



Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Degree one Milnor K -invariants of groups of multiplicative type

Alexander Wertheim

Department of Mathematics, University of California, Los Angeles, CA
90095-1555, USA



ARTICLE INFO

Article history:

Received 22 June 2020

Available online 17 March 2021

Communicated by V. Srinivas

Keywords:

Cohomological invariants

Algebraic groups

ABSTRACT

Let G be a commutative affine algebraic group over a field F , and let $H: \mathbf{Fields}_F \rightarrow \mathbf{AbGrps}$ be a functor. A (homomorphic) H -invariant of G is a natural transformation $\text{Tors}(-, G) \rightarrow H$, where $\text{Tors}(-, G)$ is the functor $\mathbf{Fields}_F \rightarrow \mathbf{AbGrps}$ taking a field extension L/F to the group of isomorphism classes of G_L -torsors over $\text{Spec}(L)$. The goal of this paper is to compute the group $\text{Inv}_{\text{hom}}^1(G, H)$ of H -invariants of G when G is a group of multiplicative type, and H is the functor taking a field extension L/F to $L^\times \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$.

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1. Introduction

Let G be an affine algebraic group over a field F (of arbitrary characteristic), and let \mathbf{Fields}_F denote the category of field extensions of F . Let

$$H: \mathbf{Fields}_F \longrightarrow \mathbf{AbGrps}$$

be a functor. In [1], an H -invariant of G is defined to be a natural transformation of set-valued functors

E-mail address: awertheim@math.ucla.edu.

$$I: \text{Tors}(-, G) \longrightarrow H$$

where $\text{Tors}(-, G)$ is the functor from \mathbf{Fields}_F to \mathbf{Sets} taking a field extension L/F to $\text{Tors}(L, G_L)$, the set of isomorphism classes of G_L -torsors over $\text{Spec}(L)$. Invariants were first introduced by Serre in [10, Section 6], where he defined invariants in the case when H is a (Galois) cohomological functor.

There is another type of invariant that one may consider, however. Namely, since any affine group scheme over F may be viewed as a functor from F -algebras to groups, we define a **type-zero** H -invariant of G to be a natural transformation of set-valued functors $G \rightarrow H$, where by G we mean the restriction of G to \mathbf{Fields}_F . We denote the group of type-zero H -invariants of G by $\text{Inv}^0(G, H)$. To distinguish the invariants introduced in the previous paragraph from type-zero invariants, we will call them **type-one** invariants, and we denote the group of type-one H -invariants of G by $\text{Inv}^1(G, H)$.

In this paper, we study type-one invariants when G is an algebraic group of multiplicative type, i.e. when G is a twisted form of a diagonalizable group. We note that every torus is a group of multiplicative type; in general, groups of multiplicative type need not be smooth or connected. We consider a slightly more restrictive class of invariants than those introduced above, however. If G is *commutative*, then for any affine F -scheme X , the pointed set $\text{Tors}(X, G)$ can be given the structure of an abelian group. Since groups of multiplicative type are commutative, we may view the functor $\text{Tors}(-, G)$ as a functor from \mathbf{Fields}_F to \mathbf{AbGrps} . Accordingly, we will focus our attention on invariants which are morphisms of *group-valued* functors. We will call such invariants **homomorphic**, and we denote the subgroup of homomorphic type-one H -invariants of G by $\text{Inv}_{\text{hom}}^1(G, H)$. Likewise, one may consider homomorphic type-zero invariants, which we will similarly denote $\text{Inv}_{\text{hom}}^0(G, H)$.

The goal of this paper is to determine $\text{Inv}_{\text{hom}}^1(G, K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})$, the group of (type-one) **degree one Milnor K -invariants** of G , where K_i^M denotes the functor sending a field extension L/F to the i^{th} Milnor K -group of L (see [6]); we recall that $K_0^M(L) = \mathbb{Z}$, $K_1^M(L) = L^\times$. For any $n \in \mathbb{N}$, let K_1^M/n denote the functor $K_1^M \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$. The embedding $\mathbb{Z}/n\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}$ induces a morphism of functors $\iota_n: K_1^M/n \rightarrow K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$; likewise, if n and m are positive integers such that n divides m , then the embedding $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ sending $[1]_n$ to $[m/n]_m$ induces a morphism of functors $\beta_{n,m}: K_1^M/n \rightarrow K_1^M/m$. One may check that the collection of functors $\{K_n^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}\}_{n \in \mathbb{N}}$ defines the data of a cycle module in the sense of Rost (see [8]), as does $\{K_n^M \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}\}_{n \in \mathbb{N}}$ for any $m \in \mathbb{N}$. The functors $K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ and K_1^M/m respectively form the first graded components of these cycle modules.

As we will explain in Section 3.1, a classical Kummer theory argument shows that there is an isomorphism $\Sigma_n: K_1^M/n \rightarrow \text{Tors}(-, \mu_{n,F})$ of group-valued functors. On the other hand, any $\chi \in \text{Hom}(G, \mu_{n,F}) = G^*[n]$ gives rise to a morphism of group-valued functors $\text{Tors}_*(\chi): \text{Tors}(-, G) \rightarrow \text{Tors}(-, \mu_{n,F})$. Thus, we may associate to any element $\chi \in G^*[n]$ a homomorphic invariant $I_\chi \in \text{Inv}_{\text{hom}}^1(G, K_1^M/n)$ which is the composition of $\text{Tors}_*(\chi)$ with Σ_n^{-1} . This leads us to our first main theorem.

Theorem A (5.4). *Let G be a group of multiplicative type over a field F . The map $\Phi(G, n): G^*[n] \rightarrow \text{Inv}_{\text{hom}}^1(G, K_1^M/n)$ sending χ to I_χ is a group isomorphism.*

The composition of any such I_χ with ι_n produces an element of $\text{Inv}_{\text{hom}}^1(G, K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})$, and so defines a group homomorphism $\tilde{\Phi}(G, n): G^*[n] \rightarrow \text{Inv}_{\text{hom}}^1(G, K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})$ for each $n \in \mathbb{N}$. Passing to the colimit as n varies, we obtain a universally induced group morphism $\Phi(G): G_{\text{tors}}^* \rightarrow \text{Inv}_{\text{hom}}^1(G, K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})$.

Theorem B (5.6). *The map $\Phi(G): G_{\text{tors}}^* \rightarrow \text{Inv}_{\text{hom}}^1(G, K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})$ is a group isomorphism.*

Our proofs of Theorems 5.4 and 5.6 depend critically on the determination of homomorphic type-zero invariants for tori with values in K_1^M/n for each $n \in \mathbb{N}$. The following result was proven by Merkurjev (cf. [3, Corollary 3.7]) in the case when the characteristic of F does not divide n ; we give a proof in this paper which holds independent of the characteristic of F .

Theorem C (4.5). *If T is an algebraic torus, then $\text{Inv}_{\text{hom}}^0(T, K_1^M/n) \cong H^0(F, T_{\text{sep}}^*/(T_{\text{sep}}^*)^n)$.*

The results we have obtained above follow a rich history of work on cohomological invariants: here are a few related recent examples. In [12], Totaro computed all mod p cohomological invariants for many important affine group schemes in characteristic p ; in particular, under the assumption that $\text{char}(F) = p > 0$, Totaro independently computed $\text{Inv}^1(G, K_1^M/p)$ for *any* affine group scheme ([12, Theorem 15.2]). The computation of invariants for smooth linear algebraic groups with values in $H^2(-, \mathbb{Q}/\mathbb{Z}(1))$ was carried out by Alexandre Lourdeaux in [2].

1.1. Acknowledgments

I would like to express my gratitude to my advisor, Alexander Merkurjev, for his advice, encouragement, and many helpful meetings. I am also grateful to Bar Roytman, David Hemminger, Will Baker, and Burt Totaro for helpful conversations.

1.2. Notation and conventions

Throughout, F denotes a fixed base field of arbitrary characteristic, and F_{sep} denotes a fixed separable closure. We put $\Gamma = \text{Gal}(F_{\text{sep}}/F)$. If G is a group scheme over F , we write G_{sep} to denote the base change of G to F_{sep} , and G^* to denote the character group of G . For an abelian group A and a positive integer n , we write $A[n]$ to denote the subgroup of n -torsion elements of A . All group schemes are affine unless otherwise indicated. For any morphism $f: X \rightarrow Y$ of affine F -schemes, the dual map of F -algebras is denoted

$f^\#: F[Y] \rightarrow F[X]$. For any group scheme G over F and any F -algebra R , we write ε_R to denote the identity element of $G(R)$. If $\varphi: Q \rightarrow Q'$ is a morphism of commutative F -group schemes, we write Q^φ to denote the image of the embedding $Q \rightarrow Q \times Q'$ induced by Id_Q and the composition of φ with the inversion map $Q' \rightarrow Q'$. For an F -scheme X , we write $\text{Tors}_*(\varphi)(X)$ to denote the morphism $\text{Tors}(X, Q) \rightarrow \text{Tors}(X, Q')$ induced by φ . Likewise, if $f: Y \rightarrow X$ is a morphism of F -schemes, we write $\text{Tors}^*(f)(Q)$ to denote the pullback morphism $\text{Tors}(X, Q) \rightarrow \text{Tors}(Y, Q)$. All torsors are considered with respect to the fppf topology. Finally, we take all reductive group schemes to be connected by convention.

2. An outline of the argument

In this section, we give a structural overview of our argument.

2.1. Resolution by tori

Recall that a group scheme G over F is said to be **diagonalizable** if the natural embedding $G^* \rightarrow F[G]^\times$ induces an isomorphism of Hopf F -algebras $F\langle G^* \rangle \rightarrow F[G]$, where $F\langle G^* \rangle$ denotes the group algebra of G^* over F . As noted in the introduction, a group scheme G over F is a **group of multiplicative type** if G_{sep} is diagonalizable over F_{sep} . The functors

$$G \longmapsto G_{\text{sep}}^*, \quad M \longmapsto \text{Spec}((F_{\text{sep}}\langle M \rangle)^\Gamma)$$

define a short exact sequence-preserving equivalence between the category of (algebraic) groups of multiplicative type over F and the category of (finitely generated) Γ -modules ([5, Theorem 12.23]). Under this equivalence, the full subcategory of diagonalizable F -group schemes is equivalent to the subcategory of Γ -modules with trivial Γ -action.

When G is an algebraic group of multiplicative type, G may be embedded in a quasplit torus P such that every G_L torsor over a field L/F is the pullback of the G -torsor $P \rightarrow P/G$ along an L -point of P/G . Indeed, since G_{sep}^* is finitely generated, it admits a surjective morphism of Γ -modules $W \rightarrow G_{\text{sep}}^*$ from a permutation Γ -module W . If S denotes the kernel of this map, then let P, T be the groups of multiplicative type respectively associated to W, S . Note that P is a quasplit torus, T is a torus, and the exact sequence

$$1 \rightarrow S \rightarrow W \rightarrow G_{\text{sep}}^* \rightarrow 1$$

of Γ -modules yields an exact sequence

$$1 \rightarrow G \xrightarrow{f} P \xrightarrow{g} T \rightarrow 1 \tag{2.1}$$

of F -group schemes. We will call such an exact sequence (2.1) a **resolution of G by tori**.

For every field extension L/F , the exact sequence on points $1 \rightarrow G(L) \rightarrow P(L) \rightarrow T(L)$ may be continued as follows. Let $\rho(L): T(L) \rightarrow \text{Tors}(L, G_L)$ be the group homomorphism sending a point $\alpha \in T(L)$ to the pullback of the G -torsor $P \rightarrow T$ along α . One may check that the sequence

$$1 \rightarrow G(L) \xrightarrow{f(L)} P(L) \xrightarrow{g(L)} T(L) \xrightarrow{\rho(L)} \text{Tors}(L, G_L) \xrightarrow{\text{Tors}_*(f_L)} \text{Tors}(L, P_L) \quad (2.2)$$

is exact; we note that this does not depend on the fact that G, P, T are of multiplicative type, and can be proven for any exact sequence of commutative group schemes. Since P_L is a quas-split torus, every P_L -torsor over $\text{Spec}(L)$ is trivial. Therefore, the map $\rho(L): T(L) \rightarrow \text{Tors}(L, G_L)$ is surjective.

The surjectivity of $\rho(L)$ allows us to relate type-one invariants for G to type-zero invariants for tori, which are well understood for certain functors H . As L varies over all field extensions of F , the morphisms $\rho(L)$ define a morphism of functors $\rho: T \rightarrow \text{Tors}(-, G)$, which gives rise to a map $\text{Inv}(\rho, H): \text{Inv}_{\text{hom}}^1(G, H) \rightarrow \text{Inv}_{\text{hom}}^0(T, H)$ given by composition with ρ . Likewise, the group homomorphism $g: P \rightarrow T$ is a natural transformation of group-valued functors, and so induces a map $\text{Inv}(g, H): \text{Inv}_{\text{hom}}^0(T, H) \rightarrow \text{Inv}_{\text{hom}}^0(P, H)$ given by composition with g . The exactness of (2.2) shows that the resulting sequence

$$1 \rightarrow \text{Inv}_{\text{hom}}^1(G, H) \xrightarrow{\text{Inv}(\rho, H)} \text{Inv}_{\text{hom}}^0(T, H) \xrightarrow{\text{Inv}(g, H)} \text{Inv}_{\text{hom}}^0(P, H) \quad (2.3)$$

is exact. To describe $\text{Inv}_{\text{hom}}^1(G, H)$, it therefore suffices to determine the image of $\text{Inv}(\rho, H)$ in $\text{Inv}_{\text{hom}}^0(T, H)$.

2.2. The argument

Fix a positive integer n , let G, P, T be as in the exact sequence (2.1), and let $H = K_1^M/n$. For any group scheme Q over F , we say that a class $V \in \text{Tors}(Q, G)$ is **normalized** if the pullback of V along $\varepsilon_F \in Q(F)$ represents the trivial class in $\text{Tors}(F, G)$. Let $\text{Tors}_{\text{nm}}(Q, G)$ denote the subgroup of normalized G -torsors over Q .

Consider the map $v_n(G): G^*[n] \rightarrow \text{Tors}_{\text{nm}}(T, \mu_{n,F})$ which sends a character $\chi \in G^*[n]$ to $(\text{Tors}_*(\chi)(T))(P \rightarrow T)$. We note that $v_n(G)$ is a group homomorphism. Indeed, for any F -scheme X , the map $\text{Tors}(X, G) \times \text{Tors}(X, G) \rightarrow \text{Tors}(X, G \times G)$ sending a pair of representatives $E_1 \rightarrow X, E_2 \rightarrow X$ to the universal map $E_1 \times E_2 \rightarrow X$ is a group isomorphism. If $\Delta_G: G \rightarrow G \times G$ denotes the diagonal map, and $m_G: G \times G \rightarrow G$ denotes the group multiplication, then up to the preceding identification, $\text{Tors}_*(m_G)(X)$ is the group operation, and $\text{Tors}_*(\Delta_G)(X)$ is the diagonal embedding. Hence, if $\chi, \chi' \in G^*[n]$, then we have $\text{Tors}_*(\chi\chi')(X) = \text{Tors}_*(\chi)(X) + \text{Tors}_*(\chi')(X)$, since $\chi\chi'$ factors as $m_{\mu_{n,F}} \circ (\chi \times \chi') \circ \Delta_G$. This argument also explains why $\Phi(G, n)$ is a group homomorphism.

Suppose we were armed with the following facts:

(1) The sequence

$$1 \rightarrow G^*[n] \xrightarrow{v_n(G)} \mathrm{Tors}_{\mathrm{nm}}(T, \mu_{n,F}) \xrightarrow{\mathrm{Tors}^*(g)(\mu_{n,F})} \mathrm{Tors}_{\mathrm{nm}}(P, \mu_{n,F}) \quad (2.4)$$

is exact.

(2) For any reductive group R over F , there is a group isomorphism

$$\tilde{\Lambda}_n(R): \mathrm{Tors}_{\mathrm{nm}}(R, \mu_{n,F}) \rightarrow \mathrm{Inv}_{\mathrm{hom}}^0(R, K_1^M/n).$$

(3) The diagram

$$\begin{array}{ccccc} G^*[n] & \xrightarrow{v_n(G)} & \mathrm{Tors}_{\mathrm{nm}}(T, \mu_{n,F}) & \xrightarrow{\mathrm{Tors}^*(g)(\mu_{n,F})} & \mathrm{Tors}_{\mathrm{nm}}(P, \mu_{n,F}) \\ \downarrow \Phi(G,n) & & \downarrow \tilde{\Lambda}_n(T) & & \downarrow \tilde{\Lambda}_n(P) \\ \mathrm{Inv}_{\mathrm{hom}}^1(G, K_1^M/n) & \xrightarrow{\mathrm{Inv}(\rho, K_1^M/n)} & \mathrm{Inv}_{\mathrm{hom}}^0(T, K_1^M/n) & \xrightarrow{\mathrm{Inv}(g, K_1^M/n)} & \mathrm{Inv}_{\mathrm{hom}}^0(P, K_1^M/n) \end{array}$$

commutes.

If these three statements hold, then an easy diagram chase using the exactness of (2.4) and (2.3) shows that $\Phi(G, n)$ is an isomorphism. The remainder of this paper is dedicated to proving these three facts, and carefully explaining why the induced map $\Phi(G)$ is an isomorphism. The remaining sections are organized as follows.

Section 3 gives a thorough treatment of $\mu_{n,F}$ -torsors, laying the ground work for facts (1) and (2). We will provide a Galois theoretic-interpretation of the group $\mathrm{Tors}_{\mathrm{nm}}(T, \mu_{n,F})$ which will allow us to interpret sequence (2.4) as an exact sequence arising in Galois cohomology in section 5. We will also prove a pullback formula for $\mu_{n,F}$ -torsors over a reductive group R which shows that normalized $\mu_{n,F}$ -torsors over R give rise to homomorphic type-zero invariants of R .

Section 4 is devoted to constructing the map $\tilde{\Lambda}_n(R)$ for any reductive group R , and proving it is a group isomorphism. We also give a description of type-zero invariants for R with values in $K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$.

The final section 5 will prove facts (1) and (3), yielding Theorem A. As noted, we will then deduce Theorem B from Theorem A via a detailed examination of $\Phi(G)$.

3. $\mu_{n,F}$ -Torsors

Throughout this section, let n denote a fixed positive integer. As indicated in the previous section, an essential ingredient in the proof of Theorem 5.4 is a robust understanding $\mu_{n,F}$ -torsors over an F -scheme X . In this section, we recall several well-known characterizations of $\mu_{n,F}$ -torsors. Our main results are Theorems 3.9, 3.11 and 3.14. Theorem 3.9

explains that when G is a smooth, connected group, $\text{Tors}(G, \mu_{n,F})$ may be identified with the kernel of the divisor map $\partial_n(G): F(G)^\times / (F(G)^\times)^n \rightarrow \text{Div}(G)/n \text{Div}(G)$. Under the further assumption that G is reductive, Theorem 3.11 proves a formula relating the pullbacks of a class in $\text{Tors}(G, \mu_{n,F})$ along points $\alpha, \beta \in G(M)$ to its pullback along the product $\alpha\beta \in G(M)$, where M is a field extension of F . Theorem 3.14 computes the Galois fixed points of $\text{Tors}_{\text{nm}}(G_{\text{sep}}, \mu_{n,F_{\text{sep}}})$ when G is geometrically integral, G_{sep}^* is torsion-free, and G_{sep} has trivial divisor class group.

3.1. The group $\Psi(A, n)$

Definition 3.1. For any commutative ring A , let $\Psi(A, n)$ denote the set of equivalence classes of pairs (\mathcal{L}, φ) , where $\mathcal{L} \in \text{Pic}(A)[n]$, φ is an A -module isomorphism $\mathcal{L}^{\otimes n} \rightarrow A$, and two pairs $(\mathcal{L}, \varphi), (\mathcal{L}', \varphi')$ are equivalent if and only if there is an isomorphism of A -modules $\rho: \mathcal{L} \rightarrow \mathcal{L}'$ such that $\varphi' \circ \rho^{\otimes n} = \varphi$.

We record the following observations about $\Psi(A, n)$, which are straightforward to check:

- (1) The tensor product induces a group operation on $\Psi(A, n)$: one defines the product of classes $[(\mathcal{L}, \varphi)], [(\mathcal{L}', \varphi')] \in \Psi(A, n)$ to be $[(\mathcal{L} \otimes_A \mathcal{L}', \varphi \otimes_A \varphi')]$, where $\varphi \otimes_A \varphi'$ really refers to the composition

$$(\mathcal{L} \otimes_A \mathcal{L}')^{\otimes n} \xrightarrow{\sim} (\mathcal{L}^{\otimes n}) \otimes_A (\mathcal{L}')^{\otimes n} \xrightarrow{\varphi \otimes_A \varphi'} A \otimes_A A \xrightarrow{\sim} A.$$

The identity class is represented by the pair (A, m) , where $m: A^{\otimes n} \rightarrow A$ is the multiplication map, and the inverse of a class $[(\mathcal{L}, \varphi)]$ is given by $[(\mathcal{L}^*, (\varphi^{-1})^*)]$, where \mathcal{L}^* is the dual bundle to \mathcal{L} , and $(\varphi^{-1})^*$ is the composition

$$(\mathcal{L}^*)^{\otimes n} \xrightarrow{\sim} (\mathcal{L}^{\otimes n})^* \xrightarrow{(\varphi^{-1})^*} A^* \xrightarrow{\sim} A.$$

- (2) For any ring morphism $f: A \rightarrow B$, extension of scalars induces a group morphism $\Psi(-, n)(f): \Psi(A, n) \rightarrow \Psi(B, n)$ sending $[(\mathcal{L}, \varphi)]$ to $[(\mathcal{L} \otimes_A B, \varphi \otimes_A \text{Id}_B)]$, where $\varphi \otimes_A \text{Id}_B$ really denotes the composition

$$(\mathcal{L} \otimes_A B)^{\otimes n} \xrightarrow{\sim} \mathcal{L}^{\otimes n} \otimes_A B \xrightarrow{\varphi \otimes_A \text{Id}_B} A \otimes_A B \xrightarrow{\sim} B.$$

In this way, the association $A \mapsto \Psi(A, n)$ defines a functor $\Psi(-, n)$ from **CommRings** to **AbGrps**.

- (3) For any positive integer m with n dividing m , there is a morphism of functors $\omega_{n,m}: \Psi(-, n) \rightarrow \Psi(-, m)$ defined for a commutative ring A by $\omega_{n,m}(A)[(\mathcal{L}, \varphi)] = [(\mathcal{L}, \varphi^{\otimes m/n})]$, where by $\varphi^{m/n}$ we mean the composition of isomorphisms

$$\mathcal{L}^{\otimes m} \xrightarrow{\sim} (\mathcal{L}^{\otimes n})^{\otimes m/n} \xrightarrow{\varphi^{\otimes m/n}} A^{\otimes m/n} \xrightarrow{\sim} A.$$

There is a convenient way to produce elements of $\Psi(A, n)$ which can be described as follows. Fix an element $y \in A^\times$, and consider the A -algebra $R_y := A[X]/\langle X^n - y \rangle$; we denote the residue class of X in R_y by $y^{1/n}$. If \mathcal{L}_y denotes the free A -submodule of R_y generated by $y^{1/n}$, one immediately sees that the “multiplication” map $\varphi_y: \mathcal{L}_y^{\otimes n} \rightarrow A$ sending $x_1 y^{1/n} \otimes \cdots \otimes x_n y^{1/n}$ to $y x_1 x_2 \cdots x_n$ is an isomorphism of A -modules, and the pair $(\mathcal{L}_y, \varphi_y)$ represents a class in $\Psi(A, n)$.

One readily checks that the map $A^\times \rightarrow \Psi(A, n)$ sending $y \in A^\times$ to $[(\mathcal{L}_y, \varphi_y)]$ is a group homomorphism whose kernel is exactly $(A^\times)^n$, and so we obtain a well-defined injective group morphism $\Delta_n(A): A^\times/(A^\times)^n \rightarrow \Psi(A, n)$. Moreover, this collection of maps is functorial in A : in other words, if \mathcal{K}^n denotes the functor from **CommRings** to **AbGrps** sending a commutative ring A to $A^\times/(A^\times)^n$, the collection of maps $\Delta_n(A)$ as A varies defines a natural transformation $\Delta_n: \mathcal{K}^n \rightarrow \Psi(-, n)$.

On the other hand, for any commutative ring A , there is a well-defined surjective group homomorphism $\Theta_n(A): \Psi(A, n) \rightarrow \text{Pic}(A)[n]$ which sends a class $[(\mathcal{L}, \varphi)] \in \Psi(A, n)$ to $[\mathcal{L}]$, and the collection of such $\Theta_n(A)$ as A varies likewise determines a natural transformation $\Theta_n: \Psi(-, n) \rightarrow \text{Pic}(-)[n]$. The relationship between Δ_n and Θ_n is explained by the following proposition.

Proposition 3.2. *For any commutative ring A , the sequence*

$$1 \rightarrow A^\times/(A^\times)^n \xrightarrow{\Delta_n(A)} \Psi(A, n) \xrightarrow{\Theta_n(A)} \text{Pic}(A)[n] \rightarrow 0$$

is exact.

Proof. The inclusion $\text{Im}(\Delta_n(A)) \subset \ker(\Theta_n(A))$ is immediate, since \mathcal{L}_y is a free A -module for any $y \in A^\times$ by construction. Suppose that $(\mathcal{L}, \varphi) \in \ker(\Theta_n(A))$, i.e. that \mathcal{L} is free. Let $\psi: \mathcal{L} \rightarrow A$ be an isomorphism of A -modules. Consider the composition of isomorphisms

$$A \xrightarrow{\varphi^{-1}} \mathcal{L}^{\otimes n} \xrightarrow{\psi^{\otimes n}} A^{\otimes n} \xrightarrow{\sim} A.$$

Every A -module isomorphism $A \rightarrow A$ is given by multiplication by some invertible element of A , so the composition above is multiplication by x for some $x \in A^\times$. Put $y = x^{-1}$, and let $\alpha: A \rightarrow \mathcal{L}_y$ be the isomorphism sending a to $ay^{1/n}$. Then one easily checks that $\alpha \circ \psi$ is an isomorphism between (\mathcal{L}, φ) and $(\mathcal{L}_y, \varphi_y)$. \square

Corollary 3.3. *If $\text{Pic}(A)[n] = 0$, then $\Delta_n(A)$ is an isomorphism.* \square

Suppose now that A is an F -algebra. To any element $[(\mathcal{L}, \varphi)]$ of $\Psi(A, n)$, one may associate a $\mathbb{Z}/n\mathbb{Z}$ -graded A -algebra $\text{Tw}(\mathcal{L}, \varphi)$. As an A -module, we set

$$\text{Tw}(\mathcal{L}, \varphi) := A \oplus \mathcal{L} \oplus \mathcal{L}^{\otimes 2} \oplus \cdots \oplus \mathcal{L}^{\otimes n-1}.$$

The multiplicative structure on $\mathrm{Tw}(\mathcal{L}, \varphi)$ is induced by the isomorphisms $\mathcal{L}^{\otimes i} \otimes_A \mathcal{L}^{\otimes j} \rightarrow \mathcal{L}^{\otimes i+j}$ for $i+j < n$, and $\mathcal{L}^{\otimes i} \otimes_A \mathcal{L}^{\otimes j} \rightarrow \mathcal{L}^{\otimes n} \otimes_A \mathcal{L}^{\otimes (n-(i+j))} \xrightarrow{\varphi \otimes \mathrm{Id}} A \otimes_A \mathcal{L}^{\otimes (n-(i+j))} \rightarrow \mathcal{L}^{n-(i+j)}$ for $i+j \geq n$. Note that the inclusion morphism $A \rightarrow \mathrm{Tw}(\mathcal{L}, \varphi)$ is faithfully flat, because $\mathrm{Tw}(\mathcal{L}, \varphi)$ is finitely generated and projective as an A -module. In fact, the dual morphism $\mathrm{Spec}(\mathrm{Tw}(\mathcal{L}, \varphi)) \rightarrow \mathrm{Spec}(A)$ is a $\mu_{n,F}$ -torsor over $\mathrm{Spec}(A)$, and we can say yet more, as the next theorem explains. Let $\lambda_n(A): \Psi(A, n) \rightarrow \mathrm{Tors}(\mathrm{Spec}(A), \mu_{n,F})$ be the set map sending $[(\mathcal{L}, \varphi)]$ to the $\mu_{n,F}$ -torsor class represented by the map $\mathrm{Spec}(\mathrm{Tw}(\mathcal{L}, \varphi)) \rightarrow \mathrm{Spec}(A)$.

Theorem 3.4. *The map $\lambda_n(A)$ is a well-defined **group** isomorphism. Moreover, as A varies over all F -algebras, the collection of maps $\lambda_n(A)$ defines a natural isomorphism $\lambda_n: \Psi(-, n) \rightarrow \mathrm{Tors}(-, \mu_{n,F})$.*

Proof. See [11, Tag 03PK]. Alternatively, see [4, page 125]. \square

We note that for any $y \in A^\times$, the universal map $A[X]/\langle X^n - y \rangle \rightarrow \mathrm{Tw}(\mathcal{L}_y, \varphi_y)$ sending \bar{X} to $y^{1/n} \in \mathcal{L}_y$ is an isomorphism of $(\mathbb{Z}/n\mathbb{Z})$ -graded A -algebras. Hence, the composition $\lambda_n(A) \circ \Delta_n(A)$ takes $y \in A^\times$ to the class of the $\mu_{n,F}$ -torsor $\mathrm{Spec}(A[X]/\langle X^n - y \rangle) \rightarrow \mathrm{Spec}(A)$. We put $\Sigma_n := \lambda_n \circ \Delta_n$.

3.2. Divisors

When A is a domain, there is another description of $\Psi(A, n)$ in terms of divisors. Let K denote the field of fractions of A , and let $\mathrm{Cart}(A)$ denote the group of invertible fractional ideals of A . Likewise, if A is a Krull domain, let $\mathrm{Div}(A)$ be the free abelian group generated by the codimension 1 points of $\mathrm{Spec}(A)$. We write $\mathrm{div}(A): \mathrm{Cart}(A) \rightarrow \mathrm{Div}(A)$ to denote the usual valuation homomorphism which sends a fractional ideal I to the formal sum of its valuations at each height one prime of A . We let $\partial(A): K^\times \rightarrow \mathrm{Div}(A)$ denote the group morphism sending $x \in K^\times$ to $\mathrm{div}(xA)$, and we write $\mathrm{Cl}(A)$ to denote the cokernel of $\partial(A)$.

Consider the set $C(A, n)$ consisting of pairs (I, f) where $I \in \mathrm{Cart}(A)$, and $f \in K^\times$ such that $I^n = fA$. The binary operation on $C(A, n)$ defined by $(I, f) \cdot (I', f') = (II', ff')$ gives $C(A, n)$ the structure of a group with identity element $(A, 1)$. There is a group homomorphism $K^\times \rightarrow C(A, n)$ sending $x \in K^\times$ to (xA, x^n) , and we set $\mathrm{Cart}(A, n)$ to be the cokernel of this morphism.

If we further assume that A is a Krull domain, then there is an analogous construction $\mathrm{Div}(A, n)$. If $D(A, n)$ denotes the group of pairs (D, g) where $D \in \mathrm{Div}(A)$ and $g \in K^\times$ such that $\partial(A)(g) = nD$, then $\mathrm{Div}(A, n)$ is defined to be the cokernel of group homomorphism $K^\times \rightarrow D(A, n)$ which sends $x \in K^\times$ to the pair $(\partial(A)(x), x^n)$. One may check that the map $\mathrm{div}(A): \mathrm{Cart}(A) \rightarrow \mathrm{Div}(A)$ described above descends to a group morphism $\mathrm{div}_n(A): \mathrm{Cart}(A, n) \rightarrow \mathrm{Div}(A, n)$, and this map is an isomorphism if A is regular.

For any element $I \in \text{Cart}(A)$, the multiplication map $I^{\otimes n} \rightarrow I^n$ is an isomorphism of A -modules, since I is projective of rank 1. Given a pair (I, f) in $C(A, n)$, we may produce a pair (I, m_f) which represents a class in $\Psi(A, n)$, where m_f is the composition of A -isomorphisms $I^{\otimes n} \xrightarrow{\sim} I^n \xrightarrow{f^{-1}} A$. One may check that the resulting set map $\Omega(A): C(A, n) \rightarrow \Psi(A, n)$ sending a pair (I, f) to $[(I, m_f)]$ is a group homomorphism.

Proposition 3.5. *The morphism $\Omega(A): C(A, n) \rightarrow \Psi(A, n)$ is surjective, and the kernel is precisely the image of the group morphism $K^\times \rightarrow C(A, n)$ sending $x \in K^\times$ to (xA, x^n) . Therefore, $\Omega(A)$ descends to a well-defined group isomorphism $\Omega_n(A): \text{Cart}(A, n) \rightarrow \Psi(A, n)$.*

Proof. Let $m: A^{\otimes n} \rightarrow A$ denote the multiplication map. If $(I, f) \in C(A, n)$ belongs to $\ker(\Omega(A))$, then there is an isomorphism $\rho: A \rightarrow I$ such that $m_f \circ \rho^{\otimes n} = m$. Then I is principal, generated by $\rho(1) =: x \in K^\times$, and

$$1 = m(1 \otimes \cdots \otimes 1) = m_f(x \otimes \cdots \otimes x) = x^n/f,$$

so $x^n = f$, and $(I, f) = (xA, x^n)$. On the other hand, for any $x \in K^\times$, the class $[(xA, m_{x^n})]$ in $\Psi(A, n)$ is trivial, via the isomorphism $A \rightarrow xA$ sending 1 to x .

Now, let the pair (\mathcal{L}, φ) represent a class in $\Psi(A, n)$. Let $I \subset K$ denote the image of \mathcal{L} under the composition of the A -embedding $\mathcal{L} \rightarrow \mathcal{L} \otimes_A K$ with a fixed K -module isomorphism $\mathcal{L} \otimes_A K \rightarrow K$. After clearing denominators, we may assume that $I \subset A \subset K$, so that I is an ideal of A . Since \mathcal{L} is projective of rank 1, I is an invertible ideal of A . Let $\alpha: \mathcal{L} \rightarrow I$ denote our A -module isomorphism of \mathcal{L} onto I . If f denotes the image of 1 under the sequence of isomorphisms $A \xrightarrow{\varphi^{-1}} \mathcal{L}^{\otimes n} \xrightarrow{\alpha^{\otimes n}} I^{\otimes n} \xrightarrow{\sim} I^n$, then one sees that $I^n = fA$, and α is an isomorphism between (\mathcal{L}, φ) and (I, m_f) . \square

The above proof shows that every element of $\Psi(A, n)$ admits a representative of the form (I, m_f) where $I \subset A$ is an invertible fractional ideal of A satisfying $I^n = fA$ for some nonzero $f \in A$. We will call such a representative an **ideal representative** of a class in $\Psi(A, n)$.

Corollary 3.6. *Let A be a normal domain with field of fractions K , and let X be a class in $\Psi(A, n)$. Let M be a domain, and let $\alpha_1, \dots, \alpha_n: A \rightarrow M$ be ring morphisms. Then one can choose an ideal representative $(\tilde{I}, m_{\tilde{f}})$ for X such that $\alpha_i(\tilde{f}) \in M \setminus \{0\}$ for each $1 \leq i \leq n$.*

Proof. For each $1 \leq i \leq n$, put $\mathfrak{p}_i = \ker(\alpha_i) \in \text{Spec}(A)$. Let S be the multiplicative subset of A defined by $S = A \setminus \bigcup_{i=1}^n \mathfrak{p}_i$; then $B := S^{-1}A$ is a semi-local ring whose maximal ideals are a subset of $\{\mathfrak{p}_i B\}_{i=1}^n$. Let (I, m_f) be an ideal representative for X , and put $J = S^{-1}I$. Since J is a B -module of constant rank 1 and B is semi-local, J is a free B -module of rank 1, hence principal. Say J is generated by $0 \neq y/z \in B$. Since $I^n = fA$, we have $J^n = fB$, whence

$$\left(\frac{y}{z}\right)^n = f \cdot u$$

for some unit $u \in B^\times$. If $u = v/w$ for $v \in A, w \in S$, we must have $v \in S$ as well. Put $g = vz/y \in K^\times$, so that

$$v^{n-1}w = g^n f,$$

and let $\tilde{I} = gI$, $\tilde{f} = v^{n-1}w \in S \subset A$. Then $\tilde{I}^n = g^n(I^n) = g^n fA = v^{n-1}wA$, and the map $I \rightarrow \tilde{I}$ given by multiplication by g is an isomorphism between (I, m_f) and $(\tilde{I}, m_{\tilde{f}})$ in $\Psi(A, n)$. Moreover, $\tilde{f} \in S$, and so $\alpha_i(\tilde{f}) \in M \setminus \{0\}$ for each i . It remains to show that $\tilde{I} \subset A$. Let $x \in I$; then $(gx)^n \in \tilde{I}^n \subset A$. But then gx is a root of $X^n - (gx)^n \in A[X]$, and so is integral over A , and therefore belongs to A . \square

Notice that if M is a field, then $\text{Pic}(M)$ is trivial, so $\Delta_n(M)$ is an isomorphism by Corollary 3.3.

Proposition 3.7. *Let A be a normal domain, and let M be a field. Let $\alpha: A \rightarrow M$ be a ring morphism. Let $X \in \Psi(A, n)$, and let $(I, m_f) \in \Psi(A, n)$ be an ideal representative for X satisfying $\alpha(f) \neq 0$. Then $(\Delta_n(M)^{-1} \circ \Psi(-, n)(\alpha))(X) = [\alpha(f)^{-1}]$.*

Proof. Consider the morphism of M -vector spaces $\tau: I \otimes_A M \rightarrow \mathcal{L}_{\alpha(f)^{-1}}$ given on simple tensors by $\tau(x \otimes z) = \alpha(x)z\alpha(f)^{-1/n}$. Since $I \otimes_A M$ and $\mathcal{L}_{\alpha(f)^{-1}}$ are both M -vector spaces of dimension 1, the map τ is an isomorphism provided it is nonzero. Indeed, this is the case, since $f \in I$, and so $\tau(f \otimes 1) = \alpha(f)\alpha(f)^{-1/n}$ is nonzero. It is straightforward to check that τ is an isomorphism between $\Psi(-, n)(\alpha)(X)$ and $\mathcal{L}_{\alpha(f)^{-1}}$. \square

Corollary 3.8. *Let A be a normal domain with field of fractions K , let M be a field, and let $\alpha: A \rightarrow M$ be a morphism of rings. Let $x \in A^\times$, and let (I, m_f) be an ideal representative for $\Delta_n(A)(x)$. Then there is a nonzero element $y \in A$ such that $I = yA$ and $y^n/f = x$ in K , and $[\alpha(x)] = [\alpha(f)^{-1}]$ in $M^\times/(M^\times)^n$. We deduce $\mathcal{K}^n(\alpha) = \Delta_n(M)^{-1} \circ \Psi(-, n)(\alpha) \circ \Delta_n(A)$.*

Proof. Since $[(I, m_f)]$ and $[(\mathcal{L}_x, \varphi_x)]$ are equal as classes in $\Psi(A, n)$, there is an isomorphism of A -modules $\rho: \mathcal{L}_x \rightarrow I$ such that $m_f \circ \rho^{\otimes n} = \varphi_x$. As \mathcal{L}_x is free, I is a (nonzero) principal ideal, generated by $y := \rho(1 \cdot x^{1/n}) \in A$. We thus have

$$x = \varphi_x(x^{1/n} \otimes \cdots \otimes x^{1/n}) = m_f(\rho^n(x^{1/n} \otimes \cdots \otimes x^{1/n})) = m_f(y \otimes \cdots \otimes y) = y^n/f$$

as claimed. Moreover, since $xf = y^n$ and $\alpha(x), \alpha(f) \in M^\times$, this forces $\alpha(y) \in M^\times$, and so $[\alpha(x)] \cdot [\alpha(f)] = [\alpha(y)^n] = [1] \in M^\times/(M^\times)^n$. \square

Suppose A is a normal domain, let K be its field of fractions, and let $\xi: A \rightarrow K$ be the canonical localization map. Fix $X \in \Psi(A, n)$, and let (I, m_f) be an ideal rep-

representative for X . By Proposition 3.7, $\Delta_n(K)^{-1}(\Psi(-, n)(\xi)(X)) = [\xi(f)^{-1}] = [1/f]$. If $\partial_n(A): K^\times/(K^\times)^n \rightarrow \text{Div}(A)/n \text{Div}(A)$ denotes the map induced by $\partial(A)$, then

$$\partial_n(A)([1/f]) = -[\partial(A)(f)] = -[\text{div}(fA)] = -[n \text{div}(I)] = 0 \in \text{Div}(A)/n \text{Div}(A)$$

Hence, the map $\Delta_n(K)^{-1} \circ \Psi(-, n)(\xi)$ takes image in $\ker(\partial_n(A)) \subset K^\times/(K^\times)^n$. If we further assume A is regular, then Theorem 3.9 shows $\Delta_n(K)^{-1} \circ \Psi(-, n)(\xi)$ (viewed by abuse of notation as a map $\Psi(A, n) \rightarrow \ker(\partial_n(A))$) is an isomorphism.

Theorem 3.9. *Let A be a regular domain, let K be its field of fractions, and let $\xi: A \rightarrow K$ be the canonical localization map. Then the map $\Delta_n(K)^{-1} \circ \Psi(-, n)(\xi): \Psi(A, n) \rightarrow \ker(\partial_n(A))$ is an isomorphism.*

Proof. First, consider the morphism $\zeta(A): \ker(\partial_n(A)) \rightarrow \text{Cl}(A)[n]$ defined as follows: if $[x] \in K^\times/(K^\times)^n$ belongs to the kernel of $\partial_n(A)$, then $\partial(A)(x) \in n \text{Div}(A)$. Define $\zeta(A)$ by sending $[x]$ to $[D]$, where D satisfies $nD = \partial(A)(x)$; note that D must be unique, since $\text{Div}(A)$ is free. This map is well-defined, because if $x' = xy^n$ for $y \in K^\times$, then $\partial(A)(xy^n) = n(D + \partial(A)(y))$, and $[D] = [D + \partial(A)(y)]$ in $\text{Cl}(A)$. We claim that the sequence

$$1 \rightarrow A^\times/(A^\times)^n \xrightarrow{\mathcal{K}^n(\xi)} \ker(\partial_n(A)) \xrightarrow{\zeta(A)} \text{Cl}(A)[n] \rightarrow 0$$

is exact. Indeed, $\mathcal{K}^n(\xi)$ is an injection because A is integrally closed in K . Moreover, if $[D] \in \text{Cl}(A)[n]$, then $nD = \partial(A)(x)$ for some $x \in K^\times$, and so $\zeta(A)([x]) = [D]$. Hence, $\zeta(A)$ is surjective.

It remains to check exactness at $\ker(\partial_n(A))$. Clearly, $\text{Im}(\mathcal{K}^n(\xi)) \subset \ker(\zeta(A))$, so suppose that $[x] \in \ker(\zeta(A))$. Then $\partial(A)(x) = n\partial(A)(y) = \partial(A)(y^n)$ for some $y \in K^\times$, whence $x = y^n \cdot x'$ for some $x' \in A^\times$, and so $[x] = [x']$ in $K^\times/(K^\times)^n$.

Now, since A is regular, $\text{div}(A): \text{Cart}(A) \rightarrow \text{Div}(A)$ is an isomorphism, and so induces an isomorphism $\text{Pic}(A)[n] \rightarrow \text{Cl}(A)[n]$. Let $\nu(A): \text{Pic}(A)[n] \rightarrow \text{Cl}(A)[n]$ be the composition of this isomorphism with the inversion automorphism $\text{Cl}(A)[n] \rightarrow \text{Cl}(A)[n]$. I claim that $\zeta(A) \circ \Delta_n(K)^{-1} \circ \Psi(-, n)(\xi) = \nu(A) \circ \Theta_n(A)$. Indeed, let X be a class in $\Psi(A, n)$, and let (I, m_f) be an ideal representative for X . Then

$$(\nu(A) \circ \Theta_n(A))(X) = \nu(A)([I]) = -[\text{div}(I)].$$

On the other hand,

$$(\zeta(A) \circ \Delta_n(K)^{-1} \circ \Psi(-, n)(\xi))(X) = \zeta(A)([1/f])$$

by Proposition 3.7. But $I^n = fA$, so $\partial(A)(1/f) = -\text{div}(I^n) = -n \text{div}(I)$, and thus $\zeta(A)(1/f) = -[\text{div}(I)]$. By Proposition 3.8, $\mathcal{K}^n(\xi) = \Delta_n(K)^{-1} \circ \Psi(-, n)(\xi) \circ \Delta_n(A)$, so we have a commutative diagram of abelian groups

$$\begin{array}{ccccccc}
1 & \longrightarrow & A^\times / (A^\times)^n & \xrightarrow{\Delta_n(A)} & \Psi(A, n) & \xrightarrow{\Theta_n(A)} & \text{Pic}(A)[n] \longrightarrow 0 \\
& & \downarrow \text{Id}_{A^\times / (A^\times)^n} & & \downarrow \Delta_n(K)^{-1} \circ \Psi(-, n)(\xi) & & \downarrow \nu(A) \\
1 & \longrightarrow & A^\times / (A^\times)^n & \xrightarrow{\kappa^n(\xi)} & \ker(\partial_n(A)) & \xrightarrow{\zeta(A)} & \text{Cl}(A)[n] \longrightarrow 0
\end{array}$$

whose rows are exact. Since $\text{Id}_{A^\times / (A^\times)^n}$ and $\nu(A)$ are isomorphisms, $\Delta_n(K)^{-1} \circ \Psi(-, n)(\xi)$ must be an isomorphism as well. \square

3.3. Pulling back torsors along products of points

Definition 3.10. Let G be an algebraic group over a field F , and let $A = F[G]$. Let $\varepsilon_F \in G(F)$ denote the identity element. We say a class $X \in \Psi(A, n)$ is **normalized** if $X \in \ker(\Psi(-, n)(\varepsilon_F))$. We denote the subgroup of $\Psi(A, n)$ consisting of normalized elements by $\Psi_{\text{nm}}(A, n)$. Likewise, we set $\text{Cart}_{\text{nm}}(A, n) = \Omega_n(A)^{-1}(\Psi_{\text{nm}}(A, n))$, and if G is smooth, then we set $\text{Div}_{\text{nm}}(A, n) = \text{div}_n(A)(\text{Cart}_{\text{nm}}(A, n))$.

We note the following properties of the subgroup $\Psi_{\text{nm}}(A, n)$:

- (1) By Theorem 3.4, one sees that $\Psi_{\text{nm}}(A, n) = \lambda_n(A)^{-1}(\text{Tors}_{\text{nm}}(G, \mu_{n, F}))$.
- (2) The assignment $A \mapsto \Psi_{\text{nm}}(A, n)$ defines a functor from the category of Hopf F -algebras to **AbGrps**. Moreover, if M/F is a field extension, and $\alpha: A \rightarrow A_M$ denotes the canonical base change morphism, then the restriction of $\Psi(-, n)(\alpha)$ to $\Psi_{\text{nm}}(A, n)$ takes image in $\Psi_{\text{nm}}(A_M, n)$.
- (3) If G is a smooth, connected group, then $\Delta_n(A)^{-1}(\Psi_{\text{nm}}(A, n)) = G^*/(G^*)^n \subset A^\times / (A^\times)^n$. This follows from Rosenlicht's theorem ([7, Theorem 3]).

Normalized elements play a key role in the following situation. Let M/F be a field extension, and fix a class $X \in \Psi(A, n)$. Consider the map $G(M) \rightarrow \Psi(M, n)$ which sends $\alpha \in G(M)$ to $\Psi(-, n)(\alpha)(X)$. Under what conditions is this map a group homomorphism? As the following theorem shows, this is the case precisely when X is normalized, provided that G is reductive.

Theorem 3.11. Let G be a reductive algebraic group over a field F . Put $A = F[G]$, and let M be a field extension of F . For any $\alpha, \beta \in G(M)$, and any class $X \in \Psi(A, n)$, we have

$$\Psi(-, n)(\alpha)(X) \cdot \Psi(-, n)(\beta)(X) = \Psi(-, n)(\alpha\beta)(X) \cdot \Psi(-, n)(\varepsilon_M)(X)$$

Proof. Let (I, m_f) be an ideal representative for X such that $\alpha(f), \beta(f), (\alpha\beta)(f), \varepsilon_M(f) \in M^\times$; this is possible by Corollary 3.6. By Proposition 3.7, it suffices to show that

$$[(\alpha\beta)(f)\varepsilon_M(f)] = [\alpha(f)\beta(f)]$$

as classes in $M^\times/(M^\times)^n$. Let $B = F[G \times_F G] = A \otimes_F A$, and let E be the field of fractions of B . Let $c, p_1, p_2: A \rightarrow B$ be the F -algebra morphisms corresponding respectively to the morphisms $G \times G \rightarrow G$ given by multiplication and projection onto each component. We note that c, p_1, p_2 are each flat, hence injective.

By [5, Theorems 16.56 and 21.84], any reductive algebraic group over a separably closed field is rational, and so the natural map $\text{Pic}(A) \oplus \text{Pic}(A) \rightarrow \text{Pic}(B)$ is an isomorphism by [9, Lemma 6.6]. Moreover, up to this identification, $\text{Pic}(c): \text{Pic}(A) \rightarrow \text{Pic}(B)$ is the diagonal embedding, and $\text{Pic}(p_i): \text{Pic}(A) \rightarrow \text{Pic}(B)$ is the embedding onto the i^{th} component. Let $J_c = c(I)B, J_i = p_i(I)B$; since c, p_1, p_2 are flat, $J_c, J_1, J_2 \in \text{Cart}(B)$, and we have $[J_c] = [J_1] + [J_2]$ as classes in $\text{Pic}(B)$. In light of the classical exact sequence

$$1 \rightarrow B^\times \rightarrow E^\times \rightarrow \text{Cart}(B) \rightarrow \text{Pic}(B) \rightarrow 0 \quad (3.1)$$

there exists $h \in E^\times$ such that $J_c = hB \cdot J_1 \cdot J_2$. Raising each side of this equation to the n^{th} power and using the relation $I^n = fA$ gives the equation $c(f)B = h^n B \cdot (f \otimes f)B$. Appealing again to (3.1), there exists $b \in B^\times$ such that $bc(f) = h^n(f \otimes f)$. Let $x, y \in B$ such that $h = x/y$, so that our equation reads $bc(f)y^n = x^n(f \otimes f)$.

Let $\omega: B \rightarrow M$ be the composition of $\alpha \otimes_F \beta: B \rightarrow M \otimes_F M$ and the multiplication map $M \otimes_F M \rightarrow M$. Then $\omega(c(f)) = (\alpha\beta)(f)$, and $\omega(f \otimes f) = \alpha(f)\beta(f)$, so applying ω to the equation above gives

$$(\alpha\beta)(f)\omega(b)\omega(y)^n = \alpha(f)\beta(f)\omega(x)^n$$

Since M is a field, $\mathfrak{p} := \ker(\omega)$ is a prime ideal of B . We know that $\alpha(f), \beta(f) \in M^\times$, so $f \otimes f$ belongs to $B \setminus \mathfrak{p}$. Hence, $h^n = bc(f)/(f \otimes f) \in B_{\mathfrak{p}}$. Since B is regular, it follows that $B_{\mathfrak{p}}$ is integrally closed in E , so $h^n \in B_{\mathfrak{p}}$ implies $h \in B_{\mathfrak{p}}$; in particular, we have $\omega(y) \neq 0$. This also forces $\omega(x) \neq 0$, since $(\alpha\beta)(f), \omega(b) \in M^\times$, so we have

$$[(\alpha\beta)(f)\omega(b)] = [\alpha(f)\beta(f)]$$

as classes in $M^\times/(M^\times)^n$. It remains to show that $\omega(b)$ and $\varepsilon_M(f)$ belong to the same class in $M^\times/(M^\times)^n$. By Rosenlicht's theorem ([7, Theorem 3]), the map $F^\times \oplus G^* \oplus G^* \rightarrow B^\times$ sending (z, χ, ρ) to $z(\chi^\sharp(t) \otimes \rho^\sharp(t))$ is an isomorphism, so b can be written as $z(g \otimes g')$ for $g, g' \in A^\times$ group-like elements, $z \in F^\times$. Then $\omega(b) = z\alpha(g)\beta(g')$, and our equation in $M^\times/(M^\times)^n$ therefore reads

$$[(\alpha\beta)(f) \cdot z \cdot \alpha(g)\beta(g')] = [\alpha(f)\beta(f)]$$

Our derivation of this equation did not depend on our choice of $\alpha, \beta \in G(M)$, only on the fact that $\alpha(f), \beta(f), (\alpha\beta)(f) \in M^\times$. In particular, since we arranged that $\varepsilon_M(f) \neq 0$, we can substitute ε_M for α or β in our equation. Plugging in $\alpha = \varepsilon_M$ and using $\varepsilon_M(g) = 1$

gives $[\varepsilon_M(f)] = [z\beta(g')]$, and likewise, plugging in $\beta = \varepsilon_M$ yields $[\varepsilon_M(f)] = [z\alpha(g)]$. Substituting both $\alpha = \varepsilon_M, \beta = \varepsilon_M$ simultaneously gives us $[z] = [\varepsilon_M(f)]$, whence $[\alpha(g)] = [\beta(g')] = 1$, and so $[z\alpha(g)\beta(g')] = [\varepsilon_M(f)]$, completing the proof. \square

Corollary 3.12. *Let G, A, M be as in the statement of Theorem 3.12. If $X \in \Psi_{\text{nm}}(A, n)$, then the map $G(M) \rightarrow \Psi(M, n)$ sending $\alpha \in G(M)$ to $\Psi(-, n)(\alpha)(X)$ is a group homomorphism. \square*

3.4. The Galois action on torsors

Suppose that our group G is smooth and connected, and $\text{Pic}(G_{\text{sep}})[n] = 0$. Then putting $A = F[G]$, $\Delta_n(A_{\text{sep}})$ is an isomorphism by Corollary 3.3, and the subgroup of $\Psi_{\text{nm}}(A_{\text{sep}}, n)$ of $\Psi(A_{\text{sep}}, n)$ is the image of $G_{\text{sep}}^*/(G_{\text{sep}}^*)^n$. Via the embedding $\Gamma \rightarrow \text{Aut}_{F\text{-alg}}(A_{\text{sep}})$, Γ acts functorially on $A_{\text{sep}}^\times/(A_{\text{sep}}^\times)^n$ and $\Psi(A_{\text{sep}}, n)$, and the map $\Delta_n(A_{\text{sep}})$ is Γ -equivariant. Since the action of Γ on $A_{\text{sep}}^\times/(A_{\text{sep}}^\times)^n$ preserves the summand $G_{\text{sep}}^*/(G_{\text{sep}}^*)^n \subset A_{\text{sep}}^\times/(A_{\text{sep}}^\times)^n$, this shows that the action of Γ on $\Psi(A_{\text{sep}}, n)$ restricts to an action on $\Psi_{\text{nm}}(A_{\text{sep}}, n)$.

Throughout this section, let $\alpha: A \rightarrow A_{\text{sep}}$ denote the canonical base change morphism. The associated map $\Psi(-, n)(\alpha): \Psi_{\text{nm}}(A, n) \rightarrow \Psi_{\text{nm}}(A_{\text{sep}}, n)$ has image in $H^0(F, \Psi_{\text{nm}}(A_{\text{sep}}, n))$. If we assume that G is geometrically integral, then $\Psi(-, n)(\alpha)$ is an embedding with image $H^0(F, \Psi_{\text{nm}}(A_{\text{sep}}, n))$; this is the content of Theorem 3.14. First, we require a lemma.

Lemma 3.13. *Let G be a smooth, geometrically integral group variety over F , and let $A = F[G]$. If $\text{Cl}(A_{\text{sep}}) = 0$, then there is an isomorphism $Z(A): H^1(F, G_{\text{sep}}^*) \rightarrow \text{Cl}(A)$.*

Proof. Let $K_s = \text{Frac}(A_{\text{sep}})$; since G is geometrically integral, $K_s = KF_{\text{sep}}$. Since $\text{Cl}(A_{\text{sep}}) = 0$, we have an exact sequence of Γ -modules

$$1 \rightarrow (A_{\text{sep}})^\times \rightarrow (K_s)^\times \xrightarrow{\partial(A_{\text{sep}})} \text{Div}(A_{\text{sep}}) \rightarrow 0$$

and therefore obtain the following long exact sequence in Galois cohomology:

$$\begin{aligned} H^0(F, (A_{\text{sep}})^\times) &\rightarrow H^0(F, K_s^\times) \rightarrow H^0(F, \text{Div}(A_{\text{sep}})) \xrightarrow{\delta} H^1(F, (A_{\text{sep}})^\times) \\ &\rightarrow H^1(F, K_s^\times) \rightarrow \cdots \end{aligned}$$

Since $\Gamma \cong \text{Gal}(KF_{\text{sep}}/K) = \text{Gal}(K_s/K)$, we have $H^1(F, K_s^\times) = 0$ by Hilbert Theorem 90. As A is regular and geometrically integral, $\text{Div}(\alpha)$ embeds $\text{Div}(A)$ onto $H^0(F, \text{Div}(A_{\text{sep}}))$. By Rosenlicht's Theorem ([7, Theorem 3]), the map $F_{\text{sep}}^\times \oplus G_{\text{sep}}^* \rightarrow A_{\text{sep}}^\times$ sending (z, χ) to $z\chi^\sharp(t)$ is an isomorphism of Γ -modules. Hence, $H^1(F, (A_{\text{sep}})^\times) \cong H^1(F, G_{\text{sep}}^*) \oplus H^1(F, (F_{\text{sep}})^\times) = H^1(F, G_{\text{sep}}^*)$. We thus have a commutative diagram

$$\begin{array}{ccccccc}
A^\times & \longrightarrow & K^\times & \xrightarrow{\partial(A)} & \operatorname{Div}(A) & \xrightarrow{\operatorname{div}(A)} & \operatorname{Cl}(A) \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \operatorname{Div}(\alpha) & & \\
H^0(F, (A_{\operatorname{sep}})^\times) & \longrightarrow & H^0(F, K_s^\times) & \xrightarrow{\partial(A_{\operatorname{sep}})} & H^0(F, \operatorname{Div}(A_{\operatorname{sep}})) & \xrightarrow{\delta} & H^1(F, G_{\operatorname{sep}}^*) \longrightarrow 0
\end{array}$$

with exact rows and vertical arrows isomorphisms. By the universal property of the cokernel, $\operatorname{Div}(\alpha)$ descends to a well-defined map $Z(A): \operatorname{Cl}(A) \rightarrow H^1(F, G_{\operatorname{sep}}^*)$ which sends the $[D] \in \operatorname{Cl}(A)$ to $\delta(\operatorname{Div}(\alpha)(D))$. By (e.g.) the Five Lemma, $Z(A)$ is an isomorphism. \square

Note that we can be more explicit in describing $Z(A)$. Let $[D] \in \operatorname{Cl}(A)$, and set $D' = \operatorname{Div}(\alpha)(D)$. Since the map $\partial(A_{\operatorname{sep}}): K_s^\times \rightarrow \operatorname{Div}(A_{\operatorname{sep}})$ is surjective, there exists $x \in K_s^\times$ such that $D' = \partial(A_{\operatorname{sep}})(x)$. One can accordingly define a cocycle $\sigma_x: \Gamma \rightarrow A_{\operatorname{sep}}^\times$ by setting $\sigma_x(\gamma) = \gamma(x)/x$ for $\gamma \in \Gamma$, and the class of σ_x in $H^1(F, A_{\operatorname{sep}}^\times) = H^1(F, G_{\operatorname{sep}}^*)$ does not depend on the choice of x . The map $Z(A)$ then takes $[D]$ to $[\sigma_x]$ in $H^1(F, G_{\operatorname{sep}}^*)$.

Theorem 3.14. *Let G be a geometrically integral, smooth group scheme over F . Put $A = F[G]$, $K = \operatorname{Frac}(A)$, and $K_s = \operatorname{Frac}(A_{\operatorname{sep}})$. Suppose that $\operatorname{Cl}(A_{\operatorname{sep}}) = 0$, and $G_{\operatorname{sep}}^*[n] = 0$. Then the natural map $\Psi(-, n)(\alpha): \Psi_{\operatorname{nm}}(A, n) \rightarrow \Psi_{\operatorname{nm}}(A_{\operatorname{sep}}, n)$ is an embedding of $\Psi_{\operatorname{nm}}(A, n)$ onto $H^0(F, \Psi_{\operatorname{nm}}(A_{\operatorname{sep}}, n))$.*

Proof. Put $\eta(A) = \operatorname{div}_n(A) \circ \Omega_n(A)^{-1} \circ \Delta_n(A)$. By Proposition 3.2, we have an exact sequence

$$1 \rightarrow G^*/(G^*)^n \xrightarrow{\eta(A)} \operatorname{Div}_{\operatorname{nm}}(A, n) \rightarrow \operatorname{Cl}(A) \xrightarrow{\cdot n} \operatorname{Cl}(A) \rightarrow 0.$$

Because $G_{\operatorname{sep}}^*[n] = 0$, there is an exact sequence of Γ -modules

$$1 \rightarrow G_{\operatorname{sep}}^* \xrightarrow{\cdot n} G_{\operatorname{sep}}^* \rightarrow G_{\operatorname{sep}}^*/(G_{\operatorname{sep}}^*)^n \rightarrow 1$$

which yields the following long exact sequence in Galois cohomology:

$$\begin{aligned}
1 \rightarrow H^0(F, G_{\operatorname{sep}}^*) &\xrightarrow{\cdot n} H^0(F, G_{\operatorname{sep}}^*) \rightarrow H^0(F, G_{\operatorname{sep}}^*/(G_{\operatorname{sep}}^*)^n) \xrightarrow{\delta} H^1(F, G_{\operatorname{sep}}^*) \\
&\xrightarrow{\cdot n} H^1(F, G_{\operatorname{sep}}^*) \rightarrow \cdots
\end{aligned}$$

We can rewrite the above (truncated) long exact sequence as

$$1 \rightarrow G^*/(G^*)^n \rightarrow H^0(F, G_{\operatorname{sep}}^*/(G_{\operatorname{sep}}^*)^n) \xrightarrow{\delta} H^1(F, G_{\operatorname{sep}}^*) \xrightarrow{\cdot n} H^1(F, G_{\operatorname{sep}}^*).$$

The boundary map δ can be described as follows: let $[u] \in H^0(F, G_{\operatorname{sep}}^*/(G_{\operatorname{sep}}^*)^n)$. Then $\gamma(u)/u \in (G_{\operatorname{sep}}^*)^n$ for any $\gamma \in \Gamma$, so let x_γ be the **unique** element of G_{sep}^* such that $x_\gamma^n = \gamma(u)/u$. Then $\delta([u])$ is the class of the cocycle $\sigma_u: \Gamma \rightarrow G_{\operatorname{sep}}^*$ which sends γ to x_γ .

Note that $\mathrm{Cl}(A_{\mathrm{sep}}) \cong \mathrm{Pic}(A_{\mathrm{sep}}) = 0$, and so $\Delta_n(A_{\mathrm{sep}})$ is an isomorphism by Corollary 3.3. Let $\tau(A)$ denote the composition

$$\mathrm{Div}_{\mathrm{nm}}(A, n) \rightarrow \Psi_{\mathrm{nm}}(A, n) \rightarrow \Psi_{\mathrm{nm}}(A_{\mathrm{sep}}, n) \rightarrow G_{\mathrm{sep}}^*/(G_{\mathrm{sep}}^*)^n$$

where the first map is $\Omega_n(A) \circ \mathrm{div}_n(A)^{-1}$, the second map is $\Psi(-, n)(\alpha)$, and the third map is $\Delta_n(A_{\mathrm{sep}})^{-1}$. Explicitly, given the class of a pair (D, g) in $\mathrm{Div}_{\mathrm{nm}}(A, n)$, $D' := \mathrm{Div}(\alpha)(D)$ is principal, since $\mathrm{Cl}(A_{\mathrm{sep}}) = 0$, so there exists $x \in K_s^\times$ such that $\partial(A_{\mathrm{sep}})(x) = D'$; $\tau(A)$ sends $[(D, g)]$ to $[x^n/g] \in G_{\mathrm{sep}}^*/(G_{\mathrm{sep}}^*)^n$. If the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & G^*/(G^*)^n & \xrightarrow{\eta(A)} & \mathrm{Div}_{\mathrm{nm}}(A, n) & \longrightarrow & \mathrm{Cl}(A) \xrightarrow{\cdot n} \mathrm{Cl}(A) \longrightarrow 0 \\ & & \downarrow \mathrm{Id}_{G^*/(G^*)^n} & & \downarrow \tau(A) & & \downarrow Z(A) \\ 1 & \longrightarrow & G^*/(G^*)^n & \longrightarrow & H^0(F, G_{\mathrm{sep}}^*/(G_{\mathrm{sep}}^*)^n) & \xrightarrow{\delta} & H^1(F, G_{\mathrm{sep}}^*) \xrightarrow{\cdot n} H^1(F, G_{\mathrm{sep}}^*) \longrightarrow 0 \end{array}$$

commutes, then $\tau(A)$ must be an isomorphism, so $\Psi(-, n)(\alpha)$ must be one as well. The last square is manifestly commutative. We have $\tau(A) \circ \eta(A) = \Delta_n(A_{\mathrm{sep}})^{-1} \circ \Psi(-, n)(\alpha) \circ \Delta_n(A)$, which is easily seen to be the inclusion $G^*/(G^*)^n \rightarrow G_{\mathrm{sep}}^*/(G_{\mathrm{sep}}^*)^n$. It remains to show that the middle square commutes.

Let the pair (D, g) represent a class in $\mathrm{Div}_{\mathrm{nm}}(A, n)$, and put $D' = \mathrm{Div}(\alpha)(D)$. Let $x \in K_s^\times$ such that $D' = \partial(A_{\mathrm{sep}})(x)$, so that $x^n = gu$ for some $u \in A_{\mathrm{sep}}^\times$. As explained above, $\tau(A)([(D, g)]) = [u] \in G_{\mathrm{sep}}^*/(G_{\mathrm{sep}}^*)^n$. For any $\gamma \in \Gamma$,

$$\frac{\gamma(u)}{u} = \frac{\gamma(x^n g^{-1})}{x^n g^{-1}} = \frac{\gamma(x^n)}{x^n} = \left(\frac{\gamma(x)}{x} \right)^n$$

because g^{-1} is Γ -invariant. Therefore, δ takes $[u]$ to the class of the cocycle $\sigma_u: \Gamma \rightarrow G_{\mathrm{sep}}^*$ defined by $\gamma \mapsto \gamma(x)/x$. On the other hand, as explained in paragraph immediately following Theorem 3.13, $Z(A)$ takes $[D]$ to the very same cocycle class, so we're done. \square

4. Type-zero invariants for reductive groups

Throughout this section, let G be a reductive algebraic group over F . Let $A = F[G]$, $K = F(G)$, and let $\xi: A \rightarrow K$ denote the generic point of G .

We are now equipped to determine the groups $\mathrm{Inv}_{\mathrm{hom}}^0(G, H)$ for $H = K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$ and $H = K_1^M/n$ for all $n \in \mathbb{N}$; this is the content of Theorems 4.7 and 4.4 respectively. A key step is the observation that, under suitable conditions, a type-zero H -invariant of G is determined by its value at ξ ; precisely, the evaluation homomorphism $\mathrm{ev}_\xi(H): \mathrm{Inv}_{\mathrm{hom}}^0(G, H) \rightarrow H(K)$ sending an invariant I to $I(K)(\xi)$ is injective. Before proving this in Proposition 4.2, we need a technical lemma. For any positive integer n , let $p_n: G \rightarrow G$ denote the n^{th} power map, which sends x to x^n for any F -algebra R and any $x \in G(R)$.

Lemma 4.1. *The map p_n is dominant.*

Proof. Since the property of dominance descends under faithfully flat base change, we may assume that our base field F is algebraically closed. By (e.g.) [5, Theorem 17.44], the union of the Cartan subgroups of G contains a dense open subset of G . Since G is reductive, the Cartan subgroups of G are precisely the maximal tori in G . But the restriction of p_n to any torus in G is surjective, and so the image of p_n contains every torus in G . \square

Proposition 4.2. *Suppose H is the d^{th} graded component of a torsion cycle module. Let $I \in \text{Inv}^0(G, H)$, and suppose $I(K)(\xi) = 1_{H(K)}$. Suppose that for any field extension L/F and any $\alpha, \beta \in G(L)$, I satisfies*

$$I(L)(\alpha)I(L)(\beta) = I(L)(\alpha\beta)I(L)(\varepsilon_L).$$

Then I is trivial.

Proof. Let L/F be a field extension, and fix $t \in G(L)$. Put $S := G_L$, and let $g: S \rightarrow G$ be the canonical base change morphism, with comorphism $f: A \rightarrow A_L$. Let $E = L(S)$, and let $\xi': A_L \rightarrow E$ be the generic point of S . Since f is injective, the composition $\xi' \circ f$ extends to a morphism $u: K \rightarrow E$ of F -algebras such that $u \circ \xi = \xi' \circ f$. Put $\xi_E := u \circ \xi$, and let n be a positive integer such that $I(E)(\xi_E)^n = I(E)(\varepsilon_E)^n = 1$.

Suppose that there exist morphisms $i: K \rightarrow E, j: L \rightarrow E$ satisfying the following two properties:

- (a) $H(j): H(L) \rightarrow H(E)$ is injective;
- (b) $G(i)(\xi) = (\xi_E)^n \cdot t_E$, where $t_E := j \circ t$.

Then we have

$$H(j)(I(L)(t)) = I(E)(t_E) = I(E)(\xi_E)^n I(E)(t_E) I(E)(\varepsilon_E)^{-n} = I(E)((\xi_E)^n \cdot t_E),$$

whence we conclude

$$H(j)(I(L)(t)) = I(E)(G(i)(\xi)) = H(i)(I(K)(\xi)) = 1_{H(E)}.$$

We therefore devote the remainder of the proof to constructing such a pair (i, j) . Let $j: L \rightarrow E$ denote the composition of the structural map $L \rightarrow A_L$ with ξ' . Since S is a smooth algebraic L -variety such that $S(L) \neq \emptyset$, $H(j)$ is injective by [3, Lemma 1.3].

To construct i , let $s: A_L \rightarrow L$ be the unique L -algebra morphism such that $t = s \circ f = g(L)(t)$, and put $s_E = j \circ s$, so that $t_E = s_E \circ f$. Let $p_{n,s}: S \rightarrow S$ be the morphism of L -schemes given by the composition of the n^{th} power map p_n with right translation by s . By Corollary 4.1, $p_{n,s}$ is dominant, and so the associated comorphism $h: A_L \rightarrow A_L$

is injective. In particular, the composition $\xi' \circ h$ extends to a morphism $v: E \rightarrow E$ of L -algebras such that $v \circ \xi' = \xi' \circ h$. Putting $i = v \circ u$, we claim that i satisfies (b). On the one hand, we have $p_{n,s}(E)(\xi') = \xi' \circ h$, but by definition of $p_{n,s}$, we also have $p_{n,s}(E)(\xi') = (\xi')^n \cdot s_E$. Accordingly, this yields

$$G(i)(\xi) = v \circ u \circ \xi = \xi' \circ h \circ f = g(E)(p_{n,s}(E)(\xi')).$$

But we compute

$$g(E)(p_{n,s}(\xi')) = g(E)((\xi')^n \cdot s_E) = g(E)(\xi')^n \cdot g(E)(s_E) = (\xi_E)^n \cdot t_E,$$

which establishes (b). \square

Corollary 4.3. *The morphism $\text{ev}_\xi(H): \text{Inv}_{\text{hom}}^0(G, H) \rightarrow H(K)$ is injective.* \square

For any fixed field extension L/F , there is a map $\Psi(A, n) \times G(L) \rightarrow L^\times / (L^\times)^n$ which sends the pair (X, y) to $\Delta_n(L)^{-1}(\Psi(-, n)(y)(X))$. If we fix a class $X \in \Psi(A, n)$ in the first argument, we obtain a set map $I_X(L): G(L) \rightarrow L^\times / (L^\times)^n$. As L varies, the collection of maps I_X determines an invariant in $\text{Inv}^0(G, K_1^M/n)$. If X is *normalized*, then Corollary 3.12 shows that I_X is homomorphic. We thus obtain a group homomorphism $\Lambda_n(G): \Psi_{\text{nm}}(A, n) \rightarrow \text{Inv}_{\text{hom}}^0(G, K_1^M/n)$. As the next theorem shows, $\Lambda_n(G)$ is in fact an isomorphism.

Theorem 4.4. *The map*

$$\Lambda_n(G): \Psi_{\text{nm}}(A, n) \rightarrow \text{Inv}_{\text{hom}}^0(G, K_1^M/n)$$

sending a class $X \in \Psi_{\text{nm}}(A, n)$ to the invariant I_X is an isomorphism.

Proof. By Lemma 3.9, the map $\Delta_n(K)^{-1} \circ \Psi(-, n)(\xi): \Psi(A, n) \rightarrow \ker(\partial_n(A))$ is an isomorphism. Thus, since $\text{ev}_\xi(K_1^M/n) \circ \Lambda_n(G)$ coincides with the restriction of $\Delta_n(K)^{-1} \circ \Psi(-, n)(\xi)$ to $\Psi_{\text{nm}}(A, n)$, $\Lambda_n(G)$ must be injective.

Now, fix an invariant $I \in \text{Inv}_{\text{hom}}^0(G, K_1^M/n)$. By Corollary 4.3, $\text{ev}_\xi(K_1^M/n)$ is injective. The sequence

$$\text{Inv}_{\text{hom}}^0(G, K_1^M/n) \xrightarrow{\text{ev}_\xi(K_1^M/n)} K^\times / (K^\times)^n \xrightarrow{\partial_n(A)} \text{Div}(A)/n \text{Div}(A)$$

is a complex by [3, Lemma 2.1], so $\text{ev}_\xi(K_1^M/n)$ has image contained in $\ker(\partial_n(A))$. Letting $X \in \Psi(A, n)$ be a class such that $\Delta_n(K)^{-1}(\Psi(-, n)(\xi)(X)) = I(K)(\xi)$, we have $I_X(K)(\xi) = I(K)(\xi)$ by construction. We must therefore have $I_X = I$ by Theorem 3.11 and Proposition 4.2. But as I is homomorphic, it must be the case that $I_X(F)(\varepsilon_F) = I(F)(\varepsilon_F)$ is the trivial class in $F^\times / (F^\times)^n$, whence X is normalized, and $\Lambda_n(G)(X) = I$. \square

Corollary 4.5. *Suppose that G is a torus, and let $\alpha: A \rightarrow A_{\text{sep}}$ be the canonical base change morphism. Then the map*

$$(\Delta(A_{\text{sep}}) \circ \Psi(-, n)(\alpha))^{-1} \circ \Lambda_n(G): H^0(F, G_{\text{sep}}^*/(G_{\text{sep}}^*)^n) \rightarrow \text{Inv}_{\text{hom}}^0(G, K_1^M/n)$$

is an isomorphism. \square

For any natural number n , let $\text{Inv}^0(G, \iota_n)$ denote the group morphism $\text{Inv}_{\text{hom}}^0(G, K_1^M/n) \rightarrow \text{Inv}_{\text{hom}}^0(G, K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})$ given by composition with ι_n . Likewise, if n and m are positive integers such that n divides m , let $\text{Inv}^0(G, \beta_{n,m})$ denote the group morphism $\text{Inv}_{\text{hom}}^0(G, K_1^M/n) \rightarrow \text{Inv}_{\text{hom}}^0(G, K_1^M/m)$ given by composition with $\beta_{n,m}$. Since $\iota_n = \iota_m \circ \beta_{n,m}$, we obtain a universal induced map

$$\text{colim}_{n \in \mathbb{N}} \text{Inv}^0(G, \iota_n): \text{colim}_{n \in \mathbb{N}} \text{Inv}_{\text{hom}}^0(G, K_1^M/n) \rightarrow \text{Inv}_{\text{hom}}^0(G, K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}).$$

Proposition 4.6. *The map $\text{colim}_{n \in \mathbb{N}} \text{Inv}^0(G, \iota_n)$ is an isomorphism.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} \text{colim}_{n \in \mathbb{N}} \text{Inv}_{\text{hom}}^0(G, K_1^M/n) & \xrightarrow{\text{colim}_{n \in \mathbb{N}} \text{ev}_{\xi}(K_1^M/n)} & \text{colim}_{n \in \mathbb{N}} \ker(\partial_n(A)) \\ \downarrow \text{colim}_{n \in \mathbb{N}} \text{Inv}^0(G, \iota_n) & & \downarrow \text{colim}_{n \in \mathbb{N}} \iota_n(K) \\ \text{Inv}_{\text{hom}}^0(G, K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\text{ev}_{\xi}(K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})} & \ker(\partial(A) \otimes_{\mathbb{Z}} \text{Id}_{\mathbb{Q}/\mathbb{Z}}) \end{array}$$

The rightmost arrow is an isomorphism, and the lower and upper horizontal arrows are injective by Corollary 4.3, so it follows that $\text{colim}_{n \in \mathbb{N}} \text{Inv}^0(G, \iota_n)$ is injective. To see that $\text{colim}_{n \in \mathbb{N}} \text{Inv}^0(G, \iota_n)$ is surjective, fix an invariant $I \in \text{Inv}_{\text{hom}}^0(G, K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})$ and let $x = I(K)(\xi) \in \ker(\partial(A) \otimes_{\mathbb{Z}} \text{Id}_{\mathbb{Q}/\mathbb{Z}})$. There exists some positive integer n and $y \in \ker(\partial_n(A))$ such that $\iota_n(K)(y) = x$. Let $Y \in \Psi(A, n)$ with $\Psi(-, n)(\xi)(Y) = \Delta_n(K)(y)$. Then the associated invariant $I_Y \in \text{Inv}^0(G, K_1^M/n)$ satisfies $(\iota_n \circ I_Y)(K)(\xi) = x = I(K)(\xi)$, and so $\iota_n \circ I_Y = I$ by Theorem 3.11 and Proposition 4.2. In particular, $((\iota_n \circ I_Y)(F))(\varepsilon_F)$ is the trivial class in $F^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$, which means that $z := I_Y(F)(\varepsilon_F)$ belongs to the kernel of $\iota_n(F): F^{\times}/(F^{\times})^n \rightarrow F^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$.

This can only be the case if $z \in \ker(\beta_{n,nd}(F))$ for some $d \in \mathbb{N}$, so fix such a d . For any field extension M/F , the diagram

$$\begin{array}{ccc}
 M^\times / (M^\times)^n & \xrightarrow{\Delta_n(M)} & \Psi(M, n) \\
 \downarrow \beta_{n,nd}(M) & & \downarrow \omega_{n,nd}(M) \\
 M^\times / (M^\times)^{nd} & \xrightarrow{\Delta_{nd}(M)} & \Psi(M, nd)
 \end{array}$$

commutes, and so putting $Y' = \omega_{n,nd}(A)(Y)$, $\Psi(-, nd)(\varepsilon_F)(Y') = \Delta_{nd}(F)(\beta_{n,nd}(F)(z))$, whence Y' is normalized. Thus, $I_{Y'} = \Lambda_{nd}(G)(Y')$ is homomorphic, and

$$((\iota_{nd} \circ I_{Y'})(K))(\xi) = \iota_{nd}(K)(\beta_{n,nd}(K)(y)) = \iota_n(K)(y) = x,$$

so $\iota_{nd} \circ I_{Y'} = I$ by Corollary 4.3. \square

Corollary 4.7. *If G is a torus, then $\text{Inv}_{\text{hom}}^0(G, K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}) \cong H^0(F, G_{\text{sep}}^* \otimes \mathbb{Q}/\mathbb{Z})$.*

Proof. If $\alpha: A \rightarrow A_{\text{sep}}$ denotes the canonical base change morphism, this follows from Proposition 4.6, Theorem 3.14, and the fact that the diagram

$$\begin{array}{ccccc}
 H^0(F, G_{\text{sep}}^* / (G_{\text{sep}}^*)^n) & \xleftarrow{\Delta_n(A_{\text{sep}})^{-1} \circ \Psi(-, n)(\alpha)} & \Psi_{\text{nm}}(A, n) & \xrightarrow{\Lambda_n(G)} & \text{Inv}_{\text{hom}}^0(G, K_1^M / n) \\
 \downarrow \cdot m/n & & \downarrow \omega_{n,m}(A) & & \downarrow \text{Inv}^0(G, \beta_{n,m}) \\
 H^0(F, G_{\text{sep}}^* / (G_{\text{sep}}^*)^m) & \xleftarrow{\Delta_m(A_{\text{sep}})^{-1} \circ \Psi(-, m)(\alpha)} & \Psi_{\text{nm}}(A, m) & \xrightarrow{\Lambda_m(G)} & \text{Inv}_{\text{hom}}^0(G, K_1^M / m)
 \end{array}$$

commutes for all $n, m \in \mathbb{N}$ with n dividing m . \square

5. Computation of degree one Milnor K -invariants of groups of multiplicative type

In this section, we determine the degree one Milnor K -invariants of an algebraic group G of multiplicative type. To begin, fix a resolution (2.1) of G by tori. Applying the snake lemma to the diagram

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & T_{\text{sep}}^* & \xrightarrow{g_{\text{sep}}^*} & P_{\text{sep}}^* & \xrightarrow{f_{\text{sep}}^*} & G_{\text{sep}}^* & \longrightarrow & 1 \\
 & & \downarrow \cdot n & & \downarrow \cdot n & & \downarrow \cdot n & & \\
 1 & \longrightarrow & T_{\text{sep}}^* & \xrightarrow{g_{\text{sep}}^*} & P_{\text{sep}}^* & \xrightarrow{f_{\text{sep}}^*} & G_{\text{sep}}^* & \longrightarrow & 1
 \end{array}$$

yields the exact sequence of Γ -modules

$$1 \rightarrow G_{\text{sep}}^*[n] \rightarrow T_{\text{sep}}^* / (T_{\text{sep}}^*)^n \rightarrow P_{\text{sep}}^* / (P_{\text{sep}}^*)^n,$$

and after taking Γ -fixed points we obtain the exact sequence

$$1 \rightarrow H^0(F, G_{\text{sep}}^*[n]) \rightarrow H^0(F, T_{\text{sep}}^*/(T_{\text{sep}}^*)^n) \rightarrow H^0(F, P_{\text{sep}}^*/(P_{\text{sep}}^*)^n)$$

of abelian groups. Let $A = F[G], B = F[P], C = F[T]$, let $g^\sharp: C \rightarrow B, f^\sharp: B \rightarrow A$ be the associated comorphisms, and let $\alpha_X: X \rightarrow X_{\text{sep}}$ denote the canonical base change morphism for $X = A, B, C$. For $Y = B, C$, let $\ell_n(Y) := \Delta_n(Y_{\text{sep}})^{-1} \circ \Psi(-, n)(\alpha_Y) \circ \lambda_n(Y)^{-1}$.

Proposition 5.1. *The diagram*

$$\begin{array}{ccccc} G^*[n] & \xrightarrow{v_n(G)} & \text{Tors}_{\text{nm}}(T, \mu_{n,F}) & \xrightarrow{\text{Tors}^*(g)(\mu_{n,F})} & \text{Tors}_{\text{nm}}(P, \mu_{n,F}) \\ \downarrow & & \downarrow \ell_n(C) & & \downarrow \ell_n(B) \\ H^0(F, G_{\text{sep}}^*[n]) & \longrightarrow & H^0(F, T_{\text{sep}}^*/(T_{\text{sep}}^*)^n) & \xrightarrow{\kappa^n(g_{\text{sep}}^\sharp)} & H^0(F, P_{\text{sep}}^*/(P_{\text{sep}}^*)^n) \end{array}$$

commutes.

Proof. The right square commutes because Δ_n and λ_n are natural transformations. To see that the left square commutes, fix $\chi \in G^*[n]$. Consider the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & T_{\text{sep}}^* & \xrightarrow{j_{\text{sep}}^*} & P_{\text{sep}}^* \times_{G_{\text{sep}}^*} \mathbb{Z}/n\mathbb{Z} & \xrightarrow{(\pi_n)_{\text{sep}}^*} & \mathbb{Z}/n\mathbb{Z} \longrightarrow 1 \\ & & \downarrow \text{Id}_{T_{\text{sep}}^*} & & \downarrow (\pi_P)_{\text{sep}}^* & & \downarrow \chi_{\text{sep}}^* \\ 1 & \longrightarrow & T_{\text{sep}}^* & \xrightarrow{g_{\text{sep}}^*} & P_{\text{sep}}^* & \xrightarrow{f_{\text{sep}}^*} & G_{\text{sep}}^* \longrightarrow 1 \end{array}$$

of Γ -modules with exact rows, and let H denote the group of multiplicative type dual to $P_{\text{sep}}^* \times_{G_{\text{sep}}^*} \mathbb{Z}/n\mathbb{Z}$. The F -group morphism $j: H \rightarrow T$ dual to j_{sep}^* is a $\mu_{n,F}$ -torsor over T , and we claim that j represents the class $v_n(G)(\chi)$. Indeed, let $\pi_n: \mu_{n,F} \rightarrow H, \pi_P: P \rightarrow H$ be the morphisms dual to $(\pi_n)_{\text{sep}}^*$ and $(\pi_P)_{\text{sep}}^*$ respectively. The morphism of T -schemes $P \times \mu_{n,F} \rightarrow H$ defined on R -points by $(x, y) \mapsto \pi_P(R)(x)\pi_n(R)(y)$ for any F -algebra R and any $x \in P(R), y \in \mu_{n,F}(R)$ is constant on G^χ -orbits. It therefore descends to a universal map $(P \times \mu_{n,F})/G^\chi \rightarrow H$ over T , which one may check is $\mu_{n,F}$ -equivariant. Now, let $y \in P_{\text{sep}}^*$ be such that $f_{\text{sep}}^*(y) = \chi$, and let $z \in T_{\text{sep}}^*$ be such that $g_{\text{sep}}^*(z) = y^n$. We must show that $\text{Spec}(C_{\text{sep}}[X]/\langle X^n - z \rangle) \rightarrow \text{Spec}(C_{\text{sep}})$ and j_{sep} are isomorphic as $\mu_{n,F_{\text{sep}}}$ -torsors over T_{sep} . Equivalently, we must exhibit an isomorphism of $\mathbb{Z}/n\mathbb{Z}$ -graded C_{sep} -algebras $s: C_{\text{sep}}[X]/\langle X^n - z \rangle \rightarrow F_{\text{sep}}[H_{\text{sep}}]$. The condition that s respect the $\mathbb{Z}/n\mathbb{Z}$ -

grading ensures that the dual morphism of schemes $H \rightarrow \operatorname{Spec}(C_{\text{sep}}[X]/\langle X^n - z \rangle)$ is $\mu_{n, F_{\text{sep}}}$ -equivariant, hence an isomorphism of $\mu_{n, F_{\text{sep}}}$ -torsors.

By construction, $F_{\text{sep}}[H_{\text{sep}}]$ is the group algebra of $H_{\text{sep}}^* = P_{\text{sep}}^* \times_{G_{\text{sep}}^*} \mathbb{Z}/n\mathbb{Z}$ over F_{sep} , and C_{sep} is likewise the group algebra $F_{\text{sep}}\langle T_{\text{sep}}^* \rangle$. The comorphism j_{sep}^\sharp corresponds to the Γ -module embedding $j_{\text{sep}}^*: T_{\text{sep}}^* \hookrightarrow H_{\text{sep}}^*$. For each $v \in \mathbb{Z}/n\mathbb{Z}$, put $Q_v := ((\pi_n)_{\text{sep}}^*)^{-1}(v)$. Note that $Q_v Q_{v'} \subset Q_{v+v'}$, and $Q_v = (y, [1]_n)^{k_v} j^*(T_{\text{sep}}^*)$, where $k_v \in \mathbb{N}$ is the unique representative for v between 0 and $n - 1$. The $(\mathbb{Z}/n\mathbb{Z})$ -grading on H_{sep} arises from the partition

$$H_{\text{sep}}^* = \coprod_{v \in \mathbb{Z}/n\mathbb{Z}} Q_v$$

by setting R_v to be the F_{sep} -subspace of $F_{\text{sep}}[H_{\text{sep}}]$ generated by Q_v . We clearly have $F_{\text{sep}}[H_{\text{sep}}] = \bigoplus_{v \in \mathbb{Z}/n\mathbb{Z}} R_v$, and $R_v R_{v'} \subset R_{v+v'}$ follows from $Q_v Q_{v'} \subset Q_{v+v'}$. Furthermore, R_v is the C_{sep} -submodule of $F_{\text{sep}}[H_{\text{sep}}]$ generated by $(y, [1]_n)^{k_v}$. With this in mind, let $s: C_{\text{sep}}[X]/\langle X^n - z \rangle \rightarrow F_{\text{sep}}[H_{\text{sep}}]$ be the universal morphism of C_{sep} -algebras sending the class of X to $(y, [1]_n)$. This respects the $(\mathbb{Z}/n\mathbb{Z})$ -grading on each C_{sep} -algebra, since $(y, [1]_n)$ belongs to the $[1]_n$ -graded component of H_{sep} , and C_{sep} embeds into each algebra as the $[0]_n$ -graded component. \square

Since all vertical arrows of the diagram in Proposition 5.1 are isomorphisms, this proves:

Corollary 5.2. *The sequence*

$$1 \rightarrow G^*[n] \xrightarrow{v_n(G)} \operatorname{Tors}_{\text{nm}}(T, \mu_{n, F}) \xrightarrow{\operatorname{Tors}^*(g)(\mu_{n, F})} \operatorname{Tors}_{\text{nm}}(P, \mu_{n, F})$$

is exact. \square

For any reductive group R over F , define $\tilde{\Lambda}_n(R): \operatorname{Tors}_{\text{nm}}(R, \mu_{n, F}) \rightarrow \operatorname{Inv}_{\text{hom}}^0(R, K_1^M/n)$ by $\tilde{\Lambda}_n(R) = \Lambda_n(R) \circ \lambda_n(F[R])^{-1}$. As noted in section 2.2, the last crucial detail in our computation of $\operatorname{Inv}_{\text{hom}}^1(G, K_1^M/n)$ is the following lemma.

Lemma 5.3. *The diagram*

$$\begin{array}{ccccc} G^*[n] & \xrightarrow{v_n(G)} & \operatorname{Tors}_{\text{nm}}(T, \mu_{n, F}) & \xrightarrow{\operatorname{Tors}^*(g)(\mu_{n, F})} & \operatorname{Tors}_{\text{nm}}(P, \mu_{n, F}) \\ \downarrow \Phi(G, n) & & \downarrow \tilde{\Lambda}_n(T) & & \downarrow \tilde{\Lambda}_n(P) \\ \operatorname{Inv}_{\text{hom}}^1(G, K_1^M/n) & \xrightarrow{\operatorname{Inv}(\rho, K_1^M/n)} & \operatorname{Inv}_{\text{hom}}^0(T, K_1^M/n) & \xrightarrow{\operatorname{Inv}(g, K_1^M/n)} & \operatorname{Inv}_{\text{hom}}^0(P, K_1^M/n) \end{array}$$

commutes.

Proof. Unwinding the definitions of $\tilde{\Lambda}_n(T)$ and $\tilde{\Lambda}_n(P)$, one sees that the commutativity of the right square is a consequence of the functoriality of the pullback map on torsors. To be precise, if $\alpha: Y \rightarrow X, \beta: Z \rightarrow Y$ are morphisms of F -schemes, then $\text{Tors}^*(\alpha \circ \beta) = \text{Tors}^*(\beta) \circ \text{Tors}^*(\alpha)$. The left square commutes because pullback operation on torsors commutes with changing the group. \square

As noted at the end of section 2.2, after a diagram chase, this proves:

Theorem 5.4. *The map $\Phi(G, n): G^*[n] \rightarrow \text{Inv}_{\text{hom}}^1(G, K_1^M/n)$ is an isomorphism.* \square

As was the case for type-zero invariants, for any natural number n , there is a group morphism $\text{Inv}^1(G, \iota_n): \text{Inv}_{\text{hom}}^1(G, K_1^M/n) \rightarrow \text{Inv}_{\text{hom}}^1(G, K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})$ given by composition with ι_n . For positive integers n, m with n dividing m , the maps $\text{Inv}^1(G, \iota_n), \text{Inv}^1(G, \iota_m)$ are compatible with the map $\text{Inv}^1(G, \beta_{n,m}): \text{Inv}_{\text{hom}}^1(G, K_1^M/n) \rightarrow \text{Inv}_{\text{hom}}^1(G, K_1^M/m)$ given by composition with $\beta_{n,m}$, and so we obtain a universal induced map

$$\text{colim}_{n \in \mathbb{N}} \text{Inv}^1(G, \iota_n): \text{colim}_{n \in \mathbb{N}} \text{Inv}_{\text{hom}}^1(G, K_1^M/n) \rightarrow \text{Inv}_{\text{hom}}^1(G, K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}).$$

Proposition 5.5. *The map $\text{colim}_{n \in \mathbb{N}} \text{Inv}^1(G, \iota_n)$ is an isomorphism.*

Proof. Set

$$\begin{aligned} u &= \text{colim}_{n \in \mathbb{N}} \text{Inv}(\rho, K_1^M/n), v = \text{colim}_{n \in \mathbb{N}} \text{Inv}(g, K_1^M/n), \\ u' &= \text{Inv}(\rho, K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}), v' = \text{Inv}(g, K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}). \end{aligned}$$

We have a commutative diagram

$$\begin{array}{ccccc} \text{colim}_{n \in \mathbb{N}} \text{Inv}_{\text{hom}}^1(G, K_1^M/n) & \xrightarrow{u} & \text{colim}_{n \in \mathbb{N}} \text{Inv}_{\text{hom}}^0(T, K_1^M/n) & \xrightarrow{v} & \text{colim}_{n \in \mathbb{N}} \text{Inv}_{\text{hom}}^0(P, K_1^M/n) \\ \downarrow \text{colim}_{n \in \mathbb{N}} \text{Inv}^1(G, \iota_n) & & \downarrow \text{colim}_{n \in \mathbb{N}} \text{Inv}^0(T, \iota_n) & & \downarrow \text{colim}_{n \in \mathbb{N}} \text{Inv}^0(P, \iota_n) \\ \text{Inv}_{\text{hom}}^1(G, K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}) & \xrightarrow{u'} & \text{Inv}_{\text{hom}}^0(T, K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}) & \xrightarrow{v'} & \text{Inv}_{\text{hom}}^0(P, K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}) \end{array}$$

whose rows are exact. Since $\text{colim}_{n \in \mathbb{N}} \text{Inv}^0(T, \iota_n)$ and $\text{colim}_{n \in \mathbb{N}} \text{Inv}^0(P, \iota_n)$ are isomorphisms by Proposition 4.6, and u, u' are injective, $\text{colim}_{n \in \mathbb{N}} \text{Inv}^1(G, \iota_n)$ is an isomorphism. \square

Theorem 5.6. *The map $\Phi(G): G_{\text{tors}}^* \rightarrow \text{Inv}_{\text{hom}}^1(G, K_1^M \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z})$ is a group isomorphism.*

Proof. Let n, m be positive integers with n dividing m , and let $\tau_{n,m}: \mu_{n,F} \rightarrow \mu_{m,F}$ be the canonical embedding. We claim that the diagram

$$\begin{array}{ccc} G^*[n] & \xrightarrow{\sigma_{n,m}} & G^*[m] \\ \downarrow \Phi(G,n) & & \downarrow \Phi(G,m) \\ \mathrm{Inv}_{\mathrm{hom}}^1(G, K_1^M/n) & \xrightarrow{\mathrm{Inv}^1(G, \beta_{n,m})} & \mathrm{Inv}_{\mathrm{hom}}^1(G, K_1^M/m) \end{array}$$

commutes, where $\sigma_{n,m}$ is the group morphism given by composition with $\tau_{n,m}$. Indeed, it is sufficient to show that $\Sigma_m(L) \circ \beta_{n,m}(L) = \mathrm{Tors}_*(\tau_{n,m})(L) \circ \Sigma_n(L)$ for any field extension L/F . Fixing $[y] \in L$, put $U = \mathrm{Spec}(L[X]/\langle X^n - y \rangle)$, $V = \mathrm{Spec}(L[X]/\langle X^m - y^{m/n} \rangle)$. The morphism of L -schemes $U \times \mu_{m,L} \rightarrow V$ defined functorially by

$$U(R) \times \mu_{m,L}(R) \rightarrow V(R), (u, z) \mapsto uz$$

for any L -algebra R is constant on $\mu_{n,L}^{\tau_{n,m}}$ -orbits, and so descends to a morphism of L -schemes $(U \times \mu_{m,L})/(\mu_{n,L}^{\tau_{n,m}}) \rightarrow V$, which one may check is $\mu_{m,L}$ -equivariant. This establishes that $\mathrm{Tors}_*(\tau_{n,m})(L)(U) = V$.

The universally induced map $\mathrm{colim}_{n \in \mathbb{N}} \Phi(G, n): G_{\mathrm{tors}}^* \rightarrow \mathrm{colim}_{n \in \mathbb{N}} \mathrm{Inv}^1(G, K_1^M/n)$ is an isomorphism, as $\Phi(G, n)$ is an isomorphism for each n . Since $\Phi(G)$ is just the composition of $\mathrm{colim}_{n \in \mathbb{N}} \Phi(G, n)$ with the $\mathrm{colim}_{n \in \mathbb{N}} \mathrm{Inv}^1(G, \iota_n)$, it is an isomorphism by Proposition 5.5. \square

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