}$:* $\mathfrak{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathfrak{N}$.

Proof. *Step* 1. Let $(\mathcal{T}, \mathcal{F}) \in \mathfrak{M}$. By using the above operation, we get for some $m \geqslant 0$, $S = \{\Delta_0, \Delta_1, \Delta_2, \ldots, \Delta_m \mid \Delta_i \subseteq \Delta\}$ and $(\mathcal{T}^{m+1}, \mathcal{F}^{m+1}) \in \mathbf{E}(\Lambda^{m+1})$. We adopt the above notations, and claim S is a 1-type part partition.

For $1 \leqslant 2i-1 \leqslant m-1$, $(\mathcal{T}^{2i-1}, \mathcal{F}^{2i-1})$ is a torsion pair on Λ^{2i-1}-mod such that $\mathcal{T}^{2i-1} \cap \mathcal{P}(\Lambda^{2i-1}) = \{0\}$, $\mathcal{F}^{2i-1} \cap \mathcal{I}(\Lambda^{2i-1}) = \operatorname{add} D(e^{2i-1}\Lambda^{2i-1})$ and $\mathcal{T}^{2i-1} \cap \mathcal{P}(\Lambda^{2i}) = \mathcal{T}^{2i} \cap \mathcal{P}(\Lambda^{2i}) = \operatorname{add} \Lambda^{2i}e^{2i}$. Thus by Corollary 3.7, we know $\mathcal{T}^{2i-1} \cap \mathcal{P}(\Lambda^{2i-1}) = \operatorname{add} \Lambda^{2i-1}e^{2i} \cap \Lambda^{2i}$-mod. But $\mathcal{T}^{2i-1} \cap \mathcal{P}(\Lambda^{2i-1}) = \{0\}$. Thus $\operatorname{add} \Lambda^{2i-1}e^{2i} \cap \Lambda^{2i}\text{-mod} = \{0\}$. So for $e \in \Delta_{2i}$, $\operatorname{Hom}(\Lambda^{2i-1}e^{2i-1}, \Lambda^{2i-1}e) \neq 0$ which means $e^{2i-1}\Lambda^{2i-1}e \neq 0$.

Similarly for $2 \leqslant 2j \leqslant m-1$, $\operatorname{add} D(e^{2j+1}\Lambda^{2j}) \cap \Lambda^{2j+1}\text{-mod} = \{0\}$. This means for $e \in \Delta_{2j+1}$, $D(e\Lambda^{2j})$ is not a Λ^{2j+1}-module. Thus $\operatorname{Hom}_{\Lambda^{2j}}(\Lambda^{2j}e^{2j}, D(e\Lambda^{2j})) \neq 0$. Therefore $e\Lambda^{2j}e^{2j} \neq 0$ by Lemma 3.9.

Define $F((\mathcal{T}, \mathcal{F})) = (S, (\mathcal{T}^{m+1}, \mathcal{F}^{m+1}))$.

Step 2. Let $m \geqslant 0$ and $(S = \{\Delta_0, \Delta_1, \Delta_2, \ldots, \Delta_m\}, (\mathcal{T}', \mathcal{F}')) \in \mathfrak{N}$. Define

$$\mathcal{T}^i = \left\langle Gen\left(\bigoplus_{i \leqslant 2k \leqslant m} \mathrm{P}_{2k}\left(\Lambda_S^{2k}\right)\right), \mathcal{T}'\right\rangle, \qquad \mathcal{F}^i = \left\langle \mathcal{F}', Cogen\left(\bigoplus_{i \leqslant 2k+1 \leqslant m} \mathrm{I}_{2k+1}\left(\Lambda_S^{2k+1}\right)\right)\right\rangle$$

for $0 \leqslant i \leqslant m+1$. Then $(\mathcal{T}^{m+1}, \mathcal{F}^{m+1}) = (\mathcal{T}', \mathcal{F}')$ is a torsion pair on Λ_S^{m+1}-mod. Now suppose for some $1 \leqslant 2i \leqslant m+1$, $(\mathcal{T}^{2i}, \mathcal{F}^{2i})$ is a torsion pair on Λ_S^{2i}-mod. By Lemma 3.2, $(\Lambda^{2i}\text{-mod}, Cogen\, D(e_S^{2i-1}\Lambda_S^{2i-1}))$ is a torsion pair on Λ_S^{2i-1}-mod. Thus by Proposition 2.6, $(\mathcal{T}^{2i}, \mathcal{F}^{2i}, Cogen\, D(e_S^{2i-1}\Lambda_S^{2i-1}))$ is a torsion 3-tuple on Λ_S^{2i-1}-mod. Thus by Theorem 2.13, $(\mathcal{T}^{2i-1}, \mathcal{F}^{2i-1}) = (\mathcal{T}^{2i}, \langle \mathcal{F}^{2i}, Cogen\, D(e_S^{2i-1}\Lambda_S^{2i-1})\rangle)$ is a torsion pair on Λ_S^{2i-1}-mod. Similarly, we have if for some $1 \leqslant 2i+1 \leqslant m+1$, $(\mathcal{T}^{2i+1}, \mathcal{F}^{2i+1})$ is a torsion pair on Λ_S^{2i+1}-mod, then $(\mathcal{T}^{2i}, \mathcal{F}^{2i})$ is a torsion pair on Λ_S^{2i}-mod. Therefore by induction, for $0 \leqslant i \leqslant m+1$, $(\mathcal{T}^i, \mathcal{F}^i)$ is a torsion pair on Λ_S^i-mod. Define $G((S, (\mathcal{T}', \mathcal{F}'))) = (\mathcal{T}^0, \mathcal{F}^0)$.

Step 3. Given $(\mathcal{T}, \mathcal{F}) \in \mathfrak{M}$, it is clear that $GF((\mathcal{T}, \mathcal{F})) = (\mathcal{T}, \mathcal{F})$.

Step 4. Let $m \geqslant 0$ and $(S = \{\Delta_0, \Delta_1, \Delta_2, \ldots, \Delta_m\}, (\mathcal{T}', \mathcal{F}')) \in \mathfrak{N}$. We adopt the denotations in Step 2, and claim that for $0 \leqslant 2i \leqslant m$, $\mathcal{T}^{2i} \cap \mathcal{P}(\Lambda_S^{2i}) = \operatorname{add} \mathrm{P}_{2i}(\Lambda_S^{2i})$.

It's clear that $\operatorname{add} \mathrm{P}_{2i}(\Lambda_S^{2i}) \subseteq \mathcal{T}^{2i} \cap \mathcal{P}(\Lambda_S^{2i})$. Now suppose there exists some $e \in \Delta - \bigcup_{0 \leqslant k \leqslant 2i} \Delta_k$ such that $\Lambda_S^{2i}e \in \mathcal{T}^{2i}$. We know $(Gen\,\mathrm{P}_{2i}(\Lambda_S^{2i}), Gen\,\mathrm{P}_{2i+2}(\Lambda_S^{2i+2}), \ldots, \mathcal{T}', \mathcal{F}', \ldots, Cogen\,\mathrm{I}_{2i+1}(\Lambda_S^{2i+1}))$ is a torsion tuple on Λ_S^{2i}-mod. By Proposition 3.4, there exists $2i+2 \leqslant 2(i+k) \leqslant m$ such that $e \in \Delta_{2(i+k)}$ since $\Lambda_S^{2i}e \in \mathcal{T}$ and $\mathcal{T}'$ has no nonzero Ext-projective modules in $\langle \mathcal{T}', \mathcal{F}' \rangle$. Since S is a 1-type part partition, $e_S^{2i+2k-1}\Lambda_S^{2i+2k-1}e \neq 0$. Thus $e_S^{2i+2k-1}\Lambda_S^{2i+2k-1}e = \mathrm{Hom}(\Lambda_S^{2i+2k-1}e, D(e^{2i+2k-1}\Lambda_S^{2i+2k-1})) \neq 0$. And hence $\mathrm{Hom}(\Lambda_S^{2i}e, D(e^{2i+2k-1}\Lambda_S^{2i+2k-1})) \neq 0$. So $\Lambda_S^{2i}e \notin \mathcal{T}^{2i}$ because $D(e^{2i+2k-1}\Lambda_S^{2i+2k-1}) \in \mathcal{F}^{2i}$. This is a contradiction.

Similarly we have $\mathcal{I}(\Lambda_S^{2i+1}) \cap \mathcal{F}^{2i+1} = \mathrm{add}\, \mathrm{I}_{2i+1}(\Lambda_S^{2i+1})$ for $1 \leqslant 2i+1 \leqslant m$. Therefore $FG((S, (\mathcal{T}', \mathcal{F}'))) = (S, (\mathcal{T}', \mathcal{F}'))$. $\square$

Dually, if we start to decompose a torsion pair from the right hand (torsion-free class), then we have the following theorem.

Theorem 3.11. *There is a one-to-one correspondence between $\mathfrak{M}$ and $\mathfrak{N}'$: $\mathfrak{M} \underset{G'}{\overset{F'}{\rightleftarrows}} \mathfrak{N}'$.*

The following proposition demonstrates the relation between the above two kinds of decomposition.

Proposition 3.12. *Let $(\mathcal{T}, \mathcal{F}) \in \mathfrak{M}$, $F((\mathcal{T}, \mathcal{F})) = (S' = \{\Delta_0', \Delta_1', \Delta_2', \ldots, \Delta_u'\}, (\mathcal{T}', \mathcal{F}'))$ and $F'((\mathcal{T}, \mathcal{F})) = (S'' = \{\Delta_0'', \Delta_1'', \Delta_2'', \ldots, \Delta_v''\}, (\mathcal{T}'', \mathcal{F}''))$. Then $(\mathcal{T}', \mathcal{F}') = (\mathcal{T}'', \mathcal{F}'')$.*

Proof. First, we give the following notations for any given $i \geqslant 0$:

$$L_{S'}^i = \langle Gen\, \mathrm{P}_{2j}(\Lambda_{S'}^{2j}) \mid 0 \leqslant 2j \leqslant \max\{u, i\} \rangle;$$

$$R_{S'}^i = \langle Cogen\, \mathrm{I}_{2j+1}(\Lambda_{S'}^{2j+1}) \mid 0 < 2j+1 \leqslant \max\{u, i\} \rangle;$$

$$L_{S''}^i = \langle Gen\, \mathrm{P}_{2j+1}(\Lambda_{S''}^{2j+1}) \mid 0 < 2j+1 \leqslant \max\{v, i\} \rangle;$$

$$R_{S''}^i = \langle Cogen\, \mathrm{I}_{2j}(\Lambda_{S''}^{2j}) \mid 0 \leqslant 2j \leqslant \max\{v, i\} \rangle.$$

It's clear that $(L_{S'}^u, \mathcal{T}', \mathcal{F}', R_{S'}^u)$ is a torsion 4-tuple on Λ-mod and $\langle L_{S'}^u, \mathcal{T}' \rangle = \mathcal{T}$, $\langle \mathcal{F}', R_{S'}^u \rangle = \mathcal{F}$. Thus $\mathcal{T}' = \mathcal{T} \cap \Lambda_{S'}^{u+1}$-mod, $\mathcal{F}' = \mathcal{F} \cap \Lambda_{S'}^{u+1}$-mod. And $(\mathcal{T}'', \mathcal{F}'')$ has the similar property. So we only need to prove $\Lambda_{S'}^{u+1}$-mod $= \Lambda_{S''}^{v+1}$-mod. For this we prove $L_{S'}^u = L_{S''}^v$, $R_{S'}^u = R_{S''}^v$ since $\Lambda_{S'}^{u+1}$-mod $= (L_{S'}^u)^{\perp} \cap {}^{\perp}(R_{S'}^u)$ and $\Lambda_{S''}^{v+1}$-mod $= (L_{S''}^v)^{\perp} \cap {}^{\perp}(R_{S''}^v)$.

We suppose $\Lambda_{S'}^{u+j} = \Lambda_{S'}^{u+1}$ and $\Lambda_{S''}^{v+j} = \Lambda_{S''}^{v+1}$ for $j \geqslant 1$, and claim that $\forall i \geqslant 0$, $L_{S'}^{2i+1} \subseteq L_{S''}^{2i+1}$, $R_{S'}^{2i} \subseteq R_{S''}^{2i}$.

For $i = 0$, $R_{S'}^0 = \{0\} \subseteq Cogen\, \mathrm{I}_0(\Lambda_{S''}^0) = R_{S''}^0$, $L_{S'}^1 = Gen\, \mathrm{P}_0(\Lambda_{S'}^0) \subseteq Gen\, \mathrm{P}_1(\Lambda_{S''}^1) = L_{S''}^1$ since $\mathrm{P}_0(\Lambda_{S'}^0) \in \mathrm{add}\, \mathrm{P}_1(\Lambda_{S''}^1)$.

Now we assume the claim holds for $0 \leqslant i \leqslant k-1$. Then $\Lambda_{S''}^{2k}$-mod is a full subcategory of $\Lambda_{S'}^{2k-1}$-mod since $\Lambda_{S'}^{2k-1}$-mod $= (L_{S'}^{2k-2})^{\perp} \cap {}^{\perp}(R_{S'}^{2k-3})$ and $\Lambda_{S''}^{2k}$-mod $= (L_{S''}^{2k-1})^{\perp} \cap {}^{\perp}(R_{S''}^{2k-2})$.

Let $0 \neq X \in \mathrm{add}\, \mathrm{I}_{2k-1}(\Lambda_{S'}^{2k-1})$. So X is Ext-injective in $\Lambda_{S''}^{2k}$-mod. The torsion pair $(\mathcal{F} \cap \Lambda_{S''}^{2k}$-mod, $R_{S''}^{2k-2})$ on $\mathcal{F}$ induces the following canonical short exact sequence $0 \to X_1 \to X \to X_2 \to 0$. Applying $\mathrm{Hom}(M, -)$ to it for $M \in \Lambda_{S''}^{2k}$-mod, we know that X_1 is Ext-injective in $\Lambda_{S''}^{2k}$-mod. Thus $X_1 \in \mathrm{add}\, \mathrm{I}_{2k}(\Lambda_{S''}^{2k})$. So $X \in R_{S''}^{2k}$. Therefore, $R_{S'}^{2k} \subseteq R_{S''}^{2k}$, and similarly, we have $L_{S'}^{2k+1} \subseteq L_{S''}^{2k+1}$.

Thus $L_{S'}^u \subseteq L_{S''}^v$, $R_{S'}^u \subseteq R_{S''}^v$ if we let $i = u + v$. Dually, $L_{S''}^v \subseteq L_{S'}^u$, $R_{S''}^v \subseteq R_{S'}^u$. This completes the proof. $\square$

4. Examples

In this section, we will characterize torsion pairs on some particular module categories. These results will be related to some work in [3,4,11,5]. In this and next section, we always assume K is an algebraically closed field of characteristic 0. If Q is a quiver and $\Delta \subseteq Q_0$ where Q_0 is the set of vertices of Q, then we denote the full sub-quiver of Q containing Δ by $Q(\Delta)$.

Definition 4.1. Let Q be a quiver, $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ a tuple of vertices such that $\Delta_1, \dots, \Delta_m \neq \phi$, $\Delta_i \cap \Delta_j = \phi$ for $i \neq j$.

(1) If for all $1 < 2i+1 \leqslant m$ and $v \in \Delta_{2i+1}$ there is a path from some vertex in Δ_{2i} to v in the sub-quiver $Q(Q_0 - \Delta_0 - \Delta_1 - \cdots - \Delta_{2i-1})$, and for $0 < 2i \leqslant m$ and $v \in \Delta_{2i}$ there is a path from v to some vertex in Δ_{2i-1} in the sub-quiver $Q(Q_0 - \Delta_0 - \Delta_1 - \cdots - \Delta_{2i-2})$, then we call $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ a 1-type part partition of Q.
The following diagram shows the relation:

$$\begin{tikzcd}
\Delta_0 & & \Delta_2 \arrow[dl, hook] \arrow[dr] & & \Delta_4 \arrow[dl, hook] & & \cdots \\
& \Delta_1 & & \Delta_3 & & \cdots &
\end{tikzcd}$$

(2) Dually, we call $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ a 2-type part partition of Q if for all $1 < 2i+1 \leqslant m$ and $v \in \Delta_{2i+1}$ there is a path from v to some vertex in Δ_{2i} in the sub-quiver $Q(Q_0 - \Delta_0 - \Delta_1 - \cdots - \Delta_{2i-1})$, and for all $0 < 2i \leqslant m$ and $v \in \Delta_{2i}$ there is a path from some vertex in Δ_{2i-1} to v in the sub-quiver $Q(Q_0 - \Delta_0 - \Delta_1 - \cdots - \Delta_{2i-2})$.
The following diagram shows the relation:

$$\begin{tikzcd}
& \Delta_1 \arrow[dr] & & \Delta_3 \arrow[dl, hook] \arrow[dr] & & \cdots & \\
\Delta_0 & & \Delta_2 & & \Delta_4 & & \cdots
\end{tikzcd}$$

If $\Delta_0 \cup \cdots \cup \Delta_m = Q_0$ we also call $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ a complete partition.

Definition 4.2. Let Q be a quiver, $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ a tuple of vertices such that $\Delta_1, \dots, \Delta_m \neq \phi$, $\Delta_i \cap \Delta_j = \phi$ for $i \neq j$. If $\forall i > 0$, Δ_{2i-1} contains all sink points in $Q(Q_0 - \Delta_0 - \Delta_1 - \cdots - \Delta_{2i-2})$, Δ_{2i} contains all source points in $Q(Q_0 - \Delta_0 - \Delta_1 - \cdots - \Delta_{2i-1})$, then we call $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ a strong 1-type part partition of Q. Dually, if $\forall i > 0$, Δ_{2i-1} contains all source points in $Q(Q_0 - \Delta_0 - \Delta_1 - \cdots - \Delta_{2i-2})$, Δ_{2i} contains all sink points in $Q(Q_0 - \Delta_0 - \Delta_1 - \cdots - \Delta_{2i-1})$, then we call $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ a strong 2-type part partition of Q.

Lemma 4.3. *Let Q be a quiver and $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ a strong 1-type part partition of Q. Then $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ is a 1-type part partition of Q.*

Dually, if $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ is a strong 2-type part partition of Q, then it is also a 2-type part partition of Q.

For a quiver Q, we denote $\mathbf{E}(KQ)$ by $\mathbf{E}(Q)$. Now we have the following theorem which is the path algebra's version of Theorem 3.10.

Theorem 4.4. *Let Q be an acyclic quiver. Then we have a bijection between the set of torsion pairs on KQ-mod and the set of the pair $(\{\Delta_0, \Delta_1, \ldots, \Delta_m\}, (\mathcal{T}', \mathcal{F}'))$, where $\{\Delta_0, \Delta_1, \ldots, \Delta_m\}$ is a 1-type part partition of Q and $(\mathcal{T}', \mathcal{F}') \in \mathbf{E}(Q(Q_0 - \Delta_0 - \Delta_1 - \cdots - \Delta_m))$.*

The dual form of the theorem is similar, we omit it here. Now let A_n be the following quiver: $1 \to 2 \to 3 \to \cdots \to n$. Applying the above theorem to the quiver A_n, we have the following theorem.

Theorem 4.5. *There exists a bijection between torsion pairs on KA_n-mod and strong 1-type complete partition sets of A_n.*

Proof. It is easy to see $\mathbf{E}(KA_m) = \phi$ for $m \geqslant 1$ since KA_m has a nonzero projective–injective module. And any 1-type part partition of A_n is a complete partition if and only if it is also a strong 1-type complete partition. The rest is clear by the above theorem. $\square$

Clearly, there is a dual form for this theorem. We also omit it here.

Proposition 4.6. *The number of torsion pairs on KA_n is the $(n+1)$-th Catalan number $C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$.*

Proof. Adding one vertex to A_n, then we get the quiver $A_{n+1}: 1 \to 2 \to 3 \to \cdots \to n \to n+1$. We have a torsion pair on KA_{n+1}-mod: $(KA_n\text{-mod}, \mathcal{P}(KA_{n+1}))$. From [10], a torsion pair $(\mathcal{T}, \mathcal{F})$ on KA_{n+1}-mod is induced by a cotilting module if and only if $\mathcal{P}(KA_{n+1}) \subseteq \mathcal{F}$. So we have a bijection between torsion pairs on KA_n-mod and torsion pairs induced by cotilting modules on KA_{n+1}-mod by Proposition 2.20. The number of torsion pairs induced by cotilting modules on KA_{n+1}-mod is well known which is the $(n+1)$-th Catalan number [5, Lemma A.1].

Let Λ be an artin algebra, $\{\mathcal{C}\} \cup \{\mathcal{C}_i,\ i \in I\}$ a set of full subcategories of Λ-mod which are closed under direct summands and extensions. If $\operatorname{Ind}\mathcal{C} = \bigcup_{i\in I} \operatorname{Ind}\mathcal{C}_i$ and $\mathcal{C}_i \cap \mathcal{C}_j = \{0\}$ for $i \neq j$, then we call $\mathcal{C}$ a direct sum of $\mathcal{C}_i$ for $i \in I$, and denote $\mathcal{C} = \bigoplus_{i\in I} \mathcal{C}_i$. The following lemma is clear. $\square$

Lemma 4.7. *Let $\mathcal{C}$ and $\mathcal{C}_i$ be defined as above. If $\operatorname{Hom}(\mathcal{C}_i, \mathcal{C}_j) = 0$ for $i \neq j$, then there exists a bijection between torsion pairs on $\mathcal{C}$ and the tuple $\{(\mathcal{T}_i, \mathcal{F}_i)\}_{i\in I}$ where $(\mathcal{T}_i, \mathcal{F}_i)$ is a torsion pair on $\mathcal{C}_i$.*

Proof. Given $(\mathcal{T}, \mathcal{F})$ a torsion pair on $\mathcal{C}$, then $(\mathcal{T} \cap \mathcal{C}_i, \mathcal{F} \cap \mathcal{C}_i)_{i\in I}$ is the corresponding tuple. Given the tuple $(\mathcal{T}_i, \mathcal{F}_i)_{i\in I}$ where $(\mathcal{T}_i, \mathcal{F}_i)$ is a torsion pair on $\mathcal{C}_i$, then $(\bigoplus_{i\in I} \mathcal{T}_i, \bigoplus_{i\in I} \mathcal{F}_i)$ is the corresponding torsion pair. $\square$

Let $\tilde{A}_n$ be a direct cycle with arrows $1 \to 2, 2 \to 3, \ldots, n-1 \to n, n \to 1$, and J the ideal of $K\tilde{A}_n$ generated by all arrows. We call a finite-dimensional $K\tilde{A}_n$-module M is an ordinary module if there exists $N \geqslant 1$ such that $J^N M = 0$. In this condition M is a $K\tilde{A}_n/J^N$-module. So if M is indecomposable, then it is uniserial and determined by its socle and length. Let $\mathcal{E}_n$ be the category of all ordinary modules. Then $\mathcal{E}_n$ is a Serre class of $K\tilde{A}_n$-mod. We denote the simple module corresponding to the vertex v_i by S_i. The following definition has been introduced in [3].

Definition 4.8. Let $\Delta \in (\tilde{A}_n)_0$. We define $Ray(\Delta) = \{M \in \mathcal{E}_n \mid socle(M) \in \operatorname{add}\bigoplus_{v_i\in\Delta} S_i\}$, $Coray(\Delta) = \{M \in \mathcal{E}_n \mid top(M) \in \operatorname{add}\bigoplus_{v_i\in\Delta} S_i\}$.

For a subcategory $\mathcal{D}$ of $\mathcal{E}_n$. Let $L_{\mathcal{D}} = \{v_i \in (\tilde{A}_n)_0 \mid \#\operatorname{Ind}(\mathcal{D} \cap Coray(\{v_i\})) = \infty\}$, $R_{\mathcal{D}} = \{v_i \in (\tilde{A}_n)_0 \mid \#\operatorname{Ind}(\mathcal{D} \cap Ray(\{v_i\})) = \infty\}$.

The following lemma is clear by the definition of torsion pairs.

Lemma 4.9. *Let $\phi \neq \Delta \subseteq (\tilde{A}_n)_0$. Then $(Coray(\Delta), \tilde{A}_n((\tilde{A}_n)_0 - \Delta)\text{-mod})$ and $(\tilde{A}_n((\tilde{A}_n)_0 - \Delta)\text{-mod}, Ray(\Delta))$ are both torsion pairs on $\mathcal{E}_n$.*

The following lemma is from [3, 4.5].

Lemma 4.10. *Let $(\mathcal{T},\mathcal{F})$ be a torsion pair on $\mathcal{E}_n$. Then $L_{\mathcal{T}} \cup R_{\mathcal{F}} \neq \phi$.*

Now we have the following proposition which gives all torsion pairs on $\mathcal{E}_n$.

Proposition 4.11. *The following are all pairwise different torsion pairs on $\mathcal{E}_n$ which are classified as two kinds.*

(1) *$(Coray(\Delta) \oplus \mathcal{T}', \mathcal{F}')$ for some $\phi \neq \Delta \subseteq (\tilde{A}_n)_0$ and $(\mathcal{T}',\mathcal{F}')$ which is a torsion pair on $\tilde{A}_n((\tilde{A}_n)_0 - \Delta)$-mod induced by a cotilting $\tilde{A}_n((\tilde{A}_n)_0 - \Delta)$-module.*
(2) *$(\mathcal{T}', \mathcal{F}' \oplus Ray(\Delta))$ for some $\phi \neq \Delta \subseteq (\tilde{A}_n)_0$ and $(\mathcal{T}',\mathcal{F}')$ which is a torsion pair on $\tilde{A}_n((\tilde{A}_n)_0 - \Delta)$-mod induced by a tilting $\tilde{A}_n((\tilde{A}_n)_0 - \Delta)$-module.*

Proof. Let $(\mathcal{T},\mathcal{F})$ be a torsion pair on $\mathcal{E}_n$ and $L_{\mathcal{T}} \neq \phi$. Then we know that $Coray(L_{\mathcal{T}}) \subseteq \mathcal{T}$ since $\mathcal{T}$ is closed under quotients. Thus $\{(Coray(L_{\mathcal{T}}), \tilde{A}_n((\tilde{A}_n)_0 - L_{\mathcal{T}})\text{-mod}), (\mathcal{T},\mathcal{F})\}$ is a chain of torsion classes. Let $\mathcal{T}' = \tilde{A}_n((\tilde{A}_n)_0 - L_{\mathcal{T}})\text{-mod} \cap \mathcal{T}, \mathcal{F}' = \mathcal{F}$. Then $(\mathcal{T}',\mathcal{F}')$ is a torsion pair on $\tilde{A}_n((\tilde{A}_n)_0 - L_{\mathcal{T}})$-mod and $\mathcal{T} = \langle Coray(L_{\mathcal{T}}), \mathcal{T}'\rangle$. Note that $\mathcal{P}(\tilde{A}_n((\tilde{A}_n)_0 - L_{\mathcal{T}})) \subseteq \mathcal{F}'$ (if not, then $L_{\mathcal{T}} \subsetneq L_{\langle Coray(L_{\mathcal{T}}),\mathcal{T}'\rangle}$). Thus $(\mathcal{T}',\mathcal{F}')$ is induced by a cotilting $\tilde{A}_n((\tilde{A}_n)_0 - L_{\mathcal{T}})$-module and $\langle Coray(L_{\mathcal{T}}), \mathcal{T}'\rangle = Coray(L_{\mathcal{T}}) \oplus \mathcal{T}'$.

On the other hand, let $\phi \neq \Delta \subseteq (\tilde{A}_n)_0$ and $(\mathcal{T}',\mathcal{F}')$ be a torsion pair on $\tilde{A}_n((\tilde{A}_n)_0 - \Delta)$-mod induced by a cotilting $\tilde{A}_n((\tilde{A}_n)_0 - \Delta)$-module. Since $(Coray(L_{\mathcal{T}}), \tilde{A}_n((\tilde{A}_n)_0 - \Delta)\text{-mod})$ is a torsion pair on $\mathcal{E}_n$ and $\langle Coray(\Delta), \mathcal{T}'\rangle = Coray(\Delta) \oplus \mathcal{T}'$, $(Coray(\Delta) \oplus \mathcal{T}', \mathcal{F}')$ is a torsion pair on $\mathcal{E}_n$ by Theorem 2.13. Meanwhile, $L_{Coray(\Delta)\oplus\mathcal{T}'} = \Delta$, $\mathcal{T}' = (Coray(\Delta) \oplus \mathcal{T}') \cap \tilde{A}_n((\tilde{A}_n)_0 - \Delta)$-mod, thus the different vertex sets Δ (or $(\mathcal{T}',\mathcal{F}')$) generate different torsion pairs.

The other half is similar. □

Since $\phi \neq \Delta$, we know $\tilde{A}_n((\tilde{A}_n)_0 - \Delta)$-mod is a direct sum of module categories of A_m-type algebras. So by Lemma 4.7 the torsion pair is easily obtained. Theorem 4.5 and its dual form give the structure of torsion pairs on A_m-type algebras. Thus we can obtain the structure of torsion pairs on $\mathcal{E}_n$.

Theorem 4.12.

(1) *There is a bijection between the set of the torsion pair $(\mathcal{T},\mathcal{F})$ on $\mathcal{E}_n$ with $L_{\mathcal{T}} \neq \phi$ and the set of the strong 1-type complete partition $\{\Delta, \Delta_1, \dots, \Delta_m\}$ of $\tilde{A}_n$ with Δ not empty.*
(2) *There is a bijection between the set of the torsion pairs $(\mathcal{T},\mathcal{F})$ on $\mathcal{E}_n$ with $R_{\mathcal{F}} \neq \phi$ and the set of the strong 2-type complete partition $\{\Delta, \Delta_1, \dots, \Delta_m\}$ of $\tilde{A}_n$ with Δ not empty.*

Proof. We only prove (1). The proof of (2) is similar.

By Proposition 4.11, a torsion pair $(\mathcal{T},\mathcal{F})$ on $\mathcal{E}_n$ with $L_{\mathcal{T}} \neq \phi$ can be uniquely written as $(Coray(\Delta) \oplus \mathcal{T}', \mathcal{F}')$ for some $\phi \neq \Delta \subseteq (\tilde{A}_n)_0$ and $(\mathcal{T}',\mathcal{F}')$ which is a torsion pair on $\tilde{A}_n((\tilde{A}_n)_0 - \Delta)$-mod induced by a cotilting $\tilde{A}_n((\tilde{A}_n)_0 - \Delta)$-module. Since $\tilde{A}_n((\tilde{A}_n)_0 - \Delta)$-mod is a direct sum of module categories of A_m-type algebras, it is uniquely corresponding to a strong 2-type complete partition $(\Delta_1, \dots, \Delta_m)$ of $\tilde{A}_n((\tilde{A}_n)_0 - \Delta)$ by the dual form of Theorem 4.5. Moreover, since $\mathcal{P}(\tilde{A}_n((\tilde{A}_n)_0 - \Delta)) \subseteq \mathcal{F}'$, all sink points of $\tilde{A}_n((\tilde{A}_n)_0 - \Delta)$ should belong to Δ_1. Thus $\{\Delta_0, \Delta_1, \dots, \Delta_m\}$ is a strong 1-type complete partition of $\tilde{A}_n$. □

5. Torsion pairs on hereditary algebras

In this section we assume Q is an acyclic quiver, denote the Auslander–Reiten translation by τ, its quasi-inverse by τ^-, let $\mathcal{P}(Q) = \mathcal{P}(KQ)$, $\mathcal{I}(Q) = \mathcal{I}(KQ)$. We try to apply the results in the formal sections to study the structure of torsion pairs on KQ-mod. The following two lemmas are well known.

Lemma 5.1. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence on kQ-mod.*

(1) *If* $\operatorname{add} A \cap \mathcal{P}(Q) = \{0\}$, *then it induces a short exact sequence* $0 \to \tau A \to \tau B \to \tau C \to 0$.
(2) *If* $\operatorname{add} C \cap \mathcal{I}(Q) = \{0\}$, *then it induces a short exact sequence* $0 \to \tau^- A \to \tau^- B \to \tau^- C \to 0$.

Proof. Note that $\tau = D\operatorname{Ext}^1_{KQ}(-, KQ)$. Thus applying $\operatorname{Hom}_{KQ}(-, KQ)$ to the exact sequence, we obtain $0 \to \tau A \to \tau B \to \tau C \to 0$. The proof of (2) is similar. □

Lemma 5.2. *Suppose $X, Y \in kQ$-mod.*

(1) *If* $\operatorname{add} X \cap \mathcal{P}(Q) = \{0\}$, *then* $\operatorname{Hom}(X, Y) \cong \operatorname{Hom}(\tau X, \tau Y)$.
(2) *If* $\operatorname{add} Y \cap \mathcal{I}(Q) = \{0\}$, *then* $\operatorname{Hom}(X, Y) \cong \operatorname{Hom}(\tau^- X, \tau^- Y)$.

We denote the set of torsion pairs on KQ-mod$(\mathcal{T}, \mathcal{F})$ such that $\mathcal{I}(Q) \subseteq \mathcal{T}$ by $\mathbf{F}_1(Q)$ and the set of torsion pairs on KQ-mod$(\mathcal{T}, \mathcal{F})$ such that $\mathcal{P}(Q) \subseteq \mathcal{F}$ by $\mathbf{F}_2(Q)$. And let $\mathbf{F}(Q) = \mathbf{F}_1(Q) \cup \mathbf{F}_2(Q)$. It is obvious that $\mathbf{E}(Q) = \mathbf{F}_1(Q) \cap \mathbf{F}_2(Q)$. As a consequence of the above two lemmas, we have the following proposition.

Proposition 5.3. *There is a one-to-one correspondence*:

$$\mathbf{F}_1(Q) \underset{\sigma}{\overset{\sigma^-}{\rightleftarrows}} \mathbf{F}_2(Q)$$

such that $\forall (\mathcal{T}', \mathcal{F}') \in \mathbf{F}_1(Q)$, $\sigma^-(\mathcal{T}', \mathcal{F}') = (\tau^-\mathcal{T}', \tau^-\mathcal{F}' \oplus \mathcal{P}(Q))$; $\forall (\mathcal{T}'', \mathcal{F}'') \in \mathbf{F}_2(Q)$, $\sigma(\mathcal{T}'', \mathcal{F}'') = (\mathcal{I}(Q) \oplus \tau\mathcal{T}'', \tau\mathcal{F}'')$.

Proof. We just prove that $\forall (\mathcal{T}', \mathcal{F}') \in \mathbf{F}_1$, $(\tau^-\mathcal{T}', \tau^-\mathcal{F}' \oplus \mathcal{P}(Q))$ is a torsion pair on KQ-mod. By Lemma 5.2, we know $\forall X \in \mathcal{T}'$, $Y \in \mathcal{F}'$, $\operatorname{Hom}(\tau^- X, \tau^- Y) \cong \operatorname{Hom}(X, Y) = \{0\}$. By Lemma 5.1, every indecomposable non-projective module has a suitable decomposition by the pair $(\tau^-\mathcal{T}', \tau^-\mathcal{F}' \oplus \mathcal{P}(Q))$. Thus it is a torsion pair on KQ-mod. □

Just like the Auslander–Reiten translation, σ^- and σ define a translation on $F(Q)$. Given $(\mathcal{T}, \mathcal{F}) \in F(Q)$, if $\mathcal{I}(Q) \subseteq \mathcal{T}$, then let $\sigma^-(\mathcal{T}, \mathcal{F}) = (\tau^-\mathcal{T}, \tau^-\mathcal{F} \oplus \mathcal{P}(Q))$; if $\mathcal{P}(Q) \subseteq \mathcal{F}$, then let $\sigma(\mathcal{T}, \mathcal{F}) = (\tau\mathcal{T} \oplus \mathcal{I}(Q), \tau\mathcal{F})$. This translation defines σ-obits for elements in $\mathbf{F}(Q)$. We use $[\mathcal{T}, \mathcal{F}]$ to denote the σ-obit of $(\mathcal{T}, \mathcal{F})$.

Definition 5.4. Let $(\mathcal{T}, \mathcal{F}) \in \mathbf{F}(Q)$. We call the elements in $[\mathcal{T}, \mathcal{F}] \cap (\mathbf{F}_2(Q) - \mathbf{F}_1(Q))$ source points of $[\mathcal{T}, \mathcal{F}]$, the elements in $[\mathcal{T}, \mathcal{F}] \cap (\mathbf{F}_1(Q) - \mathbf{F}_2(Q))$ sink points of $[\mathcal{T}, \mathcal{F}]$.

The following corollary is obvious and indicates that we can consider the σ-obit of elements in $\mathbf{E}(Q)$ to continue the decomposition in Theorem 4.4.

Lemma 5.5. *Let $(\mathcal{T}, \mathcal{F}) \in \mathbf{F}(Q)$. Then $[\mathcal{T}, \mathcal{F}]$ has at most one source point and at most one sink point. If $(\mathcal{T}, \mathcal{F})$ is a source point, then $\mathcal{F} \cap \mathcal{I}(Q) \neq \phi$. If $(\mathcal{T}, \mathcal{F})$ is a sink point, then $\mathcal{T} \cap \mathcal{P}(Q) \neq \phi$.*

We denote by $\mathcal{P}_\infty(Q)$ the full subcategory of preprojective KQ-modules, by $\mathcal{I}_\infty(Q)$ the full subcategory of preinjective KQ-modules, by $\mathcal{R}(Q)$ the full subcategory of regular KQ-modules.

Proposition 5.6. *Let $(\mathcal{T}, \mathcal{F}) \in \mathbf{F}(Q)$. Then*:

(1) *$[\mathcal{T}, \mathcal{F}]$ has a source point but no sink point $\Leftrightarrow$ for every $(\mathcal{T}', \mathcal{F}') \in [\mathcal{T}, \mathcal{F}]$, $\mathcal{I}_\infty(Q) \cap \mathcal{F}' \neq \phi$ and $\mathcal{P}_\infty(Q) \subseteq \mathcal{F}'$.*

(2) $[\mathcal{T},\mathcal{F}]$ *has a sink point but no source point* $\Leftrightarrow$ *for every* $(\mathcal{T}',\mathcal{F}') \in [\mathcal{T},\mathcal{F}]$, $\mathcal{P}_\infty(Q) \cap \mathcal{T} \neq \phi$ *and* $\mathcal{I}_\infty(Q) \subseteq \mathcal{T}'$.
(3) $[\mathcal{T},\mathcal{F}]$ *has a sink point and a source point* $\Leftrightarrow$ *for every* $(\mathcal{T}',\mathcal{F}') \in [\mathcal{T},\mathcal{F}]$, $\mathcal{I}_\infty(Q) \cap \mathcal{F} \neq \phi$ *and* $\mathcal{P}_\infty(Q) \cap \mathcal{T} \neq \phi$.
(4) $[\mathcal{T},\mathcal{F}]$ *has no sink point and no source point* $\Leftrightarrow$ *for every* $(\mathcal{T}',\mathcal{F}') \in [\mathcal{T},\mathcal{F}]$, $\mathcal{I}_\infty(Q) \subseteq \mathcal{T}'$, *and* $\mathcal{P}_\infty(Q) \subseteq \mathcal{F}'$.

We denote the set of torsion pairs $(\mathcal{T},\mathcal{F})$ on KQ-mod such that $\mathcal{I}_\infty(Q) \subseteq \mathcal{T}$, and $\mathcal{P}_\infty(Q) \subseteq \mathcal{F}$ by $\mathbf{H}(Q)$. So it is obvious that $\mathbf{H}(Q) \subseteq \mathbf{E}(Q)$. We denote the set of torsion pairs on $\mathcal{R}(Q)$ by $\mathbf{R}(Q)$. We have the following obvious lemma.

Lemma 5.7. *There is a one-to-one correspondence*:

$$\mathbf{H}(Q) \underset{F^-}{\overset{F}{\rightleftarrows}} \mathbf{R}(Q)$$

such that $\forall (\mathcal{T},\mathcal{F}) \in \mathbf{H}(Q)$, $F((\mathcal{T},\mathcal{F})) = (\mathcal{T} \cap \mathcal{R}(Q), \mathcal{F} \cap \mathcal{R}(Q))$; $\forall (\mathcal{T}',\mathcal{F})' \in \mathbf{R}(Q)$, $F^-((\mathcal{T}',\mathcal{F}')) = (\mathcal{T}' \oplus \mathcal{I}_\infty(Q), \mathcal{F}' \oplus \mathcal{P}_\infty(Q))$.

Suppose $(\mathcal{T},\mathcal{F}) \in \mathbf{F}(Q) \cap \mathbf{E}(Q)$ and $[\mathcal{T},\mathcal{F}]$ has at least one sink point or one source point. We define the following operation Φ:

Case 1. If $[\mathcal{T},\mathcal{F}]$ has a sink point, then we denote the sink point by $\Phi((\mathcal{T},\mathcal{F}))$;

Case 2. If $[\mathcal{T},\mathcal{F}]$ has a source point but no sink point, then we denote the source point by $\Phi((\mathcal{T},\mathcal{F}))$.

Then we can apply the operations defined in Theorem 3.10 to $\Phi((\mathcal{T},\mathcal{F}))$ to continue the decomposition. For any torsion pair on KQ-mod we use the two kinds of operation alternatively. At last we get a new torsion pair in $\mathbf{H}(Q')$ for some sub-quiver Q' of Q. This process is unique, and invertible by Theorem 4.4 and Proposition 5.3.

From now on we suppose Q is an acyclic quiver with a Euclid ground graph. We study the structure of all the torsion pairs in $\mathbf{R}(Q)$. The following definition and two lemmas are from [7].

Definition 5.8. Suppose $X \in KQ$-mod. Then Q is regular uniserial if there are regular submodules $0 = X_0 \subset X_1 \subset \cdots \subset X_r = X$ and these are the only regular submodules of X.

Lemma 5.9. *If* $\theta : X \to Y$ *with* X, Y *regular* KQ*-modules, then* $\mathrm{Im}(\theta)$, $\mathrm{Ker}(\theta)$ *and* $\mathrm{Coker}(\theta)$ *are regular.*

Lemma 5.10. *Every indecomposable regular* KQ*-module is regular universal.*

As a consequence we have

Corollary 5.11. *If* KQ *is a Euclid-type algebra,* X *is a regular module, then the quotient modules of* X *forms a chain*: $X = X^r \twoheadrightarrow \cdots \twoheadrightarrow X^1 \twoheadrightarrow X^0$.

Corollary 5.12. *Let* KQ *be a Euclid-type algebra,* $f : X \to Y$ *an injective homomorphism such that* X *is a maximal regular submodule of the indecomposable regular module of* Y*. Then* f *is an irreducible morphism.*

Proof. X is indecomposable by Lemma 5.10. Suppose $g : X \to Z$, $h : Z \to Y$ satisfy $f = hg$. Then by Lemma 5.10, there is an indecomposable direct summand Z' of Z such that $\exists g' : X \to Z'$, $h : Z' \to Y$ such that $h'g'$ is an injective morphism. Therefore Z' is a regular module since ${}^\perp\mathcal{R}(Q) = \mathcal{I}_\infty(Q)$, $\mathcal{R}(Q)^\perp = \mathcal{P}_\infty(Q)$. Since X, Y and Z' are regular universal, h' is an isomorphism or g' is an isomorphism. $\square$

Now let $\mathcal{R}(Q) = \bigoplus_{i\in I} \mathcal{R}_i(Q)$ where $\{\mathcal{R}_i(Q),\ i \in I\}$ is a set of minimal additive categories containing a connected component consisting of regular modules in AR-quiver of KQ. We denote the set of torsion pairs on $\mathcal{R}_i(Q)$ by $\mathbf{R}_i(Q)$. By Lemma 4.7, we have the following.

Corollary 5.13. *There exists a bijection between $\mathbf{R}(Q)$ and the set of tuples $\{(\mathcal{T}_i, \mathcal{F}_i)\}_{i\in I}$ with $(\mathcal{T}_i, \mathcal{F}_i) \in \mathbf{R}_i(Q)$.*

Proof. Let $X \in \mathcal{R}_i(Q)$. Then all regular submodules and all regular quotient modules of X are in $\mathcal{R}_i(Q)$ by the above corollary and its dual form. So we know if $i \neq j$, then $\mathrm{Hom}(X, Y) = 0$, $\forall X \in \mathcal{R}_i(Q)$ and $Y \in \mathcal{R}_j(Q)$. The rest is clear by Lemma 4.7. $\square$

By [7], a nonzero indecomposable regular KQ-module is called a regular simple module if and only if it has no non-trivial regular submodules. And the number of nonzero regular submodules of an indecomposable regular module is called its regular length.

Now we start to demonstrate $\mathbf{R}_i(Q)$. Suppose $\mathcal{R}_i(Q)$ has n regular simple modules: $S_1, S_2, \ldots, S_{n-1}, S_n$ where $S_{i+1} = \tau S_i$ for $1 \leqslant i \leqslant n-1$ and $\tau S_n = S_1$. Let $\tilde{A}_n$ be the quiver in Section 4 and $S'_1, S'_2, \ldots, S'_n$ are the correspondent simple modules to the vertices. Then we construct a map: $\overline{F}(S'_i) = S_i$. Then $\overline{F}$ induces a one-to-one correspondence $\mathcal{E}_n \to \mathcal{R}_i(Q)$ such that if $X \in \mathcal{E}_n$ and is indecomposable with the length m and top S'_i, then $F(X)$ is the indecomposable regular module with the regular length m and top S_i. We have the following proposition.

Proposition 5.14. *F induces a one-to-one correspondence between the set of torsion pairs on $\mathcal{E}_n$ and $\mathbf{R}_i(Q)$.*

Proof. Since the modules in $\mathcal{R}_i(Q)$ are regular universe, we have the following assertions

(1) $\forall X, Y \in \mathcal{E}_n$, $\mathrm{Hom}(X, Y) = 0$ if and only if $\mathrm{Hom}(F(X), F(Y)) = 0$;
(2) Suppose $Y \in \mathcal{E}_n$ and X is a submodule of Y. Then $F(Y/X) = F(Y)/F(X)$.

Thus F induces a one-to-one correspondence between the set of torsion pairs on $\mathcal{E}_n$ and $\mathbf{R}_i(Q)$. $\square$

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References

[1] I. Assem, O. Kerner, Constructing torsion pairs, J. Algebra 185 (1996) 19–41.
[2] I. Assem, D. Simson, A. Skowronski, Elements of the Representation Theory of Associative Algebras, vol. 1: Techniques of Representation Theory, London Math. Soc. Stud. Texts, vol. 65, Cambridge Univ. Press, Cambridge, 2006.
[3] K. Baur, A.B. Buan, R.J. Marsh, Torsion pairs and rigid objects in tubes, arXiv:1112.6132v1 [math.RT], 2011.
[4] K. Baur, R.J. Marsh, A geometric model of tube categories, J. Algebra 362 (2012) 178–191.
[5] A.B. Buan, H. Krause, Tilting and cotilting for quivers of type $\tilde{A}_n$, J. Pure Appl. Algebra 190 (1–3) (2004) 1–21.
[6] T. Bridgeland, Stability conditions on triangulated categories, Ann. of Math. 166 (2007) 317–346.
[7] W. Crawley-Boevey, Lectures on representations of quivers, available at http://www.amsta.leeds.ac.uk/~pmtwc/quivlecs.ps.
[8] S.E. Dickson, A torsion theory for abelian categories, Trans. Amer. Math. Soc. 121 (1966) 223–235.
[9] W. Geigle, H. Lenzing, Perpendicular categories with applications to representations and sheaves, J. Algebra 144 (1991) 273–343.
[10] M. Hoshino, Tilting modules and torsion theories, Bull. London Math. Soc. 14 (1982) 334–336.
[11] T. Holm, P. Jørgensen, M. Rubey, Ptolemy diagrams and torsion pairs in the cluster category of Dynkin type A_n, J. Algebraic Combin. 34 (2011) 507–523.
[12] A.N. Rudakov, Stability for an abelian category, J. Algebra 197 (1997) 231–245.