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# Coverings of graded pointed Hopf algebras

Esther Beneish <sup>a,b</sup>, William Chin <sup>a,b,\*</sup><sup>a</sup> University of Wisconsin–Parkside, Kenosha, WI, United States<sup>b</sup> DePaul University, Chicago, IL, United States

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## ABSTRACT

We introduce the concept of a covering of a graded pointed Hopf algebra. The theory developed shows that coverings of a bosonized Nichols algebra can be concretely expressed by biproducts using a quotient of the universal coalgebra covering group of the Nichols algebra. If there are enough quadratic relations, the universal coalgebra covering is given by the bosonization by the enveloping group of the underlying rack.

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## 1. Introduction

Nichols algebras play a crucial role in the classification results for pointed Hopf algebras over abelian groups, which vastly generalize the theory of quantized enveloping algebras. They appear in several recent investigations of pointed Hopf algebras over non-abelian groups. Nichols algebras are seen to be fundamental objects, appearing in the study of the cohomology of flag manifolds. See e.g., [2,3,10,13,8] and references therein.

A Nichols algebra  $B(V)$  is constructed from a braided vector space  $V$ , where  $V$  in turn is, in the most studied cases, a Yetter–Drinfeld module over some group  $G$ . The Nichols algebra depends only on the braiding  $c : V \otimes V \rightarrow V \otimes V$  and there may be

\* Corresponding author.

many groups yielding the braiding. In this paper, we study groups that can arise given a fixed Yetter–Drinfeld module  $V \in {}^G_G\mathcal{YD}$ .

We assume that  $V$  is a link-indecomposable Yetter–Drinfeld module over a group  $G$ , and that  $X$  is the corresponding rack with cocycle  $q : X \times X \rightarrow \mathbb{k}^\times$ .  $V$  also takes the form  $\bigoplus_i M(g_i, \rho_i) \cong (\mathbb{k}X, c^q)$  as braided vector spaces. That is, we are assuming that  $V$  is of rack type with braiding  $c = c^q : V \otimes V \rightarrow V \otimes V$ . Here  $g_i \in G$  and  $\rho_i$  is a one-dimensional representation of the centralizer of  $g_i$  in  $G$ . The equivalence of the two constructions of  $V$  is explained by [2, Theorem 4.14] (in greater generality).

The theory of coalgebra coverings in [5] yields indecomposable coalgebra coverings of  $B(V)$ . Such coverings take the form  $B(V) \rtimes \mathbb{k}G \rightarrow B(V)$  where  $G$  is a homomorphic image of the universal coalgebra covering group  $\tilde{G}$  for  $B(V)$  and  $\rtimes$  is the smash coproduct (or “co-smash product”) coalgebra. Let  $G_X$  denote the enveloping group of the rack  $X$  (see [2,10]). We have in general for a braiding of rack type that  $V \in {}^{\mathbb{k}G_X}_{\mathbb{k}G_X}\mathcal{YD}$  and therefore there is a surjection  $\tilde{G} \rightarrow G_X$  and a corresponding coalgebra surjection  $B(V) \rtimes \mathbb{k}\tilde{G} \rightarrow B(V) \rtimes \mathbb{k}G_X$ . We show in Corollary 3.6 that a certain quotient  $B(V) \# \hat{G}$  of  $B(V) \# \tilde{G}$  serves as the universal Hopf covering of  $B(V) \# G$ . Thus we have a Hopf algebra covering of the graded Hopf algebra in the sense exhibited in [14] arising from a central extension

$$1 \rightarrow Z \rightarrow \hat{G} \rightarrow G \rightarrow 1.$$

It is known by [1, Lemma 3.4] that  $G_X$  also is universal for  $V$  so we have in fact that  $\hat{G} = G_X$ .

The corresponding rack braiding  $c : X \times X \rightarrow X \times X$  decomposes into  $c$ -orbits  $\mathcal{O} = \mathcal{O}_{x,y}$ . Recall that with  $c = c^q : V \otimes V \rightarrow V \otimes V$ ,  $\ker(1 + c)$  defines the quadratic term  $B(V)(2)$ . When  $\ker(1 + c) \cap V_{\mathcal{O}} \neq 0$  for each nontrivial  $c$ -orbit  $\mathcal{O}$ , we say that  $B(V)$  has a full set of quadratic relations. (We are ignoring the orbits of the form  $\{(x, x)\}$ .) We show in Theorem 4.1 that, for a Nichols algebra with a full set of quadratic relations,  $G_X = \tilde{G}$ . Thus  $B(V) \# G_X \rightarrow B(V)$  is the universal coalgebra covering, which factors through the Hopf algebra surjection  $B(V) \# \mathbb{k}G_X \rightarrow B(V) \# \mathbb{k}G$ . This means that every group  $G$  such that  $B(V) \rtimes \mathbb{k}G$  is an indecomposable coalgebra arises as a homomorphic image of  $G_X$ . We have thus a universal Hopf covering for the Nichols algebra  $B(V)$ . The use of  $G_X$  as the “principal” grading group is mentioned in [1, Lemma 3.4] and suggested in [9].

In case  $G_X = \tilde{G}$ , we remark that the relations in the definition of  $\tilde{G}$  of degree greater than two are superfluous. In all known examples with finite dimensional  $B(V)$  we find that  $G_X = \tilde{G}$ , and we conjecture that this is always the case.

To compare with [10], recall that  $B(V)$  is said to have many quadratic relations if the  $\dim \ker(1 + c) \geq d(d - 1)/2$  where  $d = \dim V$ . The two notions pertaining to quadratic relations do not appear to be comparable, see for example 5.2.

We are interested in finite racks  $X$  that are unions of conjugacy classes of the group  $G$  and we assume that  $X$  generates  $G$ . In this latter case the Yetter–Drinfeld module is said to be link-indecomposable, so the covering Hopf algebras will also be link-indecomposable

(i.e. indecomposable as coalgebra). We provide families of link-indecomposable Hopf algebras corresponding to the covering Hopf algebras of finite-dimensional Nichols algebras. These Hopf algebras are finite-dimensional when the covering group is finite. This addresses the question posed in [17] by including all finite dimensional covering Hopf algebras of known examples of link indecomposable bosonized Nichols algebras. Specifically, the families of finite-dimensional Hopf algebras produced arise from the finite homomorphic images  $G_X$  by finite index subgroups of its center  $Z(G_X)$ . Some of these examples are given in [2, Section 6].

Our main references for racks, Nichols algebras, and pointed Hopf algebras are [2,3]; also see [10,13].

It would be interesting to try to remove the condition that  $\dim \rho = 1$  from our results. Another direction would be to examine more general situations in which  $\tilde{G} = G_X$ . It might also be of interest to extend results here to liftings to nongraded pointed Hopf algebras.

**2. Path coalgebras and pointed coalgebras**

We refer to [4,5] for basics of pointed coalgebras and path coalgebras. The **path coalgebra**  $\mathbb{k}Q$  of a quiver  $Q$  is defined to be the span of all paths in  $Q$  with coalgebra structure

$$\Delta(p) = \sum_{p=p_2p_1} p_2 \otimes p_1 + t(p) \otimes p + p \otimes s(p)$$

$$\varepsilon(p) = \delta_{|p|,0}$$

where  $p_2p_1$  is the concatenation  $a_t a_{t-1} \dots a_{s+1} a_s \dots a_1$  of the subpaths  $p_2 = a_t a_{t-1} \dots a_{s+1}$  and  $p_1 = a_s \dots a_1$  ( $a_i \in Q_0$ ). Here  $|p| = t$  denotes the length of  $p$  and the starting vertex of  $a_{i+1}$  is required to be the end of  $a_i$ . Thus the vertices  $Q_0$  are group-like elements, and if  $a$  is an arrow  $g \leftarrow h$ , with  $g, h \in Q_0$ , then  $a$  is a  $(g, h)$ -skew primitive, i.e.,  $\Delta a = g \otimes a + a \otimes h$ . It follows that  $\mathbb{k}Q$  is pointed with coradical  $(\mathbb{k}Q)_0 = \mathbb{k}Q_0$  and the degree one term of the coradical filtration is  $(\mathbb{k}Q)_1 = \mathbb{k}Q_0 \oplus \mathbb{k}Q_1$ . Moreover the coradical grading  $\mathbb{k}Q = \bigoplus_{n \geq 0} \mathbb{k}Q_n$  is given by path length. The path coalgebra may be identified with the cotensor coalgebra  $C(\mathbb{k}Q_1) = \bigoplus_{n \geq 0} (\mathbb{k}Q_1)^{\square n}$  of the  $\mathbb{k}Q_0$ -bicomodule  $\mathbb{k}Q_1$ , cf. [18].

Define the (*Gabriel-* or *Ext-*) *quiver* of a pointed coalgebra  $B$  to be the directed graph  $Q = Q_B$  with vertices  $Q_0$  corresponding to group-likes and  $\dim_{\mathbb{k}} P_{h,g}$  arrows from  $h$  to  $g$ , for all  $h, g \in Q_0$ . We will view  $B$  as a subcoalgebra of the path coalgebra of its Gabriel quiver, with the same degree one coradical term. If  $B$  has a unique group-like element with a space of primitives of dimension  $n$ , then the quiver is the  $n$ -loop quiver. The path coalgebra is the cofree pointed irreducible coalgebra (and the path algebra is the free algebra on  $n$  generators).

The indecomposable components (“blocks”) of  $B$  are coalgebras that are the direct sums of injective indecomposables having socles in a given graph component. Therefore

$B$  is indecomposable as a coalgebra if and only if it is “link-indecomposable”, if and only if its quiver is connected as a graph. In [17] it is shown that a pointed coalgebra is a crossed product over the principal block subcoalgebra, i.e. the one containing  $1_{G(B)}$ .

### 3. Coverings

We first summarize results from [5] concerning coverings of pointed coalgebras. An analogous version for bound quivers finite-dimensional algebras can be found in [12] and [15].

Let  $B \subseteq \mathbb{k}Q$  be a subspace. Let  $b = \sum_{i \in I} \lambda_i p_i \in B(x, y)$  with  $x, y \in Q_0$  and distinct paths  $p_i$ . We say that  $b$  is a *minimal element* of  $B$  if  $\sum_{i \in I'} \lambda_i p_i \notin B(x, y)$  for every nonempty proper subset  $I' \subset I$  and  $|I| \geq 2$ . Clearly every element of  $B$  is a linear combination of paths and minimal elements. Let  $\min(B)$  denote the set of minimal elements of  $B$ .

For an admissible ideal  $I$  of a path algebra, let  $A = \mathbb{k}Q/I$  denote an admissible quotient of the path algebra  $\mathbb{k}Q$  with ideal of relations  $I$ . Let  $B \subseteq \mathbb{k}Q$  be an admissible subcoalgebra of the path coalgebra  $\mathbb{k}Q$ .

Fix a base vertex  $x_0 \in Q_0$ . We define a symmetric relation  $\sim$  on paths by declaring  $p \sim q$  if there is a minimal element  $b = \sum_{i \in I} \lambda_i p_i \in B(x, y)$  where the  $p_i$  are distinct paths,  $\lambda_i \in \mathbb{k}$ ,  $x, y \in Q_0$  and  $p = p_1, q = p_2$ . We define  $N(B, x_0)$  to be the subgroup of  $\pi_1(B, x_0)$  generated by equivalence (homotopy) classes of walks  $w^{-1}p^{-1}qw$  where  $p, q$  are paths in  $Q(x, y)$  with  $p \sim q$  and  $w$  is a walk from  $x_0$  to  $x$ .

It is easy to see that  $N(B, x_0)$  is a normal subgroup of  $\pi_1(B, x_0)$ . Explicitly, if  $w^{-1}p^{-1}qw$  is closed walk as above and  $[v] \in \pi_1(B, x_0)$  where  $v$  is a closed walk at  $x_0$ , then  $wv$  is a path from  $x_0$  to  $x$  and  $[v^{-1}w^{-1}p^{-1}qwv] = [(wv)^{-1}p^{-1}q(wv)] \in N(B, x_0)$ .

Consider a Galois covering  $F : \tilde{Q} \rightarrow Q$  of quivers with automorphism group  $G$  and lifting  $L$ . Let  $\tilde{B}$  denote the  $\mathbb{k}$ -span of  $\{L(b)|L \text{ a lifting, } b \in B \text{ a minimal element or a path}\}$ . We say that the restriction  $F : \tilde{B} \rightarrow B$  is a *Galois coalgebra covering* if every minimal element of  $B$  can be lifted to  $\tilde{B}$  in the following sense: for every minimal element  $b \in B(x, y)$  with  $x, y \in Q_0$  and  $\tilde{x} \in F^{-1}(x)$ , there exists  $\tilde{y} \in \tilde{Q}_0$  and a minimal element  $\tilde{b} \in \tilde{B}(\tilde{x}, \tilde{y})$  such that  $F(\tilde{b}) = b$ . All coverings in this paper are assumed to be Galois.

Let  $B \subseteq \mathbb{k}Q$  be a pointed coalgebra. Then there exists a Galois coalgebra covering  $F : \tilde{B} \rightarrow B$ , the *universal coalgebra covering* of  $B \subseteq \mathbb{k}Q$ , such that for every Galois coalgebra covering  $F' : B' \rightarrow B$ , there exists a Galois coalgebra covering  $E : \tilde{B} \rightarrow B'$  such that the following diagram commutes.

$$\begin{array}{ccc}
 \tilde{B} & \xrightarrow{E} & B' \\
 & \searrow F & \swarrow F' \\
 & & B
 \end{array}$$

The fundamental example of a covering is given as follows. Let  $B \subseteq \mathbb{k}Q$  be a homogeneous admissible subcoalgebra with respect to the grading given by an arrow weighting

$\delta : Q_1 \rightarrow G$ . If  $b = \sum_{i \in I} \lambda_i p_i \in B(x, y)$  is a minimal element, then it is necessarily homogeneous in the  $G$ -grading. Consider the canonical map  $F : \mathbb{k}Q \rtimes \mathbb{k}G \rightarrow \mathbb{k}Q$  defined by  $F(p \rtimes g) = p$  and consider the restriction to  $B \rtimes \mathbb{k}G \rightarrow B$ . Then under the identification of  $\mathbb{k}Q \rtimes \mathbb{k}G$  with  $\mathbb{k}\tilde{Q}$  we easily see that  $\tilde{B} = B \rtimes \mathbb{k}G$ . The liftings of minimal element  $b \in B$  are given by  $b \rtimes g$  with  $g \in G$ .

**Theorem 3.1.** (See [5].) *The following are equivalent for a subcoalgebra  $B \subseteq \mathbb{k}Q$  and Galois quiver covering  $F : \tilde{Q} \rightarrow Q$ .*

- (a)  *$B$  is a homogeneous subcoalgebra of  $\mathbb{k}Q$ .*
- (b)  *$N(B, x_0) \subseteq F_*(\pi_1(\tilde{Q}, \tilde{x}_0))$  for all  $x_0 \in Q, \tilde{x}_0 \in \tilde{Q}$  with  $F(\tilde{x}_0) = x_0$ .*
- (c)  *$F : \tilde{B} \rightarrow B$  is a Galois coalgebra covering.*
- (d)  *$B$  is a homogeneous subcoalgebra of  $\mathbb{k}Q$  and the grading is connected.*

**Theorem 3.2.** (See [5].) *The universal covering of the coalgebra  $B \subseteq \mathbb{k}Q$  is isomorphic to  $B \rtimes \mathbb{k}\tilde{G} \rightarrow B$  where  $\tilde{G} = \frac{\pi_1(\tilde{Q}, \tilde{x}_0)}{N(B, x_0)}$ ,  $(\tilde{Q}, \tilde{x}_0)$  is the universal covering quiver of  $(Q, x_0)$  (with base vertices  $\tilde{x}_0$  and  $x_0$ ), and  $B \rtimes \mathbb{k}\tilde{G}$  is indecomposable as a coalgebra if  $B$  is.*

### 3.1. Hopf coverings

The idea of Hopf covering comes from the following simple observation.

**Lemma 3.3.** *Let  $B(V)$  be a Nichols algebra with a link-indecomposable Yetter–Drinfeld module  $V \in {}_{\mathbb{k}G}^{\mathbb{k}G}\mathcal{YD}$ . Let  $Z$  be a normal subgroup of  $G$  acting trivially on  $V$ . Then  $Z$  is central in  $G$ . Let  $\bar{V}$  be a copy of  $V$  graded by  $\bar{G} = G/Z$  via  $\bar{V}_{gZ} = \bigoplus_{t \in gZ} V_t$  for  $gZ \in \bar{G}$ , and make  $\bar{V}$  a  $\bar{G}$ -module by factoring by  $Z$ . Then  $\bar{V} \in {}_{\bar{G}}^{\bar{G}}\mathcal{YD}$  and  $V$  is isomorphic to  $\bar{V}$  as a braided vector space. Therefore  $B(V)$  and  $B(\bar{V})$  are isomorphic as braided Hopf algebras and there is a surjection  $B(V) \# \mathbb{k}G \rightarrow B(\bar{V}) \# \mathbb{k}\bar{G}$  on bosonized Nichols algebras given by  $b \# g \mapsto b \# gZ$ , for all  $b \in B(V), g \in G$ .*

**Proof.** We just point out that  $Z$  being central is equivalent to the condition  $z.V_g = V_{zgz^{-1}} = V_g$  for all  $g \in G, z \in Z$ .

The conclusions of the lemma hold even in the case that  $V$  is not finite-dimensional. Also if  $V$  is not assumed to be link-indecomposable in the lemma above, then we may replace “ $Z$  is central in  $G$ ” by “ $Z$  is in the centralizer of  $\{g \in G | V_g \neq 0\}$ ”.  $\square$

**Definition 3.4.** When  $\tilde{A} = \bigoplus_{n \geq 0} \tilde{A}_n = R \# \mathbb{k}\tilde{G}$  and  $A = \bigoplus_{n \geq 0} A_n = R \# \mathbb{k}G$  are coradically graded pointed Hopf algebras, which are bosonizations of the braided graded Hopf algebra  $R$  as in [3], and  $f : \tilde{G} \rightarrow G$  is a group surjection, we say that the Hopf algebra map  $F : \tilde{A} \rightarrow A$  is a *Hopf covering* if  $F(r \# \tilde{g}) = r \# f(g)$  for all  $r \in R$  and  $\tilde{g}$ . We say that  $\tilde{A}$  is a *covering Hopf algebra* of  $A$ , with covering group  $\tilde{G}$ . A universal Hopf covering is one that is universal among Hopf coverings of  $A$ .

The following result specifies the Hopf coverings of a bosonization of a Nichols algebra.

**Theorem 3.5.** *Let  $B(V)\#\mathbb{k}G$  be the bosonization for a link-indecomposable  $V \in {}_{\mathbb{k}G}^{\mathbb{k}G}\mathcal{YD}$ . Let  $\tilde{G}$  be the universal coalgebra covering group of  $B(V)$  and write  $G = \tilde{G}/N$  where  $N \triangleleft \tilde{G}$ . The covering Hopf algebras of  $B(V)\#\mathbb{k}G$  are of the form  $B(V)\#\mathbb{k}\tilde{G}/M$  where  $M \triangleleft \tilde{G}$  and  $[N, \tilde{G}] \subseteq M \subseteq N$ .*

**Proof.** We have the universal coalgebra covering

$$B(V) \rtimes \mathbb{k}\tilde{G} \rightarrow B(V) \rtimes \mathbb{k}G \rightarrow B(V)$$

by [5], see Theorem 3.1 and [5]. So we may write  $G = \tilde{G}/N$ ,  $N \triangleleft \tilde{G}$ . Now consider the set of homomorphic images  $H = \tilde{G}/M$  of  $\tilde{G}$  where  $M \in \mathcal{C}$ , and  $\mathcal{C}$  is defined as

$$\begin{aligned} \mathcal{C} &= \{M \triangleleft \tilde{G} \mid N \supseteq M \text{ and } N/M \text{ central in } \tilde{G}/M\} \\ &= \{M \triangleleft \tilde{G} \mid [N, \tilde{G}] \subseteq M\} \end{aligned}$$

using the group theoretic commutator. It is evident that the unique minimal element of  $\mathcal{C}$  is just  $[N, \tilde{G}]$ . Now for such a group  $H$ , it follows from the lemma that  $B(V)\#\mathbb{k}H$  is a bosonization where the action is lifted from the action of  $G$ , and the grading is inherited from the  $\tilde{G}$  grading. Thus we obtain the Hopf coverings  $B(V)\#\mathbb{k}\tilde{G}/M$  of  $B(V)\#\mathbb{k}G$ .  $\square$

**Corollary 3.6.** *Let  $B(V)\#\mathbb{k}G$  be the bosonization for link-indecomposable  $V \in {}_{\mathbb{k}G}^{\mathbb{k}G}\mathcal{YD}$ . Let  $\tilde{G}$  be the universal coalgebra covering group of  $B(V)$  and write  $G = \tilde{G}/N$  where  $N \triangleleft \tilde{G}$ . The universal covering Hopf algebra of  $B(V)\#\mathbb{k}G$  is  $B(V)\#\mathbb{k}\tilde{G}/[N, \tilde{G}]$ .*

#### 4. Racks and Nichols algebras

Let  $B(V)$  be a Nichols algebra generated by the Yetter–Drinfeld module  $V \in {}_{\mathbb{k}G}^{\mathbb{k}G}\mathcal{YD}$ . We utilize the description of  $B(V)$  as a subcoalgebra of the cotensor coalgebra  $C(V)$  of the bicomodule  $V$ . Namely the graded component  $B(V)(n)$  is the image of the quantum symmetrizer  $\mathfrak{S}_n = \sum_{\sigma \in \mathbb{S}_n} \hat{\sigma}$  where  $\hat{\cdot}$  denotes the Matsumoto section  $\mathbb{S}_n \rightarrow \mathbb{B}_n$ . In particular, the quadratic summand  $B(V)(2) \subset V \otimes V$  is the image of  $1 + c$ . For a fixed basis of  $V$ , we obtain an embedding  $B \hookrightarrow \mathbb{k}Q$  where  $Q = Q_V$  is the  $\dim V$ -loop quiver with arrows labeled by basis elements of  $V$  and single vertex  $1_{B(V)}$ . In fact, we choose a basis of  $G$ -homogeneous elements. Fixing this basis of homogeneous elements of  $V$ , we thus identify  $C(V)$  with the path coalgebra  $\mathbb{k}Q$ . When  $\dim \rho_i = 1$  for all  $i$ , the basis can be chosen to correspond to a union of conjugacy classes of  $G$ .

As in [19,20]; cf. [3]  $B(V)$  can be constructed as both the subcoalgebra  $\bigoplus_{n \geq 0} \text{Im } \mathfrak{S}_n$  of the cotensor coalgebra  $\mathbb{k}Q = C(V) = k1 \oplus V \oplus V \otimes V \oplus \dots$ , and as a homomorphic image of the tensor coalgebra  $T(V) = \mathbb{k}^Q$  modulo the ideal  $\bigoplus_{n \geq 0} \ker \mathfrak{S}_n$ .

4.1. Coverings of Nichols algebras

We assume that  $V = (V, c = c^q) = \mathbb{k}X$  is a Yetter–Drinfeld module with finite rack  $X = (X, \triangleright)$ , with rack structure map  $\triangleright: X \times X \rightarrow X$  and 2-cocycle  $q: X \otimes X \rightarrow \mathbb{k}^\times$  is as in [2], where we insist on a one-dimensional image for  $q$ . By [2, Theorem 4.14] such a braided vector space arises as a Yetter–Drinfeld module a one-dimensional module  $\rho$  over the centralizer  $G_g$  of a fixed chosen element  $g \in G$ . Fix a basis  $\{v_x | x \in X\}$  for  $V$  where  $v_x \in V_x$  for all  $x$ . Note that the assumption entails that the subspaces  $V_x = \mathbb{k}v_x, x \in X$  are one-dimensional. The braiding  $c = c^q: V \otimes V \rightarrow V \otimes V$  is defined by  $c(v_x \otimes v_y) = q(x, y)v_y \otimes v_x$ . We shall use the same symbol for the map  $c: X \times X \rightarrow X \times X, c(x, y) = (x \triangleright y, x)$ , as in [10]. The Yetter–Drinfeld module is thus  $\bigoplus_i M(g_i, \rho_i) = \bigoplus_i \mathbb{k}G \otimes_{\mathbb{k}G_{g_i}} \rho_i$ . The group  $G$  can be chosen to be finite if the subgroup of  $\mathbb{k}^\times$  generated by the  $q(x, y)$  is finite and  $X$  is finite, cf. [2, Theorem 4.14]. We shall investigate groups that give rise to braided vector spaces  $(V, c^q)$  and the Nichols algebra  $B(V)$ .

We follow the set-up as in [10]. We have the braided vector space  $V = (\mathbb{k}X, c)$ , and we let  $\mathcal{O} (= \mathcal{O}_{x,y})$  denote  $c$ -orbit (of  $(x, y)$ ) in  $X \times X$ . Set  $V_{\mathcal{O}} = \sum_{(s,t) \in \mathcal{O}} V_s \otimes V_t$  and note that  $\theta_i := c^i(v_x \otimes v_y), i = 0, 1, \dots, m - 1$  is a basis for  $V_{\mathcal{O}}$ .

The enveloping group  $G_X$  of a rack  $X$  is defined to be the quotient of the free group on generators  $\{g_x | x \in X\}$  by the relations

$$g_x g_y = g_{x \triangleright y} g_x$$

$x, y \in X$ .

Let  $G$  be a group with link-indecomposable  $V \in \mathbb{k}_G^G \mathcal{YD}$ . Then  $V = (\mathbb{k}X, c)$  for a rack  $X \subset G$  where  $X$  is a union of conjugacy classes of  $G$  and  $\text{Supp}_G V = X$  generates  $G$ . Since the defining relations of  $G_X$  hold in  $G$ , there is a surjective group homomorphism  $G_X \rightarrow G$ .

**Theorem 4.1.** *Let  $B(V)$  be a Nichols algebra with a full set of quadratic relations. Then*

- (a)  $G_X$  is the coalgebra covering group of  $B(V)$ ,
- (b)  $B(V) \# \mathbb{k}G_X$  is the universal covering coalgebra of  $B(V) \subset \mathbb{k}Q_V$ , and
- (c)  $B(V) \# \mathbb{k}G_X \rightarrow B(V) \# \mathbb{k}G$  is the universal Hopf covering of  $B(V) \subset \mathbb{k}Q_V$ .

**Proof.** As mentioned above,  $B(V)$  has quadratic component  $B(V)(2) = \text{Im}(1 + c) \subseteq V \otimes V$ . For  $v_x \in V_x, v_y \in V_y$  with  $x \neq y \in X$  we have  $c(x \otimes y) = q(x, y)(x \triangleright y) \otimes x$ . We need to see that

$$(1 + c)(v_x \otimes v_y) = v_x \otimes v_y + q(x, y)v_{x \triangleright y} \otimes v_x$$

is a minimal element of  $\text{Im}(1 + c)$ . For then we obtain the relation  $[v_x][v_y] = [v_{x \triangleright y}][v_x]$  in  $\pi_1(Q, 1_{B(V)})$  per the definition of  $\tilde{Q}_{B(V)} = \frac{\pi_1(B(V), 1)}{N(B(V), 1)}$ . By [1, Lemma 3.4]  $V \in \mathbb{k}_{G_X}^{G_X} \mathcal{YD}$  so there is a group surjection  $\tilde{G} \rightarrow G_X$  with  $[v_x] \mapsto g_x$ . It follows that  $\tilde{G} \simeq G_X$ .

We claim that  $1 + c : V_{\mathcal{O}} \rightarrow V_{\mathcal{O}}$  is onto (and thus bijective) if and only if  $c^m(v_i \otimes v_j) \neq (-1)^m(v_i \otimes v_j)$ . Let  $m$  be the order of  $c$  in  $\mathcal{O} \subset X \times X$ . The restriction of  $1 + c$  to  $V_{\mathcal{O}}$  is given by the  $m \times m$  matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \lambda \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 \cdots & 0 \\ 0 & \ddots & \ddots & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

with respect to the basis  $\{\theta_i\}$ , where  $\lambda \in \mathbb{k}^\times$  is such that  $c^m(v_i \otimes v_j) = \lambda v_i \otimes v_j$ . One can see that the determinant is  $1 + (-1)^{m-1}\lambda$ , so  $1 + c$  is bijective if and only if  $\lambda \neq (-1)^{m-1}$ . When  $1 + c$  is not a bijection, then it is easy to see that the  $m - 1$  elements

$$\theta_0 + \theta_1, \quad \theta_1 + \theta_2, \quad \cdots, \quad \theta_{m-2} + \theta_{m-1}$$

form a basis for the image of  $1 + c$  restricted to  $V_{\mathcal{O}}$ . It follows that the  $\theta_i$  are minimal elements.  $\square$

**Remark 4.2.** When  $\lambda = (-1)^{m-1}$ ,  $\ker(1 + c) = \mathbb{k} \sum_{n=0}^{m-1} (-1)^n \theta_n$  as noted in [10]. When  $\lambda \neq (-1)^{m-1}$  then  $\ker(1 + c)|_{\mathcal{O}} = 0$ .

**Remark 4.3.** The universal covering Hopf algebra was seen to be  $B(V) \# \tilde{G} / [\tilde{G}, N]$  in Theorem 3.6. Therefore  $\tilde{G} / [\tilde{G}, N] \simeq G_X$  for all choices of  $N$  and it follows that  $[\tilde{G}, N] = \ker(\tilde{G} \rightarrow G_X)$ . In case  $N$  is central, we get the result  $G_X = \tilde{G}$  as in the conclusion of the theorem. But if there is not a full set of quadratic relations, we may still get  $G_X = \tilde{G}$ , as can be shown for Nichols algebras of finite Cartan type, e.g. last example in 5.3.

### 5. Examples

#### 5.1. Rank 1

Let  $G = C_m = \langle K \rangle$  be the cyclic group of order  $m$  and let  $q$  be an  $m$ th root of 1. Let  $H$  be the Hopf algebra with generators  $E, K$ , where  $K$  is group-like and  $E$  is  $(K, 1)$  skew-primitive, and with relations  $KE = qEK, E^m = 0$ . Then  $H$  is the bosonization  $B(\mathbb{k}E) \# \mathbb{k}C_m$  where  $B(\mathbb{k}E) = \mathbb{k}[E]/(E^m)$  with  $G$ -comodule structure given by  $\delta(E) = K \otimes E$  and  $G$ -module structure given by  $KE = qE$ . Then  $\mathbb{k}E \in {}^{\mathbb{k}G}_{\mathbb{k}G} \mathcal{YD}$  and  $B(\mathbb{k}E) \# C_m$  is a Hopf algebra. Its quiver is a directed cycle  $Q_m$  of length  $m$ . The finite Hopf coverings of  $B(\mathbb{k}E) \# \mathbb{k}C_m$  have underlying coalgebra coverings  $B(\mathbb{k}E) \# C_n \rightarrow B(\mathbb{k}E) \# C_m$  with  $m|n$ . Setting  $n = \infty$  (so  $G = \langle g \rangle$  is infinite cyclic) results in the universal covering Hopf algebras whose quiver is of type  $\mathbb{A}_\infty^\infty$  with unidirectional arrows. These Hopf algebras appear in [7].

One may also consider the path coalgebras of the cyclic quivers  $Q_n$ , and quivers of type  $A_\infty$  with unidirectional arrows for  $m \in \mathbb{Z}^+ \cup \{\infty\}$ . As in [6] the path coalgebras can be furnished with a Hopf algebra structure depending on an  $m$ th root of unity  $q$  where  $m|n$  or  $n = \infty$ . The Hopf coverings for fixed  $m$  (and an  $m$ th root of 1) are  $\mathbb{k}Q_n \rightarrow \mathbb{k}Q_m$  where  $m|n$  or  $n = \infty$ . The Hopf algebras in the previous paragraph are the bosonizations of Nichols algebras for these infinite dimensional Hopf algebras.

5.2. *Fomin–Kirillov algebras*

Let  $X$  be the rack of transpositions of  $S_n$ ,  $n > 2$  and consider the versions of  $V = (\mathbb{k}X, c^q)$  for the cocycles as in [10]. Then the quadratic version  $\hat{B}_2(V)$  is the algebra from [8,16]; cf. [3]. It is known that  $B(V) = \hat{B}_2(V)$  is finite dimensional for  $n \leq 5$ , but this problem has been open for  $n > 5$  for more than a decade.

One can enumerate the orbits (including orbits of size 1) of  $c$  on  $X \times X$  as follows:

Orbits of size	#orbits
1	$\binom{n}{2} = d$
2	$\frac{1}{2} \binom{n}{2} \binom{n-2}{2}$
3	$2 \binom{n}{3}$

The total number of  $c$ -orbits is

$$f(n) = \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)(n-3)}{8} + \frac{n(n-1)(n-2)}{3}$$

$$= \frac{1}{24}n(3n^3 - 10n^2 + 21n - 14).$$

The number of orbits in excess of  $\binom{d}{2}$  is

$$f(n) - \binom{d}{2} = \frac{1}{24}n(3n^3 - 10n^2 + 21n - 14)$$

$$- \frac{1}{8}n(n-1)(n-2)(n+1)$$

$$= -\frac{1}{6}n(n-1)(n-5)$$

There are fewer quadratic relations ( $=$  #orbits) than  $\binom{d}{2}$  (recall  $d = \binom{n}{2}$ ) for  $n > 5$ . There is a full set of quadratic relations, but not “many” in the sense of [10].

$n$	#orbits in $X \times X = f(n)$	#orbits $- \binom{d}{2}$
3	5	2
4	17	2
5	45	0
6	100	-5

### 5.3. Nonabelian group type

We adopt the list of some racks from e.g. [10,11,16,13]. We will consider the racks  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$ ,  $\mathcal{T}$  and the affine racks  $\text{Aff}(5, 2)$ ,  $\text{Aff}(5, 3)$ ,  $\text{Aff}(7, 5)$ ,  $\text{Aff}(7, 3)$  with 2-cocyles as in the references which result in finite-dimensional Nichols algebras.

Let  $\mathcal{S}_n$  denote the rack of transpositions in  $\mathbb{S}_n$ ,  $n \geq 3$  (as above). Let  $\mathcal{B}$  be the rack of 4-cycles in  $\mathbb{S}_4$ . Let  $\mathcal{D}_4$  denote the rack of 4 reflections (transpositions) in the dihedral group  $\mathbb{D}_4$  (order 8). Hence  $\mathcal{D}_3 = \mathcal{S}_3$ ,  $\mathcal{A} = \mathcal{S}_4$  and  $\mathcal{C} = \mathcal{S}_5$ . Let  $d = \dim V$ . All but the last two rows have indecomposable racks corresponding to irreducible Yetter–Drinfeld modules. All but the last row have Nichols algebras with a full set of quadratic relations. The last two rows have decomposable racks. The example over  $\mathbb{D}_4$  is from [16, Example 6.5]; since the center of  $D_4$  acts trivially on the Yetter–Drinfeld module  $V$ , the braiding reduces to the Klein 4-group  $\mathbb{V}$  and the  $\mathbb{D}_4$  bosonization is a double cover of the smaller Hopf algebra over  $\mathbb{V}$ .

The two newer examples in [13, Prop. 32, 36] (over  $\mathcal{D}_3$  and  $\mathcal{T}$  in rows 2 and 5 in table below, respectively) do have a full set of quadratic relations, but do not have relations of the form  $x^2$ .

In the last row, the Nichols algebra of finite Cartan type of rank 2 is seen to have no quadratic relations (where the root of unity has order  $> 2$ ) because the Serre relations are cubic. It can be shown that  $G_X = \tilde{G}$  and is free abelian for Nichols algebras of finite Cartan type.

The computation of the enveloping groups and their centers was done with the aid of GAP, or done by hand. The fact concerning  $G_{\mathcal{S}_n}$  and its center is from [1, Prop. 3.2].

Rack $X$	rank $d$	$Z(G_X)$	$G_X/Z(G_X)$	#orbits	#QR
$\mathcal{S}_n$	$\binom{n}{2}$	$C_\infty$	$\mathbb{S}_n$	$f(n)$	$f(n)$
$\mathcal{D}_3$	3	$C_\infty$	$\mathbb{S}_n$	5	2
$\mathcal{B}$	6	$C_\infty$	$\mathbb{S}_4$	17	17
$\mathcal{T}$	4	$C_\infty \times C_2$	$\mathbb{A}_4$	8	8
$\mathcal{T}$	4	$C_\infty \times C_2$	$\mathbb{A}_4$	8	4
$\text{Aff}(5, 2)$	5	$C_\infty$	$C_5 \rtimes C_4$	10	10
$\text{Aff}(5, 3)$	5	$C_\infty$	$C_5 \rtimes C_4$	10	10
$\text{Aff}(7, 3)$	7	$C_\infty$	$C_7 \rtimes C_6$	21	21
$\text{Aff}(7, 5)$	7	$C_\infty$	$C_7 \rtimes C_6$	21	21
$\mathcal{D}_4$	4	$C_\infty \times C_\infty \times C_2$	$C_2 \times C_2$	4	4
rank 2	2	$C_\infty \times C_\infty$		2	0

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