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One-sided orthogonality, orthomodular spaces, quantum sets, and a class of Garside groups

Carsten Dietzel [a], Wolfgang Rump [a,*], Xia Zhang [b,1]

[a] *Institute for Algebra and Number Theory, University of Stuttgart, Pfaffenwaldring 57, D-70550 Stuttgart, Germany*
[b] *School of Mathematical Sciences, South China Normal University, Shipai, Guangzhou, 510631, PR China*

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The classification of orthomodular lattices by means of their structure group is extended to non-symmetric orthogonality relations. Spaces with a non-hermitian sesquilinear form, quantum sets, and a new class of Garside groups are given as examples.

* Corresponding author.
E-mail addresses: carstendietzel@gmx.de (C. Dietzel), rump@mathematik.uni-stuttgart.de (W. Rump), xzhang@m.scnu.edu.cn (X. Zhang).

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Introduction

In [43], Theorem 2, it is shown that every orthomodular lattice X admits a group $G(X)$ with a right invariant lattice order as a complete invariant, so that X can be retrieved from $G(X)$ without use of operations other than multiplication and $\wedge$. The projection lattice X of a von Neumann algebra is a special case. By Dye's theorem [19], it characterizes the algebra up to *-symmetry and trivial summands of type I_2. In [42] the connection to $G(X)$ was extended to unbounded lattices, which can be given by an abstract orthogonality relation in the sense of Janowitz [25]. The link between the orthogonality lattice X and the *structure group* $G(X)$ was obtained by representing X as an *L-algebra*, a structure which encapsulates the underlying quantum logic.

In this paper, we extend the connection in two respects. First and foremost, we drop the symmetry of the orthogonality relation. For example, this is quite natural if orthogonality is induced by the sesquilinear form of a Frobenius algebra. Secondly, we do not assume beforehand that X is a lattice. Since $\wedge$ does not occur in Janowitz' axioms, we start with a $\vee$-semilattice. It will turn out, however, that X is indeed a lattice. Therefore, we call it a $\perp$*-lattice*.

The translation into an L-algebra means that the structure of a $\perp$-lattice is expressed by a single binary operation $\rightarrow$ which can be conceived as quantum-logical implication. Every $\perp$-lattice can thus be understood as a special type of L-algebra (Theorem 1). An alternative characterization is given for bounded $\perp$-lattices (Proposition 4). Here the bi-orthogonal is an automorphism of the L-algebra.

At this point, it should be mentioned that L-algebras, while conceptually bound to algebraic logic, interact deeply with classical structures in geometry, topology, number theory, and analysis. They occur in connection with lattice-ordered groups [38], Garside groups [14,13,15,40], non-commutative prime factorization [44], solutions to the Yang–Baxter equation [20,28,40], para-unitary groups [17], and von Neumann algebras [40].

A special class of L-algebras arises as intervals $[u^{-1}, 1]$ in *right ℓ-groups*, that is, groups with a right invariant lattice order, where u is a strong order unit. These L-algebras admit an equational description, and their ambient group coincides with the structure group. They have been called *right bricks* [40]. The most classical example of a right brick is the unit interval in the additive group of real numbers, an MV-algebra which is fundamental in measure theory. Mundici's equivalence between MV-algebras and unital (two-sided) abelian ℓ-groups, which he applied to the classification of AF C*-algebras, is based on such type of embedding. Dvurečenskij's non-commutative extension [18] was the next step toward the 'right bricks into right ℓ-groups' embedding theorem ([40], Theorem 3). We show that every bounded $\perp$-lattice and its corresponding L-algebra is a right brick (Proposition 5). This enables us to embed every (not necessarily bounded) $\perp$-lattice X into its structure group (Theorem 2) which will turn out to be a complete invariant of X.

To recover X from its enveloping structure group $G(X)$, we introduce *right singular* elements in a right ℓ-group and describe them in the presence of a strong order unit. For two-sided ℓ-groups, singular elements are important in connection with complete groups [2]. In the one-sided case, new phenomena arise. We show that every $\perp$-lattice X coincides with the set of right singular elements in the positive cone of its structure group $G(X)$ and characterize the class of right ℓ-groups obtained in this way (Theorem 4). The bounded $\perp$-lattices then correspond to the right ℓ-groups with a *very strong* order unit.

Extending the correspondence to the categorical level, we interpret the embedding $X \hookrightarrow G(X)$ as a universal group-valued measure. Following this paradigm, we are able to define group-valued measures on an arbitrary L-algebra. For a $\perp$-lattice X, the universal property identifies group-valued measures on X with group homomorphisms on the structure group (Theorem 3). As an L-algebra is given by a single operation $\to$, it is natural to define a morphism $f\colon X \to Y$ of L-algebras to be a map which satisfies

$$f(x \to y) = f(x) \to f(y)$$

for all $x, y \in X$. Similarly, a morphism of right ℓ-groups should be a group homomorphism which satisfies $f(x \wedge y) = f(x) \wedge f(y)$. For two-sided ℓ-groups, the property $f(x \vee y) = f(x) \vee f(y)$ would be a consequence. Taking this for granted, the category of right ℓ-groups becomes a full subcategory of the category of L-algebras (Proposition 9). With the most natural morphisms, we show that the category of $\perp$-lattices is equivalent to the category of right ℓ-groups of singular type, and that both categories are full subcategories of the category of L-algebras (Theorem 5).

The last section is devoted to examples. First, we show that $\perp$-lattices arise in connection with non-degenerate sesquilinear forms on a vector space over a skew-field with involution. A space H with such a form $\langle\ \rangle$ (with an extra condition in the infinite-dimensional case) will be called *orthomodular*, generalizing the same-named concept in the hermitian case [24]. We show that the one-dimensional subspaces of an orthomodular space form an L-algebra which determines the space. More generally, we introduce *quantum sets* as sets with an irreflexive binary relation $\perp$ which gives rise to a $\perp$-lattice. Together with an external unit element, every quantum set is an L-algebra. For an orthomodular space X, the associated projective space $\mathbb{P}(X)$ is a quantum set (Proposition 14).

Finally, we show that the structure group of a finite $\perp$-lattice is a Garside group (Proposition 16). Prototypical examples of Garside groups [13] are braid groups and their siblings [5,16]. A gamut of examples can be found in [15]. In particular, every finite quantum set generates a Garside group. An explicit example of a 5-element quantum set will be discussed at the end of the paper.

1. One-sided orthogonality

Let X be a $\vee$-semilattice with smallest element 0. For a binary relation $\perp$ on X, consider the following axioms:

$$x \perp x \iff x = 0 \tag{0}$$

$$x \leqslant y \perp z \implies x \perp z \tag{1}$$

$$x \perp y \geqslant z \implies x \perp z \tag{1'}$$

$$x, y \perp z \implies (x \vee y) \perp z \tag{2}$$

$$x \perp y, z \implies x \perp (y \vee z) \tag{2'}$$

$$x \leqslant y \implies (\exists\, z \in X \colon x \perp z \text{ and } x \vee z = y) \tag{3}$$

$$x \leqslant y \implies (\exists\, z \in X \colon z \perp x \text{ and } x \vee z = y). \tag{3'}$$

In contrast to Janowitz [25], our orthogonality operation need not be symmetric. Therefore, Janowitz' main axioms (see (7)–(9) in [42]) split into pairs (1)–(1′), (2)–(2′), and (3)–(3′). So the whole system (0)–(3′) is left–right symmetric with respect to the orthogonality relation.

For $x \perp y$ in X, we call x *left orthogonal* to y and y *right orthogonal* to x. If $x \perp y$ and $z \leqslant x, y$, we can apply (1) and (1′) to obtain $z \perp z$. So (0) yields $z = 0$, which shows that $x \perp y$ implies that $x \wedge y = 0$. Our first aim is to put the axioms into an operational form.

Proposition 1. *The element z in* (3) *and* (3′) *is unique.*

Proof. By symmetry, it is enough to verify this for (3). Let $z' \in X$ be another element with $x \perp z'$ and $x \vee z' = y$. Then $t := z \vee z'$ satisfies $x \perp t$ by virtue of (2′), and $x \vee t = y$. By (3), there is an element $u \in X$ with $z \perp u$ and $z \vee u = t$. Since $x \perp u$ holds by (1′), the implication (2) yields $y = (x \vee z) \perp u$. Hence (1) gives $u \perp u$. So (0) implies that $u = 0$. So $z = t$, and by symmetry, $z' = t = z$. □

Note that all axioms (0)–(3) have been used in the preceding proof. By (3) and Proposition 1, every pair $x, y \in X$ determines a unique element $y \smallsetminus x \in X$ with $x \perp (y \smallsetminus x)$ and

$$x \vee (y \smallsetminus x) = x \vee y. \tag{4}$$

Thus $y \smallsetminus x = (x \vee y) \smallsetminus x$ can be viewed as a right orthogonal complement of x in the interval $[0, x \vee y]$:

$$\begin{tikzcd}[column sep=small] & x \vee y \arrow[dl, no head] \arrow[dr, no head] & \\ x \arrow[dr, no head] & & y \smallsetminus x \arrow[dl, no head] \\ & 0 & \end{tikzcd} \tag{5}$$

If X is bounded, that is, X has a greatest element 1, each element $x \in X$ has a *right orthogonal* $x^{\perp} := 1 \smallsetminus x$.

Corollary 1. *For $x, y \in X$, the following are equivalent:*

(a) $x \perp y$
(b) $y \smallsetminus x = y$
(c) $\exists\, z \in X : z \smallsetminus x \geqslant y$.

If X is bounded, condition (c) can be replaced by $x^{\perp} \geqslant y$.

Proof. (a) $\Rightarrow$ (b) follows by Proposition 1. The implication (b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (a): By (1′), $x \perp (z \smallsetminus x) \geqslant y$ implies that $x \perp y$.

If X is bounded, $x \perp y$ implies that $x \perp (x^{\perp} \vee y)$. So Proposition 1 yields $x^{\perp} \vee y = x^{\perp}$. Conversely, $x^{\perp} \geqslant y$ gives (c) with $z = 1$. $\square$

Corollary 2. *If X is bounded, the following equivalence holds for $x, y \in X$:*

$$x \leqslant y \iff y^{\perp} \leqslant x^{\perp}.$$

Proof. By Corollary 1, we have $x^{\perp} \geqslant y^{\perp} \Leftrightarrow x \perp y^{\perp} \Leftrightarrow (x \vee y) \perp y^{\perp} \Leftrightarrow (x \vee y)^{\perp} \geqslant y^{\perp}$. Hence $x \leqslant y$ implies that $x^{\perp} \geqslant y^{\perp}$.

Conversely, assume that $y^{\perp} \leqslant x^{\perp}$. Since $y \perp (x \smallsetminus y)$, we have $y^{\perp} \geqslant x \smallsetminus y$. Thus $x^{\perp} \geqslant x \smallsetminus y$, which gives $x \perp (x \smallsetminus y)$. So (2) implies that $(x \vee y) \perp (x \smallsetminus y)$. Since $x \vee y \geqslant x \smallsetminus y$, (1) gives $(x \smallsetminus y) \perp (x \smallsetminus y)$. Hence (0) yields $x \smallsetminus y = 0$. By Eq. (4), we obtain $x \leqslant y$. $\square$

Up to here, we have not used the last axiom (3′). It says that for a fixed $y \in X$, every $x \in X$ with $x \leqslant y$ is of the form $y \smallsetminus z$ for some $z \leqslant y$. Now the interval $[0, y]$ satisfies all axioms (0)–(3′). Therefore, Corollary 2 implies that the map $x \mapsto y \smallsetminus x$ is an order-reversing bijection $[0, y] \xrightarrow{\sim} [0, y]$. If X itself is bounded, then (3′) implies that the map $x \mapsto x^{\perp}$ is bijective. If $x \mapsto {}^{\perp}x$ denotes the inverse map, this gives

$${}^{\perp}(x^{\perp}) = ({}^{\perp}x)^{\perp} = x.$$

So we obtain

Corollary 3. *The $\vee$-semilattice X is a lattice. If X is bounded,*

$$x^{\perp} \vee y^{\perp} = (x \wedge y)^{\perp}, \qquad x^{\perp} \wedge y^{\perp} = (x \vee y)^{\perp}$$

holds for all $x, y \in X$.

Definition 1. A $\vee$-semilattice $(X; \perp)$ satisfying (0)–(3′) will be called an *orthogonality lattice* or simply a $\perp$-*lattice*.

We will represent any $\perp$-lattice as an L-algebra [38]. Recall that an element 1 of a set X with a binary operation $\rightarrow$ is said to be a *logical unit* [38] if

$$x \to 1 = 1, \qquad 1 \to x = x \tag{6}$$

and

$$x \to x = 1 \tag{7}$$

hold for all $x \in X$. By (7), a logical unit is unique. If X has a logical unit 1 and satisfies

$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z) \tag{8}$$

$$x \to y = y \to x = 1 \implies x = y \tag{9}$$

for all $x, y, z \in X$, then X is said to be an *L-algebra* [38]. The relation

$$x \leqslant y \; :\Longleftrightarrow \; x \to y = 1$$

defines a partial order of X such that 1 is the greatest element. If the elements of X are interpreted as logical propositions, and $\to$ as implication, then 1 stands for a true proposition, while (9) characterizes logical equivalence (=) between propositions. Beyond algebraic logic, L-algebras naturally arise in connection with Garside groups and solutions to the Yang–Baxter equation [40].

Every L-algebra X embeds into an L-algebra $C(X)$ which is a $\wedge$-semilattice satisfying

$$x \to (y \wedge z) = (x \to y) \wedge (x \to z) \tag{10}$$

$$(x \wedge y) \to z = (x \to y) \to (x \to z), \tag{11}$$

such that every element of $C(X)$ is of the form $x_1 \wedge \cdots \wedge x_n$ with $x_i \in X$. Up to isomorphism, the embedding $X \hookrightarrow C(X)$ is unique [39]. Note that Eq. (11) implies Eq. (8) by the commutativity of $\wedge$. An L-algebra X which coincides with $C(X)$ will be called *$\wedge$-closed*. More generally, an equational structure $(X; \to)$ satisfying Eqs. (6) and (10)–(11) is said to be a *semibrace* [39]. By [39], Corollary 1 of Theorem 3, an L-algebra X is $\wedge$-closed if and only if X satisfies Eqs. (10)–(11).

A semibrace need not be an L-algebra. The concept is derived from that of a *brace* [37], a ring-like structure equivalent to a semibrace where $\wedge$ is a group operation. As the operation $\wedge$ is idempotent for an L-algebra, adding Eq. (7) to a semibrace comes close to a tropicalization [29].

To make a $\perp$-lattice X into an L-algebra, we define

$$x \to y := y \smallsetminus x \tag{12}$$

for $x, y \in X$. Since $x \smallsetminus x = 0$, the logical unit will be 0. So the L-algebra axioms can be rewritten as follows:

$$x \diagdown x = 0 \diagdown x = 0, \qquad x \diagdown 0 = x \tag{13}$$

$$(x \diagdown z) \diagdown (y \diagdown z) = (x \diagdown y) \diagdown (z \diagdown y) \tag{14}$$

$$x \diagdown y = y \diagdown x = 0 \implies x = y. \tag{15}$$

For example, the set-theoretic difference satisfies (13)–(15). A set X with a binary operation $\diagdown$ satisfying (13)–(15) will be called a *positive L-algebra*, or simply an *L^+-algebra*. For an L^+-algebra $(X; \diagdown)$ we introduce the opposite partial order

$$x \succcurlyeq y \;:\Longleftrightarrow\; y \diagdown x = 0.$$

Thus each L-algebra $(X; \rightarrow)$ corresponds to a positive L-algebra $(X^{\mathrm{op}}; \diagdown)$, and vice versa. Accordingly, we say that an L^+-algebra $(X; \diagdown)$ is *$\curlyvee$-closed* if $(X^{\mathrm{op}}, \rightarrow)$ is $\wedge$-closed.

Alternatively, there is a multiplicative version $x \backslash y$ of the difference operation $y \diagdown x$ in an L^+-algebra. Here $x \backslash y$ stands for a left quotient with denominator x. Accordingly, 0 has to be replaced by 1. So $(X; \backslash)$ is identical with $(X; \rightarrow)$ except that the partial order is inverted.

Proposition 2. *Every $\perp$-lattice X is a $\curlyvee$-closed L^+-algebra.*

Proof. To make X into an L^+-algebra, we replace $\leqslant$ by $\preccurlyeq$ and $\vee$ by $\curlyvee$. Then Eqs. (10) and (11) can be rewritten as

$$(x \curlyvee y) \diagdown z = (x \diagdown z) \curlyvee (y \diagdown z) \tag{16}$$

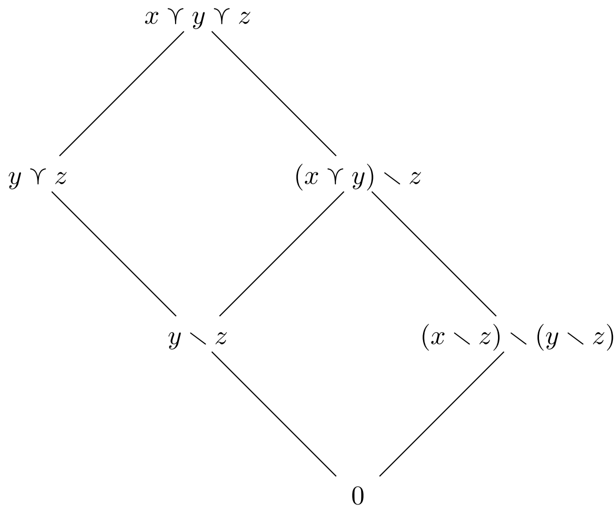
$$x \diagdown (y \curlyvee z) = (x \diagdown z) \diagdown (y \diagdown z). \tag{17}$$

For $x, y, z \in X$, we have $z \perp (x \diagdown z), (y \diagdown z)$. Hence $(2')$ gives

$$z \perp (x \diagdown z) \curlyvee (y \diagdown z).$$

By Eq. (4), $z \curlyvee (x \diagdown z) \curlyvee (y \diagdown z) = x \curlyvee z \curlyvee y$, which proves Eq. (16). Again by Eq. (4), we obtain

$$\begin{aligned} y \curlyvee z \curlyvee \big((x \diagdown z) \diagdown (y \diagdown z)\big) &= z \curlyvee (y \diagdown z) \curlyvee \big((x \diagdown z) \diagdown (y \diagdown z)\big) \\ &= z \curlyvee (y \diagdown z) \curlyvee (x \diagdown z) = z \curlyvee \big((y \curlyvee x) \diagdown z\big) \\ &= x \curlyvee y \curlyvee z. \end{aligned}$$



Furthermore, $(y \smallsetminus z) \perp (x \smallsetminus z) \smallsetminus (y \smallsetminus z)$. By (2′) and (4), we have

$$z \perp (x \smallsetminus z) \curlyvee (y \smallsetminus z) \succcurlyeq (x \smallsetminus z) \smallsetminus (y \smallsetminus z).$$

Thus (2) gives $y \curlyvee z = z \curlyvee (y \smallsetminus z) \perp (x \smallsetminus z) \smallsetminus (y \smallsetminus z)$. Whence Eq. (17) follows.

For any $x \in X$, there is a unique element $y \in X$ with $0 \curlyvee y = x$, namely, $y = x$. So (3) implies that $0 \perp x$, which yields $x \smallsetminus 0 = x$. On the other hand, there is a unique $y \in X$ with $x \perp y$ and $x \curlyvee y = x$. Since $x \wedge y = 0$, we get $y = 0$. Thus $x \perp 0$. This proves Eqs. (13). The diagram (5) shows that $y \smallsetminus x = 0$ is equivalent to $y \preccurlyeq x$. Whence X is a $\curlyvee$-closed L^+-algebra. □

2. *L*-algebras with orthogonality

As mentioned above, L-algebras arise in three forms, originally with an implicational arrow $\rightarrow$ as operation and a greatest element 1. In algebraic contexts, the operation may stand for difference $\smallsetminus$ or (left) divisibility $\backslash$. (We don't consider symmetric counterparts like right-divisibility $/$ here.) In many cases, L-algebras arise as subalgebras of right ℓ-groups (see Section 3). In their logical form $(X; \rightarrow, 1)$ they occur in the negative cone, 1 being the unit element of the group. The inversion $x \mapsto x^{-1}$ makes a right ℓ-group into a left one, with a new lattice order $\preccurlyeq$ (see Section 3) which coincides with $\leqslant$ for a two-sided ℓ-group. Accordingly, the left counterparts of the lattice operations $\vee$ and $\wedge$ are written as $\curlyvee$ and $\curlywedge$, respectively. Under inversion, the L-algebra $(X; \rightarrow, 1, \leqslant)$ is mapped to an L^+-algebra $(X^{\mathrm{op}}; \backslash, 1, \preccurlyeq)$, that is, $\leqslant$ is changed into $\succcurlyeq$ (see [40], Section 2). In its additive form, an L^+-algebra $(X; \backslash, 1)$ is written as $(X; \smallsetminus, 0)$, with difference $x \smallsetminus y$ replacing the quotient $y \backslash x$. For those working with Garside groups, where the quotient notation is used within the positive cone, we remark that inversion $x \mapsto x^{-1}$ allows a quick plunge into the negative cone, translating everything correctly into L-algebraic terms.

Whenever possible, we stick to L-algebras in their original form. Proposition 2 then states that a $\wedge$-closed L-algebra is obtained from the opposite of any $\perp$-lattice, where $\perp$ is replaced by a dual relation $\top$ with respect to a greatest element 1, and the lattice operation $\vee$ in (0)–(3′) is changed into $\wedge$. Such a structure will be called a $\top$-*lattice*. Thus every $\perp$-lattice $X = (X; \perp, \vee, 0)$ can be regarded as a $\top$-lattice $X^{\mathrm{op}} = (X^{\mathrm{op}}; \top, \wedge, 1)$, and vice versa. In a $\top$-lattice, the diagram (5) looks as follows:

$$\begin{tikzcd}[column sep=small] & 1 \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & x \to y \arrow[dl, dash] \\ & x \wedge y & \end{tikzcd} \tag{18}$$

The reader may translate the proof of Proposition 2 into $\top$-lattices and L-algebras, to make the logic behind orthogonality more visible.

Proposition 3. *Every* $\top$*-lattice* X *satisfies*

$$x \to (x \to y) = x \to y.$$

Proof. This follows by the diagram (18) since $x \top x \to (x \to y)$ and

$$x \wedge \big(x \to (x \to y)\big) = x \wedge (x \to y) = x \wedge y. \quad \square$$

By the left–right symmetry of (0)–(3′), there is a second $\wedge$-closed L-algebra structure $(X; \rightsquigarrow)$ on any $\top$-lattice X:

$$\begin{tikzcd}[column sep=small] & 1 \arrow[dl, dash] \arrow[dr, dash] & \\ x \rightsquigarrow y \arrow[dr, dash] & & x \arrow[dl, dash] \\ & x \wedge y & \end{tikzcd}$$

For a bounded $\top$-lattice, we have a *right orthogonal* $x^{\top} := x \to 0$ and a *left orthogonal* ${}^{\top}x := x \rightsquigarrow 0$ such that

$$^{\top}(x^{\top}) = ({}^{\top}x)^{\top} = x. \tag{19}$$

By Corollary 1 of Proposition 1, the orthogonality relation in a $\top$-lattice satisfies

$$x \top y \iff x \to y = y \iff x^{\top} \leqslant y.$$

Definition 2. We define an *L-algebra with orthogonality* or simply an *OL-algebra* to be a $\wedge$-closed L-algebra X such that the following are satisfied for $x, y, z \in X$:

(a) If $x \to z \leqslant y$, then $x \to y = y$.
(b) If $x \leqslant y$, there is a unique $z \geqslant x$ with $z \to x = y$.

For an OL-algebra X and $x, y \in X$, we define orthogonality by

$$x \top y \ :\Longleftrightarrow \ x \to y = y.$$

Theorem 1. *Every OL-algebra is a $\top$-lattice, and vice versa. For $x, y, z \in X$ with $z \leqslant x \wedge y$, the following equations hold:*

$$x \wedge (x \to y) = x \wedge y. \tag{20}$$

$$(x \wedge y) \to z = (x \to z) \vee (y \to z). \tag{21}$$

Proof. Let X be an OL-algebra. For $x, y \in X$, we have $x \to y \leqslant x \to y$. So Definition 2(a) gives

$$x \to (x \to y) = x \to y.$$

Hence $(x \wedge y) \to (x \to y) = (x \to y) \to \big(x \to (x \to y)\big) = 1$ by Eq. (11), which yields $x \wedge y \leqslant x \to y$. On the other hand, $\big(x \wedge (x \to y)\big) \to y = \big(x \to (x \to y)\big) \to (x \to y) = 1$. So we obtain Eq. (20).

Next we show that

$$x \leqslant y \iff y \to z \leqslant x \to z \tag{22}$$

holds for $z \leqslant x \wedge y$. Thus $y \to z \leqslant y \to x$. By Eq. (20), $x \leqslant y$ yields

$$y \to z = (y \to x) \wedge (y \to z) \leqslant (y \to x) \to (y \to z) = (x \to y) \to (x \to z) = x \to z.$$

Conversely, assume that $y \to z \leqslant x \to z$. By Definition 2(b), there exists a unique $t \geqslant y \to z$ in X with $t \to (y \to z) = x \to z$. Thus (a) yields $y \to t = t$. Hence

$$(y \wedge t) \to z = (y \to t) \to (y \to z) = t \to (y \to z) = x \to z.$$

By Eq. (20), $z = y \wedge z \leqslant y \to z \leqslant t$. So we obtain $(y \wedge t) \to z = x \to z$, with $z \leqslant y \wedge t$. Therefore, the uniqueness part of (b) yields $x = y \wedge t \leqslant y$. Thus (22) is verified.

The inequality $\geqslant$ in Eq. (21) follows immediately by (22). To prove the converse, assume that $(x \to z) \vee (y \to z) \leqslant t$ for some $t \in X$. By Definition 2(a), this implies that $x \to t = y \to t = t$. Since $u := x \wedge y \wedge t \leqslant t$, condition (b) gives a unique $v \geqslant u$ in X with $v \to u = t$. Hence $x \to u \leqslant x \to t = t = v \to u$ and $u \leqslant x \wedge v$. Thus (22) yields

$v \leqslant x$. By symmetry, $v \leqslant x \wedge y$. Furthermore, $z = x \wedge z \leqslant x \to z \leqslant t$. Again by (22), we obtain $(x \wedge y) \to u \leqslant v \to u$. Whence $(x \wedge y) \to z \leqslant (x \wedge y) \to u \leqslant v \to u = t$. So the proof of Eq. (21) is complete.

Now let us verify the axioms (0)–(3′), adapted to a $\top$-lattice. First, we have $x \top x \Leftrightarrow x \to x = x \Leftrightarrow x = 1$, which gives (0). We set $t := x \wedge y \wedge z$. If $x \geqslant y \top z$, then Eq. (21) gives

$$z = y \to z \geqslant y \to t = (x \wedge y) \to t = (x \to t) \vee (y \to t) \geqslant x \to t.$$

So Definition 2(a) implies (1). If $x \top y \leqslant z$, then $x \to y = y \leqslant z$, and thus (a) gives (1′). To verify (2), assume that $x, y \top z$. Then Eq. (21) yields

$$(x \wedge y) \to t = (x \to t) \vee (y \to t) \leqslant (x \to z) \vee (y \to z) = z.$$

Thus $(x \wedge y) \top z$. Condition (2′) follows by Eq. (10), and (3) follows by Eq. (20). Furthermore, (b) and (a) imply (3′).

Conversely, assume that X is a $\top$-lattice. By Proposition 2, this implies that X is a $\wedge$-closed L-algebra. So it remains to verify (a) and (b) of Definition 2. If $x \to z \leqslant y$, then $x \top (x \to z) \leqslant y$. Thus (1′) yields (a). Furthermore, (b) is an immediate consequence of (3′) and Proposition 1. Thus X is an OL-algebra. □

Corollary 1. *Let X be a bounded OL-algebra. Then $\to$ is given by the Sasaki arrow [45]:*

$$x \to y = (x \wedge y) \vee x^{\top}. \tag{23}$$

Furthermore, X satisfies the orthomodular law:

$$x \leqslant y \implies (x \vee y^{\top}) \wedge y = (x \vee {}^{\top}y) \wedge y = x. \tag{24}$$

Proof. The inequality $(x \wedge y) \vee x^{\top} \leqslant x \to y$ follows by Eq. (20). Conversely, assume that $(x \wedge y) \vee x^{\top} \leqslant t$ for some $t \in X$. Then Definition 2(a) with $z = 0$ implies that $x \to t = t$. Hence $1 = (x \wedge y) \to t = (x \to y) \to (x \to t)$. Thus $x \to y \leqslant x \to t = t$, which proves Eq. (23). To verify (24), assume that $x \leqslant y$. By Eq. (23), this gives $y \to x = x \vee y^{\top}$. By left–right symmetry, (24) follows by Eq. (20). □

Corollary 2. *The map $x \mapsto x^{\top\top}$ is an automorphism for any bounded OL-algebra X.*

Proof. For $x, y \in X$, we have $x^{\top} \leqslant x \to y$. Hence $x^{\top\top\top} \leqslant (x \to y)^{\top\top}$, and thus $x^{\top\top} \top (x \to y)^{\top\top}$. Furthermore, Corollary 3 of Proposition 1 gives $x^{\top\top} \wedge (x \to y)^{\top\top} = (x \wedge (x \to y))^{\top\top} = (x \wedge y)^{\top\top} = x^{\top\top} \wedge y^{\top\top}$. So (3) yields $(x \to y)^{\top\top} = x^{\top\top} \to y^{\top\top}$. □

Corollary 3. *Let X be a bounded lattice with a permutation $x \mapsto x^{\top}$ and its inverse $x \mapsto {}^{\top}x$. Then X is a $\top$-lattice if and only if the following are satisfied for $x, y \in X$:*

(a) $x \leqslant y \iff x^{\top} \geqslant y^{\top}$
(b) $x \wedge x^{\top} = 0$
(c) $x \leqslant y \implies (x \vee y^{\top}) \wedge y = (x \vee {}^{\top}y) \wedge y = x$.

Proof. By Corollary 2 of Proposition 1, every $\top$-lattice satisfies (a). With $y = 0$, Eq. (20) gives (b), while (c) follows by Corollary 1.

Conversely, assume that (a)–(c) hold. Define $x \top y :\Leftrightarrow x^{\top} \leqslant y$. Then $x \top x$ implies that $x^{\top} = 0$, which proves condition (0) for a $\top$-lattice. If $x \geqslant y$ and $y^{\top} \leqslant z$, then (a) gives $x^{\top} \leqslant y^{\top} \leqslant z$, which yields (1). Furthermore, (1′) and (2′) are trivial. If $x^{\top} \leqslant z$ and $y^{\top} \leqslant z$, then (a) implies that $(x \wedge y)^{\top} = x^{\top} \vee y^{\top} \leqslant z$. Thus (2) is valid. The implications (3) and (3′) follow by (c). Thus X is a $\top$-lattice. □

Note that (b) implies that $x \vee x^{\top} = 1$. Corollary 3 shows that an orthomodular lattice [27] is the same as a $\top$-lattice with an involutive map $x \mapsto x^{\top}$. For bounded OL-algebras, the conditions of Definition 2 can be simplified:

Proposition 4. *A $\wedge$-closed L-algebra is a bounded OL-algebra if and only if X has a smallest element 0 such that the following are satisfied for all $x, y \in X$, where $x^{\top} := x \to 0$:*

(a) *The map $x \mapsto x^{\top}$ is bijective.*
(b) $x^{\top} \leqslant y^{\top} \implies y \leqslant x$.
(c) $x^{\top} \leqslant y \implies x \to y = y$.

Proof. The necessity follows by Corollary 2 of Proposition 1 and Definition 2(a).

Conversely, assume that (a)–(c) are satisfied. Then Definition 2(a) follows by (c). So we get Eq. (20) as in the proof of Theorem 1. To verify Definition 2(b), assume that $x \leqslant y$. If $x \mapsto {}^{\top}x$ denote the inverse map of $x \mapsto x^{\top}$, then $z := {}^{\top}y \vee x$ satisfies $z^{\top} = y \wedge x^{\top} \leqslant y$. Hence $z \to y = y$. Furthermore, $x \leqslant y$ gives $y^{\top} \leqslant x^{\top}$. So $y \to x^{\top} = x^{\top}$. By Eq. (20), $y^{\top} \vee (y \wedge x^{\top}) \leqslant y \to x^{\top}$. Assume that $y^{\top} \vee (y \wedge x^{\top}) \leqslant t$ for some $t \in X$. Then $y^{\top} \leqslant t$ gives $y \to t = t$, and

$$1 = (y \wedge x^{\top}) \to t = (y \to x^{\top}) \to (y \to t),$$

which yields $y \to x^{\top} \leqslant y \to t = t$. So $y \to x^{\top} \leqslant y^{\top} \vee (y \wedge x^{\top})$. Hence $x^{\top} = y \to x^{\top} = y^{\top} \vee (y \wedge x^{\top})$, and thus $x = y \wedge z$. Consequently, $z \to x = z \to (y \wedge z) = z \to y = y$. So z satisfies (b) of Definition 2, except for the uniqueness.

To show that z is unique, let $z, t \geqslant x$ be elements in X with $z \to x = t \to x = y$. Then $(z \wedge t)^{\top} = z^{\top} \vee t^{\top} \leqslant y$. Furthermore, there is an element $u \geqslant z \wedge t$ with $u \to (z \wedge t) = z$. Hence $u^{\top} \leqslant z$ and $u^{\top} \leqslant (z \wedge t)^{\top} \leqslant y$, which yields $u^{\top} \leqslant z \wedge y = z \wedge (z \to x) = z \wedge x \leqslant z \wedge t \leqslant u$. Thus (c) gives $u = u \to u = 1$. So $t \geqslant z \wedge t = u \to (z \wedge t) = z$. By symmetry, $z = t$. □

3. The structure group

Recall that a group G with a partial order $\leqslant$ is said to be *right partially ordered* [21] if

$$x \leqslant y \implies xz \leqslant yz$$

holds for $x, y, z \in G$. If $(G; \leqslant)$ is a lattice, G is called a *right ℓ-group* [40]. With the partial order

$$x \preccurlyeq y \;:\Longleftrightarrow\; y^{-1} \leqslant x^{-1},$$

every right ℓ-group becomes a *left ℓ-group*, that is, a right ℓ-group with respect to the opposite multiplication. So there are two lattice structures on any right ℓ-group G, which coincide if and only if G is a lattice-ordered group (an *ℓ-group* for short).

Right ℓ-groups with a total order, introduced by Paul Conrad [11] as *right ordered groups*, are important in algebraic topology [3,4,34,35,47,10]. In particular, right orderability of 3-manifolds is of interest here [4]. Garside groups [14,13,15] are special right ℓ-groups which need not be right orderable. The structure group [20,28] of a finite involutive set-theoretic solution of the Yang–Baxter equation is Garside [7,9], but right orderable only in case of a multipermutation solution [8,1].

The *negative cone*

$$G^- := \{x \in G \mid x \leqslant 1\}$$

of a right ℓ-group G is an L-algebra with respect to each of the two operations

$$x \to y := yx^{-1} \wedge 1, \qquad x \rightsquigarrow y := (y^{-1}x \vee 1)^{-1}. \tag{25}$$

The induced partial orders on G^- are

$$x \leqslant y \Longleftrightarrow x \to y = 1, \qquad x \preccurlyeq y \iff x \rightsquigarrow y = 1.$$

An element $u \in G$ is said to be *normal* [40] if $uG^-u^{-1} = G^-$, or equivalently,

$$x \leqslant y \iff ux \leqslant uy$$

for all $x, y \in G$. In the context of Garside groups, normal elements are also called *balanced* [22]. The normal elements of a right ℓ-group G form an ℓ-group ([41], Proposition 5), the *quasi-centre* $N(G)$ of G. A normal element $u \geqslant 1$ in $N(G)$ is said to be a *strong order unit* [40] if each $x \geqslant 1$ satisfies $x \leqslant u^n$ for some $n \in \mathbb{N}$. Note that normal elements $u \in G$ satisfy

$$u \leqslant x \iff u \preccurlyeq x, \qquad x \leqslant u \iff x \preccurlyeq u$$

for all $x \in G$.

Mundici [31] proved that an abelian ℓ-group with a distinguished strong order unit u is equivalent to an MV-algebra [6], given by the interval $[1, u]$. This result was generalized to non-commutative ℓ-groups by Dvurečenskij [18]. For our present purpose, we need an extension to right ℓ-groups [40], where the rôle of the MV-algebra is taken by a *right brick*, a set X with binary operations $\rightarrow$ and $\rightsquigarrow$ such that $(X; \rightarrow)$ and $(X; \rightsquigarrow)$ are L-algebras with the same logical unit 1 and a simultaneous smallest element 0. With $x^\top := x \rightarrow 0$ and ${}^\top x := x \rightsquigarrow 0$, the following are required to hold for all $x, y \in X$:

$$({}^\top x)^\top = x \tag{26}$$

$${}^\top(x^\top \rightarrow y^\top) = ({}^\top x \rightarrow {}^\top y)^\top \tag{27}$$

$$x \rightarrow (x \rightsquigarrow y)^\top = y \rightarrow (y \rightsquigarrow x)^\top \tag{28}$$

$$x \rightsquigarrow {}^\top(x \rightarrow y) = y \rightsquigarrow {}^\top(y \rightarrow x). \tag{29}$$

In [40] it was shown that the symmetric counterparts

$$
\begin{aligned}
{}^\top(x^\top) &= x \\
{}^\top(x^\top \rightsquigarrow y^\top) &= ({}^\top x \rightsquigarrow {}^\top y)^\top
\end{aligned}
$$

of (26) and (27) are also valid in a right brick. By [40], Theorem 3, every right brick X embeds into a right ℓ-group G with a strong order unit u such that $X \cong [u^{-1}, 1]$. Eq. (27) can also be written as

$$(x \rightarrow y)^{\top\top} = x^{\top\top} \rightarrow y^{\top\top},$$

and the symmetry in Eqs. (28)–(29) with respect to x and y is explained by

$$
\begin{aligned}
x^\top \vee y^\top &= x \rightarrow (x \rightsquigarrow y)^\top \\
{}^\top(x \wedge y) &= x \rightsquigarrow {}^\top(x \rightarrow y).
\end{aligned}
$$

Proposition 5. *Every bounded OL-algebra is a right brick.*

Proof. Eq. (26) is a consequence of Proposition 4. Eq. (27) follows by Corollary 2 of Theorem 1. By definition, $x \rightsquigarrow y$ is given by the relations $(x \rightsquigarrow y) \top x$ and $(x \rightsquigarrow y) \wedge x = x \wedge y$. Hence $(x \rightsquigarrow y)^\top \leqslant x$. By (24), this implies that $\big((x \rightsquigarrow y)^\top \vee x^\top\big) \wedge x = (x \rightsquigarrow y)^\top$. Since $(x \rightsquigarrow y) \wedge x = x \wedge y$, we have $(x \rightsquigarrow y)^\top \vee x^\top = (x \wedge y)^\top$. Thus

$$(x \rightsquigarrow y)^\top = (x \wedge y)^\top \wedge x.$$

So Eq. (10) yields $x \to (x \rightsquigarrow y)^{\top} = x \to \big((x \wedge y)^{\top} \wedge x\big) = (x \to (x \wedge y)^{\top}) \wedge (x \to x) = x \to (x \wedge y)^{\top}$. Since $x \wedge y \leqslant x$, we have $x^{\top} \leqslant (x \wedge y)^{\top}$. Hence $x \top (x \wedge y)^{\top}$, and thus $x \to (x \wedge y)^{\top} = (x \wedge y)^{\top}$. So we obtain $x \to (x \rightsquigarrow y)^{\top} = (x \wedge y)^{\top}$, which proves Eq. (28). Now Eq. (29) follows by the left–right symmetry of (0)–(3′). □

To take profit from Proposition 5, we recall the construction of the structure group $G(X)$ of an L-algebra X. For details, we refer to [38]. First, an L-algebra X is said to be *self-similar* [38] if for any $x \in X$, the map $y \mapsto (x \to y)$ is a bijection from the lower set $\downarrow x := \{y \in X \mid y \leqslant x\}$ onto X. Equivalently, this means that X has an associative multiplication with unit element 1 such that the following are satisfied for $x, y, z \in X$:

$$x \to yx = y \tag{30}$$

$$xy \to z = x \to (y \to z) \tag{31}$$

$$(x \to y)x = (y \to x)y. \tag{32}$$

It is easily checked that these equations imply that $(X; \to)$ is an L-algebra. For $x, y \in X$, the product xy is the unique element $z \leqslant y$ with $y \to z = x$. By [38], Proposition 4, every self-similar L-algebra is $\wedge$-closed, with

$$x \wedge y = (x \to y)x = (y \to x)y. \tag{33}$$

Up to isomorphism, any L-algebra X admits a unique embedding into a self-similar L-algebra $S(X)$, generated as a monoid by X, the *self-similar closure* of X. Eq. (30) shows that the monoid $S(X)$ is right cancellative, and Eq. (32) implies the left Ore condition. So $S(X)$ has a group of left fractions $G(X)$, the *structure group* [38] of the L-algebra X, and there is a natural map $q_X \colon X \to G(X)$. The structure group, introduced by Etingof et al. [20] in connection with solutions to the Yang–Baxter equation, is a special case ([40], Section 4). If X is a right brick, q_X is an embedding so that $S(X)$ identifies with the negative cone of $G(X)$.

Definition 3. Let G be a right ℓ-group. We call a strong order unit $u \in G$ *very strong* if the partial orders $\leqslant$ and $\preccurlyeq$ coincide on the interval $[1, u]$.

As a consequence of Proposition 5, we obtain

Proposition 6. *Let X be a bounded OL-algebra. There exists a right ℓ-group $G(X)$ with a very strong order unit u such that the interval $[u^{-1}, 1] \hookrightarrow G(X)$ is isomorphic to X.*

Proof. By Proposition 5 and [40], Theorem 3, X embeds into a right L-group $G(X)$ with a strong order unit u such that X can be identified with the interval $[u^{-1}, 1]$ in $G(X)$. For $x, y \in [1, u]$ we have $x^{-1}, y^{-1} \in [u^{-1}, 1]$ and $(x^{-1})^{\top} = u^{-1}x \wedge 1 = u^{-1}x$. Hence $x \preccurlyeq y \iff y^{-1} \leqslant x^{-1} \iff (x^{-1})^{\top} \leqslant (y^{-1})^{\top} \iff u^{-1}x \leqslant u^{-1}y \iff x \leqslant y$. Thus u is a very strong order unit. □

Under the map $x \mapsto x^{-1}$, the interval $X = [u^{-1}, 1]$ in $G(X)$ is transformed into an L^+-algebra $[1, u] \subset G(X)$ which is isomorphic to X^{op}.

Theorem 2. *Let X be an OL-algebra. Then $G(X)$ is a right ℓ-group. In particular, $G(X)$ is a two-sided group of fractions of $S(X)$, with $S(X) = G(X)^-$.*

Proof. For any $x \in X$, the upper set $\uparrow x := \{y \in X \mid x \leqslant y\}$ is a $\top$-lattice, hence a bounded OL-algebra, an L-subalgebra of $(X; \rightarrow)$ and $(X; \rightsquigarrow)$. Note that by (3′), the two operations are related by

$$(x \rightarrow y) \rightsquigarrow y = x,$$

for $x \geqslant y$ in X. In the proof of [40], Theorem 3, it was shown that the self-similar closure $S(Y)$ of any right brick Y coincides with the negative cone of the structure group $G(Y)$. So $S(Y)$ is a right ℓ-cone in the sense of [40], Definition 2. Thus Proposition 5 implies that $S(\uparrow x) = G(\uparrow x)^-$ for all $x \in X$. By [38], Theorem 3, $S(\uparrow x)$ can be identified with the submonoid of $S(X)$ generated by $\uparrow x$. So the direct union $S(X) = \varinjlim S(\uparrow x)$ is a right ℓ-cone. Thus [40], Theorem 1, yields $S(X) = G(X)^-$. In particular, $G(X)$ is a right ℓ-group. Since any right ℓ-cone is left and right cancellative and satisfies the left and right Ore condition, $G(X)$ is a two-sided group of fractions of $S(X)$. $\square$

4. Measures on L^+-algebras

Let X be an L^+-algebra. In what follows, we write the operation $\smallsetminus$ multiplicatively, that is, we pass to the operation

$$x \backslash y := y \smallsetminus x$$

with 1 instead of 0. We say that X is *self-similar* if the L-algebra X^{op} is self-similar. So there is a natural embedding $X \hookrightarrow S(X)$ into a self-similar L^+-algebra $S(X)$ which is obtained from $S(X^{\mathrm{op}})$ by inverting the partial order and the multiplication:

$$S(X)^{\mathrm{op}} = S(X^{\mathrm{op}}).$$

We call $S(X)$ the *self-similar closure* of X. Eqs. (30)–(32) in $S(X^{\mathrm{op}})$ translate into the following equations in $S(X)$:

$$x \backslash xy = y \tag{34}$$

$$xy \backslash z = y \backslash (x \backslash z) \tag{35}$$

$$x(x \backslash y) = y(y \backslash x). \tag{36}$$

Note that for the difference operation, Eq. (35) takes the more natural form

$$z \diagdown xy = (z \diagdown x) \diagdown y,$$

which partly explains why it is natural to invert the multiplication in passing from $S(X^{\mathrm{op}})$ to $S(X)$. More importantly, this convention enables us to identify $G(X)$ with $G(X^{\mathrm{op}})$. Indeed, there is a commutative diagram

$$\begin{tikzcd} X^{\mathrm{op}} \arrow[r, "q_X"] \arrow[d, "1_X"] & G(X^{\mathrm{op}}) \arrow[d, "i"] \\ X \arrow[r, "q_X^+"] & G(X) \end{tikzcd}$$

where the identity map 1_X inverts the partial order and the multiplication, while $i(a) := a^{-1}$ for $a \in G(X^{\mathrm{op}})$. The map i carries the negative cone $G(X)^-$ to the *positive cone* $G(X)^+ := \{a \in G(X) \mid a \geqslant 1\}$, with

$$a \backslash b := (a^{-1} \to b^{-1})^{-1}.$$

The partial order $\leqslant$ in $G(X)^-$ is transformed into the partial order $\succcurlyeq$ in $G(X)^+$, in accordance with our convention to denote the partial order of an L^+-algebra X by $\preccurlyeq$, and to speak of $\curlyvee$-closed L^+-algebras. We call $G(X)$ the *structure group* of X. It is equipped with the canonical map $q_X^+ \colon X \to G(X)$, given by $q_X^+ = iq_X$. In case that the structure group $G(X)$ of an L^+-algebra X is abelian, we use the operation $\diagdown$ in X instead of its multiplicative counterpart.

Example. Classical logic is encapsulated in the Boolean algebra $\mathbb{B} := \{0, 1\} = \mathfrak{P}(\varnothing)$, the subobject classifier of the topos of sets [30]. With respect to set-theoretic difference, $\mathbb{B}$ is an L^+-algebra. Its self-similar closure is the L-algebra $(\mathbb{N}; \diagdown, +)$ of natural numbers, with

$$a \diagdown b := \begin{cases} a - b & \text{for } a \geqslant b \\ 0 & \text{for } a < b. \end{cases}$$

The structure group $G(\mathbb{B})$ is $\mathbb{Z}$, the additive group of integers.

Definition 4. Let X be an L^+-algebra. We say that x is *left orthogonal* to y and y is *right orthogonal* to x, written $x \perp y$, if $x \backslash y = y$.

Viewed in $S(X)$, the internal condition $x \perp y$ can be expressed, more symmetrically, by $xy = x \curlyvee y$. Since $x(x \backslash y) = x \curlyvee y$, the equivalence follows since $S(X)$ is left cancellative. Thus $x \perp y$ expresses a certain disjointness of the factors in xy.

The following result suggests a generalization to arbitrary L-algebras.

Proposition 7. *Let X be a $\perp$-lattice. Then $x \perp y$ holds in X if and only if the product $xy \in S(X)$ belongs to X.*

Proof. If $x \perp y$, then $xy = x \curlyvee y \in X$. Conversely, let $x, y \in X$ be elements with $xy \in X$. By Proposition 3 and Eq. (34), $x \backslash y = x \backslash (x \backslash xy) = x \backslash xy = y$. □

Definition 5. Let M be a monoid, and let X be a (positive) L-algebra. We define a *measure* of X with values in M to be a map $\mu \colon X \to M$ satisfying $\mu(1) = 1$ and

$$\mu(xy) = \mu(x)\mu(y)$$

for all $x, y \in X$ with $xy \in X$.

For example, the embedding $X \hookrightarrow S(X)$ of an L-algebra X into its self-similar closure $S(X)$, as well as the natural map $q_X \colon X \to G(X)$, are measures on X.

Theorem 3. *Let M be a monoid, and let X be a $\curlyvee$-closed L^+-algebra. Every measure $\mu \colon X \to M$ admits a unique extension to a measure on $S(X)$ with values in M.*

Proof. For an integer $n \in \mathbb{N}$, let $S_n(X) \subset S(X)$ be the subset of all $a \in S(X)$ which can be written as $a = x_1 \cdots x_n$ with $x_i \in X$. Thus $S_0(X) = \{1\}$ and $S_1(X) = X$. We proceed by induction on n. By Eq. (35) and [38], Proposition 5, the implication

$$b \in S_n(X) \implies a \backslash b \in S_n(X)$$

holds for all $a \in S(X)$ and $n \in \mathbb{N}$. Hence $S_n(X)$ is an L^+-subalgebra of $S(X)$. By [38], Proposition 5, the equation

$$a \backslash bc = (a \backslash b)\big((b \backslash a) \backslash c\big) \tag{37}$$

holds in $S(X)$.

Now assume that μ has been extended to some $S_n(X)$ with $n \geqslant 1$, such that $\mu(xa) = \mu(x)\mu(a)$ holds for all $x \in X$ and $a \in S_n(X)$ with $xa \in S_n(X)$. Then any element of $S_{n+1}(X)$ is of the form xa with $x \in X$ and $a \in S_n(X)$. Hence, an extension to $S_{n+1}(X)$ must satisfy

$$\mu(xa) = \mu(x)\mu(a),$$

which proves the uniqueness assertion of the theorem. Thus, assume that $xa = yb$ for some $x, y \in X$ and $a, b \in S_n(X)$. We show that

$$\mu(x)\mu(a) = \mu(y)\mu(b). \tag{38}$$

By Eqs. (35) and (37),

$$xa\backslash yb = a\backslash(x\backslash yb) = a\backslash(x\backslash y)\big((y\backslash x)\backslash b\big) = \big(a\backslash(x\backslash y)\big)\big(((x\backslash y)\backslash a)\backslash((y\backslash x)\backslash b)\big).$$

Hence $xa \succcurlyeq yb$ is equivalent to $a \succcurlyeq x\backslash y$ and $(x\backslash y)\backslash a \succcurlyeq (y\backslash x)\backslash b$. So $xa = yb$ if and only if $(x\backslash y)\backslash a = (y\backslash x)\backslash b$ together with $a \succcurlyeq x\backslash y$ and $b \succcurlyeq y\backslash x$. Thus $c := (x\backslash y)\backslash a = (y\backslash x)\backslash b \in S_n(X)$. By Eq. (33), this gives $(x\backslash y)c = (x\backslash y) \curlyvee a = a$, and similarly, $(y\backslash x)c = b$. Thus, by assumption, $\mu(a) = \mu(x\backslash y)\mu(c)$ and $\mu(b) = \mu(y\backslash x)\mu(c)$. Since X is $\curlyvee$-closed, this implies that $\mu(x)\mu(a) = \mu(x)\mu(x\backslash y)\mu(c) = \mu(x \curlyvee y)\mu(c)$. By symmetry, this proves Eq. (38). Therefore,

$$\mu(xa) := \mu(x)\mu(a) \tag{39}$$

extends μ unambiguously to $S_{n+1}(X)$.

Now assume that $x \in X$ and $a \in S_{n+1}(X)$ such that $xa \in S_{n+1}(X)$. To complete the inductive step, we have to show that $\mu(xa) = \mu(x)\mu(a)$. Choose $y \in X$ and $b \in S_n(X)$ with $xa = yb$. As above, we have $c := (y\backslash x)\backslash b \in S_n(X)$. Hence $\mu(b) = \mu((y\backslash x)c) = \mu(y\backslash x)\mu(c)$, and thus $\mu(xa) = \mu(y)\mu(b) = \mu(y)\mu(y\backslash x)\mu(c) = \mu(x)\mu(x\backslash y)\mu(c) = \mu(x)\mu((x\backslash y)c) = \mu(x)\mu(a)$. So Eq. (39) holds for all $x \in X$ and $a \in S(X)$.

Finally, let $a, b \in S(X)$ be arbitrary. We prove that

$$\mu(ab) = \mu(a)\mu(b).$$

For $a \in X$, this follows by Eq. (39). Thus, assume that $a = xc \in S_{n+1}(X)$ and $c \in S_n(X)$. Then $\mu(ab) = \mu(xcb) = \mu(x)\mu(cb)$. By induction, we can assume that $\mu(cb) = \mu(c)\mu(b)$. So we obtain $\mu(ab) = \mu(x)\mu(c)\mu(b) = \mu(a)\mu(b)$. □

Note that $S(S(X)) \cong S(X)$ holds since $S(X)$ is self-similar [38]. Therefore, a measure on $S(X)$ is just a monoid homomorphism.

Corollary. *Every measure $\mu\colon X \to G$ on a $\curlyvee$-closed L^+-algebra X with values in a group G admits a unique extension to a group homomorphism $\mu\colon G(X) \to G$, and any G-valued measure on X arises in this way.*

Proof. This follows since $G(X)$ is a group of right fractions of $S(X)$. □

5. Right singular right ℓ-groups

Theorem 3 and its corollary show that the natural map $q_X\colon X \hookrightarrow G(X)$ of a $\curlyvee$-closed L^+-algebra X is universal among the group-valued measures $\mu\colon X \to G$. We will use this fact to establish a connection between $\perp$-lattices and right ℓ-groups.

Definition 6. Let G be a right ℓ-group. We call an element $s \in G^-$ *right singular* if the implication

$$s \leqslant ab \implies ab = a \wedge b$$

holds for all $a, b \in G^-$. The set of right singular elements will be denoted by $X_r(G)$. We say that G is of *right singular type* if $X_r(G)$ is closed with respect to $\wedge$ and generates G as a group, such that $x \leqslant y$ implies $x \preccurlyeq y$ for all $x, y \in X_r(G)$.

By Eqs. (33), $s \in G^-$ is right singular if and only if $s \leqslant ab$ implies that a is right orthogonal to b. For convenience, we extend the definition to positive elements, defining $s \in G^+$ to be *right singular* if s^{-1} is right singular, or equivalently, if

$$ab \preccurlyeq s \implies ab = a \curlyvee b$$

holds for all $a, b \in G^+$. For ℓ-groups, this definition coincides with the classical one, except that we do not exclude the unit element (see [2], 11.2.9, or [12], Definition 6.9). Moreover, the set of singular elements in the positive cone of an ℓ-group G is $\vee$-closed. By [12], Proposition 55.11, an ℓ-group G is of right singular type if and only if G is a Specker group, hence a subgroup of some cardinal power $\mathbb{Z}^I$ (see [12], Theorem 55.14).

Proposition 8. *Let G be a right ℓ-group, and let $u \in G^+$ be a strong order unit. Then any right singular element $s \in G^-$ is of the form $s = x_1 \wedge \cdots \wedge x_n$ with $u^{-1} \leqslant x_i \leqslant 1$.*

Proof. There is an integer $n \geqslant 2$ with $u^{-n} \leqslant s$. Since $u^{-1}(u^{1-n} \vee s) = u^{-n} \vee u^{-1}s \leqslant s$, we have $x := s(u^{1-n} \vee s)^{-1} \in [u^{-1}, 1]$ and $u^{1-n} \leqslant u^{1-n} \vee s \leqslant 1$. Hence $s = x(u^{1-n} \vee s)$ implies that $s = x \wedge (u^{1-n} \vee s)$. Now the statement follows by induction. $\square$

Next we show that the structure group of an OL-algebra is a complete invariant.

Theorem 4. *Let X be an OL-algebra. The structure group $G(X)$ is of right singular type, with $X_r(G(X)) = X$, and any right ℓ-group of right singular type arises in this way.*

Proof. Let X be an OL-algebra. By Theorem 2, $G(X)$ is a right ℓ-group, a two-sided group of fractions of its negative cone $S(X)$. Let $s \in G(X)^-$ be right singular. Suppose that $s \notin X$. Then $s = xa$ with $x \in X \smallsetminus \{1\}$ and $a \in S(X)$. Hence $s = x \wedge a$. So a is again right singular. By induction, we can assume that $a \in X$. Hence $s \in X$, which proves that $X_r(G(X)) \subset X$. Conversely, assume that $x \in X$ and $x \leqslant ab$ for some $a, b \in S(X)$. By [38], Proposition 5, this gives $1 = x \to ab = \big((b \to x) \to a\big)(x \to b)$. Thus $x \leqslant b$ and $b \to x \leqslant a$, and Proposition 6 implies that $a, b \in X$. By Theorem 1, $b \to a = a$. So we obtain $ab = (b \to a)b = a \wedge b$. Whence x is right singular. Thus $X_r(G(X)) = X$.

In particular, $X_r(G(X))$ is $\wedge$-closed and generates $G(X)$ as a group. Let $x, y \in X_r(G(X))$ be singular elements with $x \leqslant y$. By Definition 2(b), there is an element $z \geqslant x$ in X with $z \to x = y$. Hence $x = z \wedge x = (z \to x)z = yz \preccurlyeq y$. This proves that $G(X)$ is of right singular type.

Conversely, let G be a right ℓ-group of right singular type. We show that $X := X_r(G)$ is an OL-algebra. Note first that X is a sublattice of G. For $x, y \in X$, this implies that $x \wedge y = (x \to y)x = (x \to y) \wedge x \leqslant x \to y$. Hence X is an L-subalgebra of G^-. For $x, y, z \in X$, assume that $x \to z \leqslant y$. Then $x \wedge z = (x \to z)x \leqslant yx$, which yields $yx = x \wedge y = (x \to y)x$. Hence $x \to y = y$. This establishes condition (a) of Definition 2. To verify (b), let $x, y \in X$ be singular elements with $x \leqslant y$. We have to find an element $z \in X$ with $z \geqslant x$ and $z \to x = y$. Such an element is unique, because $x = z \wedge x = (z \to x)z = yz$. Since $x \preccurlyeq y$, we have $x = x \curlywedge y = y(y \rightsquigarrow x)$. So the element $z := y \rightsquigarrow x \in X$ satisfies $x = yz$. Hence $x \leqslant z$ and $x = yz = y \wedge z$. By Eqs. (25), we thus obtain $z \to x = z \to (y \wedge z) = z \to y = yz^{-1} \wedge 1 = (y \wedge z)z^{-1} = xz^{-1} = y$. Thus X is an OL-algebra.

For any pair $x, y \in X$, we have $(x \rightsquigarrow y) \top x$ and $(x \rightsquigarrow y) \wedge x = x \wedge y$. Hence $(x \rightsquigarrow y) \to x = x$, and thus $x(x \rightsquigarrow y) = \big((x \rightsquigarrow y) \to x\big)(x \rightsquigarrow y) = (x \rightsquigarrow y) \wedge x = x \wedge y$. By symmetry, we obtain $x(x \rightsquigarrow y) = y(y \rightsquigarrow x)$, that is, $x^{-1}y = (x \rightsquigarrow y)(y \rightsquigarrow x)^{-1}$. If S denotes the set of all products $x_1 \cdots x_n \in G$ with $x_i \in X$, this implies that every element of G is of the form ab^{-1} with $a, b \in S$. By [40], Theorem 1, $(G^-; \to)$ is a self-similar L-algebra. Hence

$$a \to bc = \big((c \to a) \to b\big)(a \to c) \tag{40}$$

holds for $a, b, c \in G^-$ (see [38], Proposition 5). Together with Eq. (31), this shows that $a \to b$ with $a, b \in S$ belongs to S. For any $c \in G^-$, the representation $c = ab^{-1}$ with $a, b \in S$ gives $a = cb$. Hence Eq. (30) yields $c = b \to a \in S$. Thus $G^- = S$. By [38], Theorem 3, $S \cong S(X)$. This proves that $G \cong G(X)$. $\square$

Corollary. *Up to isomorphism, there is a one-to-one correspondence between bounded OL-algebras and right ℓ-groups with a right singular very strong order unit.*

Proof. By Proposition 6, every bounded OL-algebra X embeds as an interval $[u^{-1}, 1]$ into its structure group $G(X)$ such that u is a very strong order unit of $G(X)$. Thus u is right singular.

Conversely, let G be a right ℓ-group with a right singular very strong order unit u. Then Proposition 8 implies that $[u^{-1}, 1] = X_r(G)$. Since u is a very strong order unit, the partial orders $\leqslant$ and $\preccurlyeq$ coincide on $[u^{-1}, 1]$. Hence G is of right singular type, and G can be identified with the structure group $G(X_r(G))$. $\square$

Let us write $\mathbf{Gr}_\ell$ for the category of right ℓ-groups, with group homomorphisms respecting the $\wedge$-semilattice structure as morphisms. By **LAlg** we denote the category of L-algebras. Morphisms are maps $f\colon X \to Y$ which satisfy

$$f(x \to y) = f(x) \to f(y)$$

for all $x, y \in X$. By Eq. (7), this implies that $f(1) = 1$. We show first that $\mathbf{Gr}_\ell$ can be regarded as a full subcategory of $\mathbf{LAlg}$.

Proposition 9. *The functor $G \mapsto G^-$ gives a full embedding $\mathbf{Gr}_\ell \hookrightarrow \mathbf{LAlg}$.*

Proof. Every morphism $f\colon G \to H$ of right ℓ-groups induces a monoid homomorphism $f^-: G^- \to H^-$, and f is determined by f^-. By Eqs. (25), f^- is a morphism of L-algebras. Conversely, let $g\colon G^- \to H^-$ be a morphism of L-algebras. For $a, b \in G^-$, Eq. (30) gives $g(a) = g(b \to ab) = g(b) \to g(ab)$. Hence $g(a)g(b) \leqslant g(ab)$. On the other hand, Eq. (40) yields

$$\begin{aligned} g(ab) \to g(a)g(b) &= \big((g(b) \to g(ab)) \to g(a)\big)\big(g(ab) \to g(b)\big) \\ &= \big(g(b \to ab) \to g(a)\big)g(ab \to b) = 1. \end{aligned}$$

Thus $g(ab) \leqslant g(a)g(b)$, which shows that g is a monoid homomorphism. Thus g extends uniquely to a group homomorphism $f\colon G \to H$.

By [40], Theorem 1, G is a two-sided group of fractions of G^-. Hence every pair of elements $a, b \in G$ admits an element $c \in G^-$ with $ac, bc \in G^-$. By Eqs. (33), g is a morphism of $\wedge$-semilattices. So f is a morphism of $\wedge$-semilattices. Whence $f \in \mathbf{Gr}_\ell$. □

In $\mathbf{Gr}_\ell$, we consider the subcategory $\mathbf{S}_r\mathbf{Gr}_\ell$ of right ℓ-groups of right singular type, with morphisms which map right singular elements to right singular elements. By $\mathbf{OLAlg}$ we denote the full subcategory of OL-algebras in $\mathbf{LAlg}$, and we write $\mathbf{OSL}$ for the category of $\top$-lattices, with morphisms $f\colon X \to Y$ satisfying $f(x \wedge y) = f(x) \wedge f(y)$ and

$$x \top y \implies f(x) \top f(y) \tag{41}$$

for all $x, y \in X$.

Theorem 5. *The categories* $\mathbf{OSL}$, $\mathbf{OLAlg}$, *and* $\mathbf{S}_r\mathbf{Gr}_\ell$, *with their natural morphisms, are mutually equivalent.*

Proof. Let us start with the more obvious equivalence $\mathbf{OSL} \approx \mathbf{OLAlg}$. By Theorem 1, the two structures live on the same set. Thus, let $f\colon X \to Y$ be a morphism of $\top$-lattices. Since f respects meets, the diagram (18) with (41) implies that f is a morphism of L-algebras. So $f \in \mathbf{OLAlg}$. Conversely, assume that $f \in \mathbf{OLAlg}$. Then (41) follows since $x \top y \Leftrightarrow x \to y = y$. As $q_Y f\colon X \to Y \hookrightarrow G(Y)$ is a measure, Eqs. (33) imply that $q_Y f$ is a morphism of $\wedge$-semilattices. Hence $f \in \mathbf{OSL}$.

Now let $f\colon X \to Y$ be a morphism of OL-algebras. Since $q_Y f\colon X \to G(Y)$ is a measure, the corollary of Theorem 3 implies that f gives rise to a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "q_X"] & Y \arrow[d, hook, "q_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

with a group homomorphism $G(f)$. By Theorem 2, $G(X)$ is a two-sided group of fractions of $S(X) = G(X)^-$. Thus $G(f)$ restricts to a monoid homomorphism $S(f): S(X) \to S(Y)$. By Eqs. (31) and (40), every element $x_1 \cdots x_n \to y_1 \cdots y_m$ with $x_i, y_j \in X$ can be represented as a term in the x_i, y_j with $\to$ and multiplication as operations. Hence $S(f)$ is a morphism of L-algebras. By Proposition 9, this implies that $G(f)$ is a morphism of right ℓ-groups. So we obtain a functor $G: \mathbf{OLAlg} \to \mathbf{S}_r\mathbf{Gr}_\ell$.

Conversely, every morphism $g: G \to H$ in $\mathbf{S}_r\mathbf{Gr}_\ell$ induces a morphism $X_r(g): X_r(G) \to X_r(H)$ of L-algebras. So the equivalence $\mathbf{OLAlg} \approx \mathbf{S}_r\mathbf{Gr}_\ell$ follows by Theorem 4. □

6. Examples

Let K be a skew-field with an involutive anti-automorphism $\lambda \mapsto \lambda^*$, and let H be a left K-vector space with a sesquilinear form $\sigma: H \times H \to K$, that is, σ is linear in the first variable, additive in the second, and $\sigma(x, \lambda y) = \sigma(x, y)\lambda^*$ holds for $x, y \in H$ and $\lambda \in K$. For a subspace U of H, we define the *left orthogonal* ${}^\perp U := \{x \in H \mid \sigma(x, U) = 0\}$ and the *right orthogonal* $U^\perp := \{x \in H \mid \sigma(U, x) = 0\}$. Assume that σ is *non-degenerate*, that is ${}^\perp H = H^\perp = 0$. If H is finite dimensional, there is a unique automorphism ν of H with

$$\sigma(x, \nu y) = \sigma(y, x)^* \tag{42}$$

for all $x, y \in H$, the *Nakayama automorphism* [26]. Two-fold application of (42) yields

$$\sigma(\nu x, \nu y) = \sigma(x, y),$$

which shows that ν is σ-invariant. By Eq. (42), every subspace U of H satisfies

$$U^\perp = {}^\perp(\nu U) = \nu({}^\perp U). \tag{43}$$

The subspaces U with $({}^\perp U)^\perp = U$ coincide with the subspaces of the form $V^\perp$. We call them *closed*. The set $X(H)$ of closed subspaces is closed with respect to intersection. So $X(H)$ is a complete lattice with set-theoretic intersection as meet. For $U \in X(H)$, Eqs. (43) give

$$\nu(U) = U^{\perp\perp}.$$

Definition 7. Let K be a skew-field with an involutive anti-automorphism $\lambda \mapsto \lambda^*$. We say that a sesquilinear form $\langle\ \rangle\colon H \times H \to K$ on a K-vector space H is *orthomodular* if $\langle\ \rangle$ admits a Nakayama automorphism ν, and

$$U^{\perp} \oplus U^{\perp\perp} = H \tag{44}$$

holds for all subspaces U of H. Accordingly, we call $H = (H; \langle\ \rangle)$ an *orthomodular space*.

By Eqs. (43), the sesquilinear form of an orthomodular space H is *anisotropic*, that is, $\langle x, x\rangle = 0$ implies $x = 0$ for all $x \in H$. Indeed, a vector $x \in H$ with $\langle x, x\rangle = 0$ satisfies $x \in {}^{\perp}(Kx) \cap ({}^{\perp}Kx)^{\perp} = \nu^{-1}(Kx)^{\perp} \cap \nu^{-1}(Kx)^{\perp\perp} = 0$. In particular, $\langle\ \rangle$ is non-degenerate. For an orthomodular space H, we introduce the orthogonality relation

$$U \perp V \ :\Longleftrightarrow\ U \subset {}^{\perp}V \iff U^{\perp} \supset V.$$

If H is *hermitian* ($\langle x, y\rangle^* = \langle y, x\rangle$ for all $x, y \in H$), the left and right orthogonals coincide. By Solèr's theorem [24], the skew-field of scalars of a hermitian orthomodular space with an infinite orthonormal sequence must be either $\mathbb{R}$ or $\mathbb{C}$, or the skew-field $\mathbb{H}$ of quaternions.

Proposition 10. *Let H be an orthomodular space. Then $X(H)$ is a bounded $\perp$-lattice.*

Proof. The axioms (0)–(2′) are easily verified. Let U, V be closed subspaces of H with $U \subset V$. Then $U \perp (U^{\perp} \cap V)$ and $U + (U^{\perp} \cap V) = (U + U^{\perp}) \cap V = V$. Since $U = ({}^{\perp}U)^{\perp}$, Definition 7 gives $U \cap U^{\perp} = 0$. Thus (3) holds. Axiom (3′) follows by symmetry. □

Remarks. 1. For example, every Frobenius algebra [32,33] is an orthomodular space. By Proposition 10, the ideals form a $\perp$-lattice. More generally, Jans [26] considers generalized Frobenius algebras A with an associative bilinear form and a Nakayama automorphism. So the ideals of A form a $\perp$-lattice if they satisfy Eq. (44).

2. In case that H is a finite dimensional vector space over a field K with an anisotropic symmetric bilinear form $\langle\ \rangle\colon H \times H \to K$, the set $X(H)$ of all subspaces of H is a modular ortho-lattice [27]. The first author proved [17] that the structure group $G(X(H))$ is isomorphic to the corresponding *pure para-unitary group*, that is, the kernel of the map $PU(H) \twoheadrightarrow U(H)$, where $U(H)$ is the automorphism group of the bilinear form, and $PU(H)$ is the group of *para-unitary* matrices over $K[t, t^{-1}]$ (see [17] for details).

For $H := \mathbb{R}^n$, equipped with the standard Euclidean inner product $\sigma(v, w) := v^{\top} w$,

$$\mathrm{PU}(H) := \left\{ M(t) \in K\left[t, t^{-1}\right]^{n\times n} : M(t)^{\top} M(t^{-1}) = 1 \right\},$$

where $M(t^{-1})$ is obtained by replacing t by t^{-1}. The pure para-unitary group is the subgroup

$$\mathrm{PPU}(H) = \{M(t) \in \mathrm{PU}(H) : M(1) = 1\}.$$

It has a right-invariant lattice-order with negative cone

$$\mathrm{PPU}(H)^- = \mathrm{PPU}(H) \cap k\left[t^{-1}\right]^{n\times n}.$$

Now consider the lattice $X_f(H)$ of finite dimensional subspaces of an orthomodular space H. Let $X_1(H)$ be the set of subspaces of dimension at most 1. Thus $\mathbb{P}(H) := X_1(H) \smallsetminus \{0\}$ consists of the points of the projective space associated to H.

Proposition 11. *Let H be an orthomodular space. Then $X_f(H)$ is a positive OL-subalgebra of $X(H)$ with $X_1(H)$ as an L^+-subalgebra.*

Proof. Let $U \in X(H)$ be finite dimensional, with a basis $\{x_1, \ldots, x_n\}$. Then the $f_i := \langle x_i, -\rangle^*$ are linear forms on H with $\operatorname{Ker} f_i = (Kx_i)^\perp$, and $U^\perp = (Kx_1)^\perp \cap \cdots \cap (Kx_n)^\perp$. Hence $\dim H/U^\perp \leqslant n$. As the elements of ${}^\perp(U^\perp)$ are linear forms on $H/U^\perp$, it follows that $U = {}^\perp(U^\perp) \in X(H)$. Thus $X_f(H)$ is a modular sublattice of $X(H)$. Therefore, X_f is a positive OL-subalgebra of $X(H)$ with

$$V \smallsetminus U = (U+V) \cap U^\perp$$

for $U, V \in X_f(H)$. If U, V are one-dimensional with $U \neq V$, then U is of codimension 1 in $U+V$. Since $[U, H] \cong [0, U^\perp]$, this implies that $(U+V) \cap U^\perp$ is one-dimensional. If $U = V$, then $V \smallsetminus U = U \cap U^\perp = 0$. Thus $X_1(H)$ is an L-subalgebra. □

We will show that $X(H)$ is completely determined by the L-algebra $X_1(H)$. Let X be a set with an irreflexive binary relation $\perp$. For subsets $A, B \subset X$, we write $A \perp B$ if $x \perp y$ for all $x \in A$ and $y \in B$. If $x \in X$ and $A \subset X$, we abbreviate $\{x\}\perp A$ and $A \perp \{x\}$ by $x \perp A$ and $A \perp x$, respectively. We define the *left* and *right orthogonal* of a subset $A \subset X$ to be

$${}^\perp A := \{x \in X \mid x \perp A\}, \qquad A^\perp := \{x \in X \mid A \perp x\}.$$

Accordingly, we write ${}^\perp x := {}^\perp\{x\}$ and $x^\perp := \{x\}^\perp$. For brevity, let us call $(X; \perp)$ a *$\perp$-set*. In particular, every subset of $(X; \perp)$ is a $\perp$-set.

Definition 8. We call a $\perp$-set X *non-degenerate* if there exists a permutation ν of X such that ${}^\perp(x^\perp) = ({}^\perp x)^\perp = x$ and $x^\perp = {}^\perp\nu(x)$ holds for all $x \in X$. The subsets of the form $A^\perp$ will be called *closed*.

Thus any subset A of a non-degenerate $\perp$-set X satisfies

$$A^\perp = {}^\perp\nu(A),$$

and A is closed if and only if $({}^{\perp}A)^{\perp} = A$ or equivalently, ${}^{\perp}(A^{\perp}) = A$. As any intersection of closed subsets is closed, the closed subsets of X form an atomistic complete lattice $L(X)$. If no confusion is possible, we also write x for the atoms $\{x\}$ of $L(X)$. We call them *points*. Since

$$A \subset B \iff B^{\perp} \subset A^{\perp} \tag{45}$$

holds for closed subsets A, B of X, the closed sets $x^{\perp}$ with $x \in X$ are maximal among the proper closed subsets. We call them *co-points*.

Proposition 12. *Let X be a non-degenerate $\perp$-set. The following are equivalent:*

(a) *For closed sets $A \subset B$ of X, the set A is closed in B.*
(b) *For closed subsets $A \subset B$ in X, we have $A = {}^{\perp}(A^{\perp} \cap B) \cap B = ({}^{\perp}A \cap B)^{\perp} \cap B$.*
(c) *$L(X)$ is a $\perp$-lattice.*

Proof. The axioms (0)–(2′) follow by (45). The implications (c) ⇒ (a) ⇒ (b) are trivial.

(b) ⇒ (c): Since $B^{\perp} \subset A^{\perp}$, we have $B^{\perp} = (B \cap A^{\perp})^{\perp} \cap A^{\perp}$, that is, $B = (B \cap A^{\perp}) \vee A$. Thus (3) holds. By symmetry, this implies (c). □

Note that the equations in (b) are equivalent to the orthomodular law (24). For a non-degenerate $\perp$-set X, we write X_1 for the set of all points together with 0. By X_f we denote the set of all elements of the form $x_1 \vee \cdots \vee x_n$ with $x_i \in X_1$.

Proposition 13. *Let X be a non-degenerate $\perp$-set satisfying the equivalent properties of Proposition 12. Then X_f is a modular sublattice of $L(X)$ if and only if X_1 is an L^+-subalgebra of $L(X)$. If these equivalent conditions hold, $X_f = C(X)$.*

Proof. Assume that X_f is a modular sublattice of $L(X)$. For distinct $x, y \in X$, Proposition 12 implies that $y \smallsetminus x := (x \vee y) \cap x^{\perp}$ is a complement of x in $[0, x \vee y]$. Hence $y \smallsetminus x \in X_1$. So X_1 is an L^+-subalgebra of $L(X)$.

Conversely, let X_1 be $\smallsetminus$-closed. For $a \in X_f$ and $x, y \in X_1$, Eq. (17) gives $y \smallsetminus (x \curlyvee a) = (y \smallsetminus a) \smallsetminus (x \smallsetminus a)$. Thus, by induction, $x \smallsetminus a \in X_1$ for all $x \in X_1$ and $a \in X_f$. Since $x \smallsetminus a$ is a complement of a in the interval $[0, a \vee x]$, the length of $[a, a \vee x]$ is 1 in X. Hence X_f is an upper semimodular semilattice. So the Jordan–Dedekind chain condition holds [36], which implies that the intervals $[0, a]$ in X with $a \in X_f$ belong to X_f. Hence X_f is a lattice. By (45), the intervals $[0, a]$ in X_f are lower semimodular. Hence X_f is modular. By Proposition 2, $L(X)$ is a $\curlyvee$-closed L^+-algebra. So $C(X_1) = X_f$. □

Definition 9. We define a *quantum set* to be a non-degenerate $\perp$-set X for which $L(X)$ is a $\perp$-lattice with X_1 as an L^+-subalgebra.

For example, every set X with the trivial orthogonality relation

$$x \perp y \ :\Longleftrightarrow\ x \neq y$$

is a quantum set. The corresponding L^+-algebra X_1 is given by

$$x \smallsetminus y := \begin{cases} 0 & \text{for } x = y \\ x & \text{for } x \neq y. \end{cases}$$

Proposition 14. *Let H be an orthomodular space. Then $\mathbb{P}(H)$ is a quantum set.*

Proof. By Proposition 10, $X(H)$ is a bounded $\perp$-lattice. Its Nakayama automorphism ν satisfies $\nu(U) = U^{\perp\perp}$ and induces a permutation on the L^+-algebra $X_1(H)$. Hence $\mathbb{P}(H)$ is a non-degenerate $\perp$-set. Since $X_f(H)$ is modular, Proposition 13 implies that $\mathbb{P}(H)$ is a quantum set. □

In accordance with [40], Definition 8, we call a (positive) L-algebra X *discrete* if $X \smallsetminus \{1\}$ is an antichain. For example, the L^+-algebra $X_1(H)$ of an orthomodular space H or the L^+-algebra X_1 of a quantum set X is discrete.

Proposition 15. *A quantum set X is uniquely determined by the L^+-algebra X_1.*

Proof. For $x, y \in X$, we have $y \smallsetminus x = (x \vee y) \cap x^\perp$. Hence $y \smallsetminus x = y \Leftrightarrow y \subset x^\perp \Leftrightarrow x \perp y$. Thus X is determined by X_1. □

Therefore, quantum sets in the sense of Definition 9 are equivalent to a special class of L^+-algebras or $\perp$-lattices. The category of quantum sets and its connection with classical set theory will be studied in a forthcoming article.

We conclude with a characterization of Garside groups arising from OL-algebras. A *quasi-Garside monoid* [15] is a left and right cancellative monoid M which admits a function $\lambda\colon M \to \mathbb{N}$, non-zero on $M \smallsetminus \{1\}$, satisfying $\lambda(ab) \geqslant \lambda(a) + \lambda(b)$, such that any pair of elements has a left and right lcm and left and right gcd. Furthermore, it is assumed that there is a *Garside element*, an element Δ for which the left and right divisors form the same set S which generates M. If S is finite, M is said to be *Garside*. The group of left fractions of a Garside monoid is said to be a *Garside group*.

Equivalently, a quasi-Garside group can be defined to be a right ℓ-group G with a strong order unit Δ such that G^+ is *bounded atomic*, which means that every $a \in G^+$ is a finite product $a = x_1 \cdots x_n$ of minimal $x_i > 1$ (atoms) such that n is bounded for each a. The following result extends [40], Theorem 3, to a wider class of Garside groups.

Proposition 16. *The structure group $G(X)$ of an OL-algebra X is a Garside group if and only if X is finite.*

Proof. Assume that X is finite. Then X is bounded, hence an interval $[u^{-1}, 1]$ in $S(X)$ with a strong order unit u. Let m be the maximal length of a chain in X. So there is no representation $s^{-1} = x_1 \cdots x_n$ with $x_i < 1$ and $n > m$. For any $a \in S(X)$ there exists an integer $n > 0$ with $s^{-n} \leqslant a$. Thus $sa \geqslant a \vee s^{1-n}$, which shows that the length of the interval $[a, a \vee s^{1-n}]$ is at most m. By induction, we infer that $S(X)$ is bounded atomic. Hence $G(X)$ is a Garside group. The converse is trivial. $\square$

As there are no anisotropic bilinear forms of rank $\geqslant 3$ over a finite field ([46], I, Chap. IV), most of the Garside groups of Proposition 16 do not arise from an orthomodular space. Those with symmetric orthogonality can be constructed via Greechie diagrams [23]. The simplest example with a non-symmetric orthogonality relation is given by the bilinear form

$$\langle x, y \rangle := x_1 y_1 + x_1 y_2 + x_2 y_2$$

in $\mathbb{F}_2^2$, where $\binom{1}{0} \perp \binom{0}{1} \perp \binom{1}{1} \perp \binom{1}{0}$.

A sesquilinear example is obtained from the bilinear form

$$\langle x, y \rangle := x_1 y_1 + \alpha x_1 y_2 + x_2 y_2$$

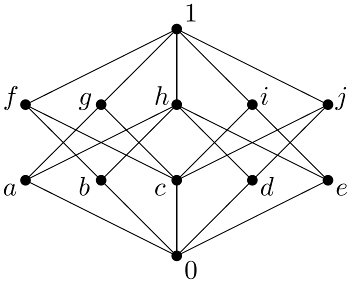
on $\mathbb{F}_4^2$, where the field $\mathbb{F}_4 = \mathbb{F}_2(\alpha)$ with $\alpha^2 = \alpha + 1$ is equipped with the Frobenius involution $\alpha \mapsto \alpha^2$. Here the orthogonality relation gives the cycle

$$\binom{1}{0} \perp \binom{\alpha}{1} \perp \binom{1}{\alpha} \perp \binom{0}{1} \perp \binom{1}{1} \perp \binom{1}{0}.$$

A wide class of Garside groups consists of the structure groups of finite quantum sets. For example, the set $X = \{a, b, c, d, e\}$ with $a^\perp = \{c, e\}$, $b^\perp = \{c, d\}$, $c^\perp = \{a, b, d, e\}$, $d^\perp = \{a, c\}$, and $e^\perp = \{b, c\}$, is a quantum set with permutation $\nu = (ab)(de)$ and L^+-algebra X_1 given by the following table:

$\diagdown$	a	b	c	d	e
a	0	d	a	a	b
b	e	0	b	a	b
c	c	c	0	c	c
d	e	d	d	0	b
e	e	d	e	a	0

The corresponding $\perp$-lattice $L(X)$ is



Since X_f is modular for any quantum set X, the structure group $G(X)$ is a modular Garside group. By [40], Theorem 5, the corresponding discrete L^+-algebra X_1 is characterized by the property

$$x \setminus y = y \setminus x \implies x = y$$

for all non-zero $x, y \in X_1$. Quantum sets thus correspond to a subclass of these L^+-algebras. For example, the L^+-algebra $X = \{0, 1, 2\}$ with $1 \setminus 2 = 2$ and $2 \setminus 1 = 1$ has no orthogonality relation between 1 and 2. So it does not belong to a quantum set.

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