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Characters, bilinear forms and solvable groups

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ABSTRACT

We prove a number of results about the ordinary and Brauer characters of finite solvable groups in characteristic 2, by defining and using the concept of the extended nucleus of a real irreducible character. In particular we show that the Isaacs canonical lift of a real irreducible Brauer character has Frobenius–Schur indicator +1. We also show that the principal indecomposable module corresponding to a real irreducible Brauer character affords a quadratic geometry if and only if each extended nucleus is a split extension of a nucleus.

1. Introduction

Let G be a finite group with irreducible 2-Brauer characters $\mathrm{IBr}(G)$. The theory of real valued characters and self-dual G-modules over a field of characteristic 2 admits some remarkable improvements if G is restricted to being solvable.

Theorem 1. *Suppose that G is solvable and $\varphi \in \mathrm{IBr}(G)$ is real valued and non-trivial. Then there exists (U, δ) such that $U \subseteq G$, $\delta \in \mathrm{IBr}(U)$, $\delta^G = \varphi$, δ is real valued*

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and $\delta(1)_2 = 2$. Moreover, the Sylow 2-subgroups of U are determined by φ up to G-conjugacy.

We use the Isaacs nucleus of a lift of φ to prove the existence of our 'extended nucleus'. The uniqueness part on the Sylow 2-subgroups of U lies much deeper and its proof relies on the new theory of **symmetric vertices** developed by the first author.

Now we turn our attention to Frobenius–Schur indicators. For a character χ of G the indicator $\nu(\chi)$ is the average value of $\chi(g^2)$ for $g \in G$. If χ is irreducible then $\nu(\chi)$ takes one of the values $+1, -1, 0$, as χ is afforded by a real representation or is real-valued but not afforded by a real representation or is not real-valued, respectively. We use the extended nucleus to answer an old question of W. Willems [12, p. 518].

Theorem 2. *Suppose that G is solvable and $\varphi \in \mathrm{IBr}(G)$ is real valued. Then G has a real representation whose character lifts φ.*

Next recall that the decomposition numbers $d_{\chi\varphi}$ are given by

$$\chi(g) = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi(g), \quad \text{for all odd order } g \in G.$$

Then $\Phi_\varphi := \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi}\chi$ is called the principal indecomposable character of φ. It is known that Φ_φ vanishes on all elements of even order. In [11] G.R. Robinson used this to show that $\nu(\Phi_\varphi) \geq 0$. This result is peculiar to $p = 2$.

Theorem 3. *Suppose that G is solvable and $\varphi \in \mathrm{IBr}(G)$ is real valued and non-trivial. Let (U, δ) be an extended nucleus and let (W, γ) be a nucleus of φ. Suppose that $U \backslash W$ contains an involution t. Then $\langle \Phi_\varphi, 1_{C(t)}^G \rangle > 0$ and thus $\nu(\Phi_\varphi) > 0$.*

2. Extended nucleus

As usual $\mathrm{Irr}(G)$ denotes the ordinary irreducible characters of G. Also χ^* denotes the restriction of a character χ to the odd order elements of G.

Let k be a field of characteristic 2 and suppose that S is a non-trivial simple self-dual kG-module. Fong's Lemma asserts that S affords a G-invariant non-degenerate symplectic bilinear form which is unique up to a non-zero scalar. As a consequence, every non-trivial real valued irreducible Brauer character of G has even degree.

Suppose that $\varphi \in \mathrm{IBr}(G)$ and G is solvable. The Fong–Swan theorem asserts that there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi^* = \varphi$. Let $H \subseteq G$ be a Hall $2'$-subgroup of G. So $|H| = |G|_{2'}$ and every odd order subgroup of G is contained in a conjugate of H. For the given φ and χ, we say that $\psi \in \mathrm{Irr}(H)$ is a Fong character of χ if $\psi(1)$ is minimal such that $\langle \chi_H, \psi \rangle \neq 0$. In that case it is known that $\langle \chi_H, \psi \rangle = 1$, $\psi(1) = \chi(1)_{2'}$ and ψ^G is the principal indecomposable character of G corresponding to φ.

Lemma 4. *Suppose that G is solvable and $\varphi \in \mathrm{IBr}(G)$ is non-trivial and real valued. Then there is $U \subseteq G$ and a real valued $\delta \in \mathrm{IBr}(U)$ such that $\delta^G = \varphi$ and $\delta(1)_2 = 2$.*

Proof. In [6] I.M. Isaacs constructed for each $\chi \in \mathrm{Irr}(G)$ a nucleus (W, γ); here $W \subseteq G$ and $\gamma \in \mathrm{Irr}(W)$ is the product of a 2-special character and a $2'$-special character and satisfies $\gamma^G = \chi$. His construction uniquely determines (W, γ) up to G-conjugacy. By definition $\mathrm{B}_{2'}(G)$ is the set of all χ for which γ is $2'$-special. Isaacs showed that $\mathrm{B}_{2'}(G)$ gives a canonical set of lifts for the irreducible Brauer characters of G.

Let $\chi \in \mathrm{B}_{2'}(G)$ with $\chi^* = \varphi$ and let (W, γ) be a nucleus of χ. Then $\bar{\chi}$ belongs to $\mathrm{B}_{2'}(G)$ as $\bar{\chi}$ has nucleus $(W, \bar{\gamma})$ and $\bar{\gamma}$ is $2'$-special. Moreover $\bar{\chi}^* = \bar{\varphi} = \varphi = \chi^*$. So $\bar{\chi} = \chi$. On the other hand, γ is non-trivial as φ is non-trivial, and $\gamma(1)$ is odd as γ is $2'$-special. So $\bar{\gamma} \neq \gamma$, using Fong's Lemma.

Now $(W, \bar{\gamma})$ is G-conjugate to (W, γ) as both are nuclei of χ, and $\mathrm{N}_G(W, \gamma) = W$ as $\gamma^{\mathrm{N}_G(W)}$ is irreducible. So the set stabilizer U of $\{\gamma, \bar{\gamma}\}$ in $\mathrm{N}_G(W)$ satisfies $|U : W| = 2$. It is clear that $\eta = \gamma^U$ is a real valued irreducible character of U with $\eta(1)_2 = 2$. Now set $\delta = \eta^*$, and notice that (U, δ) satisfies what is required. $\square$

Our next result proves a precise form of Theorem 2, thus answering Willems question:

Theorem 5. *Suppose that G is solvable and $\varphi \in \mathrm{IBr}(G)$ is real valued. Let $\chi \in \mathrm{B}_{2'}(G)$ be the Isaacs canonical lift of φ. Then $\nu(\chi) = +1$.*

Proof. We may assume that φ is non-trivial. Let (W, γ) and (U, η) be as in the previous lemma. So $|U : W| = 2$, γ is $2'$-special, $\gamma^U = \eta$ and $\eta_W = \gamma + \bar{\gamma}$. Now $U = W\langle u \rangle$ where $u \in U \backslash W$ and $u^2 \in W$. We can and do assume that u is a 2-element. Set $C = \langle u \rangle$ and $D = \langle u^2 \rangle$, so that $U = WC$ and $C \cap W = D$.

We claim that $\langle \gamma_D, 1_D \rangle$ is odd. To show this, we may assume that $D \neq 1$. Let ζ generate the cyclic group $\mathrm{Irr}(D)$ and set $q = |D|$. Then the rational characters in $\mathrm{Irr}(D)$ are $\zeta^0 = 1_D$ and $\zeta^{q/2}$. Now γ is 2-rational as it is $2'$-special. So γ_D is rational and hence

$$\gamma_D = m_0 \zeta^0 + m_{q/2} \zeta^{q/2} + \sum_{i=1}^{q/2-1} m_i (\zeta^i + \bar{\zeta}^i), \quad \text{for non-negative integers } m_i.$$

Then clearly $\det \gamma(u^2) = (-1)^{m_{q/2}}$. But $o(\gamma)$ is odd, as γ is $2'$-special. So $m_{q/2}$ is even. Then $\langle \gamma_D, 1_D \rangle = m_0 \equiv \gamma(1) \pmod 2$. The claim follows, as $\gamma(1)$ is odd.

The previous paragraph implies that $\langle \eta, 1_C^U \rangle$ is odd, as

$$\langle \eta_C, 1_C \rangle = \langle (\gamma^U)_C, 1_C \rangle = \langle \gamma_D, 1_D \rangle.$$

As 1_C^U is afforded by an $\mathbb{R}$-representation of U, this implies that η is afforded by an $\mathbb{R}$-representation of U. So finally $\chi = \eta^G$ is afforded by an $\mathbb{R}$-representation of G. $\square$

Note that it can easily happen that a real irreducible Brauer character of a solvable group has a lift to an ordinary character with Frobenius–Schur indicator -1. For example, let G be the non-abelian group $C_3 \rtimes C_4$ and let $\varphi \in \mathrm{IBr}(G)$ with $\varphi(1) = 2$. Then $\Phi_\varphi = \chi_1 + \chi_2$, where $\chi_1, \chi_2 \in \mathrm{Irr}(G)$ are real valued and $\chi_1^* = \chi_2^* = \varphi$. Now $\nu(\Phi_\varphi) = 0$ (see [9, Theorem 2]). So we can choose notation so that $\nu(\chi_1) = +1$ and $\nu(\chi_2) = -1$.

3. Symmetric vertices and extended nucleus

For the moment k is a field of arbitrary characteristic p. Let $H \subseteq G$. Following [5] a kG-module M is H-projective if it is a direct summand of an induced module $\mathrm{Ind}_H^G(L)$, for some kH-module L. Suppose that M is indecomposable. Following [2] a vertex of M is a minimal $V \subseteq G$ such that M is V-projective. The vertices of M are p-subgroups of G which are determined up to G-conjugacy. Now a V-source of M is an indecomposable kV-module Z such that M is a direct summand of $\mathrm{Ind}_H^G(Z)$. Then Z is a direct summand of $\mathrm{Res}_V^G(M)$, and Z is uniquely determined by M and V up to $N_G(V)$-conjugacy.

Recall that the dual of a left kG-module M is the left kG-module $M^* = \mathrm{Hom}_{kG}(M, k)$. Here if $f : M \to k$ and $g \in G$, we set $(gf)(m) := f(g^{-1}m)$, for all $m \in M$. Now $M \cong M^*$ as kG-modules if and only if there exists a G-invariant non-degenerate bilinear form $b : M \times M \to k$. We say that b is symmetric if $b(m_1, m_2) = b(m_2, m_1)$, alternating if $b(m_1, m_2) = -b(m_2, m_1)$ and symplectic if $b(m_1, m_1) = 0$, for all $m_1, m_2 \in M$. If $p \neq 2$ alternating is the same as symplectic and no symplectic form is symmetric. If $p = 2$ alternating is the same as symmetric and all symplectic forms are symmetric but not all symmetric forms are symplectic.

Let (L, c) be a symmetric kH-module. Now $\mathrm{Ind}_H^G(L) = \sum_{gH} g \otimes L$ as k-vector spaces, where $g \otimes L$ is a k^gH-module. The obvious isomorphism $H \cong {}^gH$ maps L to $g \otimes L$. So $g \otimes L$ inherits a ${}^gH$-invariant non-degenerate form ${}^gc$ from c. The induced symmetric kG-module $\mathrm{Ind}_H^G(L, c)$ is the orthogonal direct sum of the symmetric k-spaces $(g \otimes L, {}^gc)$.

Following [10] a symmetric kG-module (M, b) is H-projective if (M, b) is an orthogonal direct summand of $\mathrm{Ind}_H^G(L, c)$, for some symmetric kH-module (L, c). Moreover a symmetric vertex of M is a minimal $T \subseteq G$ such that there exists a T-projective symmetric kG-module (M, b). Analogous concepts exist for alternating kG-modules.

For the remainder of this section k is a perfect field of characteristic 2 which is a splitting field for all subgroups of G. We simplify our exposition by referring to both symplectic and non-symplectic symmetric forms as symmetric forms. In practice symplectic forms are more important than non-symplectic symmetric forms, because the isometry group of a symmetric form is closely related to a symplectic group.

Example 6. There is a unique non-trivial simple kD_{12}-module, where D_{12} is the dihedral group of order 12. Its projective cover P affords a 2-dimensional space of D_{12}-invariant symmetric bilinear forms. It can be shown that each non-central C_2-subgroup of D_{12} is a symmetric vertex of P. As there are two D_{12}-conjugacy classes of such subgroups, this shows that symmetric vertices are not uniquely determined up to G-conjugacy.

However, the first author proved the following result in [10]:

Proposition 7. *The symmetric vertices of a self-dual simple kG-module S are uniquely determined up to G-conjugacy. Let b be a symmetric form and let (V, Z) be a vertex-source pair of S. Then S has a symmetric vertex $T \supseteq V$ and exactly one of (i) or (ii) holds:*

(i) *$T = V$ and b is non-degenerate on a submodule of $\mathrm{Res}_V^G(S)$ isomorphic to Z. Moreover $\mathrm{Ind}_V^G(Z) \cong S \oplus Q$, where Q has no summands isomorphic to S.*
(ii) *$|T : V| = 2$ and $\mathrm{Ind}_V^T(Z)$ affords a non-degenerate T-invariant symmetric form c. For any such form c, (S, b) is an orthogonal direct summand of $\mathrm{Ind}_T^G(\mathrm{Ind}_V^T(Z), c)$.*

We shall see in Lemma 9 that only (ii) occurs when G is solvable and S is non-trivial.

L. Puig has shown that if G is solvable then the source Z of a simple module S is an endo-permutation module constructed from tensor products of endo-trivial modules of quotients of a vertex (cf. [8, Abstract]). As a consequence of the classification of torsion endo-trivial modules for p-groups and [1], the sources are self-dual unless a vertex has a generalized quaternion quotient. We present an example of a solvable group with a simple self-dual module which has a non-self-dual source, as this seems to be a relatively uncommon phenomenon:

Example 8. Let E be an extra-special group of order 27 and exponent 3. Then $\mathrm{Aut}(E) \cong \mathrm{GL}(2, 3)$ has a Sylow 2-subgroup T which is semi-dihedral of order 16. Set $G = E \rtimes T$. The centralizer of $Z(E)$ in T is a quaternion group V of order 8. Let k be a field extension of $\mathbb{F}_4$. Then kE has a faithful 3-dimensional module, which extends to a simple $kE \rtimes V$-module M. Now $M^T = M^* \ncong M$. So $S = \mathrm{Ind}_{E \rtimes V}^G(M)$ is a self-dual simple kG-module with vertex V. Moreover S has V-source $Z := \mathrm{Res}_V^{E \rtimes V}(M)$. As Z is a 3-dimensional endo-trivial kV-module, Z is not self-dual [1, p. 322]. So S has symmetric vertex T.

Theorem 1 is a consequence of our next lemma and the uniqueness of symmetric vertices proved in Proposition 7.

Lemma 9. *Suppose that G is solvable and $\varphi \in \mathrm{IBr}(G)$ is non-trivial and real valued. Let $U \subseteq G$ and $\delta \in \mathrm{IBr}(U)$ be such that δ is real valued, $\delta^G = \varphi$ and $\delta(1)_2 = 2$. Let S be the simple kG-module whose Brauer character is φ. Then each Sylow 2-subgroup T of U is a symmetric vertex of S.*

Proof. Let (W, γ) be the Isaacs nucleus of the lift of δ in $\mathrm{B}_{2'}(U)$, let S_U be the simple kU-module with Brauer character δ and let S_W be the simple kW-module with Brauer character γ^*. Recall that $S_W \ncong S_W^*$ as $\gamma(1)$ is odd. Let (V, Z) be a vertex source pair of S_W. Then it is clear that (V, Z) is a vertex source pair of S and S_U.

We claim that V is not a symmetric vertex of S. For otherwise $Z \cong Z^*$ by the first statement in Proposition 7(i). So Z is a V-source of S_W^*. In particular S_W and S_W^*

are non-isomorphic components of $\mathrm{Ind}_V^W(Z)$. Now $\mathrm{Ind}_W^G(S_W) \cong S \cong S^* \cong \mathrm{Ind}_W^G(S_W^*)$. So S occurs at least twice as a direct summand of $\mathrm{Ind}_V^G(Z)$. This contradicts the second statement in Proposition 7(i), which proves our claim.

We can apply the previous paragraph to S_U. So V is not a symmetric vertex of S_U. Then by Proposition 7(ii), S_U has a symmetric vertex $T \supseteq V$ with $|T : V| = 2$. Now V is a Sylow 2-subgroup of W, as $\dim(S_W)$ is odd. But $|U : W| = 2$. So T is a Sylow 2-subgroup of U. Now let b_U be a symmetric form on S_U. Then $(S, b) \cong \mathrm{Ind}_U^G(S_U, b_U)$ as b is unique up to isometry. Moreover b_U is T-projective. So it follows from the transitivity of induction of forms that b is T-projective. Since $|T : V| = 2$, we deduce that T is a symmetric vertex of S. $\square$

4. Projective indecomposable modules and orthogonal forms

Temporarily let k be a field of arbitrary characteristic p. The study of bilinear and quadratic forms on projective kG-modules has attracted some interest. There are ring-theoretic criteria for a projective indecomposable kG-module to be of quadratic type (have a non-degenerate G-invariant quadratic form). These are due to Landrock and Manz [7] for $p \neq 2$, and to Gow and Willems [3] for $p = 2$.

Recall that the Jacobson radical $J(kG)$ of kG is the annihilator of all simple kG-modules and the contragredient map ^{o} is the k-algebra involutary anti-automorphism of kG such that $g^o = g^{-1}$, for all $g \in G$.

Proposition 10 *(Landrock–Manz). Suppose that $p \neq 2$ and P is a projective indecomposable kG-module. Then P is of quadratic type if and only if there is a primitive idempotent e in kG such that $P \cong kGe$ and $e^o = e$.*

From now on k is a perfect field of characteristic $p = 2$. From [3], if P is the projective cover of a non-trivial simple kG-module then each G-invariant symmetric form on P is the polarization of a G-invariant quadratic form on P. In particular each such form is symplectic. Now a primitive idempotent $e \in kG$ satisfies $e^o = e$ if and only if kGe is the projective cover of the trivial kG-module. So Proposition 10 is wrong for $p = 2$, and is replaced by:

Proposition 11 *(Gow–Willems). Suppose that e is a primitive idempotent in kG. Then kGe is of quadratic type if and only if there is an involution $t \in G$ such that $e^oe^t \notin J(kG)$.*

If $e^oe^t \notin J(kG)$ there is a unique idempotent $f \in kG$ such that $kGe = kGf$ and $f^o = f^t$.

Parts of this result are only implicit in [3, Section 3].

G.R. Robinson showed in [11] that if Φ is a principal indecomposable character of G then $\nu(\Phi) = \sum_t \langle \Phi, 1_{C_G(t)}^G \rangle$ where t ranges over 1 and the conjugacy classes of involutions in G. The first author showed in [9, Corollaries 5.2 and 6.5]:

Lemma 12. *Suppose that e is a primitive idempotent in kG and $t \in G$ is an involution such that $e^o e^t \notin J(kG)$. Let Φ be the principal indecomposable character of kGe. Then $\langle \Phi, 1_{C_G(t)}^G \rangle > 0$. In particular $\nu(\Phi) > 0$, if kGe has a quadratic geometry.*

For G solvable, we aim to directly relate the Gow–Willems criterion to the extended nucleus and symmetric vertex of the corresponding simple modules. We begin with a very general remark, which holds for an arbitrary field k:

Lemma 13. *Suppose that N is a normal subgroup of G. Then $J(kN) = J(kG) \cap kN$.*

Proof. Let S be a simple kG-module. Then $\mathrm{Res}_N^G(S)$ is semi-simple, by Clifford's theorem. So $J(kN) \subseteq J(kG) \cap kN$. Conversely, let S_N be a simple kN-module. Then $\mathrm{Res}_N^G \mathrm{Ind}_N^G(S_N) = \sum_{gN \subseteq G} S_N^g$ by Mackey's formula. This implies that S_N is a direct summand of $\mathrm{Res}_N^G(S)$, for some simple kG-module S. So $J(kN) \supseteq J(kG) \cap kN$. □

We also need a result from [4]:

Lemma 14. *Suppose that (M, b) is a symmetric kG-module and $M = M_1 \dot{+} \dots \dot{+} M_t$ is a decomposition of M as an internal direct sum of indecomposable kG-modules M_i. Then for each i, either b is non-degenerate on M_i or there exists $j \neq i$ such that $M_j \cong M_i^*$ and b is non-degenerate on $M_i \dot{+} M_j$.*

Let $P_G(M) = P(M)$ denote the projective cover of a kG-module M. Theorem 3 is a consequence of Lemmas 9 and 12 and our next result:

Theorem 15. *Suppose G is solvable and S is a self-dual simple kG-module with a vertex and symmetric vertex $V \subseteq T$. Then $P(S)$ is of quadratic type if and only if $T : V$ splits.*

Proof. If S is trivial, then $T = V$ and it is easy to see that $P(S)$ has a quadratic geometry. So from now on S is non-trivial. Let φ be the Brauer character of S and let $\chi \in \mathrm{B}_{2'}(G)$ with $\chi^* = \varphi$. Also let (W, γ) and (U, δ) be as in Lemma 4. So $\delta(1)_2 = 2$ and $\delta^G = \varphi$.

Let S_U be the self-dual simple kU module whose Brauer character is δ. As $\mathrm{Ind}_U^G(S_U) = S$, Frobenius–Nakayama reciprocity implies that $\mathrm{Res}_U^G(P(S)) = P(S_U) \oplus Q$, where no component of Q is isomorphic to $P(S_U)^* \cong P(S_U)$.

Suppose first that $P(S)$ is of quadratic type. Then $P(S_U)$ is of quadratic type, by the previous paragraph and Lemma 14. Now $\mathrm{Res}_W^U(S_U) = S_W \oplus S_W^*$, where S_W is the simple kW-module whose Brauer character is γ^*. Let e be a primitive idempotent in kW such that $kWe \cong P(S_W)$. Then e is still primitive in kU and indeed $kUe \cong P(S_U)$. So according to Proposition 11, there is an involution $t \in U$ such that $e^o e^t \notin J(kU)$.

We claim that $t \notin W$. For suppose otherwise. Then $e^o e^t \in kW$. But Lemma 13 implies that $e^o e^t \notin J(kW)$. So $P(S_W) \cong kWe$ is of quadratic type and in particular $S_W \cong S_W^*$.

This contradiction proves the claim. We have shown that U splits over W. So T splits over V, by Lemma 9.

Suppose now that T splits over V. Let t be any involution in $T \backslash V$ and let H be a Hall $2'$-subgroup of G such that $H \cap W$ is a Hall $2'$-subgroup of W. As $\chi = \gamma^G$ we have

$$\chi(1)_{2'} = [G:W]_{2'}\gamma(1) = |H : H \cap W|\gamma(1) = (\gamma_{H\cap W})^H(1).$$

Moreover $\langle \chi, (\gamma_{H\cap W})^G\rangle \geq \langle \gamma, (\gamma_{H\cap W})^H\rangle \geq 1$. So $\gamma_{H\cap W}$ is a Fong character for γ and $(\gamma_{H\cap W})^H$ is a Fong character for χ. Then $\Phi_\delta = (\gamma_{H\cap W})^U$ and $\Phi_\varphi = (\gamma_{H\cap W})^G$ are the principal indecomposable characters of S_U and S, respectively. In particular $\Phi_\delta^G = \Phi_\varphi$. It follows from this that $\mathrm{Ind}_U^G(P(S_U)) = P(S)$. So to complete the proof we need only show that $P(S_U)$ is of quadratic type.

We can and do assume that $U = G$, $S_U = S$ and thus $|G : W| = 2$. Set $N = \mathrm{O}_{2'}(G)$.

Suppose first that N acts trivially on S. Set $L = O_{2',2}(G)$ and $\overline{G} = G/L$. Then S can be identified (by deflation) with an irreducible $k\overline{G}$-module. As $k\overline{G}$-module it has vertex $\overline{V}$ and symmetric vertex $\overline{T}$. Now $\bar{t}$ is an involution in $\overline{T}\backslash\overline{V}$ and $|G/L| < |G|$. So by induction on $|G|$ there is a primitive idempotent $\bar{e} \in k\overline{G}$ such that $k\overline{G}\bar{e} \cong P_{\overline{G}}(S)$ and $\bar{e}^{\bar{t}} = \bar{e}^o$.

The map $x^\sigma := tx^ot$, for $x \in kG$, is an involutory k-algebra anti-automorphism of kG. The kernel of the projection map $kG \to k\overline{G}$ is $\mathrm{sp}\{g(1-\ell) \mid g \in G, \ell \in L\}$. It is easy to check that this is σ-invariant. So σ induces the involutory k-algebra anti-automorphism $\bar{x}^\sigma = \bar{t}\bar{x}^o\bar{t}$ on $k\overline{G}$.

Notice that $\bar{e}^\sigma = \bar{e}$. By idempotent lifting [10, Lemma 2.1] there is a primitive idempotent $e \in kG$ such that $e^\sigma = e^o$ and $\bar{e}$ is the image of e in $k\overline{G}$. Then Proposition 11 implies that $kGe \cong P(S)$ is of quadratic type. This completes the case $N \subseteq \ker(S)$.

Let $\theta \in \mathrm{Irr}(N|\gamma)$. By the work above we may assume that θ is non-trivial. In particular $\theta \neq \bar{\theta}$. Set $m := \langle \chi_N, \theta\rangle = \langle \gamma_N, \theta\rangle$. Then m is odd, as it divides $\gamma(1)$. Let Z be the simple kN-module whose Brauer character is θ. Then Z occurs m times as a direct summand of the semisimple kN-module $\mathrm{Res}_N^G(S)$. So by Frobenius–Nakayama reciprocity, $P(S)$ occurs m times as a direct summand of the projective kG-module $\mathrm{Ind}_N^G(Z)$.

Now $m = |W : \mathrm{N}_W(\theta)|$ is odd. So $\mathrm{N}_W(\theta)$ contains a Sylow 2-subgroup of W. Moreover θ is G-conjugate to $\bar{\theta}$ as both belong to $\mathrm{Irr}(N|\chi)$. So $|\mathrm{N}_G(\theta, \bar{\theta}) : \mathrm{N}_G(\theta)| = 2$. As $|G : W| = 2$, it follows that $\mathrm{N}_G(\theta, \bar{\theta})$ contains a Sylow 2-subgroup of G. So we can and do assume that T is a Sylow 2-subgroup of $\mathrm{N}_G(\theta, \bar{\theta})$ and $V = T \cap \mathrm{N}_G(\theta)$. In particular $\theta^t = \bar{\theta}$.

Consider the group $E := N\langle t\rangle$, which is a degree 2-extension of N. Then $\mathrm{Ind}_N^E(Z)$ is a simple kE-module which is self-dual as its Brauer character is θ^E. So it affords a non-degenerate E-invariant symplectic bilinear form which is $\langle t\rangle$-projective. As $P(S)$ occurs with odd multiplicity m in $\mathrm{Ind}_N^G(Z) = \mathrm{Ind}_E^G(\mathrm{Ind}_N^E(Z))$, we deduce that $P(S)$ affords a non-degenerate G-invariant symplectic bilinear form which is $\langle t\rangle$-projective. In particular $P(S)$ is of quadratic type and there is a primitive idempotent $e \in kG$ such that $e^t = e^o$ and $P(S) \cong kGe$. This completes the proof of the theorem. $\square$

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