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On the Number of Injective Indecomposable Modules

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For every natural number m, there exists a noncommutative valuation ring R with a completely prime ideal P so that there are exactly m nonisomorphic indecomposable injective right R-modules with P as associated prime ideal.

Key Words: indecomposable injective modules; valuation rings.

1. INTRODUCTION

Let R be a ring with identity. We will denote by spec(R) the set of all prime ideals of R and by Ind(R) a set of representatives of the isomorphism classes of indecomposable, injective right R-modules. Matlis in [12, Proposition 3.1] establishes a one-to-one correspondence between spec(R) and Ind(R) for R commutative and noetherian by assigning the injective hull $E(R/P)$ of R/P to the prime ideal P of R. We mention a few instances in which this result has been extended and generalized. If R is right noetherian, but not necessarily commutative, a mapping φ from Ind(R) onto spec(R) can be defined by defining $\varphi(E) = \text{Ass}(E)$ for E in Ind R; see [10, 3.60 Theorem], where Ass(E) consists of a single prime

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ideal of R and $\mathrm{Ass}(E(R/P)_R) = P$ for a prime ideal P of R. Krause in [9] gives the exact extra conditions on R for this mapping φ to be one-to-one. Facchini in [6] shows that there is a one-to-one correspondence between $\mathrm{Ind}(R)$ and $\mathrm{spec}(R)$ for a commutative integral domain R for which any indecomposable injective R-module is uniserial over its endomorphism ring, if and only if R is a Prüfer domain with $P \neq P^2$ for every nonzero prime ideal P of R. This last condition, that no nonzero prime ideal is idempotent, characterizes therefore the commutative valuation domains V so that every injective indecomposable V-module E is isomorphic to $E(R/P)$ for some prime ideal P of V; see also [15, 16, 18]. Törner showed in [17] that for invariant valuation domains R with maximum condition on prime ideals, either exactly one, two, or infinitely many nonisomorphic injective indecomposable modules exist which are associated with the same prime ideal P of R.

We consider in this paper not necessarily commutative valuation rings R, i.e., subrings R of a skew field F such that $x \in F \setminus R$ implies $x^{-1} \in R$. If E_R is an injective indecomposable right R-module, then $P(E) = \{p \in R \mid \exists\ 0 \neq m \in E \text{ with } mp = 0\}$ is a completely prime ideal of R, the associated prime of E; see Lemma 2.3. We are particularly interested in the subset $\mathscr{E}(P)$ of $\mathrm{Ind}(R)$ consisting of all E in $\mathrm{Ind}(R)$ with $P(E) = P$ for a given completely prime ideal of R.

If $P = (0)$, then the quotient skew field of R, $Q(R) = E(R/(0)) = F_R$, is the only indecomposable injective module with P as associated prime. The remaining completely prime ideals P are divided into four types: (i) P is a limit prime; i.e., $P = \bigcup P_i$, $P \supset P_i$, is the union of completely prime ideals P_i properly contained in P; or (ii) $P \supset\!\!\!\cdot\ P'$ has a lower neighbor P' in the lattice of completely prime ideals of R. We then say $P \supset\!\!\!\cdot\ P'$ is a prime segment of R. Prime segments $P \supset\!\!\!\cdot\ P'$ are further classified in Lemma 3.1: (ii, a) If $aP = Pa$ for $a \in P \setminus P'$, then the prime segment is called invariant. (ii, b) If there are no further ideals of R between P and P', then the prime segment $P \supset\!\!\!\cdot\ P'$ is simple. (ii, c) If there exists a prime ideal Q of R with $P \supset Q \supset P'$, then the prime segment is called exceptional. Every nonzero completely prime ideal P of R falls into exactly one of the four categories (i), (ii, a), (ii, b), or (ii, c). Limit primes are called not branched prime ideals in [8].

Corollary 3.5 characterizes the completely prime ideals $P \neq (0)$ of R with $|\mathscr{E}(P)| = 1$ as those for which $P \neq P^2$. This is equivalent to the condition given in the case (ii, a) with the additional requirement that the factor ring $R_P/P'R_P$ of the localization R_P of R at P is a principal ideal ring.

In Theorem 3.8 it is proved that for a completely prime ideal P of R of type (ii, a), only the cases $|\mathscr{E}(P)| = 1$, 2, or ∞ can occur.

As the main result of this paper, we show in Theorem 4.1 that there exists for any $m \geq 2$ a valuation ring R with exactly one prime segment $J(R) \supset (0)$, which is simple, so that R has exactly m indecomposable injective nonisomorphic torsion right R-modules; i.e., $|\mathscr{E}(J(R))| = m$. Here, $J(R)$ is the maximal ideal, the Jacobson radical of R. We do need some results about limit primes P of a valuation ring R and we show that $|\mathscr{E}(P)| = 2$ or ∞ in this case is possible; see Lemmas 3.9 and 3.10 and Example 3.11. We do not know the possible values for $|\mathscr{E}(P)|$ if P is a limit prime or if P has a lower neighbor P' in the lattice of completely prime ideals and the segment $P \supset P'$ is exceptional. However, it follows from these results that no prime ideal Q of R which is not completely prime is associated to an injective indecomposable right R-module. These prime ideals Q are exactly the prime ideals of R for which R/Q is a non-Goldie ring.

2. INJECTIVE INDECOMPOSABLE MODULES, RELATED RIGHT IDEALS, AND ASSOCIATED PRIME IDEALS

Throughout this paper, R will be a valuation ring; that is, R is a subring of a skew field F such that $a \in F \setminus R$ implies $a^{-1} \in R$. A valuation ring R has a unique maximal ideal $J(R) = R \setminus U(R)$ with $U(R)$ the group of units of R. Since the lattice of right ideals of R is a chain, it follows that any right ideal I of R is irreducible and that a right R-module E is an injective indecomposable right R-module if and only if $E \cong E(R/I)$, where $E(R/I)$ is the injective hull of R/I for a right ideal $I \neq R$.

If E is an injective indecomposable right R-module that contains an element m such that $mr = 0$ implies $r = 0$ for $r \in R$, then $E \cong E(R)$. This module is isomorphic to F_R, since R is essential in the torsion-free and divisible module F_R. With this argument used in [10, 3.25], modified for right Ore domains, it follows that torsion-free divisible right R-modules are injective. We can therefore restrict ourselves to injective indecomposable right R-modules of the form $E(R/I)$ with I a right ideal of R not equal to R or (0). These modules are torsion modules.

If $I \neq R$ is a nonzero right ideal of R, then $a^{-1}I := \{r \in R \mid ar \in I\}$ for $a \in R$. It follows from [4] that $E(R/I_1) \cong E(R/I_2)$ for right ideals I_1, I_2 if and only if $s_1^{-1}I_1 = s_2^{-1}I_2$ for elements $s_j \in R \setminus I_j$. We write $I_1 \sim I_2$ or $I_1 \sim_R I_2$ in this case and say that I_1 and I_2 are *related*. It follows from the next result that this condition takes on a particularly easy form if R is a valuation ring.

LEMMA 2.1. *The following conditions are equivalent for right ideals I_1 and I_2 of R that are different from* (0) *and R*:

(a) $I_2 = t^{-1}I_1$ *for some* $t \in R \setminus I_1$.

(b) $tI_2 = I_1$ *for some* $t \in R \setminus I_1$.

(c) $rI_2 = I_1$ *for some* $r \in R$.

Proof. To show that (a) implies (b), assume $tI_2 \subset I_1$ and $b \in I_1 \setminus tI_2$. Then $bR \subset tR$, since $t \notin I_1$, $b = tb_1$ for $b_1 \in R$, $b_1 \in t^{-1}I_1 = I_2$, and $b = tb_1 \in tI_2$, a contradiction that shows $I_1 = tI_2$. It remains to show that (c) implies (a). The element r is not zero and is not in I_1, since otherwise $r = rt_1$ for $t_1 \in I_2 \subseteq J(R)$ implies $r(1 - t_1) = 0$ and $r = 0$. We have $I_2 \subseteq r^{-1}I_1$. If $b \in r^{-1}I_1$, then $rb \in I_1 = rI_2$ and $b \in I_2$, and $I_2 = r^{-1}I_1$, $r \in R \setminus I_1$ follows. ∎

COROLLARY 2.2. *Two right ideals* I_1 *and* I_2 *of* R, *different from* (0) *and* R, *are related if and only if either* $I_1 = aI_2$ *or* $aI_1 = I_2$ *for an element* $a \in R$.

Proof. If $s_1^{-1}I_1 = s_2^{-1}I_2 = I$, then $I_1 = s_1 I$ and $I_2 = s_2 I$, where $s_i \in R \setminus I_i$. Since R is a valuation ring, either $s_1 = as_2$ or $s_2 = as_1$ for some $a \in R$. Then $I_1 = aI_2$ in the first case and $I_2 = aI_1$ in the second. ∎

Let E be an indecomposable injective right R-module. We define $P(E) = \{p \in R \mid \exists\ 0 \neq m \in E \text{ with } mp = 0\}$. It follows from the next lemma that $P = P(E)$ is a completely prime ideal of R, the *associated prime of* E. Isomorphic injective indecomposable right R-modules E_1 and E_2 have identical associated prime ideals $P(E_1) = P(E_2)$. It is the topic of this paper to obtain some information on the set $\mathscr{E}(P)$ of all nonisomorphic indecomposable injective right R-modules E with $P(E) = P$. Since it also follows from the next lemma that $P(E(R/P)) = P$ for a completely prime ideal $P \neq (0)$ of R, we obtain $|\mathscr{E}(P)| \geq 1$.

For a right ideal I of R which is not equal to (0) or R, we define the sets $P_r(I) = \{p \in R \mid \exists\ t \notin I \text{ with } tp \in I\}$ and $S(I) = \{s \in R \mid Is = I\}$; again we say that $P_r(I)$ is *the associated prime* ideal of I.

LEMMA 2.3. *Let* R *be a valuation ring, let* $I \neq R$, (0) *be a right ideal of* R, *and let* E *be a torsion, indecomposable, injective right* R-*module. Then*

(i) $P_r(I) = R \setminus S(I)$ *is a completely prime ideal.*

(ii) $P_r(I) = \bigcup_{I' \sim I} I'$ *is the union of all right ideals* I' *of* R *that are related to* I.

(iii) *If* $E = E(R/I)$, *then* $P(E) = P_r(I)$.

Proof. That $P_r(I) = R \setminus S(I)$ follows if we rewrite the definitions of these sets with the help of Lemma 2.1. We have $P_r(I) = \{p \in R \mid Ip^{-1} \supset I\}$ and $S(I) = \{s \in R \mid Is^{-1} = I\}$. Since $Ir^{-1} \supseteq I$ for any $r \in R$, it follows immediately that $P_r(I) = R \setminus S(I)$. For $p \in P_r(I)$ and $r \in R$, we have $I \supset Ip$ and $I \supset Irp$, and $I \supset Ipr$ follows. This shows that $P_r(I)$ is an ideal, and since its complement in R is multiplicatively closed, it follows that $P_r(I)$ is completely prime.

In order to prove (ii), we rewrite $P_r(I)$ again: We obtain from the definition that $P_r(I) = \bigcup_{t \in R \setminus I} t^{-1}I$. This implies that $P_r(I) = \bigcup_{I' \sim I} I'$. To see this, we observe first that $I' \sim I$ implies either $I' = aI$ for $0 \neq a \in R$, and I' is then contained in the ideal $P_r(I)$ that contains I, or $I' = t^{-1}I$ for some $t \in R \setminus I$; hence $\bigcup_{I' \sim I} I' \subseteq P_r(I) = \bigcup_{t \in R \setminus I} t^{-1}I \subseteq \bigcup_{I' \sim I} I'$, and the equality $P_r(I) = \bigcup_{I' \sim I} I'$ follows.

In the proof of (iii), we show first that $P_r(I)$ is contained in $P(E)$. If $p \in P_r(I)$, then there exists $0 \neq t + I = m \in R/I$ with $mp = 0$, and hence $p \in P(E)$. Conversely, if $q \in P(E)$, then $mq = 0$ for some $0 \neq m \in E(R/I)$. Hence, $q \in I' = \{r \in R \mid mr = 0\}$ and $E(mR) \cong E(R/I') \cong E(R/I)$. Therefore, $I' \sim I$ and $q \in I' \subseteq P_r(I)$. ▮

COROLLARY 2.4. *Let P be a nonzero completely prime ideal in R. Then $|\mathscr{E}(P)|$ is equal to the number of equivalence classes with respect to the related condition in the set of nonzero right ideals $I \neq R$ of R with $P_r(I) = P$.*

3. PRIME SEGMENTS AND LIMIT PRIMES

As we assume throughout the paper, let R be a valuation ring, let $I \neq (0)$, R, be a right ideal of R, and let $P = P_r(I)$ be the associated completely prime ideal. Then $(0) \subset P \subseteq J(R) \subset R$.

We recall first some results about prime segments of R. Either $P = \bigcup_{i \in \Lambda} P_i$, $P_i \subset P$ completely prime ideals in R, and we say that P is a *limit prime*, or there exists a completely prime ideal P' with $P \supset P'$ but no further completely prime ideal of R properly between P and P'. We then say that $P \supset P'$ is a *prime segment* of R. Prime segments correspond to jumps as defined for totally ordered groups; see [7]. We say that R has rank n if R has exactly n nonzero completely prime ideals.

The next result can be found in [3]; see also [1].

LEMMA 3.1. *Assume that $P \supset P'$ is a prime segment in the valuation ring R. Then exactly one of the following alternatives occurs*:

(a) *The prime segment $P \supset P'$ is invariant*; *i.e.*, *$aP = Pa$ for $a \in P \setminus P'$.*

(b) *The prime segment $P \supset P'$ is simple*; *i.e.*, *there are no ideals of R properly between P and P'.*

(c) *The prime segment $P \supset P'$ is exceptional*; *i.e.*, *there exists a prime ideal Q of R with $P \supset Q \supset P'$. Then there are no ideals properly between P and Q and $\bigcap Q^n = P'$.*

It follows that $P = P^2$ in the cases (b) and (c). Of course, if $P = \bigcup_{i \in \Lambda} P_i$, $P_i \subset P$, is a limit prime, then $P_i \subset P^2$ for all $i \in \Lambda$ and $P = P^2$ in this case as well.

The next result shows that for the computation of $|\mathscr{E}(P)|$, we can always assume that $P = J(R)$ is the maximal ideal of R.

Let P be a completely prime ideal of the valuation ring R. Then $S = R \setminus P$ is a multiplicatively closed subset of R, and for $r \in R$, $s \in S$, there exists $t \in R$ with either $r = st$ or $s = rt$, and r, t are in S in the second case. It follows that the localization $R_P = RS^{-1} = \{rs^{-1} \mid r \in R, s \in S\}$ exists and that $R_P = R \cup S^{-1}$.

LEMMA 3.2. *Assume that I_1 and $I_2 \neq (0)$ and R are right ideals of R with $P_r(I_1) = P_r(I_2) = P$. Then*

(a) *$I_i R_P = I_i$ for $i = 1, 2$, and I_1 and I_2 are related in R if and only if they are related in R_P.*

(b) *For every completely prime ideal $\tilde{P} \subset P$ of R, there exist right ideals $\tilde{I}_i \sim I_i$ in R for $i = 1, 2$ with $\tilde{P} \subset \tilde{I}_i \subseteq P$.*

Proof. Since $P_r(I_i) = P$ implies $I_i s = I_i$ for all $s \in S = R \setminus P$, it follows that $I_i s^{-1} = I_i$ and $I_i R_P = I_i$ for $i = 1, 2$. If $I_1 \sim I_2$ in R, then $aI_1 = I_2$ or $I_1 = aI_2$ for some $a \in R$ and $I_1 \sim I_2$ in R_P. Conversely, if $bI_1 = I_2$ for $b \in R_P$, then either $b \in R$ or $b = s^{-1}$ for $s \in S$. It follows that $I_1 \sim I_2$ in R. The part (b) follows from Lemma 2.3(ii). ∎

COROLLARY 3.3. *Let $P \neq (0)$ be a completely prime ideal of R. Then $|\mathscr{E}_R(P)| = |\mathscr{E}_{R_P}(P)| = |\mathscr{E}_{\overline{R}}(\overline{P})|$ for every completely prime ideal $\hat{P} \subset P$ of R with $\overline{R} = R_P/\hat{P}$ and $\overline{P} = PR_P/\hat{P}$.*

Proof. We saw in the previous lemma that classes of related right ideals of R with associated prime ideal P correspond to classes of right ideals of R_P with associated prime ideal $PR_P = P$. Since, conversely, every right ideal $\overline{I} \neq (0)$, R_P, in R_P is also a right ideal of R, the first equation follows. The second equation is an immediate consequence of part (b) of Lemma 3.2. ∎

In the case where P has a lower neighbour P' in the chain of completely prime ideals of R, it is sufficient to determine $|\mathscr{E}_{R_P/P'R_P}(\overline{P})|$, where $\overline{R} = R_P/P'Rp$ is now a rank-1 valuation ring with $J(\overline{R}) = \overline{P}$ the only nonzero completely prime ideal.

It follows from Lemma 3.1 that a rank-1 valuation ring R falls into one of the following three categories:

(a) R is invariant; i.e., $aR = Ra$ for all $a \in R$.

(b) R is nearly simple; i.e., R, $J(R)$, and (0) are the only ideals of R.

(c) R is exceptional; i.e., there exists a prime ideal Q in R which is not completely prime (see also [5]).

We can now assume that R is a valuation ring with $P = J(R)$. The principal right ideals $I = aR \neq (0), R$ are all related and $aRs = aR$ is possible only for $s \in U(R)$; hence $P_r(I) = J$.

If we assume that $|\mathscr{E}_R(P)| = 1$, then $P = aR$ for some $a \in R$, since $P_r(P) = P$, and $P \neq P^2 = a^2R$ follows; see below. This proves one half of the next result.

LEMMA 3.4. *Let P be a nonzero completely prime ideal in the valuation ring R. Then $|\mathscr{E}(P)| = 1$ if and only if $P \neq P^2$.*

Proof. It remains to prove that $|\mathscr{E}(P)| = 1$ if $P \neq P^2$. We can assume that $P = J(R)$ and that $\bigcap P^n = (0)$. The last assumption is justified since $\bigcap P^n = P'$ is a completely prime ideal [3] and $P \neq P^2$ implies that $P \supset P'$ is a prime segment. It follows that $P = aR$ for $a \in P \setminus P^2$ and that $P^n = a^nR$, since P is a two-sided ideal. Hence, for every nonzero right ideal $I \neq R$, there exists an integer n with $a^nR \supseteq I \supset a^{n+1}R$ and $I = a^nR$ follows. ∎

If $P = J(R) = aR$ for $a \in R$, then $Ra = aR$. Otherwise, $Ra \subset aR$ and there exist elements $r \in R$, $j \in P$, with $ar \in aR \setminus Ra$ and $a = jar$. However, $j = ar'$ for $r' \in R$ leads to $a = ar'ar = a^2r''r$ for $r'' \in R$ and $a = 0$, a contradiction. It follows that R is invariant if R is a rank-1 valuation ring with $P \neq P^2$ and $P = J(R)$.

COROLLARY 3.5. *The following conditions are equivalent for a nonzero completely prime ideal P of a valuation ring R:*

(a) $|\mathscr{E}(P)| = 1$.

(b) $P \neq P^2$.

(c) *P has a lower neighbour P' in the lattice of completely prime ideals of R, the segment $P \supset P'$ is invariant, and $R_P/(P'R_P)$ is a right and left principal ideal ring.*

COROLLARY 3.6. *Let R be a valuation ring. There exists a one-to-one correspondence between* spec R *and the set of isomorphism classes of injective indecomposable R-modules if and only if $P^2 \neq P$ for all completely prime ideals $P \neq (0)$ of R.*

Remark 3.7. A valuation ring R as in Corollary 3.6 has no limit primes and only invariant prime segments; in particular, spec R consists of completely prime ideals only. However, R itself is not necessarily invariant if the rank of R is greater than 1.

We saw that a completely prime ideal P of R with $P^2 \neq P$ has a lower neighbour $P' = \bigcap P^n$ and that the segment $P \supset P'$ is invariant. The value group of the rank-1 valuation ring $R_P/(P'R_P)$ is isomorphic to $(\mathbb{Z}, +)$ and we say that the *segment* $P \supset P'$ is *discrete*.

If $P \supset P'$ is invariant, then the *value group* $G(P, P')$ of $R_P/(P'R_P)$ is isomorphic to a subgroup of $(\mathbb{R}, +)$ (see [3] and below).

We next prove the following result:

THEOREM 3.8. *Let R be a valuation ring and let $P \supset P'$ be an invariant prime segment of R. Then*:

(a) $|\mathscr{E}(P)| = 1$ *if and only if* $P^2 \neq P$, *if and only if* $G(P, P') \cong (\mathbb{Z}, +)$.

(b) $|\mathscr{E}(P)| = 2$ *if and only if* $G(P, P') \cong (\mathbb{R}, +)$.

(c) $|\mathscr{E}(P)| = \infty$ *if and only if* $(\mathbb{Z}, +) \ncong G(P, P') \ncong (\mathbb{R}, +)$.

Proof. It follows from Corollary 3.3 that we can assume that R is an invariant rank-1 valuation ring with $P = J(R) \supset P' = (0)$. Part (a) of the theorem follows from Corollary 3.5.

The group $G(P, P')$ is equal to $(\{\alpha R \mid \alpha \in F^*\}, \cdot)$, where F is the skew field of quotients of R and $(\alpha R) \cdot (\beta R) = \alpha\beta R$ defines the operation. This is a group, since R is invariant; it is totally ordered if we define $\alpha R \geq \beta R$ if and only if $\alpha R \subseteq \beta R$; and it is archimedean since R has rank 1. It follows by Hölder's Theorem [7] that $G(P, P')$ is isomorphic to a subgroup of $(\mathbb{R}, +)$ as ordered groups.

If $G(P, P') \ncong \mathbb{Z}$, no smallest positive element exists in this group, P is not finitely generated as a right ideal, and hence $P^2 = P$.

We can then consider the group $\Gamma = (\{\alpha P \mid \alpha \in F^*\}, \cdot)$ with $\alpha P \cdot \beta P = \alpha\beta P$, $\alpha, \beta \in F^*$, and $\alpha P \geq \beta P$ if and only if $\alpha P \subseteq \beta P$.

The group Γ is order-isomorphic to $G(P, P')$ and an order monomorphism w exists from Γ into $(\mathbb{R}, +)$. If $I \neq F$ is any not finitely generated R-module in F, then $I = \bigcup_{\alpha \in I} \alpha P$ and we can define $w_1(I) = \inf\{w(\alpha P) \mid \alpha \in I\} \in \mathbb{R}$.

It follows (with arguments similar to the arguments in [17]) that w_1 is an extension of w and defines an order isomorphism between the group of not finitely generated right R-submodules $\neq F$ of F and $(\mathbb{R}, +)$. That $\Gamma_1 = (\{I \subset F \mid I \text{ not finitely generated right } R\text{-module}\}, \cdot)$, with $I_1 \cdot I_2 = \{\sum a_i b_i \mid a_i \in I_1,\ b_i \in I_2\}$ as operation, is a group, follows from the fact that w_1 is a bijective mapping from Γ_1 to $\mathbb{R}$ with $w_1(I_1 I_2) = w_1(I_1) + w_1(I_2)$ for $I_1, I_2 \in \Gamma_1$. The element $I_\rho \in \Gamma_1$ that corresponds to $\rho \in \mathbb{R}$ is given by $I_\rho = \bigcup_{\rho \leq w(\alpha P)} \alpha P$.

The group $w(\Gamma)$ is dense in $\mathbb{R} = w_1(\Gamma_1)$. If $I_1 \supseteq I_2$ are right ideals in Γ_1, i.e., not finitely generated, then $I_1 \sim I_2$ if and only if $aI_1 = I_2$ for $a \in R$ if and only if $aPI_1 = I_2$. Therefore, $w_1(aPI_1) = w(aP) + w_1(I_1) = w_1(I_2)$. We can formulate this in the following form: Two right ideals $I_1, I_2 \in \Gamma_1$ are related if and only if $w_1(I_1) - w_1(I_2) \in w(\Gamma)$.

The statement (b) of the theorem follows immediately. $|\mathscr{E}(P)| = 2$ if and only if $P = P^2$ and $\Gamma \cong \mathbb{R}$. The two related classes which determine two nonisomorphic indecomposable R-modules are $\{aR \mid 0 \neq a \in J\}$ and $\{aP \mid 0 \neq 0\}$. Every right ideal of R is of the form aR or aP for $a \in R$.

To prove (c), it must be shown that there exist infinitely many related classes of infinitely generated right ideals in R if $\Gamma \not\cong \mathbb{R}$ and $P = P^2$. We assume on the contrary that $w(\Gamma)$ is dense in $\mathbb{R}$ and that there are only finitely many, say, n related classes of infinitely generated right ideals in R. Let I be any right ideal with $0 < w_1(I) < 1/(2n!)$ in $\mathbb{R}$. Since not all the powers I^i, $i = 1, 2, \ldots, n, n+1$, are unrelated, there exist $n + 1 \geq m > k \geq 1$ with $aI^k = I^m$ and $I^{m-k} = aP$. This implies that $w_1(I^{m-k}) \in w(\Gamma)$. It follows that $w_1(I^{n!}) = n!(w_1(I)) \in w(\Gamma)$. However, for every real number $0 < r < 1/(2n!)$ there exists a right ideal I in Γ_1 with $w_1(I) = r$ and $n! \cdot r \in w(\Gamma)$, and all real numbers t with $0 \leq t < 1/2$ are in $w(\Gamma)$. Hence, $w(\Gamma) = \mathbb{R}$ and we are in case (b) and have two classes of related right ideals only. ▌

In contrast to the result in the last theorem, we will construct for every $m \geq 2$ nearly simple rings R with $|\mathscr{E}(J(R))| = m$.

Dubrovin [5] has constructed a rank-1 valuation ring R with an exceptional prime segment $J(R) \supset (0)$. The principal right ideals of this ring are given in the form $t^{\rho}R$ with $0 \leq \rho \in \mathbb{R}$ and $t^{\rho_1}R \supseteq t^{\rho_2}R$ if and only if $\rho_1 \leq \rho_2$. It follows that there are exactly two related classes of right ideals $I \neq R, (0)$: The principal right ideals $\{aR \mid 0 \neq a \in J(R)\}$ and the infinitely generated ones $\{aP \mid 0 \neq a \in R\}$ with $P = J(R)$. We have no further information about $\mathscr{E}(J(R))$ for R an exceptional rank-1 valuation ring.

Finally, we consider limit primes $P = \bigcup_{i \in \Lambda} P_i$, $P \supset P_i$, $i \in \Lambda$, in a valuation ring R. Let R be a valuation ring, let $J = J(R)$ be the maximal ideal of R, and let $P = \bigcup P_i$ be a limit prime. We consider for any $a \in J$ the intersection $P(a) := \bigcap_{a \in P} P$ of all completely prime ideals P of R containing a. Then $P(a)$ is a completely prime ideal. The union $P'(a) := \bigcup_{a \notin Q} Q$ of all completely prime ideals Q of R not containing a is also a completely prime ideal $P'(a)$ and $P(a) \supset P'(a)$ is a prime segment. Hence, we can assume that, in the representation $P = \bigcup_{i \in \Lambda} P_i$ of a limit prime, every P_i has a lower neighbor P_i' by replacing P_i by $P(a_i)$ for $a_i \in P \setminus P_i$. We can also assume that Λ is a well-ordered index set with $P_i \subset P_j$ for $i < j$. In fact, we will only consider limit primes P where $\Lambda = \mathbb{N}$; i.e., $P = \bigcup_{i \in \mathbb{N}} P_i$, $P \supset P_i \supset P_i'$. We will then say that P is given in *standard form*. The next result provides us with a family of right ideals I in a valuation ring R with limit prime P, so that $P_r(I) = P$.

LEMMA 3.9. *Assume that $P = \bigcup_{i \in \mathbb{N}} P_i$ is a limit prime of a valuation ring R given in standard form. For each $i \geq 1$, choose a nonzero $a_i \in P_i \setminus P_i'$*

and $a_0 \neq 0 \in R$. *Then* $I = \bigcap I_n$ *with* $I_n = a_0 a_1 \cdots a_n R$ *is a right ideal of* R *with* $I \neq cP$ *for all* c *in* R *and* $P_r(I) = P$ *provided* $I \neq (0)$.

Proof. To show that $P \subseteq P_r(I)$, it is enough to prove that $a_n a_{n+1}^2 \in P_r(I)$ for $n \geq 1$, since $a_n a_{n+1}^2 \in P_n \setminus P_{n-1}$. The element $t_n = a_0 a_1 \ldots a_{n-1}$ is not contained in I since $a_n \in J(R)$. We have $a_{n+1} = a_{n+2} \ldots a_{n+k} r_k$ for some $r_k \in R$ and $k \geq 2$ since $a_{n+2} \ldots a_{n+k} \notin P_{n+1}$. Therefore, $t_n(a_n a_{n+1}^2) = (a_0 a_1 \ldots a_{n-1} a_n a_{n+1}) a_{n+2} \ldots a_{n+k} r_k$ for all $k \geq 2$. This shows that $t_n(a_n a_{n+1}^2) \in I$ and $a_n a_{n+1}^2 \in P_r(I)$, and $P \subseteq P_r(I)$ follows.

To prove that $P_r(I) \subseteq P$, we assume that $tp \in I$ for $t \in R \setminus I$, $0 \neq p \in R$. Then there exists an integer n and $r \in R$ with $a_0 a_1 \ldots a_n = tr$ and $tp = a_0 a_1 \ldots a_{n+1} v = tra_{n+1} v$ for some $v \in R$. Hence, $p = ra_{n+1} v \in P_{n+1} \subseteq P$ and $P_r(I) = P$.

It remains to prove that $I \neq cP$ for $c \in R$. Otherwise, we have $I = cP$ and $c \notin I$. Hence, there exist an integer n and r in R with $a_0 a_1 \ldots a_n = cr$. Since $a_{n+1} \in P$, it follows that $a_0 \ldots a_{n+1} = cra_{n+1} \in cP = I$, a contradiction that proves $I \neq cP$. ∎

We will prove one more result about the right ideals associated with a limit prime $P = \bigcup_{i \in \mathbb{N}} P_i$ in standard form. By Corollary 3.3 we can assume that $P = J(R)$ is the maximal ideal in the valuation ring R. The ideal P is not finitely generated as a right ideal, since $P = P^2$. The principal right ideals $\{aR \mid 0 \neq a \in J\}$ form one related class associated with P. A proper nonzero right ideal I of R is not finitely generated if and only if $I = IP$, since then for every $a \in I$ exists $b \in I$ with $aR \subset bR \subset I$. Let I be a not finitely generated ideal with $P_r(I) = P$. Then $I = IP = I(\bigcup_{n \in \mathbb{N}} P_n) = \bigcup_{n \in \mathbb{N}} IP_n$, and we will show that $P_r(IP_n) = P_n$. Since $P_n s = P_n$ for all $s \in P \setminus P_n$, we have $P_r(IP_n) \subseteq P_n$. Since $P_r(I) = P \supset P_n$, there exists by Lemma 2.3(ii) a right ideal I' in R related to I and $P \supseteq I' \supset P_n$. Hence IP_n and $I'P_n$ are related and $I'P_n \supset P_n'$, the lower neighbour of P_n. It follows that $P_r(IP_n) \supset P_n'$ and $P_r(IP_n) = P_n$.

We now assume that I_1 and I_2 are not finitely generated right ideals of R associated with P and that $|\mathscr{E}(P_n)| \leq 2$ for all $n \geq n_0$. This happens for invariant prime segments $P_n \supset P_n'$ if and only if the corresponding groups are isomorphic either to $\mathbb{Z}$ or $\mathbb{R}$ (see Theorem 3.8). Then $I_1 P_n$ and $I_2 P_n$ are both associated with P_n. If $P_n^2 \neq P_n$, these two right ideals are related, since there exists only one related class of right ideals associated with P_n. If $P_n^2 = P_n$, then $(I_i P_n) P_n = I_i P_n$ for $i = 1, 2$ and then again $I_1 P_n$ and $I_2 P_n$ are related, since $|\mathscr{E}(P_n)| = 2$ and neither $I_1 P_n$ nor $I_2 P_n$ is of the form aR_{P_n} for $a \in P_n$ and $n \geq n_0$. It follows that there exists for $n \geq n_0$ an element $a_n \in R$ with $a_n I_1 P_n = I_2 P_n$, or $I_1 P_n = b_n I_2 P_n$ for some $b_n \in R$. Since $sP_n = P_n$ for $s \notin P_n$, it follows that $P_m P_n = P_n$ for $m > n$ and $a_m I_1 P_m = I_2 P_m$ implies $a_m I_1 P_n = I_2 P_n$. We can therefore assume that there exist elements a_n in R with $a_n I_1 P_n = I_2 P_n$ for all $n \geq n_0$ and

$a_m I_1 P_n = a_n I_1 P_n$ for $m \geq n$. It is not possible to conclude from this that I_1 and I_2 are related, but we obtain the following result if we add an extra condition.

LEMMA 3.10. *Let $P = \bigcup_{i \in \mathbb{N}} P_i$ be a limit prime given in standard form. Assume that $|\mathscr{E}(P_n)| \in \{1, 2\}$ for $n \geq n_0$. Further assume that for right ideals $I_1 \supseteq I_2 \neq (0)$ of R associated with P there exists $a \in R$ with $aI_1 P_n = I_2 P_n$ whenever there exist elements $a_n \in R$ with $a_n I_1 P_n = I_2 P_n$ and $a_m I_1 P_n = a_n I_1 P_n$ for $m \geq n \geq n_0$. Then $|\mathscr{E}(P)| = 2$.*

Proof. Since $P = P^2$, we have $|\mathscr{E}(P)| \geq 2$. The arguments before the lemma and the assumptions show that any two right ideals I_1, I_2 with $I_i P = I_i$ for $i = 1, 2$ and associated with P are related. ∎

We conclude this section with examples that show that $|\mathscr{E}(P)| = 2$ and $|\mathscr{E}(P)| = \infty$ are possible for a limit prime P.

If $(G, \geq)$ is any ordered group, then $D = \mathbb{Q}((G))$, the Malcev–Neumann ring of generalized power series, is a skew field. It consists of elements α of the form $\alpha = \Sigma g a_g$ with $g \in G$, $a_g \in \mathbb{Q}$, and the support $\operatorname{supp}(\alpha) = \{g \mid a_g \neq 0\}$ well ordered. The subring $V = \{\Sigma g a_g \in D \mid e \leq g\}$ of D is an invariant valuation ring and G is the associated group. The field $\mathbb{Q}$ of rational numbers can be replaced by any skew field in this construction.

We will write $\operatorname{Pos}(G)$ for the set $\{g \in G \mid e \leq g\}$, and a right ideal I_0 of $\operatorname{Pos}(G)$ is a nonempty subset I_0 of $\operatorname{Pos}(G)$ with $I_0 \operatorname{Pos}(G) \subseteq I_0$. The principal right ideals $I \neq (0)$ of V have the form $I = gV$ for some $g \in \operatorname{Pos}(G)$ and $(I \cap \operatorname{Pos}(G))V = I$ for any right ideal I of V.

EXAMPLE 3.11. Let $\hat{G}$ be the subgroup of the direct product $\prod_{i \in \mathbb{Z}} \mathbb{Z}_i$ of copies of $(\mathbb{Z}, +)$ consisting of all elements $g = (z_i)$ with well-ordered support $\{i \mid z_i \neq 0\}$. Then $e = (0)$ is the identity of $\hat{G}$, for every $e \neq g = (z_i) \in \hat{G}$ there exists $\operatorname{lind}(g) = \min\{i \mid z_i \neq 0\}$, the *leading index* of g, and $\hat{G}$ is linearly ordered lexicographically from the left: $\operatorname{Pos}(\hat{G}) = \{g = (z_i) \in \hat{G} \mid z_{i_0} > 0 \text{ for } \operatorname{lind}(g) = i_0\} \cup \{e\}$.

The group $\hat{G}$ contains as subgroup the group $G = \sum_{i \in \mathbb{Z}} \mathbb{Z}_i$, the direct sum of the $\mathbb{Z}_i$, $i \in \mathbb{Z}$, and G is also lexicographically ordered: $\operatorname{Pos}(G) = G \cap \operatorname{Pos}(\hat{G})$. Thus there exists the valued field (D, V) with G as associated value group.

Let $P \neq (0)$ be a prime ideal of V. Then $P = (P \cap \operatorname{Pos}(G))V$ and P is generated as a right V-ideal by $P_0 = P \cap \operatorname{Pos}(G)$. If $\{\operatorname{lind}(g) \mid g \in P\}$ has no maximum, then $P = J(V)$ is the maximal ideal of V. Otherwise, $\{\operatorname{lind}(g) \mid g \in P\}$ has a maximum, say n, and $P = P_n$ is in this case the prime ideal of V generated by $\{g \in \operatorname{Pos}(G) \mid \operatorname{lind}(g) \leq n\}$. If we write e_n for the element $e_n = (z_i)$ with $z_n = 1$ and $z_i = 0$ otherwise, then $P_n =$

$\langle g_k^{(n)} \mid g_k^{(n)} = e_n - ke_{n+1},\ k \in \mathbb{N}\rangle$; i.e., P_n is also generated by the elements $g_k^{(n)}$ as a right ideal of V.

To prove this, we assume that $g = (z_i) \in P_n$ with $\operatorname{lind}(g) = n$ and z_n minimal. If $z_n > 1$, then $(g - e_n) + (e_n) \in P$ with $e_n, g - e_n \notin P$, a contradiction, since P is a prime ideal. We observe that addition in G corresponds to multiplication in D and we will also use below multiplicative notation for the operation in G. Similarly, if there exists a minimum $h \in \mathbb{Z}$ with $e_n + he_{n+1} \in P_n$, then $[e_n + (h-1)e_{n+1}] + [e_{n+1}] \in P_n$ leads to a contradiction.

It follows that $J = P = \bigcup_{n \in \mathbb{N}} P_n$ is a limit prime, and $P_n \supset P_{n-1}$ is an invariant prime segment with $(\mathbb{Z}, +)$ as associated group for every n. Hence $|\mathscr{E}(P_n)| = 1$ for all n. We like to show $|\mathscr{E}(P)| = \infty$ and use Lemma 3.9.

We will choose $a_0 = g_0$ arbitrarily in $\operatorname{Pos}(G)$ and $a_i \in \operatorname{Pos}(G)$ with $\operatorname{lind}(a_i) = i$ for $i = 1, 2, \dots$. The conditions of the lemma are then satisfied. For each index m there are only finitely many elements $a_i = (z_t^{(i)})$ in the set $\{a_0, a_1, a_2, \dots\}$ such that $z_m^{(i)} \neq 0$. The element $\hat{g} = a_0 + a_1 + a_2 + \cdots$ therefore exists in the group $\hat{G}$.

We want to show that $\hat{g} \in \hat{G}$ corresponds to $I = \bigcap_{n=1}^{\infty} a_0 a_1 \dots a_n V$ in V. The element $g \in G$ is contained in I if and only if $g \geq \hat{g}$ in $\hat{G}$. To prove this, we have $g = \hat{g}\beta$ for $\beta \in \operatorname{Pos}(\hat{G})$ and $\hat{g} = a_0 \dots a_n \cdot \gamma_n$ for $\gamma_n = a_{n+1} \cdots \in \operatorname{Pos}(\hat{G})$, if $g \geq \hat{g}$. Hence, $g = a_0 \dots a_n \cdot \gamma_n \cdot \beta$ and $\gamma_n \beta = (a_0 \dots a_n)^{-1} g \in G \cap \operatorname{Pos}(\hat{G})$ for all n; it follows that $g \in \bigcap a_0 \dots a_n V$.

Conversely, if $g = (d_i) \in I \cap \operatorname{Pos}(G)$, then $g = h_n \cdot g_n$ for $h_n = a_0 \dots a_n$ and $g_n \in \operatorname{Pos}(G)$ for all $n \geq 1$. If $g \not\geq \hat{g} = (c_i)$ in $\hat{G}$, then there exists an index i_0 with $c_{i_0} > d_{i_0}$ and $c_i = d_i$ for $i < i_0$. This leads to a contradiction, since $g = h_{i_0} \cdot g_{i_0}$ and $\hat{g} = h_{i_0}\gamma_{i_0}$ with $g_{i_0} \in \operatorname{Pos}(G)$, $\gamma_{i_0} \in \operatorname{Pos}(\hat{G})$, and $\operatorname{lind}(\gamma_{i_0}) = i_0 + 1$. This implies $d_{i_0} \geq c_{i_0}$.

It follows that the ideal I corresponds to a unique element $\hat{g} \in \hat{G}$ since G is dense in $\hat{G}$. If we define ideals I_n by choosing $a_0 = e$ and $a_i = n \cdot e_i$ for $i = 1, 2, \dots,$ then I_n corresponds to $\hat{g}_n = (\dots, 0, n, n, \dots)$ for $n = 1, 2, 3, \dots$.

Every nonzero element $r \in V$ can be written as $r = g_0 \cdot u$, $g_0 \in \operatorname{Pos}(G)$, and u a unit in V. Then $r \cdot I_n$ corresponds to $g_0 \hat{g}_n$ and it follows immediately that no two of the ideals I_n are related in V. By Lemma 3.9 we have $P_n(I_n) = P$ and therefore $|\mathscr{E}(P)| = \infty$.

The previous example gives an indication of how to obtain a limit prime ideal P in a valuation ring V with $|\mathscr{E}(P)| = 2$.

EXAMPLE 3.12. We construct a valued field $(\hat{D}, \hat{V})$ with $\hat{G}$ in place of G, where the notation is as in the previous example. The valuation ring $\hat{V}$ has a limit prime ideal $\hat{P} = J(\hat{V})$ and $\hat{P} = \bigcup_{n \in \mathbb{N}} \hat{P}_n$; the prime ideals

$\neq (0)$ of $\hat{V}$ are $\hat{P}$ and $\hat{P}_n$ for $n \in \mathbb{Z}$. We now apply the considerations that lead to Lemma 3.10.

If $I_1 \supseteq I_2$ are not finitely generated ideals with $P_r(I_1) = \hat{P} = P_r(I_2)$, then there exist elements $a_n \in \hat{V} \cap \operatorname{Pos}(\hat{G})$ with $a_n I_1 \hat{P}_n = I_2 \hat{P}_n$, and $a_m I_1 \hat{P}_n = a_n I_1 \hat{P}_n$ for $m \geq n$. Here we use the fact that $\hat{P}_n \neq \hat{P}_n^2$ and hence $|\mathscr{E}(\hat{P}_n)| = 1$. We can also assume that $a_n \in P_n$ for some $n > n_0$, since otherwise $I_1 \hat{P}_n = I_2 \hat{P}_n$ for all n implies $I_1 = I_2$. Then $a_n^{-1} a_{n+1} = t_n \in \hat{D}$ with $t_n I_1 \hat{P}_n = I_1 \hat{P}_n$. It follows that either t_n or $t_n^{-1} \in S_l(I_1 \hat{P}_n)$, which is equal to $S_r(I\hat{P}_n) = \hat{V} \setminus \hat{P}_n$, since $\hat{V}$ is commutative.

Therefore, $a_{n+1} = a_n t_n$ for elements $a_{n+1}, a_n \in \operatorname{Pos}(\hat{G})$ with $\operatorname{lind}(a_i) \leq i$ and $t_n \in \hat{G}$ with $\operatorname{lind}(t_n) > n$. As in the previous example, it follows that $\hat{g} = a_n t_n t_{n+1} \dots$ is an element in $\operatorname{Pos}(\hat{G})$, and $\hat{g} I_1 \hat{P}_n = I_2 \hat{P}_n$ for all $n \geq n_0$ and $\hat{g} I_1 = I_2$ follows. There is only one related class of infinitely generated ideals of $\hat{V}$ that belongs to $\hat{P}$ and $|\mathscr{E}(\hat{P})| = 2$ follows.

Example 3.13. We replace $\hat{G}$ in the previous example by the corresponding subgroup of the product $\prod_{i \in \mathbb{Z}} \mathbb{R}_i$ with $\mathbb{R}_i = (\mathbb{R}, +)$. This group is called $\hat{G}$ again, and we obtain $\hat{D} \supset \hat{V} \supset \hat{P} = \bigcup_{n \in \mathbb{N}} \hat{P}_n$ as before; here, $\hat{P}_n = \langle \epsilon_i \cdot e_n \mid \text{for } 0 < \epsilon_i \in \mathbb{R} \text{ and } \lim \epsilon_i = 0 \rangle$. The group associated with $\hat{P}_n \supset \hat{P}'_n = \hat{P}_{n-1}$ is $(\mathbb{R}, +)$. However, if $I_1 \supset I_2$ are nonzero ideals of $\hat{V}$ with $I_i \hat{P} = I_i$ for $i = 1, 2$, then $I_1 \hat{P}_n \supseteq I_2 \hat{P}_n$ are related $\hat{P}_n$-ideals by Theorem 3.8(b). With the arguments as in the previous example, we obtain $|\mathscr{E}(\hat{P})| = 2$ in this case as well.

4. NEARLY SIMPLE VALUATION RINGS

In this section we prove the following result:

Theorem 4.1. *Given any integer $m \geq 2$, there exists a nearly simple valuation ring R with $|\mathscr{E}(J(R))| = m$.*

Proof. We prove the result first in the case where $m = 2n + 1$ is an odd number ≥ 3. Let $\Lambda_n = \mathbb{Z}^n = \{(t_1, \dots, t_n) \mid t_i \in \mathbb{Z}\}$ be the direct sum of n copies of $\mathbb{Z}$. This set, which is used as an index set in the next step, is lexicographically ordered from the left. We denote by $\hat{G} = \hat{G}_n$ the subgroup of $\prod_{\lambda \in \Lambda_n} \mathbb{Z}_\lambda$ consisting of all elements $g = (z_\lambda)$ with well-ordered support; the groups $\mathbb{Z}_\lambda$ are $(\mathbb{Z}, +)$ and $\hat{G}$ is ordered lexicographically from the left. We will use multiplicative notation for the operation in $\hat{G}$. We will write e_i for the element in Λ_n with 1 at the ith component and zeros elsewhere, and will also write e_λ for the element in $\hat{G}$ with 1 at the λ-component and 0 elsewhere.

The group $\hat{G}$ admits automorphisms σ_i for $i = 1, \dots, n$ with $\sigma_i(z_\lambda) = (z'_\lambda)$ and $z'_\lambda = z_{\lambda - e_i}$. All σ_i's shift the components of an element $g = (z_\lambda)$

$\in \hat{G}$ to the right. The subgroup of the automorphism group of $\hat{G}$ generated by the σ_i's is isomorphic to $\mathbb{Z}^n$.

As in Example 3.12, there exists a valued field (D, V) with $\hat{G}$ as value group and $D = \mathbb{Q}((\hat{G}))$; we dropped the hat for D and V. Every ideal $I \neq (0)$ of V is generated by the set $I \cap \mathrm{Pos}(\hat{G})$. The automorphisms σ_i, $i = 1, \dots, n$, can be extended to D by defining $\sigma_i(\Sigma g a_g) = \Sigma \sigma_i(g) a_g$ for $\Sigma g a_g \in D$. These automorphisms of D have the property that $\sigma_i(V) = V$ for all i; we say that the σ_i's are *compatible* with (D, V).

We determine in the next step the prime ideals $P \neq (0)$ of V. If $C_1(P) := \{t_1 \mid \mathrm{lind}(g) = (t_i) \in \mathbb{Z}^n,\ g \in P \cap \hat{G}\}$ has no maximal element, then $P = J(V) = J = P_\infty$, the maximal ideal of V. If, on the other hand, $t_1^{(0)} = \max C_1(P)$, then we consider $C_2(P) := \{t_2 \mid \mathrm{lind}(g) = (t_1^{(0)}, t_2, \dots, t_n),\ g \in P \cap \hat{G}\}$. Either $C_2(P)$ has no maximal element and we write $P := P_{(t_1^{(0)}, \infty)}$, or $t_2^{(0)} = \max C_2(P)$.

Repeating this process, it follows that P is equal either to $P_{(t_1^{(0)}, \dots, t_i^{(0)}, \infty)} = P_{(\mu_0, \infty)}$ with $\mu_0 = (t_1^{(0)}, \dots, t_i^{(0)}) \in \mathbb{Z}^i$ for $i < n$, or P is equal to P_{λ_0} for $\lambda_0 = (t_1^{(0)}, \dots, t_n^{(0)}) \in \mathbb{Z}^n$.

The prime ideal $P_{(\mu_0, \infty)}$ with $\mu_0 = (t_1^{(0)}, \dots, t_i^{(0)}) \in \mathbb{Z}^i$, $i < n$, is a limit prime in V generated by the set $p_k^{(\mu_0)} = e_{(\mu_0, k, 0, \dots, 0)} \in \hat{G} \mid k = 1, 2, \dots\}$. The prime ideal P_{λ_0} is generated by the elements $g_k = e_{\lambda_0}(e_{\lambda_0 + e_n})^{-k}$ for $k = 1, 2, \dots$; see Examples 3.11, 3.12. Every P_{λ_0} has a lower neighbor $P'_{\lambda_0} = P_{\lambda_0 - e_n}$ in the chain of prime ideals of V.

We have $P = P_{(\mu_0, \infty)} = \bigcup_{k \in \mathbb{N}} P_{(\mu_0, k, 0, \dots, 0)}$ for the limit prime $P = P_{(\mu_0, \infty)}$. It follows that $|\mathscr{E}_V(P_{\lambda_0})| = 1$ since $P_{\lambda_0}^2 \neq P_{\lambda_0}$ for all $\lambda_0 \in \Lambda$. To prove that $|\mathscr{E}_V(P)| = 2$, we denote $(\mu_0, k, 0, \dots, 0)$ by λ_k and obtain for ideals $J \supseteq I_1 \supset I_2 \neq (0)$ with $P_r(I_1) = P_r(I_2) = P$ and $I_1 P = I_1$, $I_2 P = I_2$, that $a_k I_1 P_{\lambda_k} = I_2 P_{\lambda_k}$. We assume further that $a_k \in P_{\lambda_k}$ for $k \geq n_0$ and $a_{k+m} = a_k t_k \dots t_{k+m-1}$ follows (see Lemma 3.10 and Example 3.12). The elements a_k can be chosen in such a way that their λth components are zero for $\lambda > \lambda_k$. The elements t_k therefore have nonzero λ-components possibly only for $\lambda_k < \lambda \leq \lambda_{k+1}$. The element $\hat{t} = t_k t_{k+1} \dots$ exists in $\hat{G}$ and $a_k \hat{t} I_1 = I_2$ in V. Hence, $|\mathscr{E}_V(P)| = 2$ for any limit prime $P = P_{(\mu_0, \infty)}$ in V.

We therefore have the following classes of related proper nonzero (right) ideals of V: There are two classes associated with $J(V)$, the proper nonzero principal ideals of V and the class represented by $J(V)$. Since $P_{\lambda_0} \neq P_{\lambda_0}^2$, there exists by Theorem 3.8 exactly one class of related ideals of V associated with P_{λ_0}, and P_{λ_0} is a representative of this class.

Finally, for $P_{(\mu_0, \infty)}$, $\mu_0 \in \mathbb{Z}^i$ for some $i \in \{1, 2, \dots, n-1\}$, we have two classes, one represented by $P_{(\mu_0, \infty)}$ and the other represented, for example, by the ideal I_{μ_0} generated by the set $\{e_{(\mu_0, 0, \dots, 0)} e^{-1}_{(\mu_0 + e_i, -k, 0, \dots, 0)} = g_k,\ k \in \mathbb{N}\}$. This last class corresponds to the proper nonzero principal ideals of the localization $V_{P_{(\mu_0, \infty)}}$ of V at $P_{(\mu_0, \infty)}$ with $I_{\mu_0} = e_{(\mu_0, 0, \dots, 0)} V_{P_{(\mu_0, \infty)}}$.

We now describe the final step of the construction. Consider the valued field (D, V) and the automorphism σ_1 of D that is compatible with V. We can therefore form the subring $V[x_1, \sigma_1]$ of $D[x_1, \sigma_1] = \{\sum a_i x_1^i \mid a_i \in D\}$ with $x_1 a = \sigma_1(a) x_1$ for $a \in D$. Then $D[x_1, \sigma]$ and $V[x_1, \sigma_1]$ are both Ore domains with $F_1 = D(x_1, \sigma_1)$ as skew field of quotients.

The set $T_1 = \{\sum c_i x_1^i \in V[x_1, \sigma_1] \mid \sum c_i V = V\}$ is a right Ore system in $V[x_1, \sigma_1]$ and $R_1 := V[x_1, \sigma_1]T_1^{-1}$ is a valuation subring of F_1. We will show below that the nonzero principal right ideals of R_1 are in one-to-one correspondence with the elements in $\operatorname{Pos}(\hat{G})$, and it follows that R_1 has no ideals besides (0), $J(R_1)$, and R_1 since x_1 is a unit in R_1 and $x_1^n a x_1^{-n} = \sigma_1^n(a)$ for $a \in D$; see also [2].

The automorphism σ_2 of D also has the property $\sigma_2(V) = V$. If we define $\sigma_2(x_1) = x_1$, this automorphism can be extended to F_1 since $\sigma_2(T_1) = T_1$ and $\sigma_1\sigma_2 = \sigma_2\sigma_1$. Therefore $\sigma_2(R_1) = R_1$, σ_2 is compatible with R_1 and we can consider the domains $R_1[x_2, \sigma_2] \subset F_1[x_2, \sigma_2] \subset F_1(x_2, \sigma_2) = F_2$, where F_2 is the skew field of quotients of $R_1[x_2, \sigma_2]$. This ring contains the Ore set $T_2 = \{\sum c_i x_2^i \in R_1[x_2, \sigma_2] \mid \sum c_i R_1 = R_1\}$, and $R_2 = R_2[x_2, \sigma_2]T_2^{-1}$ is a valuation subring of F_2. Again, R_2 is a nearly simple valuation subring of F_2 and its principal nonzero right ideals are in one-to-one correspondence with the elements in $\operatorname{Pos}(\hat{G})$ by Lemma 4.2 below.

Repeating this step, we obtain a nearly simple valuation ring $R = R_n$ of the skew field $F = F_n = D(x_1, \sigma_1)\dots(x_n, \sigma_n)$. We want to show that $|\mathscr{E}(J(R))| = 2n + 1$.

The next lemma describes the nonzero principal right ideals of R_k for $k = 1, \dots, n$ and will be used to determine the classes of related right ideals of R.

LEMMA 4.2. *Let $k \in \{1, 2, \dots, n\}$. Then*:

(i) *Every nonzero principal right ideal rR_k is of the form gR_k for an element $g \in \operatorname{Pos}(\hat{G})$.*

(ii) *Let $g_1, g_2 \in \operatorname{Pos}(\hat{G})$. Then $g_1 R_k = g_2 R_k$ if and only if $g_1 = g_2$.*

(iii) *For every $0 \neq r \in R_k$ and every right ideal $I = \bigcup g_j R_k$, $g_j \in \operatorname{Pos}\hat{G}$, $j \in \mathbb{N}$, there exist $h \in \operatorname{Pos}(\hat{G})$ and $(u_1, \dots, u_k) \in \mathbb{Z}^k$ such that $hrI = x_1^{u_1} \cdots x_k^{u_k} I$ or $rI = h x_1^{u_1} \cdots x_k^{u_k} I$.*

Proof. (i) The claims are true for $R_0 = V \subset D = F_0$. We assume that they are true for $k - 1$ and that $k \geq 1$. We have $R_k = R_{k-1}[x_k, \sigma_k]T_k^{-1}$ with $x_k r = \sigma_k(r) x_k$, $r \in R_{k-1}$, and $T_k = \{\sum c_i x_k^i \in R_{k-1}[x_k, \sigma_k] \mid \exists i_0$ with $c_{i_0} \in U(R_{k-1})\}$. An element $0 \neq r \in R_k$ has the form $r = (\sum a_i x_k^i)(\sum c_i x_k^i)^{-1}$ with $a_i, c_i \in R_{k-1}$ for all i, not all a_i are zero, and $c_{i_0} \in U(R_{k-1})$ for at least one index i_0. It follows that $rR_k = (\sum a_i x_k^i)R_k =$

$a_{i'}R_k = gR_k$ for an index i' with $a_{i'}R_{k-1} \supseteq a_iR_{k-1}$ for all i and $a_{i'}R_{k-1} = gR_{k-1}$ for some $g \in \mathrm{Pos}(\hat{G})$ by induction. This proves (i).

We also use induction to prove $R_k \cap \hat{G} = \mathrm{Pos}(\hat{G})$, which implies (ii). Assume that $(\Sigma a_i x_k^i)(\Sigma c_i x_k^i)^{-1} = g \in \hat{G}$ with $a_i, c_i \in R_{k-1}$, $c_{i_0} \in U(R_{k-1})$. Then $\Sigma a_i x_k^i = \Sigma g c_i x_k^i$, and $a_{i_0} = gc_{i_0}$, $g = a_{i_0}c_{i_0}^{-1} \in R_{k-1} \cap \hat{G} = \mathrm{Pos}(\hat{G})$ follows.

To prove (iii), we consider first $r = \Sigma a_i x_k^i$, $a_i \in R_{k-1}$, and $I = gR_k$, $g \in \mathrm{Pos}(\hat{G})$. Then $rgR_k = a_{i'}x_k^{i'}gR_k = a_{i'}\sigma_k^{i'}(g)R_k$ for an index i' with $a_{i'}\sigma_k^{i'}(g)R_{k-1} \supseteq a_i\sigma_k^i(g)R_{k-1}$ for all i. By induction, there exist $h \in \mathrm{Pos}(\hat{G})$, $(u_1, \dots, u_{k-1}) \in \mathbb{Z}^{k-1}$ with either $hrgR_k = x_1^{u_1} \cdots x_{k-1}^{u_{k-1}}x_k^{i'}gR_k$ or $rgR_k = hx_1^{u_1} \cdots x_{k-1}^{u_{k-1}}x_k^{i'}gR_k$. If $I = \bigcup g_jR_k$, $g_jR_k \subset g_{j'}R_k$ for $j < j' \in \mathbb{N}$, is not finitely generated, then there exists an infinite subset S of $\mathbb{N}$ such that $rg_jR_k = a_i'x_k^{i'}g_jR_k$ for the same i' and all j in S. Then $I = \bigcup g_jR_{k, j \in S}$, and $rI = \bigcup rg_jR_k = a_i'(x_k^{i'}I)$, and the result follows by induction since $a_i' \in R_{k-1}$ and every right ideal in R_k is generated by elements in $\hat{G}$. Finally, if $r = (\Sigma c_i x_k^i)^{-1}$, $c_i \in R_{k-1}$, $c_{i_0} \in U(R_{k-1})$, then $rI = I'$ and $I = (\Sigma c_i x_k^i)I'$. The result follows from the first part.

We would like to determine the classes of related nonzero, proper right ideals I of R. Such a right ideal is of the form $I = I_0R$ for $I_0 = I \cap \mathrm{Pos}(\hat{G})$ and we know that I is related either to some P_{λ_0} for some $\lambda_0 \in \Lambda$, to $P_{(\mu_0, \infty)}R$, or to $e_{(\mu_0, 0, \dots, 0)}V_{P_{(\mu_0,\infty)}}R$ for $\mu_0 \in \mathbb{Z}^i$, $i = 0, \dots, n-1$. Here we have for $i = 0$ that $P_{(\mu_0, \infty)}R = P_\infty R = J(R)$ is the maximal ideal of R, and $I_\phi = e_{(0,0,\dots,0)}R$ represents the class of proper nonzero principal right ideals of R. So far we have only used multiplication by elements in $\mathrm{Pos}(\hat{G})$ as operation on the set of all nonzero, proper right ideals I of R. If we also consider multiplication by the elements $x_i^{\pm 1}$, we obtain $x_1^{u_1}x_2^{u_2} \cdots x_n^{u_n}P_{\lambda_0}R = P_{(u_1, \dots, u_n) + \lambda_0}R$ for $(u_1, u_2, \dots, u_n) \in \Lambda = \mathbb{Z}^n$; it follows that $P_{\lambda_1}R \sim P_{\lambda_2}R$ for $\lambda_1, \lambda_2 \in \Lambda$. Further, $P_{(\mu_0, \infty)}R \sim P_{(\mu_0', \infty)}R$ for any $\mu_0, \mu_0' \in \mathbb{Z}^i$ for some $i = 1, \dots, n-1$, since

$$x_1^{u_1}x_2^{u_2} \cdots x_n^{u_n}P_{(\mu_0, \infty)}R = P_{((u_1, \dots, u_i) + \mu_0, \infty)}R.$$

We also have $x_1^{u_1}x_2^{u_2} \cdots x_n^{u_n}I_{\mu_0} = e_{((u_1, \dots, u_i) + \mu_0, u_{i+1}, \dots, u_n)}V_{P_{((u_1, \dots, u_i) + \mu_0, \infty)}}R \sim I_{(u_1, \dots, u_i) + \mu_0}$. Finally, multiplication by the $x_i^{\pm 1}$ from the left reproduces the maximal ideal $J(R)$ and maps a proper nonzero principal right ideal of R into another.

It remains to prove that the $2n + 1$ classes of related proper nonzero right ideals of R represented by the $2n + 1$ types of right ideals as listed above are indeed distinct. Let I_1 and I_2 be right ideals of R, related in R, say $rI_1 = I_2$ but representing two different types. It follows from Lemma 4.2(iii) that there exists $(u_1, \dots, u_n) \in \mathbb{Z}^n$ and $h \in \mathrm{Pos}(\hat{G})$ so that $rI_1 = hx_1^{u_1} \cdots x_n^{u_n}I_1 = I_2$ or that $hrI_1 = hI_2 = x_1^{u_1} \cdots x_n^{u_n}I_1$. However, $x_1^{u_1} \cdots x_n^{u_n}I_1$ is of the same type as I_1, as we saw above, and these last equations would imply that $I_2 \cap V$ and $x_1^{u_1} \cdots x_n^{u_n}I_1 \cap V$ are related in V, a contradiction.

For this last argument, the fact $hI_2 \cap V = h(I_2 \cap V)$, which can be proved by induction, was used.

This shows that $|\mathscr{E}(J(R)| = 2n + 1$ since $J(R)$ is the only nonzero completely prime ideal in R.

We must now consider the case where $m = 2n$, $n \geq 1$, is even. We will deal with the case $m = 2$ separately at the end. We repeat the construction as in the first part with $\Lambda_n = \mathbb{Z}^n$, but $\hat{G}$ this time is the subgroup of elements $g \in \prod_{i \in \Lambda} \mathbb{R}_i$ with well-ordered support. The components $\mathbb{R}_i$ of the direct product are isomorphic to $(\mathbb{R}, +)$. Again there is a valued field (D, V) with $\hat{G}$ as associated value group, and the prime ideals of V are of the form P_{λ_0}, $\lambda_0 \in \Lambda_n$, or $P_{(\mu_0, \infty)}$, $\mu_0 \in \mathbb{Z}^i$ for $i = 0, \dots, n-1$.

It follows from Theorem 3.8(b) that $|\mathscr{E}_V(P_{\lambda_0})| = 2$, in contrast to the first case, with P_{λ_0} generated by $\{g_k = (1/2^k)e_{\lambda_0},\ k \in \mathbb{N}\}$ as representative of one class and I_{λ_0} generated by $\{g_k = e_{\lambda_0}(e_{\lambda_0} + e_n)^{-k},\ k \in \mathbb{N}\}$ representing the other class; here $P_{\lambda_0}^2 = P_{\lambda_0}$.

It follows from Lemma 3.10 that $|\mathscr{E}_V(P_{\mu_0, \infty})| = 2$ for all limit prime ideals including the maximal ideal $P_\infty = J(V)$. The representatives of these two classes are $P_{(\mu_0, \infty)}$ generated as before by $\{g_k = e_{(\mu_0, k, 0, \dots, 0)},\ k \in \mathbb{N}\}$ for $\mu_0 \in \mathbb{Z}^i$, $i = 0, 1, \dots, n-1$, and I_{μ_0} generated by $\{g_k = e_{\mu_0} e^{-1}_{(\mu_0 + e_i, -k, 0, \dots, 0)},\ k \in \mathbb{N}\}$ for $\mu_0 \in \mathbb{Z}^i$, $i = 1, \dots, n-1$. Of course, any proper principal (right) ideal $\neq (0)$ of V represents the other class associated with $J(V)$.

We use the automorphisms $\sigma_1, \dots, \sigma_n$ of $\hat{G}$ defined as in the first part and obtain a nearly simple valuation ring R in $F = F_n = D(x_1, \sigma_1) \dots (x_n, \sigma_n)$ with $|\mathscr{E}_R(J(R))| = 2n + 2$.

Finally, we consider the case $m = 2$. Let $D = \mathbb{Q}(t^r \mid r \in \mathbb{R})$ be a field which is isomorphic to the field of quotients of the group ring $\mathbb{Q}[\mathbb{R}]$ where $\mathbb{R} = (\mathbb{R}, +)$. Then D contains V the t-adic valuation ring and admits an automorphism σ, defined by $\sigma(t^r) = t^{r/2}$. Then $\sigma(V) = V$ and σ is compatible with V. Hence $V[x, \sigma]$ contains the right Ore system $T = \{\sum c_i x^i \in V[x, \sigma] \mid \exists\ i_0 \text{ with } c_{i_0} \in U(V)\}$, and we define $R := V[x, \sigma]T^{-1}$. The nonzero principal right ideals aR of R are of the form $aR = t^r R$ for $0 \leq r \in \mathbb{R}$. Since $x^n \cdot t^r R = t^{r/2^n} R$, it follows that R is nearly simple. Every right ideal of R is either principal or of the form $aJ(R)$, $0 \neq a \in R$. Hence, $|\mathscr{E}(J(R))| = 2$. ∎

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