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# Finiteness conditions on the Ext-algebra of a cycle algebra

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## Abstract

Let  $A$  be a finite-dimensional algebra given by quiver and monomial relations. In [E.L. Green, D. Zacharia, Manuscripta Math. 85 (1994) 11–23] we see that the Ext-algebra of  $A$  is finitely generated only if all the Ext-algebras of certain cycle algebras overlying  $A$  are finitely generated. Here a cycle algebra  $\Lambda$  is a finite-dimensional algebra given by quiver and monomial relations where the quiver is an oriented cycle. The main result of this paper gives necessary and sufficient conditions for the Ext-algebra of such a  $\Lambda$  to be finitely generated; this is achieved by defining a computable invariant of  $\Lambda$ , the smo-tube. We also give necessary and sufficient conditions for the Ext-algebra of  $\Lambda$  to be Noetherian.

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**Keywords:** Ext-algebra; Finite generation; Noetherian; Monomial algebra; Cycle algebra

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## Introduction

Let  $\mathcal{Q}$  be an oriented cycle with  $n$  vertices and  $n$  arrows. Label the vertices with the natural ordering  $1, \dots, n$  so that there is one arrow from  $i$  to  $i + 1$  for  $1 \leq i < n$  and one arrow from  $n$  to  $1$ . Let  $\mathbb{k}$  be an algebraically closed field,  $\mathcal{I}$  an admissible ideal of the path

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algebra  $\mathbb{k}Q$  with a minimal generating set  $\rho$  of  $m$  monomial relations, which we fix, such that  $\mathbb{k}Q/\mathcal{I}$  is a finite-dimensional algebra. We say that a path in  $\mathbb{k}Q$  is a *relation* if and only if it is in the set  $\rho$ . Hereinafter we let  $\Lambda = \mathbb{k}Q/\mathcal{I}$ , reading paths from left to right. Let  $\underline{r}$  denote the Jacobson radical of  $\Lambda$ , and let  $\bar{\Lambda} = \Lambda/\underline{r}$ . Then the Ext-algebra of  $\Lambda$ , denoted  $E(\Lambda)$ , is the graded  $\mathbb{k}$ -algebra  $\bigoplus_{i=0}^{\infty} \text{Ext}^i(\bar{\Lambda}, \bar{\Lambda})$ , with multiplication the Yoneda product.

The main result of this paper, Theorem 4.14, gives necessary and sufficient conditions for  $E(\Lambda)$  to be finitely generated when  $Q$  is an oriented cycle. We also present a fast method for determining if these conditions hold. This is motivated by [4, Proposition 1.6], which states that the Ext-algebra of an arbitrary monomial algebra given by quiver and relations is finitely generated only if the Ext-algebras of certain cycle algebras are all finitely generated. Cycle algebras have an oriented cycle as underlying quiver and therefore are exactly those we study here. Using the same machinery that we use for finite generation of  $E(\Lambda)$ , we also give necessary and sufficient conditions for the Ext-algebra of  $\Lambda$  to be Noetherian.

The paper is structured as follows. In Section 1 we present a convenient way of representing the basis elements of  $E(\Lambda)$ : the smo-tube  $\mathcal{T}_\Lambda$ . This is improved in Section 2, where Theorems 2.3, 2.5, and 2.11 give conditions on the smo-tube that speed-up its calculation. Section 3 demonstrates the existence of natural constraints on  $\mathcal{T}_\Lambda$ , culminating in Theorem 3.10, which serves to reduce computational work even further. This section paves the way for Section 4 where our main result, Theorem 4.14, gives necessary and sufficient conditions for the finite generation of  $E(\Lambda)$ . We then give some special cases of this result. In Section 5 we give necessary and sufficient conditions for  $E(\Lambda)$  to be a Noetherian ring.

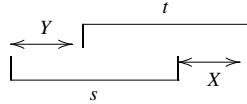
## Background and preliminaries

We now give some further notation that will be required. Let  $p$  be a path in  $\mathbb{k}Q$ . We denote by  $\ell(p)$  the length of  $p$  and by  $\text{o}(p)$  and  $\text{t}(p)$  the start and end vertices of  $p$ , respectively. A path  $q$  is an *initial* subpath of  $p$  if  $p = qs$  for some path  $s \in \mathbb{k}Q$ . A path  $q$  is a *terminal* subpath of  $p$  if  $p = rq$  for some path  $r \in \mathbb{k}Q$ .

Define a cycle algebra  $\Lambda$  to be a quotient of a path algebra that has an oriented cycle  $Q$  for a quiver, and that has monomial relations. Then we have a very nice description of the Yoneda product for  $E(\Lambda)$  via maximal overlaps of these relations. We recall the basic definitions from [3] and recursively define certain sets of paths denoted  $\mathcal{Q}_{z+1}$ . Let  $\mathcal{Q}_0$  be the set of trivial paths of  $Q$ ,  $\mathcal{Q}_1$  the set of arrows and set  $\mathcal{Q}_2 = \rho$ . Let  $B$  be the usual basis of  $\mathbb{k}Q$  consisting of all finite paths, and let  $\mathcal{M} = \{b \in B : \text{no subpath of } b \text{ lies in } \rho\}$ . For  $z \geq 1$ , a path  $p$  in  $\mathbb{k}Q$  is a  $z$ -prechain if  $p = qwu$ , where  $q \in \mathcal{Q}_{z-1}$ ,  $qw \in \mathcal{Q}_z$ ,  $u \in \mathcal{M} - \mathcal{Q}_0$ , and  $wu$  has a subpath in  $\mathcal{Q}_2$ . Call a  $z$ -prechain a  $z$ -chain if no proper initial subpath is a  $z$ -prechain. Then  $\mathcal{Q}_{z+1}$  is defined as the set of all  $z$ -chains,  $z \geq 1$ .

These  $z$ -chains correspond to the paths of maximal overlap sequences of [2], [4] and also [1]. The terminology is understood in the following way. Let  $s, t \in \rho$ . The relation

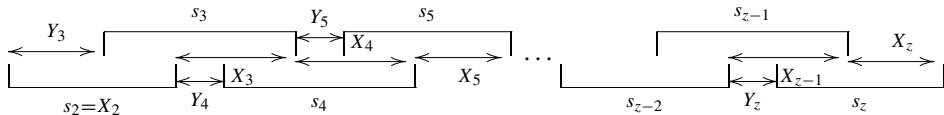
$t$  is said to *overlap* the relation  $s$  if there are paths  $X, Y$  in  $\mathbb{k}Q$  such that  $Yt = sX$ , with  $1 \leq \ell(Y) < \ell(s)$  and  $1 \leq \ell(X) < \ell(t)$ . This is illustrated in the following diagram:



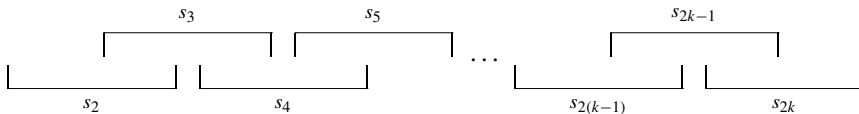
It is then clear that if  $t$  overlaps  $s$ , the path  $sX$  is a 2-prechain. Moreover, we say  $t$  *maximally overlaps*  $s$  if  $t$  overlaps  $s$  and further, there are only 2 subpaths of  $Yt$  that are relations: the proper initial subpath  $s$  and the proper terminal subpath  $t$ . If  $t$  maximally overlaps  $s$ , then  $(s, t)$  is a maximal overlap sequence and  $sX$  is the underlying path of  $(s, t)$ . In this case the path  $sX$  is a 2-chain and so  $sX \in Q_3$ . More generally, for  $z \geq 3$  and  $s_2, s_3, \dots, s_z \in \rho$ ,  $(s_2, s_3, \dots, s_z)$  is a maximal left overlap sequence with underlying path  $X_2X_3 \cdots X_z$  if

- (i)  $X_2 = s_2$ ;
- (ii)  $1 \leq \ell(X_i) < \ell(s_i)$  for  $i = 3, \dots, z$ ;
- (iii) there exist paths  $Y_3, \dots, Y_z$  with  $X_iX_{i+1} = Y_{i+1}s_{i+1}$  for  $i = 2, \dots, z-1$ ;
- (iv) there are only 2 subpaths of  $Y_3s_3$  that are relations: the proper initial subpath  $s_2$  and the proper terminal subpath  $s_3$ ;
- (v) for  $i = 4, \dots, z$ , there is only 1 subpath of  $Y_is_i$  that is a relation: the proper terminal subpath  $s_i$ .

The above conditions are visualised, thus:



The path  $X_2X_3 \cdots X_z$  is a  $(z-1)$ -chain and is thus an element of  $Q_z$ . The *degree* of the maximal left overlap sequence  $(s_2, s_3, \dots, s_z)$  is  $z$ . Throughout the paper a maximal left overlap sequence will be used interchangeably with its underlying path and will usually be illustrated, thus:



Note that if for some  $z \geq 3$ ,  $P^z = (s_2, s_3, \dots, s_z)$  is a maximal left overlap sequence with underlying path  $X_2X_3 \cdots X_z$ , then so is  $P^{z-1} = (s_2, s_3, \dots, s_{z-1})$  with underlying path  $X_2X_3 \cdots X_{z-1}$ . We call  $X_z$  the *path of unoverlapped arrows* of  $P^z$ . Note that  $P^z = P^{z-1}X_z$ .

There is an analogous concept of a maximal right overlap sequence. In fact, from [1] we know that the underlying path of a maximal left overlap sequence of degree  $z$  is also the un-

derlying path of a maximal right overlap sequence of degree  $z$  and vice versa. Henceforth, we refer to such a path as the underlying path of a maximal overlap sequence; however the construction will always be considered as from the left. We comment further in Section 5.

These maximal overlap sequences are of major importance, since they describe a minimal projective resolution of  $\bar{\Lambda}$ . Following our notation, for  $z \geq 0$  the  $z$ th projective  $\mathcal{P}_z$  in such a resolution is given in [2] as

$$\mathcal{P}_z = \bigoplus_{p \in \mathcal{Q}_z} e_p \Lambda,$$

where  $e_p$  is the trivial path at  $t(p)$ . For each  $p \in \mathcal{Q}_z$  there is a corresponding element  $\varepsilon_p^z$  in  $\text{Ext}_{\bar{\Lambda}}^z(\bar{\Lambda}, \bar{\Lambda})$ . This element  $\varepsilon_p^z$  is represented by the  $\Lambda$ -homomorphism  $h_p^z \in \text{Hom}(\mathcal{P}_z, \bar{\Lambda})$  given by

$$h_p^z(e_q \lambda) = \begin{cases} 0 & \text{if } p \neq q \text{ in } \mathbb{k}\mathcal{Q}, \\ e_p \bar{\lambda} & \text{if } p = q \text{ in } \mathbb{k}\mathcal{Q}, \end{cases}$$

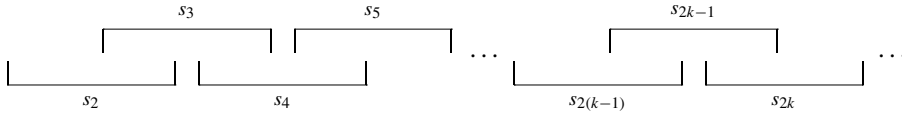
where  $\bar{\lambda}$  is the image of  $\lambda$  under the canonical surjection  $\Lambda \rightarrow \bar{\Lambda}$ . Each set  $\mathcal{Q}_z$  is identified with a  $\mathbb{k}$ -basis of  $\text{Ext}_{\bar{\Lambda}}^z(\bar{\Lambda}, \bar{\Lambda})$  in the obvious way by taking  $p$  in  $\mathcal{Q}_z$  to  $\varepsilon_p^z$  in  $\text{Ext}_{\bar{\Lambda}}^z(\bar{\Lambda}, \bar{\Lambda})$ .

The set  $\mathcal{G}_z := \{\varepsilon_p^z : p \in \mathcal{Q}_z\}$  is a basis for  $\text{Ext}_{\bar{\Lambda}}^z(\bar{\Lambda}, \bar{\Lambda})$  and from [4] we have that the union of all the  $\mathcal{G}_z$ , for  $z \geq 0$ , forms a multiplicative basis for  $E(\Lambda)$ . This means that for  $\varepsilon_p^z \in \mathcal{G}_z$  and  $\varepsilon_q^w \in \mathcal{G}_w$ , either  $\varepsilon_p^z \varepsilon_q^w = 0$ , or  $\varepsilon_p^z \varepsilon_q^w \in \mathcal{G}_{z+w}$ . The one-to-one correspondence between  $\mathcal{Q}_z$  and  $\mathcal{G}_z$ , for each  $z \geq 0$ , given in [4], means that for the remainder of the paper we may deal with maximal overlap sequences as if they themselves form the multiplicative basis of  $E(\Lambda)$ . With this identification, a maximal overlap sequence of degree 0 is a trivial path, and a maximal overlap sequence of degree 1 is an arrow. If  $P^{z_1}$  and  $Q^{z_2}$  are maximal overlap sequences of degree  $z_1$  and  $z_2$ , respectively, then  $P^{z_1} Q^{z_2}$  represents a non-zero element of  $E(\Lambda)$  if and only if the product of paths  $P^{z_1} Q^{z_2}$  in  $\mathbb{k}\mathcal{Q}$  is the underlying path of a maximal overlap sequence of degree  $z_1 + z_2$ . In this case  $P^{z_1} Q^{z_2}$  represents an element in  $E(\Lambda)$  of degree  $z_1 + z_2$ .

In particular, with this description we can avoid the lifting of maps usually associated to the Yoneda product. Therefore, maximal overlap sequences are fundamental to the results in this paper.

The definitions of the above paragraphs work for a general quiver  $\mathcal{Q}$  with monomial relations, but in this paper we restrict  $\mathcal{Q}$  to an oriented cycle. This is a special case where, given a  $(z-1)$ -chain  $p$ , there is at most one  $z$ -chain of the form  $pr$ . We can thus form a sequence of  $z$ -chains defined as follows. Let  $v$  be the start vertex of some relation  $r$ , and for  $z \geq 2$  let  $A_v^z$  be the unique  $(z-1)$ -chain, if it exists, starting at  $v$ . Note that  $A_v^2 = r$ . Then we say the sequence  $A_v := (A_v^z)_{z \geq 2}$  is the *extending sequence* of  $\Lambda$  starting at  $v$ . We define  $\sigma(A_v)$  as being the vertex  $v$ . The suffix will often be omitted if the start vertex itself

is clear from the context or if it is unspecified. Thus  $A_v$  is formed from the sequence of maximal overlaps



with  $\sigma(s_2) = v$  and  $A_v^z$  being the maximal overlap sequence  $(s_2, s_3, \dots, s_z)$ . We also define the *lower half* of  $A_v$ ; this is the sequence of relations  $(s_{2j})_{j \geq 1}$ , where each  $s_{2j}$  is the unique relation such that  $t(s_{2j}) = t(A_v^{2j})$ . The *upper half* of  $A_v$  is the sequence of relations  $(s_{2j+1})_{j \geq 1}$ , where each  $s_{2j+1}$  is the unique relation such that  $t(s_{2j+1}) = t(A_v^{2j+1})$ . We may also define the *maximum degree* attained by  $A_v$ , denoted  $\maxdeg A_v$ . If the sequence  $A_v$  terminates at some degree  $z \geq 2$  (that is  $A_v^z$  is a  $(z-1)$ -chain but  $A_v^{z+1}$  does not exist), then  $\maxdeg A_v = z$ ; it is defined as  $\infty$  otherwise. Note that  $\maxdeg A_v$  must always be at least 2.

## Monomial algebras and cycle algebras

Here we review the results that show the fundamentality of cycle algebras in the study of the Ext-algebra of a monomial algebra. We draw our material from [4] and from discussion with the authors of [4].

**Definition 0.1** ([4]). Let  $B$  be a monomial algebra and let  $Q$  be a quiver consisting of a single oriented cycle in the quiver  $\Gamma$  of  $B$ . Note that we allow  $Q$  to go through the same vertex or arrow of  $\Gamma$  more than once, so certain vertices or arrows of  $\Gamma$  may have several copies appearing in  $Q$ . Let  $f : Q \rightarrow \Gamma$  be a map of quivers, that is,  $f$  sends vertices to vertices and arrows to arrows, so that if  $\eta$  is an arrow from  $i$  to  $j$  then  $f(\eta)$  is an arrow from  $f(i)$  to  $f(j)$ . We take the relations on  $Q$  by pulling back the relations on  $\Gamma$ , that is, a path in  $Q$  is a relation if its image in  $\Gamma$  is a relation in  $B$ . Let  $Z_Q$  be the algebra with quiver  $Q$  and with the above relations on  $Q$ . Then  $Z_Q$  is said to be a *cycle algebra overlying  $B$* .

For oriented cycles  $Q$  and  $Q'$ ,  $Z_{Q'}$  is a *finite covering* of  $Z_Q$  if there is a surjective quiver map  $Q' \rightarrow Q$  that takes relations to relations and each relation in  $Q$  lifts to one in  $Q'$ . Then we say that a cycle algebra  $Z_{Q'}$  overlying  $B$  is a *minimal cycle algebra overlying  $B$*  if, for all cycle algebras  $Z_Q$ , whenever  $Z_{Q'}$  is a finite covering of  $Z_Q$  then  $Z_Q \cong Z_{Q'}$ .

The reason we consider minimal cycle algebras is due to the following result from [4]. For completeness we provide a more detailed proof.

**Proposition 0.2** ([4]). Let  $Q$  be the quiver of a minimal cycle algebra  $Z_Q$  and let  $Q'$  be the quiver of a cycle algebra  $Z_{Q'}$  overlying  $Z_Q$ . Then  $E(Z_{Q'})$  is finitely generated if and only if  $E(Z_Q)$  is finitely generated.

**Proof.** Let  $\mathcal{Q}$  have  $q$  vertices and  $q$  arrows. Then there exists a positive integer  $d$  such that  $\mathcal{Q}'$  has  $dq$  vertices and  $dq$  arrows.

Let  $E(Z_{\mathcal{Q}})$  be finitely generated, with  $b_1, \dots, b_l$  a complete set of generators from our usual basis described in the previous section. Each  $b_i$  in  $E(Z_{\mathcal{Q}})$  lifts to one of  $d$  different paths in  $\mathcal{Q}'$ . Since the relations on  $\mathcal{Q}'$  are taken from  $\mathcal{Q}$ , we get that each  $b_i$  corresponds to a set of  $d$  basis elements of  $E(Z_{\mathcal{Q}'})$ , denoted  $b'_i = \{b'_{i,1}, \dots, b'_{i,d}\}$ . We will now show that  $E(Z_{\mathcal{Q}'})$  is finitely generated with generating set  $\bigcup_{i=1}^l b'_i$ .

Let  $a'$  be an element from the usual basis of  $E(Z_{\mathcal{Q}'})$ . The underlying path in  $\mathcal{Q}'$  of  $a'$  corresponds to a path  $a$  in  $\mathcal{Q}$ . Since the relations on  $\mathcal{Q}'$  are taken from  $\mathcal{Q}$  we have that  $a$  is a basis element of  $E(Z_{\mathcal{Q}})$ . Write  $a$  as a finite product of the  $b_i$ 's. For each copy of a generator that appears in this product there is a single natural choice  $1 \leq j \leq d$  so that  $a'$  is written as a non-zero product of the  $b'_{i,j}$ 's. We can be certain of obtaining a non-zero product because the relations of  $\mathcal{Q}'$  are taken from  $\mathcal{Q}$ . Since  $a'$  was arbitrary,  $E(Z_{\mathcal{Q}'})$  is finitely generated.

Similarly, if we assume that  $E(Z_{\mathcal{Q}'})$  is finitely generated, then an arbitrary basis element  $a$  in  $E(Z_{\mathcal{Q}})$  corresponds to some  $a'$  in  $E(Z_{\mathcal{Q}'})$ , and thus it is shown that  $E(Z_{\mathcal{Q}})$  is finitely generated.  $\square$

We now consider the following claim from [4].

**Claim** ([4, Proposition 1.6]). *Let  $B$  be a monomial algebra. Then the Ext-algebra  $E(B)$  is finitely generated if and only if the  $\mathbb{k}$ -algebras  $E(Z_{\mathcal{Q}})$  are finitely generated for all minimal cycle algebras  $Z_{\mathcal{Q}}$  overlying  $B$ .*

We give a proof for one direction stated in the following proposition.

**Proposition 0.3.** *Let  $B$  be a monomial algebra and let the Ext-algebra  $E(B)$  be finitely generated. Then the  $\mathbb{k}$ -algebras  $E(Z_{\mathcal{Q}})$  are finitely generated for all minimal cycle algebras  $Z_{\mathcal{Q}}$  overlying  $B$ .*

**Proof.** Let  $B$  be a monomial algebra with quiver  $\Gamma$  and let  $E(B)$  be finitely generated with (basis) generators  $b_1, \dots, b_l$ . Also let  $Z_{\mathcal{Q}}$  be an overlying minimal cycle algebra of  $B$ . Reordering if necessary, let  $b_1, \dots, b_k$  be precisely those generators of  $E(B)$  whose underlying paths lie on the closed path in  $\Gamma$  that is the image of  $\mathcal{Q}$ . Since the relations of  $Z_{\mathcal{Q}}$  are lifted from  $B$ , we have that  $b_1^*, \dots, b_k^*$  are corresponding basis elements of  $E(Z_{\mathcal{Q}})$ .

Now let  $a^*$  be a basis element of  $E(Z_{\mathcal{Q}})$ . Then we have  $a$  as the corresponding basis element of  $E(B)$ , so  $a$  can be written as a product of  $b_1, \dots, b_l$ . However, if  $b_i$  is a subpath of  $a$ , then  $b_i^*$  is a path in  $\mathcal{Q}$ , thus  $a$  is a product of  $b_1, \dots, b_k$  and so  $a^*$  is a product of  $b_1^*, \dots, b_k^*$ .

Since  $a^*$  was arbitrary, we get that  $E(Z_{\mathcal{Q}})$  is finitely generated.  $\square$

The details of the proof of this proposition are not given in [4]. Following discussion with the authors of [4], we state here that the reverse implication of the claim above is false. They have provided a counter-example which we give as Example 11 in Section 4, where we can treat it more fully.

However, it is the direction proved in Proposition 0.3 above that is most useful to us. With it, it is clear that finding just one overlying cycle algebra with infinitely generated Ext-algebra gives us the Ext-algebra of  $B$  infinitely generated. Hence, studying the Ext-algebras of cycle algebras is fundamental. Of course,  $B$  has finite global-dimension if its quiver has no oriented cycles: then  $E(B)$  is trivially finitely generated.

We note here that a similar result exists for the respective Ext-algebras being Noetherian. In Section 5 we look at when the Ext-algebra of a cycle algebra is Noetherian, and Proposition 5.5 is the Noetherian analogue to Proposition 0.3.

## 1. The smo-tube

We now turn exclusively to the study of the Ext-algebra of a monomial cycle algebra  $\Lambda$ , that is  $\Lambda = \mathbb{k}Q/\mathcal{I}$ , where  $Q$  is an oriented cycle and  $\mathcal{I}$  has monomial generators. In this section we give the basic definition of the smo-tube and show how it relates to the extending sequences of  $\Lambda$ . We begin by giving some notation that will be used throughout the paper.

If  $v$  and  $w$  are vertices let  $v \rightarrow w$  denote the path in  $\mathbb{k}Q$  from  $v$  to  $w$  with length in the range 0 to  $n - 1$  inclusive. The arrow will only ever be used with this precise meaning. Let  $v$  be a vertex,  $z$  some integer. Then  $v + z$  is a vertex where the addition is integer addition modulo  $n$ . We identify the vertex  $v$  with the trivial path  $e_v$  in  $\mathbb{k}Q$  of length 0 at  $v$ . Let  $p$  be a path in  $\mathbb{k}Q$ . If  $p \in \mathbb{k}Qx\mathbb{k}Q$  for some path  $x$  of length 0 (or 1), then we say that  $x$  is a *vertex in  $p$*  (respectively *arrow in  $p$* ). Likewise if  $p \in JxJ$ , where  $J$  is the 2-sided ideal of  $\mathbb{k}Q$  generated by the arrows, we say that  $x$  is a *vertex* (respectively *arrow*) *strictly in  $p$* .

Label the relations  $r_1, \dots, r_m$  such that the concatenation of  $m$  paths  $H_1 \rightarrow H_2 \rightarrow \dots \rightarrow H_m \rightarrow H_1$  has length  $n$ , where  $H_i = o(r_i)$ . Likewise let  $T_i = t(r_i)$ . Define  $m + 1 = 1$ . We call  $H_i$  the *head vertex* or *head*, and  $T_i$  the *tail vertex* or *tail* of the relation  $r_i$ . An arrow is called a *head arrow* of  $r_i$  if its start vertex is  $H_i$ . Let  $\alpha_i$  be the head arrow of  $r_i$ , so  $o(\alpha_i) = H_i$ , and let  $\omega_i$  be the arrow that ends at  $T_i$ , so  $t(\omega_i) = T_i$ . A head vertex  $H$  is said to *follow* a tail vertex  $T$  if the path  $T \rightarrow H$  is of minimal length among all paths starting at  $T$  and ending at a head vertex. If a tail vertex  $T$  is also a head vertex  $H$  then we also say  $H$  follows  $T$ .

Our first results give us some control over the behaviour of maximal overlap sequences. Proofs of the first two results use induction and are left to the reader.

**Proposition 1.1.** *Let  $\Lambda$  have  $m$  relations  $r_1, \dots, r_m$  with the path  $H_1 \rightarrow \dots \rightarrow H_m \rightarrow H_1$  of length  $n$ . Then the path  $T_1 \rightarrow \dots \rightarrow T_m \rightarrow T_1$  is also of length  $n$ .*

**Proposition 1.2.** *The path of unoverlapped arrows at the end of an odd-degree maximal overlap sequence always has length less than or equal to  $n$ .*

**Lemma 1.3.** *Let  $P^{2i}$  be a maximal overlap sequence of degree  $2i$  with last relation  $s_{2i}$ . If  $\ell(s_{2i}) > n$ , then one can always overlap  $P^{2i}$  on the right with another relation.*

**Proof.** If  $i = 1$ , the result follows since a relation of length greater than  $n$  has to overlap itself. For  $i > 1$ ,  $P^{2i}$  takes the form:

$$P^{2i} = \begin{array}{c} \overbrace{\hspace{1.5cm}}^{s_3} \\ \underbrace{\hspace{1.5cm}}_{s_2} \end{array} \cdots \begin{array}{c} \overbrace{\hspace{1.5cm}}^{s_{2i-1}} \\ \underbrace{\hspace{1.5cm}}_{s_{2(i-1)}} \end{array} \begin{array}{c} \overbrace{\hspace{1.5cm}}^{q} \\ \underbrace{\hspace{1.5cm}}_{s_{2i}} \end{array} \begin{array}{c} \overbrace{\hspace{1.5cm}}^p \\ \underbrace{\hspace{1.5cm}}_{s_{2i}} \end{array}$$

If there are  $n$  or more unoverlapped arrows at the end of  $P^{2i}$ , then we are done, so set  $p$  equal to the path of unoverlapped arrows and suppose  $\ell(p) < n$ ; let  $q$  be such that  $s_{2i} = qp$ . Now, since  $\ell(s_{2i}) > n$ , we must have at least two copies of the head arrow of  $s_{2i}$  appearing in  $s_{2i}$ . By Proposition 1.2, only one copy can appear in  $q$  and so we must have at least one copy in  $p$ .  $\square$

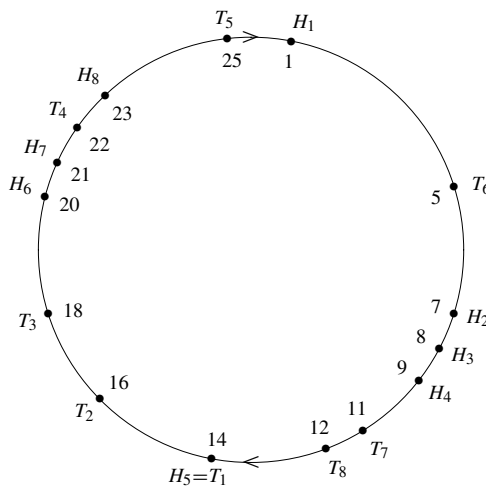
We will use the following example throughout the paper to illustrate our method.

**Example 1.** Let  $\mathcal{Q}$  be an oriented cycle with 25 vertices labelled  $1, \dots, 25$ . Label an arrow  $\eta_i$  if it starts at vertex  $i$ . Let  $\mathcal{I} = \langle r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8 \rangle$ , where

$$\begin{array}{llll} r_1 = \eta_1 \cdots \eta_{13}, & r_2 = \eta_7 \cdots \eta_{15}, & r_3 = \eta_8 \cdots \eta_{17}, & r_4 = \eta_9 \cdots \eta_{21}, \\ r_5 = \eta_{14} \cdots \eta_{24}, & r_6 = \eta_{20} \cdots \eta_4, & r_7 = \eta_{21} \cdots \eta_{10}, & r_8 = \eta_{23} \cdots \eta_{11}, \end{array}$$

and let  $\Lambda = \mathbb{k}\mathcal{Q}/\mathcal{I}$ .

We thus have head vertices  $H_1 = 1, H_2 = 7, H_3 = 8$ , etc., and tail vertices  $T_1 = 14, T_2 = 16, T_3 = 18$ , etc. The following diagram illustrates which vertices are head or tail vertices in our example:

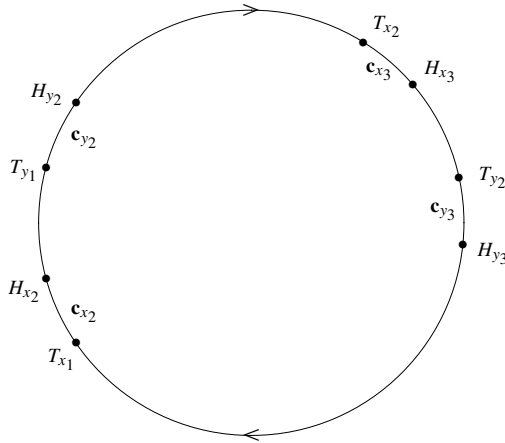


Note that  $H_2$  is the head vertex that follows  $T_6$ , and  $H_5$  follows  $T_7, T_8$ , and  $T_1$ .



**Proof.** If  $\Lambda$  has precisely one repetition, and this is of order 1, we are immediately done. We treat all other cases together. Thus let  $R_x$  be a repetition of order  $k$  with connective paths  $\mathbf{c}_{x_2}, \mathbf{c}_{x_3}, \dots, \mathbf{c}_{x_{k+1}}$ , let  $R_y$  be a repetition of order  $l$  with connective paths  $\mathbf{c}_{y_2}, \mathbf{c}_{y_3}, \dots, \mathbf{c}_{y_{l+1}}$ . Recall that  $\mathbf{c}_{x_2}$  is the path  $T_{x_1} \rightarrow H_{x_2}$  and  $\mathbf{c}_{y_2}$  is the path  $T_{y_1} \rightarrow H_{y_2}$ . Relabelling if necessary, suppose that  $\mathbf{c}_{x_2}$  and  $\mathbf{c}_{y_2}$  are adjacent on the quiver, that is they are the only distinct connective paths of  $R_x$  and  $R_y$ , respectively, that are subpaths of the

path  $T_{x_1} \rightarrow H_{y_2}$ . We will show that  $\mathbf{c}_{x_3}$  and  $\mathbf{c}_{y_3}$  are the only connective paths of  $R_x$  and  $R_y$ , respectively, that are subpaths of the path  $T_{x_2} \rightarrow H_{y_3}$ :



Let  $\sigma \subseteq \rho$  be the set of relations whose indices appear in  $R_x$  or  $R_y$ . In particular, the head vertex of each of these relations is the end vertex of a connective path of either  $R_x$  or  $R_y$ . By hypothesis, we have no head vertex indexed by a relation in  $\sigma$  in the path  $H_{x_2} \rightarrow H_{y_2}$  except the start and end vertices. Thus by Proposition 1.1, there is no tail vertex indexed by a relation in  $\sigma$  in the path  $T_{x_2} \rightarrow T_{y_2}$  except the start and end vertices. Hence  $\mathbf{c}_{x_3}$  and  $\mathbf{c}_{y_3}$  are the only connective paths of  $R_x$  or  $R_y$  that are subpaths of the path  $T_{x_2} \rightarrow H_{y_3}$ . Inductively it follows that  $k + 1 = l + 1$ , giving equality of the orders of  $R_x$  and  $R_y$ .  $\square$

**Remark.** The above proof gives us some insight not present in the statement of the lemma: we will thus refer to the proof itself later in the paper. In particular, we note here a consequence. Suppose that  $\Lambda$  has two or more repetitions, and let  $\mathbf{c}_{x_2}$  and  $\mathbf{c}_{y_2}$  be any two distinct connective paths, with no other connective paths in the path  $T_{x_1} \rightarrow H_{y_2}$ . Then  $\mathbf{c}_{x_2}$  and  $\mathbf{c}_{y_2}$  are in different repetitions.

The reader may like to note that diagrams of the sort in the proof above can be drawn to illustrate most of the proofs in this paper.

We now define the smo-tube, a combinatorial description of the maximal overlap sequences of  $\Lambda$ .

### Definition 1.7.

- (1) A *degeneration path* is a path  $T_p \rightarrow H_q$ , denoted  $\mathbf{d}_q$ , with no head or tail vertices strictly in the path  $\mathbf{d}_q$  and such that, unless  $\mathbf{d}_q$  is of zero length,  $\mathbf{o}(\mathbf{d}_q)$  is not a head vertex and  $\mathbf{t}(\mathbf{d}_q)$  is not a tail vertex. Then  $H_q$  follows  $T_p$  and we call  $T_p$  and  $H_q$  respectively *degeneration tail* and *head* vertices. Notice that every connective path has exactly one degeneration path as a terminal subpath: this means

that every repetition index is also the index of a degeneration head vertex. Let  $D$  be the set of degeneration paths of  $\Lambda$ , with elements labelled  $\mathbf{d}_{q_1}, \mathbf{d}_{q_2}, \dots, \mathbf{d}_{q_{|D|}}$  so that  $p_1 < p_2 < \dots < p_{|D|}$  with respect to the ordering of the relations of  $\Lambda$ , where  $T_{p_i} = \mathfrak{o}(\mathbf{d}_{q_i})$  for  $1 \leq i \leq |D|$ .

- (2) Place the smo-sequence of  $T_{p_i}$  in the  $i$ th row of an array where row 1 is at the bottom. We call this array the *smo-array*.

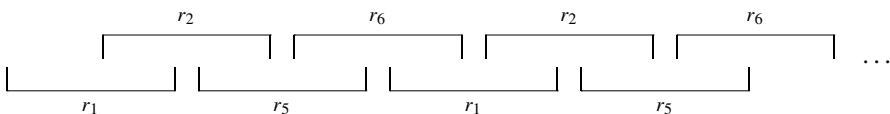
In practice one need only write down the first  $L$  columns, for  $L = M + \lambda + 1$ , where the  $M$ th column is the first to contain only repetition indices and  $\lambda$  is the order of the repetitions. We can bound the size of  $L$  as follows. Consider any row  $i$  in the smo-array that has entry  $(i, M - 1)$  not a repetition index. Since there are  $m$  relations, at least  $\lambda$  of which are repetition relations, we have  $M - 1 \leq m - \lambda$ . This yields a bound of  $L = M + \lambda + 1 \leq m + 2$ . Since  $|D| \leq m$ , we can have no more than  $m(m + 2)$  entries in the first  $L$  columns of the smo-array.

Fix the above definitions of  $\lambda$ ,  $M$ , and  $D$  for the remainder of the paper. We consider the top row of the above array to be joined to the bottom, and so, once the flags of the next definition have been placed, we will call this array the *smo-tube* and denote it  $\mathcal{T}_\Lambda$ . In this spirit we will refer to the  $j$ th column as *band  $j$* . Henceforth *entry  $(i, j)$*  refers to the entry of the smo-tube (or array) in row  $i$ , band  $j$ . Thus, if  $(a_k)$  is an smo-sequence such that  $a_s = (i, j)$ , for some  $s, i, j$ , then  $a_{s+1} = (i, j + 1)$ .

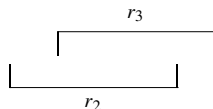
**Example 3.** In Example 1 we have 5 degeneration paths:  $\eta_{25}$ ,  $\eta_5\eta_6$ ,  $e_{14}$ ,  $\eta_{18}\eta_{19}$ , and  $\eta_{22}$ . We thus have the smo-array:

	1	2	3	4	5	6	.....
5	6	2	6	2	6	2	.....
4	5	1	5	1	5	1	.....
3	4	8	5	1	5	1	.....
2	3	6	2	6	2	6	.....
1	1	5	1	5	1	5	.....

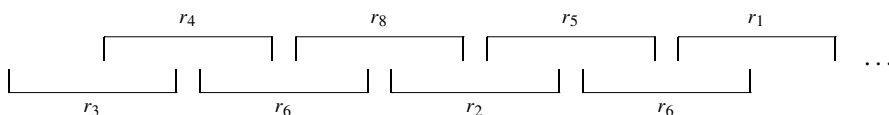
Here we have  $M = 3$  and  $\lambda = 2$ , giving  $L = 6$ . Let us look again at extending sequences to see how they relate to the smo-array. The extending sequence  $A_{H_1}$  has lower half  $(r_1, r_5, r_1, r_5, \dots)$  and upper half  $(r_2, r_6, r_2, r_6, \dots)$ , yielding the overlaps



However, the extending sequence  $A_{H_2}$  terminates almost immediately, that is at degree 3, with the maximal overlap sequence

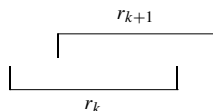


It is clear that the reason for this degree 3 termination is that  $T_2$  is not the start of a degeneration path. We also view the overlaps from which  $A_{H_3}$  is formed:



Note that in all but position 1, the upper half of  $A_{H_1}$  is the same as the lower half of  $A_{H_3}$ . This occurs because the same head vertex,  $H_6$ , follows  $T_2$  and  $T_3$ .

The above example well illustrates what happens for general  $\Lambda$ , inasmuch as two things go. Firstly, if  $T_k$  is not a degeneration tail vertex then  $A_{H_k}$  terminates at degree 3. This is clear since if  $T_k$  is not a degeneration tail vertex then it is immediate that we have no head arrows in the path  $T_k \rightarrow T_{k+1}$ . By Proposition 1.2, this path is equal to the path of unoverlapped arrows in the maximal overlap sequence



This maximal overlap sequence therefore cannot be overlapped by a relation on the right. Note that  $r_k$  can always be overlapped by  $r_{k+1}$  if  $T_k$  is not a degeneration tail vertex. This is why we include only degeneration tail vertices in the first band of the smo-array: all other tail vertices give rise to extending sequences that terminate at degree 3.

Secondly, suppose that  $T_a$  and  $T_b$  are degeneration tail vertices. If  $\Lambda$  has only one degeneration path take  $T_a = T_b$ , otherwise take  $T_a$  and  $T_b$  such that the path  $T_a \rightarrow T_b$  is of positive length (so  $T_a \neq T_b$ ) and contains no other degeneration tail vertices. We consider the upper-half of  $A_{H_a}$ ,  $(r_{a+1}, r_{f_\Lambda(a+1)}, r_{f_\Lambda^2(a+1)}, \dots)$ , and the lower-half of  $A_{H_b}$ ,  $(r_b, r_{f_\Lambda(b)}, r_{f_\Lambda^2(b)}, \dots)$ . Let us show that  $r_{f_\Lambda(a+1)} = r_{f_\Lambda(b)}$ . If  $\Lambda$  has only one degeneration path, then this is immediate since  $H_{f_\Lambda(b)}$  is the head vertex that follows all tail vertices, and thus follows  $T_{a+1}$ . So suppose  $T_b$  is different from  $T_a$ , as above, and consider  $T_{a+1}$ . By hypothesis there cannot be a head vertex in the path  $T_{a+1} \rightarrow T_b$ , other than possibly  $T_b$  itself, else we would have a degeneration tail vertex strictly in the path  $T_a \rightarrow T_b$ . Hence we have that  $H_{f_\Lambda(b)}$  is the head vertex that follows both  $T_{a+1}$  and  $T_b$ , giving  $r_{f_\Lambda(a+1)} = r_{f_\Lambda(b)}$ . It follows immediately that  $r_{f_\Lambda^k(a+1)} = r_{f_\Lambda^k(b)}$  for all  $k \geq 1$ , and hence that the upper-half

of  $A_{H_a}$  is identical to the lower-half of  $A_{H_b}$  in all places but the first. For this reason we include only the lower halves of extending sequences  $A_{H_k}$ , where  $T_k$  is a degeneration tail vertex, as rows in the smo-array. We then get the upper halves automatically from the row above. This simplifies matters a great deal, as long as we keep track of what is really happening in band 1.

The above reasoning has shown that the smo-array contains all the information needed to build the infinite extending sequences of  $\Lambda$ . However, with the smo-array as it stands, we are unable to tell which, if any, of the extending sequences terminate. The problem is that smo-sequences are infinite whilst extending sequences can be finite, terminating in a maximal overlap sequence that cannot be overlapped on the right by a relation. We solve this problem by introducing *flags* to the smo-array. The position of a flag in some row  $i$  of  $T_\Lambda$  indicates that the extending sequence with lower half associated to row  $i$  terminates; the exact point of termination depends on the type of flag. For an extending sequence  $A_v$  of  $\Lambda$ , Theorem 1.10 gives the exact values of  $\maxdeg(A_v)$  for each position and type of flag. The following definition gives the rules for marking the smo-array with the different types of flag. Recall that  $\alpha_k$  is the first arrow of the relation  $r_k$ .

**Definition 1.8.**

- (0) *Flags of type 0.* For  $1 \leq i \leq |D|$ , entry  $(i, j)$  is marked flag0 if and only if  $j = 1$ ,  $\ell(r_{(i,1)}) \leq n$  and  $r_{(i,1)}$  contains no head arrow other than  $\alpha_{(i,1)}$ .
- (1) *Flags of type 1.* If  $|D| = 1$ , then no entry is marked flag1.  
If  $|D| > 1$ , then for  $1 \leq i \leq |D|$  and  $2 \leq j \leq L$ , entry  $(i, j)$  of the smo-tube is marked flag1 if and only if  $(i, j) \neq (i+1, j)$  and no head arrow lies in the path  $T_{(i,j)} \rightarrow T_{(i+1,j)}$ . Note that if  $i = |D|$  then  $i+1 = 1$ .
- (2) *Flags of type 2.* If  $|D| = 1$ , take  $(2, 1) = (1, 1) + 1$ , and  $(2, j) = (1, j)$  for  $j \geq 2$ . If  $|D| > 1$  and  $i = |D|$ , then take  $i+1 = 1$ . Let  $N(i, j)$  be the number of occurrences of  $\alpha_{(1,1)}$  in  $r_{(i,j)}$ .
  - (i) For  $2 \leq i \leq |D|$ ,  $2 \leq j \leq L$ , entry  $(i, j)$  is marked flag2 if and only if  $\sum_{k=1}^{j-1} N(i+1, k) = \sum_{k=1}^j N(i, k)$  and
    - \* for  $j = 2$  we have  $(i, 1) + 1 \neq (i, 2)$  and no head arrow lies in the path  $T_{(i,1)+1} \rightarrow T_{(i,2)}$ ;
    - \* for  $3 \leq j \leq L$  we have  $(i, j) \neq (i+1, j-1)$  and no head arrow lies in the path  $T_{(i+1,j-1)} \rightarrow T_{(i,j)}$ .
  - (ii) Entry  $(1, j)$ , for  $2 \leq j \leq L$ , is marked flag2 if and only if  $1 + \sum_{k=1}^{j-1} N(2, k) = \sum_{k=1}^j N(1, k)$  and
    - \* for  $j = 2$  we have  $(1, 1) + 1 \neq (1, 2)$  and no head arrow lies in the path  $T_{(1,1)+1} \rightarrow T_{(1,2)}$ ;
    - \* for  $3 \leq j \leq L$  we have  $(1, j) \neq (2, j-1)$  and no head arrow lies in the path  $T_{(2,j-1)} \rightarrow T_{(1,j)}$ .

**Example 4.** In Example 1 the only entry to be marked with a flag is entry  $(3, 2)$ : it is a flag1. We return to this example in Section 2 with more detail.

The above definition is a very slow way to calculate flags and so Section 2 will present three theorems which speed up this calculation. However, we do have the following corollary of Lemma 1.3.

**Corollary 1.9.** *If all relations  $r$  are such that  $\ell(r) > n$ , then no flag0s or flag2s can be placed in the smo-tube.*

We complete this section by proving that the flags defined in Definition 1.8 really do give us precisely the termination points of the finite extending sequences.

**Theorem 1.10.** *Let  $A$  be an extending sequence with  $\mathfrak{o}(A) = H_a$  and let  $\mathcal{T}_A$  be the smo-tube of  $A$ . Then*

- (1)  $\maxdeg A = 2$  if and only if  $a = (i, 1)$  for some row  $i$  of  $\mathcal{T}_A$ , and  $(i, 1)$  is marked flag0;
- (2)  $\maxdeg A = 3$  if and only if  $a \neq (i, 1)$  for all rows  $i$ ;
- (3)  $\maxdeg A = 2j$ , some  $j \geq 2$ , if and only if  $a = (i, 1)$  for some row  $i$ , and  $(i, j)$  is marked flag2 and is the first flagged entry in row  $i$ ;
- (4)  $\maxdeg A = 2j + 1$ , some  $j \geq 2$ , if and only if  $a = (i, 1)$  for some row  $i$ , and  $(i, j)$  is marked flag1 and is the first flagged entry in row  $i$ ;
- (5)  $\maxdeg A = \infty$  if and only if  $a = (i, 1)$  for some unflagged row  $i$ .

**Proof.** Let us look in turn at the different ways in which an extending sequence might terminate.

(1) Let  $A$  be an extending sequence; by definition  $A$  attains at least degree 2. The degree 2 maximal overlap sequence is just a single relation,  $r_k$  say. Thus by previous reasoning,  $A$  terminating at degree 2 is equivalent to  $k$  equalling  $(i, 1)$ , for some row  $i$  in  $\mathcal{T}_A$ , and  $r_k$  not having any relation overlapping it. This in turn is equivalent to  $k$  equalling  $(i, 1)$ , for some row  $i$  in  $\mathcal{T}_A$ , and  $r_k$  containing no head arrows other than  $\alpha_k$  once at the start: exactly the condition needed to mark entry  $(i, 1)$  with a flag0.

(2) We have seen already how  $\mathcal{T}_A$  excludes precisely those extending sequences that terminate at degree 3.

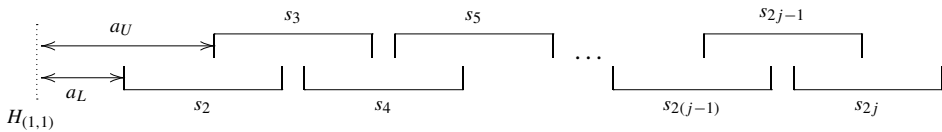
(3) Whether the extending sequence  $A$  terminates at degree  $2j$ , for some  $j \geq 2$ , is, a priori, more difficult to determine. We know already that this condition is equivalent to the degree  $2j$  maximal overlap sequence of  $A$ ,

$$A^{2j} = \begin{array}{ccccccc} & & s_3 & & s_5 & & \dots & & s_{2j-1} & & \\ & \overbrace{\hspace{1.5cm}} & & \overbrace{\hspace{1.5cm}} & & \dots & & \overbrace{\hspace{1.5cm}} & & \overbrace{\hspace{1.5cm}} & \\ s_2 & \boxed{\phantom{000}} & s_4 & \boxed{\phantom{000}} & \dots & \boxed{\phantom{000}} & s_{2(j-1)} & \boxed{\phantom{000}} & s_{2j} & \end{array} \quad \begin{array}{c} \leftarrow p \rightarrow \end{array}$$

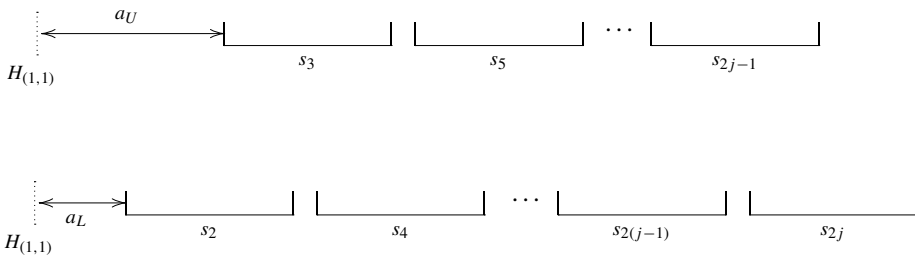
having no head arrows in the path of unoverlapped arrows  $p$ . The problem is that  $p$  may have length less, greater or equal to  $n$ . If  $\ell(p) \geq n$ , then there will be a head arrow in  $p$ , and we can conclude that  $A$  does not terminate at degree  $2j$ . If  $\ell(p) < n$ , then  $p$  equals the path  $t(s_{2j-1}) \rightarrow t(s_{2j})$ . We then just need to check the path  $t(s_{2j-1}) \rightarrow t(s_{2j})$  for head arrows to determine whether or not  $A$  terminates at degree  $2j$ . We calculate whether  $\ell(p) < n$

with a counting argument. Let  $A^{2j}$  be the maximal overlap sequence shown above, with  $\circ(A^{2j}) = \circ(s_2)$ .

Suppose first that the lower half of  $A^{2j}$  is not in row 1 of  $\mathcal{T}_A$ . Consider the two paths in  $\mathbb{k}\mathcal{Q}$  that consist of  $A^{2j-1}$  and  $A^{2j}$  with the path  $a_L$  appended to the start of each, where  $\ell(a_L) < n$ ,  $\circ(a_L) = H_{(1,1)}$ ,  $\mathfrak{t}(a_L) = \circ(s_2)$  and  $a_U$  is the path as shown below:



Let  $U = a_L A^{2j-1}$  and  $L = a_L A^{2j}$ . We visualise the paths  $U$  and  $L$  respectively in the following natural way:



Write  $U = c^k q$ , where  $c$  is the cycle such that  $\ell(c) = n$ ,  $\circ(c) = \mathfrak{t}(c) = H_{(1,1)}$  and  $q$  is such that  $\ell(q) < n$ ,  $\circ(q) = H_{(1,1)}$ ,  $\mathfrak{t}(q) = \mathfrak{t}(s_{2j-1})$ . Then  $L = Up = c^k qp$ . We will count the number of occurrences of  $\alpha_{(1,1)}$  in  $U$  and in  $L$ . Then, if  $\alpha_{(1,1)}$  occurs more often in  $L$  than in  $U$ , we can conclude that at least one  $\alpha_{(1,1)}$  is in  $p$ . Since  $\alpha_{(1,1)}$  is a head arrow we would know that  $A$  does not terminate at degree  $2j$ . Otherwise, if  $\alpha_{(1,1)}$  occurs the same number of times in  $U$  as it does in  $L$ , then we know it cannot occur in  $p$ . Thus  $\ell(p) < n$  and so we check the path  $\mathfrak{t}(s_{2j-1}) \rightarrow \mathfrak{t}(s_{2j})$  for head arrows as detailed above.

For any finite path  $v \in \mathbb{k}\mathcal{Q}$  let  $N(v)$  be the number of occurrences of  $\alpha_{(1,1)}$  in  $v$ . Then

$$N(L) = 1 + \sum_{z=1}^j N(s_{2z}) \quad \text{and} \quad N(U) = 1 + \sum_{z=2}^j N(s_{2z-1}).$$

For firstly the “gaps” between the relations (the paths  $Y_z$ ,  $4 \leq z \leq 2j$ , in the Preliminaries section) contain no head arrows by maximality of the construction of  $A^{2j}$ , and secondly, since  $\circ(s_2) \neq H_{(1,1)}$  the paths  $a_U$  and  $a_L$  contain exactly one copy of  $\alpha_{(1,1)}$ . Thus

$$\sum_{z=1}^j N(s_{2z}) = \sum_{z=2}^j N(s_{2z-1}) \iff N(L) = N(U) \implies \ell(p) < n; \quad \text{and}$$

$$\sum_{z=1}^j N(s_{2z}) > \sum_{z=2}^j N(s_{2z-1}) \iff N(L) > N(U) \implies p$$

contains a head arrow. Note that the function  $N$  here is an extension to all paths of the function of the same name in Definition 1.8. We leave it to the reader to verify that  $N(s_3) = N(i+1, 1)$ , where  $s_2 = r_{(i,1)}$ . This shows that marking entry  $(i, j)$  with a flag2 via Definition 1.8, part (2i), is equivalent to the corresponding maximal overlap sequence terminating at degree  $2j$ .

The case where  $\sigma(s_2) = H_{(1,1)}$  is almost identical to above; the only change is that now  $\ell(a_L) = 0$  and we have  $N(a_L) = 0$ ,  $N(a_U) = 1$ . This is left to the reader.

(4) This part follows part (3) above, but without the necessity for the counting argument, since by Proposition 1.2 the path of unoverlapped arrows  $p$  is always such that  $\ell(p) \leq n$ .

(5) This part is also left in the hands of the reader.  $\square$

## 2. Calculating flags

In this section we present three theorems that speed up calculation of the smo-tube with its flags: for this reason the section becomes rather technical. Note that certain of the auxiliary results presented here will be drawn upon throughout the remainder of the paper. We will illustrate calculation of the smo-tube with two examples.

**Lemma 2.1.** *For any entry  $(i, j)$  in an smo-tube, we have that the concatenation of paths  $H_{(i,j)} \rightarrow H_{(i+1,j)} \rightarrow \cdots \rightarrow H_{(|D|,j)} \rightarrow H_{(1,j)} \rightarrow \cdots \rightarrow H_{(i-1,j)} \rightarrow H_{(i,j)}$  is of length  $n$ .*

**Proof.** Using Proposition 1.1 and the discussion in Definition 1.7, the result follows by induction on  $j$ .  $\square$

**Lemma 2.2.** *If  $x$  is some repetition index, then for all  $j$  greater or equal to 2, there exists some row  $i$  in  $\mathcal{T}_A$  such that  $x = (i, j)$ .*

**Proof.** Recall that band 1 contains the indices of all degeneration tail vertices. By definition, the next entry in the smo-sequence of a such a tail vertex is the index of a degeneration head. We get all degeneration head indices this way; these appear in band 2. As remarked in Definition 1.7, each repetition index is also a degeneration head index, and so each repetition index appears in band 2. Immediately we get that each repetition index appears in band  $j$ , for all  $j \geq 2$ .  $\square$

**Theorem 2.3.** *If an entry  $(i, j)$  of  $\mathcal{T}_A$  is assigned a flag1, then  $(i, j) \neq (i+1, j)$  and  $(i, j+1) = (i+1, j+1)$ .*

*Moreover, if  $\Lambda$  has more than one repetition index, then  $(i, j)$  is assigned a flag1 if and only if  $(i, j) \neq (i+1, j)$  and  $(i, j+1) = (i+1, j+1)$ .*



**Proof.** If entry  $(i, j)$  is marked with a flag1, then by definition  $(i + 1, j) \neq (i, j)$  and there is no head arrow in the path  $T_{(i,j)} \rightarrow T_{(i+1,j)}$ . This means that  $H_{(i,j+1)}$  follows both  $T_{(i,j)}$  and  $T_{(i+1,j)}$  giving  $H_{(i,j+1)} = H_{(i+1,j+1)}$  and so  $(i, j + 1) = (i + 1, j + 1)$ .

Conversely suppose that  $(i, j)$  is such that  $(i, j) \neq (i + 1, j)$  and  $(i, j + 1) = (i + 1, j + 1)$ ; this means that either the path  $T_{(i,j)} \rightarrow T_{(i+1,j)}$  or the path  $T_{(i+1,j)} \rightarrow T_{(i,j)}$  is free from head arrows. Suppose also that  $\Lambda$  has more than one repetition index. By Lemma 2.2, all repetition indices occur in band  $j + 1$  so, since  $\Lambda$  has more than one repetition index, there must be a third row in  $\mathcal{T}_\Lambda$ , row  $k$  say, such that  $(i + 1, j + 1) \neq (k, j + 1)$ . This gives us  $(i + 1, j) \neq (k, j)$  and  $(i, j) \neq (k, j)$ . Thus we have that  $T_{(i,j)} \rightarrow T_{(i+1,j)} \rightarrow T_{(k,j)} \rightarrow T_{(i,j)}$  is a path of length  $n$  by Lemma 2.1 and Proposition 1.1. Since a different head vertex follows  $T_{(k,j)}$  than follows  $T_{(i,j)}$  and  $T_{(i+1,j)}$ , there is a head vertex, namely  $H_{(k,j+1)}$ , in the path  $T_{(i+1,j)} \rightarrow T_{(i,j)}$ . Thus there must no head arrow in the path  $T_{(i,j)} \rightarrow T_{(i+1,j)}$ , and so  $(i, j)$  will be assigned a flag1.  $\square$

**Proposition 2.4.** *The number of rows in  $\mathcal{T}_\Lambda$  that do not have a flag1 is equal to the number of distinct repetition indices of  $\Lambda$ .*

**Proof.** Recall that the  $M$ th band of  $\mathcal{T}_\Lambda$  is the first to contain only repetition indices. If  $\Lambda$  has  $\mu \geq 2$  repetition indices, then by Lemma 2.2 there must be some row  $i$  such that  $(i, M + 1) \neq (i + 1, M + 1)$ . By Lemma 1.6, we get  $(i, j) \neq (i + 1, j)$ ,  $\forall j \geq 1$ . Thus by Theorem 2.3, row  $i$  will never be marked flag1. Lemmas 2.1 and 2.2 give us exactly  $\mu$  rows  $i$  in  $\mathcal{T}_\Lambda$  such that  $(i, M + 1) \neq (i + 1, M + 1)$ , which gives us at least  $\mu$  rows without a flag1 by above. By Theorem 2.3, any row  $i$  with  $(i, M + 1) = (i + 1, M + 1)$  has a flag1. This gives us precisely the same number of unflagged rows as we have repetition indices.

Suppose then that  $\Lambda$  has only one repetition index. We get our result immediately if  $|D| = 1$ , so suppose  $|D| \geq 2$ . All entries of band  $M$  are equal to the same repetition index,  $x$  say. From the proof of Lemma 2.2, band 2 contains at least 2 distinct indices and so we have  $M \geq 3$ . Now, all entries of band  $M - 1$  index tail vertices that are followed by  $H_x$ , so by Lemma 2.1 and Proposition 1.1 there exists exactly one row,  $i$  say, such that the path  $T_{(i,M-1)} \rightarrow T_{(i+1,M-1)}$  contains a head arrow. It follows from Theorem 2.3 that row  $i$  will not get a flag1. Let row  $k$  be different to row  $i$ . Then there exists  $j$ , with  $1 \leq j \leq M - 1$ , such that  $(k, j) \neq (k + 1, j)$  and  $(k, j + 1) = (k + 1, j + 1)$ . We will show  $(k, j)$  is marked flag1. To seek a contradiction suppose it is not. Then by definition,  $T_{(k,j)} \rightarrow T_{(k+1,j)}$  contains a head arrow. Since both  $T_{(k,j)}$  and  $T_{(k+1,j)}$  are followed by  $H_{(k,j+1)}$ , this means the path  $T_{(k+1,j)} \rightarrow T_{(k,j)}$  must contain no head arrows. Hence by Lemma 2.1, we must have  $j = M - 1$ , but since  $k \neq i$ , we get a contradiction. Thus entry  $(k, j)$  is marked flag1 for all  $k \neq i$ .  $\square$

**Theorem 2.5.** *If entry  $(i, 1)$  of  $\mathcal{T}_\Lambda$  is marked flag0, then  $(i, 1) + 1 = (i, 2)$  modulo  $m$ .*

*Moreover, if  $\ell(r_{(i,1)}) \leq n$ , entry  $(i, 1)$  is marked flag0 if and only if  $(i, 1) + 1 = (i, 2)$  modulo  $m$ .*

**Proof.** Suppose  $(i, 1)$  is marked flag0. This means no head arrow lies in  $r_{(i,1)}$  except  $\alpha_{(i,1)}$  once at the start. Therefore  $H_{(i,1)+1}$  is the head vertex that follows  $T_{(i,1)}$ , and hence  $(i, 1) + 1 = (i, 2)$ .

Conversely, suppose  $(i, 1) + 1 = (i, 2)$  and that  $\ell(r_{(i,1)}) \leq n$ . We have that  $H_{(i,1)+1}$  is the head vertex that follows  $T_{(i,1)}$  and so the path  $H_{(i,1)} \rightarrow T_{(i,1)} \rightarrow H_{(i,1)+1}$  must have length less than or equal to  $n$ . Suppose  $r_{(i,1)}$  is not the only relation, else we are done. This means  $(i, 1) \neq (i, 1) + 1$  and so  $\alpha_{(i,1)}$  is the only head arrow in the path  $H_{(i,1)} \rightarrow H_{(i,1)+1}$ . Since  $\ell(r_{(i,1)}) \leq n$ ,  $r_{(i,1)}$  must be an initial subpath of  $H_{(i,1)} \rightarrow H_{(i,1)+1}$ . Thus  $r_{(i,1)}$  contains no head arrow other than  $\alpha_{(i,1)}$ . Hence  $(i, 1)$  gets marked with a flag0.  $\square$

To prove a similar theorem concerning flag2s, we need access to a few more results. Recall that  $\lambda$  is the order of the repetitions.

**Proposition 2.6.** *If  $\lambda \geq 2$ , and  $r_{x_1}$  and  $r_{x_2}$  are two distinct relations, then the path  $H_{x_1} \rightarrow T_{x_1} \rightarrow T_{x_2} \rightarrow H_{x_2} \rightarrow H_{x_1}$  has length greater than  $n$ .*

**Proof.** We proceed by contradiction. Assume that  $r_{x_1}$  and  $r_{x_2}$  are distinct relations such that the path  $H_{x_1} \rightarrow T_{x_1} \rightarrow T_{x_2} \rightarrow H_{x_2} \rightarrow H_{x_1}$  is of length  $n$ . Since  $\lambda \geq 2$ ,  $H_{x_2}$  cannot follow  $T_{x_2}$ . We therefore have a third relation,  $r_{x_3}$  say, distinct from  $r_{x_1}$  and  $r_{x_2}$ , such that  $H_{x_3}$  is in the path  $T_{x_2} \rightarrow H_{x_2} - 1$ . By Proposition 1.1,  $T_{x_3}$  must lie in the path  $T_{x_1} \rightarrow T_{x_2}$ .

It is clear that each time this argument is applied to  $r_{x_1}$  and  $r_{x_i}$ , for some  $i \geq 2$ , we get a new relation  $r_{x_{i+1}}$  distinct from all the others. Since  $\mathcal{I}$  has a fixed finite generating set, we get our contradiction.  $\square$

**Corollary 2.7.** *Suppose  $\lambda \geq 2$  and let  $r_k$  be a relation such that  $cn < \ell(r_k) \leq (c+1)n$ , for some non-negative integer  $c$ . Then  $cn < \ell(r) \leq (c+1)n$  for all relations  $r$ .*

**Proof.** Let  $r_k$  be a relation such that  $cn < \ell(r_k) \leq (c+1)n$ , for some positive integer  $c$  and let  $r_l$  be a relation such that  $(c-1)n < \ell(r_l) \leq cn$ . To ensure that  $r_l$  is not a subpath of  $r_k$  it is clear that the path  $H_k \rightarrow T_k \rightarrow T_l \rightarrow H_l \rightarrow H_k$  must have length  $n$ . This contradicts the above proposition.  $\square$

**Definition 2.8.** Let  $r_{y_1}, \dots, r_{y_l}$  be the repetition relations of  $\Lambda$ , ordered such that the concatenation of  $l$  paths  $H_{y_1} \rightarrow H_{y_2} \rightarrow \dots \rightarrow H_{y_l} \rightarrow H_{y_1}$  is of length  $n$ . Then we call the path  $H_{y_i} \rightarrow H_{y_{i+1}}$  the *repetition path*  $\mathbf{b}_{y_{i+1}}$ , where  $l+1=1$ . Clearly every arrow in  $\mathbb{k}\mathcal{Q}$  is in precisely one repetition path.

**Example 5.** In Example 1 the repetition relations are  $r_1, r_2, r_5$ , and  $r_6$ ; the repetition paths are  $\eta_1 \cdots \eta_6, \eta_7 \cdots \eta_{13}, \eta_{14} \cdots \eta_{19}$ , and  $\eta_{20} \cdots \eta_{25}$ .

**Lemma 2.9.** *Suppose  $\lambda \geq 2$ . Let  $H_l$  be a degeneration head vertex with  $r_l$  not a repetition relation, and let  $H_l$  and  $T_l$  lie in the repetition path  $\mathbf{b}_a$ , with  $T_l \neq \mathbf{o}(\mathbf{b}_a)$ . Then  $\mathbf{t}(\mathbf{b}_a) = H_a$  is the head vertex that follows  $T_l$ .*

**Proof.** Let the setup be as above and let  $H_k = \mathbf{o}(\mathbf{b}_a)$ , so that  $\mathbf{b}_a$  is the path  $H_k \rightarrow H_a$ . To seek a contradiction suppose  $H_a$  does not follow  $T_l$ . Firstly,  $T_l$  lies in the path  $H_k + 1 \rightarrow H_l - 1$ , otherwise the path  $H_l \rightarrow T_l \rightarrow T_k \rightarrow H_k \rightarrow H_l$  would have length  $n$ , since  $T_k$  is a repetition tail vertex. This would contradict Proposition 2.6. Now  $\lambda \geq 2$ , so  $H_a$  may not

follow  $T_a$  and so  $T_a$ , since it is a repetition tail vertex, is in the path  $H_a \rightarrow H_k$ . By the ordering imposed by Proposition 1.1, we get that the path  $H_a \rightarrow T_a \rightarrow T_k \rightarrow H_k \rightarrow H_a$  is of length  $n$ . This contradicts Proposition 2.6 and we get our result.  $\square$

**Lemma 2.10.** *Suppose  $\lambda \geq 2$  and let  $(i, j)$  be an entry of  $\mathcal{T}_\Lambda$ . Then*

- (1) *if  $j = 2$ , the path  $H_{(i,1)} \rightarrow H_{(i,1)+1} \rightarrow H_{(i,2)} \rightarrow H_{(i+1,2)} \rightarrow H_{(i,1)}$  has length  $n$ ;*
- (2) *if  $j \geq 3$ , the path  $H_{(i,j-1)} \rightarrow H_{(i+1,j-1)} \rightarrow H_{(i,j)} \rightarrow H_{(i+1,j)} \rightarrow H_{(i,j-1)}$  has length  $n$ .*

**Proof.** (1) Suppose that  $j = 2$ . Since  $H_{(i,2)}$  and  $H_{(i+1,2)}$  are both degeneration head vertices, the path  $T_{(i,1)} \rightarrow H_{(i,2)} \rightarrow T_{(i,1)+1} \rightarrow H_{(i+1,2)} \rightarrow T_{(i,1)}$  is of length  $n$ . Also  $T_{(i+1,2)}$  must be in the path  $H_{(i+1,2)} + 1 \rightarrow T_{(i,1)}$  since  $\lambda \geq 2$ . Now,  $T_{(i,2)}$  must be in the path  $T_{(i,1)} \rightarrow T_{(i+1,2)}$  else we contradict Proposition 2.6 with  $r_{(i+1,2)}$  and  $r_{(i,2)}$ . Since  $\lambda \geq 2$ ,  $T_{(i,2)} \neq T_{(i,1)}$  so in fact  $T_{(i,2)}$  is in the path  $T_{(i,1)+1} \rightarrow T_{(i+1,2)}$ . Thus the path  $T_{(i,1)} \rightarrow T_{(i,1)+1} \rightarrow T_{(i,2)} \rightarrow T_{(i+1,2)} \rightarrow T_{(i,1)}$  has length  $n$  and Proposition 1.1 yields our result.

(2) Suppose now that  $j \geq 3$ . We show that the path  $H_{(i,j-1)} \rightarrow H_{(i+1,j-1)} \rightarrow H_{(i,j)} \rightarrow H_{(i+1,j)} \rightarrow H_{(i,j-1)}$  has length  $n$ . Note that if  $(i, j-1) = (i+1, j-1)$ , then  $(i, j) = (i+1, j)$  and we immediately get our result. Thus we assume  $(i, j-1) \neq (i+1, j-1)$ . Consider the two vertices  $H_{(i,j-1)}$  and  $H_{(i+1,j-1)}$ , the terminating vertices of the degeneration paths  $\mathbf{d}_{(i,j-1)}$  and  $\mathbf{d}_{(i+1,j-1)}$ , respectively.

We first wish to place  $H_{(i,j)}$  in the path  $H_{(i+1,j-1)} \rightarrow H_{(i,j-1)}$ . We assume  $(i, j) \neq (i+1, j-1)$  and  $(i, j) \neq (i, j-1)$ . To seek a contradiction suppose  $H_{(i,j)}$  is in the path  $H_{(i,j-1)} + 1 \rightarrow H_{(i+1,j-1)} - 1$ . Then by Lemmas 2.1 and 2.2,  $\mathbf{d}_{(i,j)}$ , and therefore  $\mathbf{d}_{(i,j-1)}$ , cannot be a repetition degeneration path. By Lemma 2.9 this means  $H_{(i,j-1)}$  and  $T_{(i,j-1)}$  cannot be in the same repetition path. Thus there must be a repetition head vertex in the path  $H_{(i,j-1)} + 1 \rightarrow H_{(i,j)} - 1$ , contradicting Lemmas 2.1 and 2.2 regarding band  $j-1$ . Hence  $H_{(i,j)}$  must lie in the path  $H_{(i+1,j-1)} \rightarrow H_{(i,j-1)} - 1$ .

It remains only to locate  $H_{(i+1,j)}$  in the path  $H_{(i,j)} \rightarrow H_{(i,j-1)}$ . Assume  $(i+1, j) \neq (i, j)$  and  $(i+1, j) \neq (i, j-1)$ . There are two cases to consider:

- (i) If  $H_{(i+1,j)}$  lies in the path  $H_{(i,j-1)} \rightarrow H_{(i+1,j-1)}$ , then so does  $T_{(i+1,j-1)}$ . Two subcases arise. If  $H_{(i+1,j-1)}$  is a repetition head, then so is  $H_{(i+1,j)}$ . This is contradicted by Lemmas 2.1 and 2.2. If  $H_{(i+1,j-1)}$  is not a repetition head, then we must have a repetition head in the path  $H_{(i+1,j)} \rightarrow H_{(i+1,j-1)}$  or Lemma 2.9 will be contradicted. However, the existence of this repetition head again contradicts Lemmas 2.1 and 2.2.
- (ii) If  $H_{(i+1,j)}$  lies in the path  $H_{(i+1,j-1)} + 1 \rightarrow H_{(i,j)} - 1$ , then so does  $T_{(i+1,j-1)}$ . Proposition 2.6 on relations  $r_{(i,j-1)}$  and  $r_{(i+1,j-1)}$  provides the contradiction.

Hence  $H_{(i+1,j)}$  must lie in the path  $H_{(i,j)} \rightarrow H_{(i,j-1)}$ . This completes the proof.  $\square$

We can at last prove our final theorem of this section.

**Theorem 2.11.** *If an entry  $(i, j)$  of  $\mathcal{T}_\Lambda$  is assigned a flag2, then*

- (1) *for  $j = 2$  we have  $(i, 1) + 1 \neq (i, 2)$  and  $(i + 1, 2) = (i, 3)$ ;*
- (2) *for  $j \geq 3$  we have  $(i + 1, j - 1) \neq (i, j)$  and  $(i + 1, j) = (i, j + 1)$ .*

*Moreover, if  $\lambda \geq 2$  and  $\ell(r) \leq n$  for all relations  $r$ , then  $(i, j)$  is assigned a flag2 if and only if the appropriate condition (1) or (2) holds.*

**Proof.** If  $(i, j)$  is assigned a flag2, then showing the appropriate condition (1) or (2) is easy.

For the reverse direction, suppose  $\lambda \geq 2$  and  $\ell(r) \leq n$  for all relations  $r$ . Suppose also that  $j \geq 3$  and the conditions from (2) hold. For a contradiction assume that no flag2 is assigned to entry  $(i, j)$ . As  $\ell(r) \leq n$  for all relations  $r$ , and by Theorem 1.10, no flag2 assigned to entry  $(i, j)$  is equivalent to the existence of a head arrow in the path of unoverlapped arrows  $T_{(i+1, j-1)} \rightarrow T_{(i, j)}$ . However, since  $T_{(i+1, j-1)}$  and  $T_{(i, j)}$  are both followed by the same head vertex, we have the path  $T_{(i, j)} \rightarrow T_{(i+1, j-1)}$  free from head arrows. As  $\lambda \geq 2$  we must have  $T_{(i+1, j)}$  in the path  $T_{(i+1, j-1)} + 1 \rightarrow T_{(i, j)} - 1$ . This means the path  $T_{(i+1, j-1)} \rightarrow T_{(i, j)} \rightarrow T_{(i+1, j)}$  has length greater than  $n$ , contradicting Lemma 2.10.

The case of  $j = 2$  is similar.  $\square$

**Example 6.** For our algebra of Example 1, we have a single flag1 in the smo-tube. There are no flag0s or flag2s. The position of the flag1 is indicated by the square box:

	1	2	3	4	5	6	.....
5	6	2	6	2	6	2	.....
4	5	1	5	1	5	1	.....
3	4	<span style="border: 1px solid black; padding: 0 2px;">8</span>	5	1	5	1	.....
2	3	6	2	6	2	6	.....
1	1	5	1	5	1	5	.....

Theorem 2.3 was used to mark the flag1: notice above that  $(3, 2) \neq (4, 2)$  but that  $(3, 3) = (4, 3)$ . Since here  $\ell(r) \leq n$ , for all relations  $r$ , and  $\lambda \geq 2$ , we may use the full equivalences of Theorems 2.5 and 2.11 to conclude that there are no flag0s or flag2s present in  $\mathcal{T}_\Lambda$ .

**Example 7.** Let us consider a different example. We keep the same quiver of 25 vertices and 25 arrows, but this time put on 6 relations:

$$\begin{aligned}
 r_1 &= \eta_1 \cdots \eta_6, & r_2 &= \eta_2 \cdots \eta_{10}, & r_3 &= \eta_4 \cdots \eta_{13}, \\
 r_4 &= \eta_{10} \cdots \eta_{15}, & r_5 &= \eta_{15} \cdots \eta_{22}, & r_6 &= \eta_{18} \cdots \eta_{24}.
 \end{aligned}$$

We get a different smo-tube; this time all rows have a flag. The position of a flag0 or flag2 is indicated by a circle; the flag1 by the square:

	1	2	3	4	5	6	7	.....	
4		6	1	4	6	1	4	6	.....
3		4	6	1	4	6	1	4	.....
2		3	5	1	4	6	1	4	.....
1		1	4	6	1	4	6	1	.....

Notice in each example that the number of rows without a flag1 is equal to the number of repetition indices, as stated in Proposition 2.4. It is no fluke that all the rows have flags in the second example above. The next section, while introducing the notion of shifts, shows that if an smo-tube has any flag0s or flag2s at all, then all rows have a flag.

### 3. Repetition shift

The aim of this section is to present some further structure of the smo-tube, the so-called repetition shift, and we see how important the repetition shift is in governing the placement of flags. This importance is shown in Theorem 3.10, which gives exact conditions for when and how the different types of flag will be present on the smo-tube. We begin with a definition. Recall that the  $M$ th band of  $\mathcal{T}_\Lambda$  is the first to contain only repetition indices and that  $f_\Lambda$  is the smo-function of  $\Lambda$ .

**Definition 3.1.** Let  $a$  and  $b$  be rows in  $\mathcal{T}_\Lambda$  that share the same repetition. Then row  $a$  is said to have a  $b$ -shift of  $N$  if  $(a, M + N) = (b, M)$  and  $N \geq 0$  is minimal with this property.

**Lemma 3.2.** Let  $R$  be a repetition of  $\Lambda$  with order  $\lambda \geq 2$ . Let  $x_1, \dots, x_\lambda$  be the repetition indices of  $R$  with the path  $H_{x_1} \rightarrow \dots \rightarrow H_{x_\lambda} \rightarrow H_{x_1}$  of length  $n$ . If  $N \geq 0$  is minimal such that  $f_\Lambda^N(x_1) = x_2$ , then  $N$  is minimal such that  $f_\Lambda^N(x_i) = x_{i+1}$ , for all  $i = 1, \dots, \lambda$ . Note that if  $i = \lambda$  we take  $i + 1 = 1$ .

**Proof.** Pick  $2 \leq k \leq \lambda$ . Let  $l$  be minimal such that  $f_\Lambda^l(x_1) = x_k$ . Thus  $f_\Lambda^{l+N}(x_1) = f_\Lambda^N(x_k)$  giving  $f_\Lambda^l(x_2) = f_\Lambda^N(x_k)$ . Now the proof of Lemma 1.6, applied  $l$  times, gives us no  $R$ -indexed head vertex in the path  $H_{f_\Lambda^l(x_1)} \rightarrow H_{f_\Lambda^l(x_2)}$  except the start and end vertices. Hence there is no  $R$ -indexed head vertex in the path  $H_{x_k} \rightarrow H_{f_\Lambda^l(x_2)}$  except the start and end vertices. Thus  $f_\Lambda^l(x_2) = x_{k+1}$ , and so  $f_\Lambda^N(x_k) = x_{k+1}$ . Minimality of  $N$  follows since  $x_1$  was arbitrary.  $\square$

Motivated by this result, we now make the following definition.

**Definition 3.3.** Let  $R$  be a repetition of  $\Lambda$ , with  $x_1$  and  $x_2$  repetition indices of  $R$  such that no  $R$ -indexed head vertex lies in the path  $H_{x_1} \rightarrow H_{x_2}$  except the start and end vertices. The *repetition shift* of  $R$  is the least positive integer  $N$  such that  $f_A^N(x_1) = x_2$ .

**Example 8.** We take the usual oriented cycle  $Q$  with 25 vertices and 25 arrows, and let  $\Lambda = \mathbb{k}Q/\mathcal{I}$ , where  $\mathcal{I}$  is generated by the 8 relations:

$$\begin{aligned} r_1 &= \eta_1 \cdots \eta_{16}, & r_2 &= \eta_3 \cdots \eta_{19}, & r_3 &= \eta_5 \cdots \eta_{21}, & r_4 &= \eta_8 \cdots \eta_{22}, \\ r_5 &= \eta_{12} \cdots \eta_{24}, & r_6 &= \eta_{14} \cdots \eta_4, & r_7 &= \eta_{19} \cdots \eta_{11}, & r_8 &= \eta_{21} \cdots \eta_{14}. \end{aligned}$$

The only flags on this smo-tube are flag 1s:

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 5 & 7 & 5 & 1 & 7 & 5 & 1 & 7 \cdots \\ 4 & 6 & \boxed{3} & 1 & 7 & 5 & 1 & 7 \cdots \\ 3 & 5 & 1 & 7 & 5 & 1 & 7 & 5 \cdots \\ 2 & 2 & \boxed{8} & 7 & 5 & 1 & 7 & 5 \cdots \\ 1 & 1 & 7 & 5 & 1 & 7 & 5 & 1 \cdots \end{array}$$

There is only one repetition here,  $R = (1, 7, 5)$ . The repetition shift  $N$  of  $R$  is equal to 2.

The repetition shift can be observed in the smo-tube by taking a flag 1-free row containing  $R$ : the  $b$ -shift of that row gives the repetition shift, where row  $b$  is the next flag 1-free row up that contains  $R$ .

**Example 9.** The algebra in Example 1 has a repetition shift of  $N = 1$ .

We use the proof of Lemma 1.6 to get the following result.

**Lemma 3.4.** Let  $\Lambda$  have a repetition with repetition shift  $N$ . Then all repetitions of  $\Lambda$  have repetition shift  $N$ .

For the remainder of the paper, fix  $N$  as the repetition shift of all the repetitions of  $\Lambda$ .

**Proposition 3.5.** Suppose  $\Lambda$  has only one repetition  $R$ , and this is of order  $\lambda \geq 2$ . Then for each  $i$ , row  $i$  of the smo-tube of  $\Lambda$  has  $(i + 1)$ -shift equal to 0 or  $N$ .

**Proof.** If we have only 2 degeneration paths then, since  $\lambda \geq 2$ , the indices of both paths must be repetition indices. So clearly  $R$  has a repetition shift of  $N = 1$ . Now, since there are only 2 rows in  $\mathcal{T}_\Lambda$ , we get immediately from Lemma 2.2 that each row  $i$  has  $(i + 1)$ -shift of 1.

Thus assume  $\Lambda$  has at least 3 degeneration paths. Let  $x_1, \dots, x_\lambda$  be the repetition indices of  $R$ , with the path  $H_{x_1} \rightarrow \dots \rightarrow H_{x_\lambda} \rightarrow H_{x_1}$  of length  $n$ . Let  $y$  and  $z$  be distinct degeneration head indices such that no degeneration head vertices lie in the path  $H_y \rightarrow H_z$  other than the start and end vertices. Then for some  $k \in \{1, \dots, \lambda\}$  (with  $\lambda + 1 = 1$ ) we have that  $H_y$  and  $H_z$  lie in the path  $H_{x_k} \rightarrow H_{x_{k+1}}$ . We will show that either  $f_\Lambda^M(y) = f_\Lambda^M(z)$  or  $f_\Lambda^{M+N}(y) = f_\Lambda^M(z)$ .

By applying Proposition 1.1  $M$  times, we get that the path

$$H_{f_\Lambda^M(x_k)} \rightarrow H_{f_\Lambda^M(y)} \rightarrow H_{f_\Lambda^M(z)} \rightarrow H_{f_\Lambda^M(x_{k+1})}$$

has length less than or equal to  $n$ . By  $M$  applications of the proof of Lemma 1.6 we have no repetition head vertices in this path except the start and end vertices. Since by the definition of  $M$ ,  $f_\Lambda^M(y)$  and  $f_\Lambda^M(z)$  are repetition indices, three possibilities occur:

- (1)  $f_\Lambda^M(y) = f_\Lambda^M(z) = f_\Lambda^M(x_k)$ ,
- (2)  $f_\Lambda^M(y) = f_\Lambda^M(z) = f_\Lambda^M(x_{k+1})$ ,
- (3)  $f_\Lambda^M(y) = f_\Lambda^M(x_k)$ ,  $f_\Lambda^M(z) = f_\Lambda^M(x_{k+1})$ .

Let  $i$  be such that  $y = (i, 2)$ , and so by hypothesis  $z = (i + 1, 2)$ . If either possibility (1) or (2) occurs, then row  $i$  has an  $(i + 1)$ -shift of 0, since  $(i, M + 2) = f_\Lambda^M(y) = f_\Lambda^M(z) = (i + 1, M + 2)$  and hence  $(i, M + \lambda) = (i + 1, M + \lambda)$ , so  $(i, M) = (i + 1, M)$ . If possibility (3) occurs, then  $f_\Lambda^{M+N}(y) = f_\Lambda^{M+N}(x_k) = f_\Lambda^M(x_{k+1}) = f_\Lambda^M(z)$ , and by a similar argument  $(i, M + N) = (i + 1, M)$ ; so row  $i$  has an  $(i + 1)$ -shift of  $N$ .  $\square$

We can bring the above results together to form the next proposition, which builds upon Proposition 2.4. First though, as a consequence of the proof of Lemmas 1.6 and 2.1 we have the following result.

**Lemma 3.6.** *If  $\lambda \geq 2$  then there is an ordering on the repetitions of  $\Lambda$ :  $R_1, \dots, R_l$  such that whenever  $(i, j) \neq (i + 1, j)$ , for some  $j \geq M$ , we have  $(i, j) \in R_k$  and  $(i + 1, j) \in R_{k+1}$ , some  $k \in \{1, \dots, l\}$  with  $l + 1 = 1$ .*

**Remark.** Notice that in the case of Example 8 we have only one repetition. This renders Lemma 3.6 somewhat trivial in that  $l = 1$ , giving an ordered list of one element. Thus  $R_{k+1} = R_k$  and the lemma is then obvious in this case.

**Proposition 3.7.** *Suppose  $\lambda \geq 2$ . If the following conditions all occur:*

- (1)  $\Lambda$  has only one repetition  $R$ ,
- (2)  $R$  has repetition shift  $N = 1$ ,
- (3)  $\ell(r) \leq n$  for all relations  $r$ ,

*then all rows in  $T_\Lambda$  are flagged. Otherwise the number of unflagged rows is equal to the number of distinct repetition indices.*

**Proof.** If conditions (1) and (2) hold, then by Proposition 3.5, for each  $i$ , row  $i$  has  $(i + 1)$ -shift either 0 or 1. If this shift is 0 then, since  $\lambda \geq 2$ , we can use Theorem 2.3 to get a flag1 in row  $i$ . If the  $(i + 1)$ -shift is 1 then by condition (3) we can use Theorems 2.5 and 2.11 to get a flag0 or flag2 in row  $i$ . Hence all the rows are flagged.

Otherwise Proposition 2.4 states there are the same number of rows without a flag1 as there are repetition indices. Thus we need to show there are no flag0s or flag2s in  $\mathcal{T}_A$  whenever one of the three conditions above fails.

If condition (1) fails, we have 2 or more repetitions. Let  $(i, j)$  be an entry of  $\mathcal{T}_A$ , with  $j \geq 3$ . Suppose for a contradiction that  $(i, j) = (i + 1, j - 1)$ ; then we have  $(i, j + M) = (i + 1, j + M - 1)$  and so, since  $\lambda \geq 2$ , this gives  $(i, j + M) \neq (i + 1, j + M)$  with  $(i, j + M)$  and  $(i + 1, j + M)$  in the same repetition. This is prohibited by Lemma 3.6. We can thus assume that  $(i, j) \neq (i + 1, j - 1)$  for all  $i, j$ , and so by Theorem 2.11 we have no flag2s. A similar argument in the case  $j = 2$  shows that  $(i, 2) \neq (i, 1) + 1$  and hence by Theorem 2.5 that there are no flag0s.

If condition (2) fails, then for all  $i$  and for all  $j \geq 3$  we have  $(i, j) \neq (i + 1, j - 1)$  and  $(i, 2) \neq (i + 1, 1) + 1$ . Theorems 2.5 and 2.11 then give us no flag0s or flag2s.

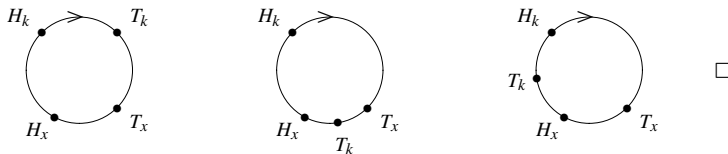
If condition (3) fails, we get our result by Corollaries 2.7 and 1.9.  $\square$

We now focus our attention to the case where the order of the repetitions is 1.

**Lemma 3.8.** *Let  $\lambda = 1$  and let  $r_x$  be a repetition relation. If  $(a - 1)n < \ell(r_x) \leq an$  for some positive integer  $a$ , then  $(a - 1)n < \ell(r_k) \leq (a + 1)n$  for all relations  $r_k$ .*

**Proof.** Let  $a \in \mathbb{Z}$  be such that  $(a - 1)n < \ell(r_x) \leq an$  and let  $r_k$  be some relation. Clearly we must have  $\ell(r_k) \leq (a + 1)n$  else  $r_x$  would be a subpath of  $r_k$ .

Now, since  $H_x$  must follow  $T_x$ , we must have  $H_k$  in the path  $H_x \rightarrow T_x$ . The following three diagrams show the possible relative positions of  $T_k$ : note that we allow  $T_x = H_x$  and  $T_k = H_x$  or  $H_k$  where appropriate. By looking at each diagram in turn it is not hard to see that we must have  $\ell(r_k) > (a - 1)n$  to prevent  $r_k$  being a subpath of  $r_x$ .



**Proposition 3.9.** *Suppose  $\lambda = 1$ . If the following conditions both occur:*

- (1)  $\Lambda$  has only one repetition relation  $r_x$ ,
- (2)  $\ell(r_x) \leq n$ ,

*then all rows in  $\mathcal{T}_A$  are flagged. Otherwise the number of unflagged rows is equal to the number of distinct repetition indices.*



**Proof.** Suppose both conditions hold and yet we have an unflagged row in the smo-tube of  $\Lambda$ . This gives rise to a degree  $2M + 1$  maximal overlap sequence ending with

$$\begin{array}{c} \dots \quad \quad \quad \overbrace{\hspace{2cm}}^{r_x} \\ \underbrace{\hspace{2cm}}_{r_x} \end{array}$$

which contradicts condition (2), since the head arrow of  $r_x$  appears here in  $r_x$  twice.

For the converse we will show that if either condition (1) or (2) fails, then the number of unflagged rows in  $\mathcal{T}_\Lambda$  is equal to the number of repetition indices.

Suppose (1) fails. Then we have  $k \geq 2$  repetition relations. We know from Proposition 2.4 that there are  $k$  rows in  $\mathcal{T}_\Lambda$  with no flag1: let row  $i$  be one of these. To seek a contradiction suppose that entry  $(i, j)$  has a flag2 and that  $j \geq 3$ . By Theorem 2.11, we have  $(i, j) \neq (i + 1, j - 1)$  and  $(i, j + 1) = (i + 1, j)$ . Thus  $(i, j + M + 1) = (i + 1, j + M)$  and so, since  $\lambda = 1$ ,  $(i, j + M + 1) = (i + 1, j + M + 1)$ . By Theorem 2.3, row  $i$  has a flag1, contradicting our hypothesis. The cases where  $j = 1$  or  $2$  are similar to the above, with the case  $j = 1$  prohibiting flag0s.

Finally, if (2) fails then Lemma 3.8 and Corollary 1.9 show that there are no flag0s or flag2s in  $\mathcal{T}_\Lambda$ . Thus by Proposition 2.4, the number of unflagged rows is equal to the number of distinct repetition indices.  $\square$

Putting the last two propositions together gives us the theorem of this section. Using Theorem 1.10, we follow with a useful corollary.

**Theorem 3.10.** *The smo-tube  $\mathcal{T}_\Lambda$  has every row flagged if and only if one of the following occurs:*

- (1)  $\lambda \geq 2$ , there is only 1 repetition,  $N = 1$ , and  $\ell(r) \leq n$  for all relations  $r$ .
- (2)  $\lambda = 1$ , there is only 1 repetition relation  $r_x$ , and  $\ell(r_x) \leq n$ .

*Otherwise the smo-tube has the same number of unflagged rows as it does distinct repetition indices.*

**Corollary 3.11.** *If  $\Lambda$  has an infinite extending sequence, then  $\mathcal{T}_\Lambda$  has no flag0s or flag2s.*  $\square$

#### 4. Finite generation of the Ext-algebra

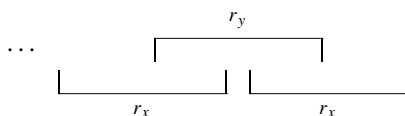
The previous sections have given us a method to identify exactly those extending sequences that are infinite in extent. In this section we bring together these results to determine precisely when  $E(\Lambda)$  is finitely generated: Theorem 4.14, our main result, does this for us. Toward the end of this section there are a number of propositions each of which, for different conditions on  $E(\Lambda)$ , give an explicit finite generating set. As noted in the introduction, we will freely interchange a maximal overlap sequence and its underlying path.

However, when we talk about products of maximal overlap sequences, we refer to the corresponding product as basis elements in  $E(\Lambda)$ . Thus the product of two maximal overlap sequences may be zero in  $E(\Lambda)$ , whilst the product of their underlying paths may be non-zero in  $\mathbb{k}\mathcal{Q}$ . We recall here that, for  $\Lambda$  a cycle algebra,  $\lambda$  is the order of the repetitions,  $N$  is the repetition shift, and  $M$  is the first band of  $\mathcal{T}_\Lambda$  to contain only repetition indices.

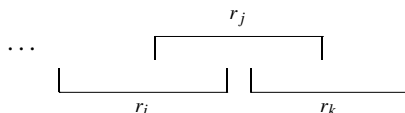
Our first three results, for certain conditions on  $\Lambda$ , give us restrictions on the behaviour of the maximal overlap sequences. These results will aid us when determining finite generation of  $E(\Lambda)$ .

**Lemma 4.1.**

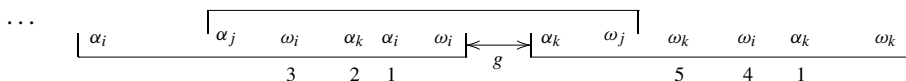
- (1) If  $\lambda \geq 2$  and  $\ell(r) > n$  for all relations  $r$ , then there are always  $n$  or more unoverlapped arrows at the end of any even-degree maximal overlap sequence.
- (2) If  $\lambda = 1$ ,  $r_x$  and  $r_y$  are repetition relations, and  $\ell(r) > n$  for all repetition relations  $r$ , then there are always  $n$  or more unoverlapped arrows at the end of any even-degree maximal overlap sequence which ends



**Proof.** (1) If the degree of the maximal overlap sequence is 2, then it is a relation and there is nothing to prove. Thus suppose the degree is  $\geq 4$ . If  $\ell(r) > 2n$  for all relations  $r$ , the result is immediate using Proposition 1.2. So by Corollary 2.7, we may suppose  $n < \ell(r) \leq 2n$ , for all relations  $r$ . Let



be the end of the even-degree maximal overlap sequence. Note that  $r_i \neq r_k$  by hypothesis. If  $r_j = r_k$ , then the result is immediate, so assume  $r_j \neq r_k$ . By Theorems 2.3 and 1.10, we may assume  $r_i \neq r_j$ . The diagram below is a copy of the one above, but with the relative positions of certain arrows marked; the order in which an arrow has been marked is indicated below that arrow. Recall that for a relation  $r_l$  we have  $\alpha_l$  as the start arrow and  $\omega_l$  as the end arrow:



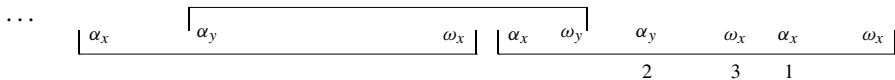
Once the start and end arrows of each relation have been marked on the diagram above, we know we can mark the other arrows for the following reasons, given by order of marking.

Note that from Lemma 2.10 and Proposition 1.1, the paths  $H_i \rightarrow H_j \rightarrow H_k \rightarrow H_i$  and  $T_i \rightarrow T_j \rightarrow T_k \rightarrow T_i$  each have length  $n$ .

- (1) As  $\ell(r_i) > n$ ,  $r_i$  contains two copies of  $\alpha_i$ . One copy must lie in the overlapped part of  $r_i$  and  $r_j$ , by Proposition 1.2 if the degree of the maximal overlap sequence is greater than or equal to 6, or trivially if the degree is 4. By the same reasoning, a copy of  $\alpha_k$  must lie in the unoverlapped part of  $r_k$ . We remark that  $\alpha_i \neq \omega_i$  since a copy of  $\alpha_j$  must lie between  $\alpha_i$  and  $\alpha_k$ , but cannot lie in the path  $g$ . At this stage we allow the possibility that  $\alpha_k = \omega_k$ .
- (2) By the note above, a copy of  $\alpha_k$  lies in  $r_i$  between  $\alpha_j$  and the copy of  $\alpha_i$  placed in 1.
- (3)  $H_k$  is the head vertex that follows  $T_i$  by maximality of the overlap sequence, so  $\omega_i$  sits as marked in the overlapped part of  $r_i$  and  $r_j$ .
- (4) Three copies of  $\omega_i$  cannot lie in  $r_j$ , but a copy of  $\omega_i$  must lie in  $r_k$ , since  $n < \ell(r_j), \ell(r_k) \leq 2n$ . Thus there is a copy of  $\omega_i$  as shown.
- (5) By the note above, a copy of  $\omega_k$  lies in  $r_k$  between  $\omega_j$  and  $\omega_i$ .

The presence of two copies of  $\omega_k$  in the unoverlapped part of  $r_k$  yields our result.

(2) is proved in a similar way with the diagram:

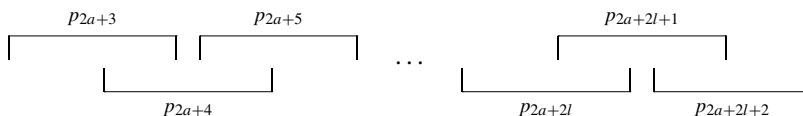


□

**Proposition 4.2.** A maximal overlap sequence  $P^{2k}$ ,  $k \geq 2$ , cannot be written as a product of maximal overlap sequences  $P^{2a+1}F^{2l+1}Q^{2b}$ , some  $a \geq 0$ ,  $l \geq M+1$ ,  $b \geq 0$ , if one of the following occurs:

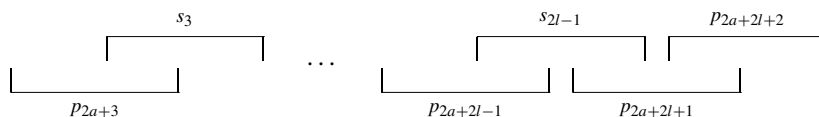
- (1)  $\lambda \geq 2$ ,  $\Lambda$  has  $\geq 3$  repetitions,
- (2)  $\lambda \geq 2$ ,  $\Lambda$  has 2 repetitions and  $N \neq 1$ ,
- (3)  $\lambda \geq 2$ ,  $\Lambda$  has only 1 repetition and  $2N \not\equiv 1 \pmod{\lambda}$ ,
- (4)  $\lambda \geq 2$ ,  $\ell(r) > n$  for all relations  $r$ ,
- (5)  $\lambda = 1$ ,  $\Lambda$  has 2 repetition relations  $r_x$  and  $r_y$ , and  $\ell(r_x), \ell(r_y) > n$ ,
- (6)  $\lambda = 1$ ,  $\Lambda$  has  $\geq 3$  repetitions.

**Proof.** (1) If  $P^{2k}$  is to be written as such a product, we at least need the product in  $\mathbb{K}Q$  of the three underlying paths to be non-zero: thus we assume this now. Suppose for a contradiction that  $P^{2k}$  can be written as the above product. Consider the underlying path of  $F^{2l+1}$ , represented thus



where the relations are from  $P^{2k}$ , in the same positions that they appear in the corresponding part of the underlying path of  $P^{2k}$ .

Let us also construct  $F^{2l+1}$  as a maximal overlap sequence, starting at  $\mathfrak{o}(p_{2a+3})$ :



Now look at  $\mathcal{T}_\Lambda$ . If  $p_{2a+2l+1} = p_{2a+2l+2}$ , then two cases arise. Either  $s_3 = p_{2a+3}$ , in which case  $p_{2a+3}$  is the only relation of  $\Lambda$  by maximality of the overlap sequence, or  $s_3 \neq p_{2a+3}$ , in which case by Theorems 2.3 and 1.10 there is no such maximal overlap sequence of degree  $2l + 1$ . Both cases give a contradiction to the hypothesis.

So assume  $p_{2a+2l+1} \neq p_{2a+2l+2}$ . Let  $f, g$  and  $h$  be integers such that  $1 \leq f, g, h \leq m$  and  $r_f = p_{2a+2l}$ ,  $r_g = p_{2a+2l+1}$  and  $r_h = p_{2a+2l+2}$ . By the remark following Lemma 1.6, the two indices  $g$  and  $h$  are in different repetitions, respectively  $R_t$  and  $R_s$  say. By Lemma 3.6, we have a special ordering on the repetitions, which says that  $R_s$  follows  $R_t$ . However, since the index  $f$  is in  $R_s$  we have that  $R_t$  follows  $R_s$ . Since there are more than two repetitions this contradicts Lemma 3.6. Hence  $P^{2k}$  cannot be written as such a product.

The cases (2)–(6) are proved in a similar way, with (4) and (5) using Lemma 4.1.  $\square$

**Lemma 4.3.** *Let  $\Lambda$  be such that  $\ell(r) \leq n$  for all repetition relations  $r$  and suppose one of the following occurs:*

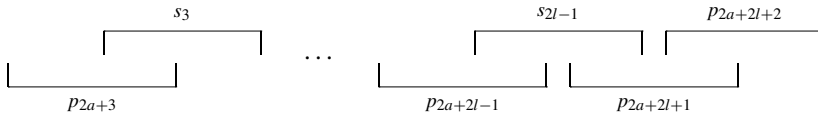
- (1)  $\lambda \geq 2$ ,  $\Lambda$  has precisely 2 repetitions,  $N = 1$ ,
- (2)  $\lambda \geq 2$ ,  $\Lambda$  has only 1 repetition,  $2N \not\equiv 1 \pmod{\lambda}$ ,
- (3)  $\lambda = 1$ ,  $\Lambda$  has precisely 2 repetition relations.

Let  $P^k$  be a maximal overlap sequence and let  $S$  be the subpath (but not necessarily a maximal overlap sequence):



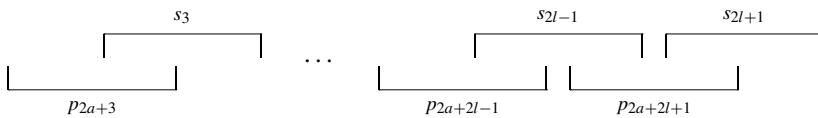
for some  $l \geq M$ . The relations above are from  $P^k$ , in the same positions that they appear in the corresponding part of the underlying path of  $P^k$ . Then the path  $S$  can be constructed as a maximal overlap sequence if and only if a maximal overlap sequence exists starting at  $\mathfrak{o}(p_{2a+3})$  of degree  $2l + 1$ .

**Proof.** If the path  $S$  can be formed as a maximal overlap sequence, then it must take the form:



which is a maximal overlap sequence of degree  $2l + 1$ .

We prove the converse in the case where condition (2) holds. Suppose there exists a maximal overlap sequence of degree  $2l + 1$  starting at  $\mathfrak{o}(p_{2a+3})$ . It will take the form:



Since  $\ell(r) \leq n$  for all relations  $r$ , it is enough to show that  $s_{2l+1} = p_{2a+2l+2}$ . It is clear that  $s_{2l+1} \neq p_{2a+2l+1}$  and  $p_{2a+2l+2} \neq p_{2a+2l+1}$ . Now in  $\mathcal{T}_\Lambda$ ,  $2N \not\equiv 1 \pmod{\lambda}$  is equivalent to having  $(i, j) = (i + 2 + k(i), j - 1)$  for every unflagged row  $i$  and for  $j \geq M$ , where  $k(i)$  is the number of flagged rows counting up from row  $i$  to the next but one unflagged row. Consider band  $a + l$  of  $\mathcal{T}_\Lambda$ . From the maximal overlap sequence  $P^k$ , and using Lemma 2.1, we can see this means that the index of  $p_{2a+2l+2}$  is the next different one in band  $a + l$  up from the index of  $p_{2a+2l+1}$ . Since  $s_{2l+1} \neq p_{2a+2l+1}$  we thus get that  $s_{2l+1} = p_{2a+2l+2}$ .

Similar arguments may be used to prove the converse where either condition (1) or (3) holds.  $\square$

The following definition and lemma are fundamental to the finite generation of  $E(\Lambda)$ .

**Definition 4.4.** Let  $e_v$  be a zero-length connective path of some repetition  $R$ . For ease of notation, we write  $R = (a_1, a_2, \dots, a_\lambda)$  and  $\mathbf{c}_{a_1} = e_v$ , with  $\mathfrak{o}(r_{a_1}) = \mathfrak{t}(r_{a_\lambda}) = v$  and  $f_\Lambda^i(a_1) = a_{i+1}$ , for all  $0 \leq i < \lambda$ . The *multiplication path* of  $\mathbf{c}_{a_1}$  is the path in  $\mathbb{k}\mathcal{Q}$ ,

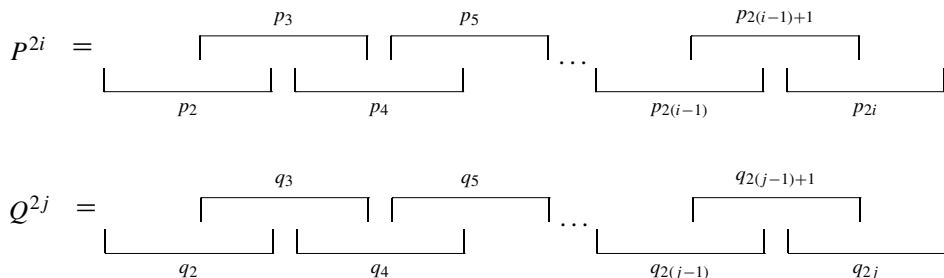
$$g_v = r_{a_1} \mathbf{c}_{a_2} r_{a_2} \mathbf{c}_{a_3} \cdots r_{a_{\lambda-1}} \mathbf{c}_{a_\lambda} r_{a_\lambda}.$$

If a multiplication path can be formed as a maximal overlap sequence then it is called a *generative multiplication path*. Note that if this is the case,  $\deg(g_v) = 2\lambda$ .

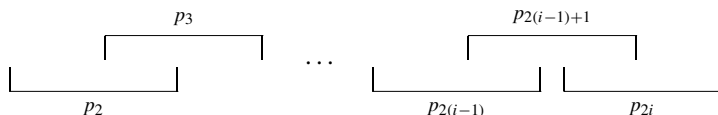
**Remark.** If  $g_v = r_{a_1} \mathbf{c}_{a_2} r_{a_2} \mathbf{c}_{a_3} \cdots r_{a_{\lambda-1}} \mathbf{c}_{a_\lambda} r_{a_\lambda}$  is a generative multiplication path, then  $r_{a_1} \mathbf{c}_{a_2} r_{a_2} \mathbf{c}_{a_3} \cdots r_{a_{i-1}} \mathbf{c}_{a_i} r_{a_i}$  is also a maximal overlap sequence, for any  $i \leq \lambda$ .

**Proposition 4.5.** Let  $\Lambda$  be such that no flag0s or flag2s are present in  $\mathcal{T}_\Lambda$  and let  $P^{2i}$  and  $Q^{2j}$  be even-degree maximal overlap sequences (of degree  $2i$  and  $2j$ , respectively) such that the product of underlying paths  $P^{2i} Q^{2j}$  is non-zero in  $\mathbb{k}\mathcal{Q}$ . Then the path  $P^{2i} Q^{2j}$  is also a maximal overlap sequence, of degree  $2i + 2j$ .

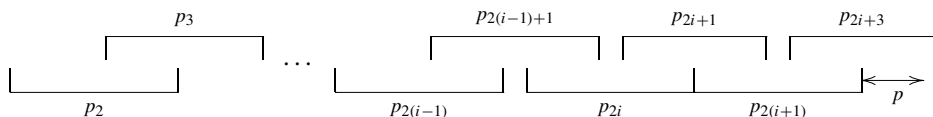
**Proof.** Consider two maximal overlap sequences  $P^{2i}$  and  $Q^{2j}$ :



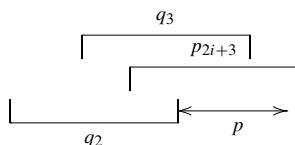
Suppose that  $P^{2i} Q^{2j}$  is non-zero as a path in  $\mathbb{k}Q$ ; then  $t(P^{2i}) = o(Q^{2j})$ . We need to build the path  $P^{2i} Q^{2j}$  as a maximal overlap sequence. We start with  $P^{2i}$  as above:



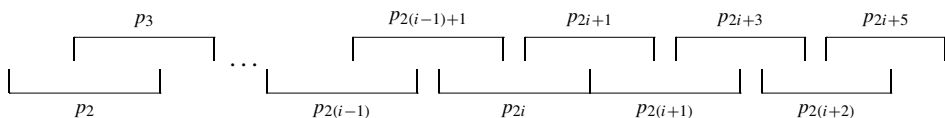
By hypothesis and Theorem 1.10, we know we may overlap with another relation,  $p_{2i+1}$  say. Since  $t(p_{2i}) = o(q_2)$ , and using the hypothesis and Theorem 1.10 again, we have the maximal overlap sequence:



where  $p_{2(i+1)} = q_2$  and  $p$  is the path of unoverlapped arrows. In the following diagram we can see the path  $p$  within  $Q^{2j}$ :



Thus  $p$  has as an initial subpath the path of unoverlapped arrows of  $Q^3$  and hence we may overlap our maximal overlap sequence above with the relation  $q_4$ . Using the hypothesis and Theorem 1.10, we get the maximal overlap sequence



where  $p_{2(i+2)} := q_4$ . We continue inductively setting  $p_{2(i+k)} = q_{2k}$  for  $1 \leq k \leq j$ .  $\square$

**Corollary 4.6.** *Let  $\Lambda$  be such that no flag0s or flag2s are present in  $\mathcal{T}_\Lambda$ . If  $g_v$  is a generative multiplication path, then  $g_v^l \in E(\Lambda)$  is non-zero for all  $l \geq 1$ .*

Once we have stated the following definition, we will be in a position to start deciding if  $E(\Lambda)$  is or is not finitely generated.

**Definition 4.7.** If  $A_w$  is an extending sequence of  $E(\Lambda)$  starting at the vertex  $w$  then, along with  $w$  and the arrow starting at  $w$ , it naturally corresponds to  $E_w := \text{Ext}_\Lambda^*(S_w, \bar{\Lambda})$ , where  $S_w$  is the simple  $\Lambda$ -module occurring at  $w$ . Then  $E_w$  is a (possibly non-unital) subring of  $E(\Lambda)$ . We also let  $E_w^{\text{ev}} := \text{Ext}_\Lambda^{2*}(S_w, \bar{\Lambda})$  be the (possibly non-unital) subring of  $E_w$  consisting of the even-degree elements.

Say that a non-unital subring  $\mathcal{R}$  of  $E(\Lambda)$  has a *finite generating set* if there is a finite subset  $\mathcal{S}$  of  $E(\Lambda)$  in which every element of  $\mathcal{R}$  may be expressed as a finite product of elements from  $\mathcal{S}$ .

A maximal overlap sequence  $a$  of degree  $z$  is said to be *in* an extending sequence  $A$  if  $a = A^z$  for some  $z \geq 2$ . A generative multiplication path  $g_v$  is *in the lower half* (respectively *upper half*) of an infinite extending sequence  $A$  if there is some degree  $z \geq 2$  and some even-degree (respectively odd-degree) maximal overlap sequence  $p$  in  $A$  such that  $A^z = pg_v$ , with the product in  $E(\Lambda)$ .

Using this definition, we get that  $E(\Lambda)$  is finitely generated as a  $\mathbb{k}$ -algebra if and only if  $E_w$  has a finite generating set for all  $w$  such that  $A_w$  is an infinite extending sequence.

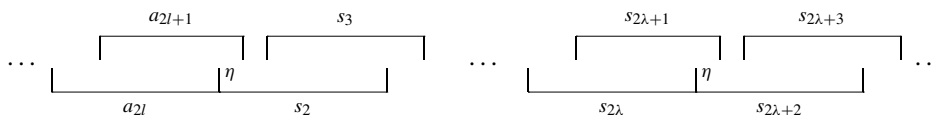
**Remark.** Once and for all we take care of the basis elements of  $E(\Lambda)$  of degree 0 and 1; from the introduction we know these correspond respectively to the vertices and arrows of  $\Lambda$ . The question we resolve here is that of whether, to have a finite generating set  $\mathcal{S}$  for  $\text{Ext}_\Lambda^{\geq 2}(\bar{\Lambda}, \bar{\Lambda})$ , we need  $\mathcal{S}$  to include elements of  $E(\Lambda)$  of degree 0 or 1. The answer is that it does not, as the next proposition shows, when used with subsequent propositions.

**Proposition 4.8.** *Let  $A_w$  be an infinite extending sequence and let  $a$  be a maximal overlap sequence in  $A_w$  of degree greater than or equal to  $2M$ . Let  $\eta$  be an arrow in  $\mathbb{k}\mathcal{Q}$ ; then  $\eta$  corresponds to a basis element of  $E(\Lambda)$  of degree 1 and we have the following:*

- (1) *If  $\deg(a)$  is even and  $a\eta \in \mathbb{k}\mathcal{Q}$  is non-zero in  $E(\Lambda)$ , then  $A_w$  has a generative multiplication path in its lower half.*
- (2) *If  $\deg(a)$  is odd and  $a\eta \in \mathbb{k}\mathcal{Q}$  is non-zero in  $E(\Lambda)$ , then  $A_w$  has a generative multiplication path in its upper half.*

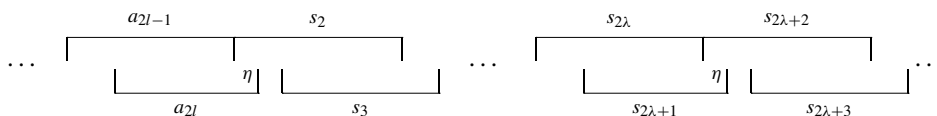
**Proof.** Let  $A_w$ ,  $a$  and  $\eta$  be as above.

(1) Let  $\deg(a)$  be even and  $a\eta$  be non-zero in  $E(\Lambda)$ ; we let  $a_{2l}$  and  $a_{2l+1}$  be the last relations of  $a$  and  $a\eta$ , respectively. Then since  $\eta$  is a path of length 1 and  $\deg(a) \geq 2M$ ,  $A_w$  must take the form



with  $s_{2\lambda+2} = s_2$  and  $s_{2\lambda+3} = s_3$ . It is immediate to see that the multiplication path starting at  $\sigma(\eta)$  is a maximal overlap sequence since clearly  $s_3$  is the relation that maximally overlaps  $s_2$ : if  $\eta$  were a longer path this need not be true. We continue building the maximal overlap sequence up to degree  $2\lambda$  in the obvious way using the relations from  $A_w$ . Thus  $A_w$  has a generative multiplication path in its lower half.

(2) Now let  $\deg(a)$  be odd with  $a\eta$  non-zero in  $E(\Lambda)$  and let  $a_{2l-1}$  and  $a_{2l}$  be the last relations of  $a$  and  $a\eta$ , respectively. Then since  $\eta$  is a path of length 1 and  $\deg(a) > 2M$ ,  $A_w$  must take the form



with  $s_{2\lambda+2} = s_2$  and  $s_{2\lambda+3} = s_3$ . As before, it is immediate that the multiplication path starting at  $\sigma(\eta) = \sigma(s_2)$  is a maximal overlap sequence and hence a generative multiplication path in the upper half of  $A_w$ .  $\square$

**Remark.** If  $\Lambda$  satisfies one of the six conditions in Proposition 4.2, then the second diagram in the proof above yields the existence of an arbitrarily long odd-degree maximal overlap sequence in the position of the  $F^{2l+1}$  from Proposition 4.2. This contradicts that proposition and so we may conclude that condition (2) of Proposition 4.8 never occurs under any of the conditions from Proposition 4.2.

In Proposition 4.2 we showed that, given one of six conditions, a maximal overlap sequence could not be written as a product with a second odd-degree factor of degree greater than or equal to  $2M + 3$  (called  $F^{2l+1}$  in Proposition 4.2). Non-existence of this factor is used as a hypothesis in part of the next proposition. The reason for this is that we want to examine the cases where we cannot use arbitrary powers of a generative multiplication path found in the upper half of an extending sequence, to get a finite generating set.

**Proposition 4.9.** *Let  $A_w$  be an infinite extending sequence of  $\Lambda$ .*

*If there is a generative multiplication path  $g_v$  in the lower half of  $A_w$ , then  $E_w^{\text{ev}}$  has a finite generating set.*

*Moreover, if no even-degree maximal overlap sequence  $A_w^{2k}$  in  $A_w$  may be written as a product of maximal overlap sequences  $A_w^{2a+1} F^{2l+1} Q^{2b}$ , for any  $a \geq 0$ ,  $l \geq M + 1$ ,  $b \geq 0$ ,*



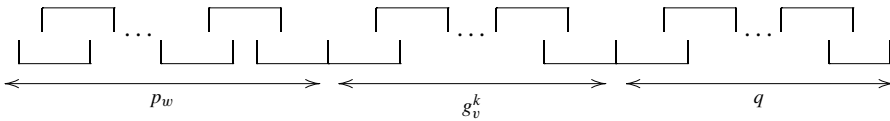
then there is a generative multiplication path  $g_v$  in the lower half of  $A_w$  if and only if  $E_w^{\text{ev}}$  has a finite generating set.

**Proof.** Let  $A_w$  be an infinite extending sequence with a generative multiplication path  $g_v$  in its lower half. Let  $G_v$  be the infinite extending sequence starting at  $v$ , so that  $g_v$  is in  $G_v$ . Define  $p_w$  as the maximal overlap sequence of least even-degree in  $A_w$  such that  $\deg(p_w) \geq 2M$  and  $t(p_w) = v$ . We take as our generating set for  $E_w^{\text{ev}}$ :

- (1) The trivial path  $e_v$ .
- (2) All even-degree maximal overlap sequences in  $A_w$  with degree less than or equal to  $\deg(p_w)$ .
- (3) The maximal overlap sequence  $g_v$  of the hypothesis.
- (4) All even-degree maximal overlap sequences in  $G_v$  that have degree less than the degree of  $g_v$ .

Let  $a$  be an even-degree maximal overlap sequence in  $A_w$ . We show how to get  $a$  from the above set by considering the degree of  $a$ :

- If  $\deg(a) \leq \deg(p_w)$ , then  $a$  is in the chosen generating set.
- If  $\deg(a) > \deg(p_w)$ , then, using the remark following Definition 4.4, we may write  $a = p_w g_v^k q$ , for some  $k \geq 0$ , and where  $q$  is a maximal overlap sequence in  $G_v$  of even-degree less than or equal to  $\deg(g_v)$ . We show this below:



The above product is non-zero in  $E(\Lambda)$  by Proposition 4.5.

Conversely, suppose that we do have some finite generating set  $\mathcal{S}$  for  $E_w^{\text{ev}}$  and that no maximal overlap sequence  $A_w^{2k}$  in  $A_w$  may be written as a product of maximal overlap sequences  $A_w^{2a+1} F^{2l+1} Q^{2b}$ , for any  $a \geq 0$ ,  $l \geq M + 1$ ,  $b \geq 0$ . We consider a maximal overlap sequence in  $A_w$  of sufficiently high even-degree such that, in any expression of it as a product of elements of  $\mathcal{S}$ , at least one element of  $\mathcal{S}$  of degree  $\geq 2$  occurs with multiplicity at least 2. Without loss of generality, we may choose a maximal overlap sequence  $a$  in  $A_w$  with  $a = h_0 d h_1 d$ , where  $d \in \mathcal{S}$ ,  $\deg(d) \geq 2$ , each  $h_i$  is a product of generators,  $i = 0, 1$ , and  $\deg(h_1) \geq 2M + 1$ . Now, since  $\deg(h_1 d) \geq 2M + 3$  and  $\deg(a)$  is even, we have by hypothesis that  $\deg(h_1 d)$  is even. Thus  $\deg(h_0 d)$  is even. Therefore, since  $t(d) = o(h_1)$ , we have that the maximal overlap sequence  $h_1 d$  is either a generative multiplication path, or some power (with multiplication in  $E(\Lambda)$ ) of a generative multiplication path, in the lower half of  $A_w$ .  $\square$

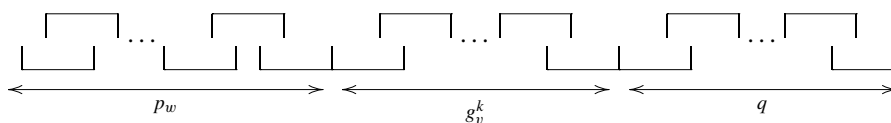
**Theorem 4.10.** *If each infinite extending sequence of  $E(\Lambda)$  contains a generative multiplication path in its lower half, then  $E(\Lambda)$  is finitely generated as a  $\mathbb{k}$ -algebra.*

**Proof.** Let  $A_w$  be an infinite extending sequence with  $g_v$ ,  $p_w$ , and  $G_v$  as in Proposition 4.9. Then by Corollary 3.11,  $\mathcal{T}_\Lambda$  has no flag0s or flag2s. Note also that  $\deg(g_v) = 2\lambda$  and  $2M \leq \deg(p_w) \leq 2M + 2\lambda - 2$ . From the first part of Proposition 4.9 we get all the even-degree maximal overlap sequences in  $A_w$  with the finite generating set for  $E_w^{\text{ev}}$  as given there. Now augment that generating set by including the following elements:

- (5) The arrow starting at  $w$ .
- (6) All odd-degree maximal overlap sequences  $s$  in  $A_w$  such that  $\deg(s) \leq \deg(p_w) + 2M + 1$ .
- (7) All odd-degree maximal overlap sequences  $t$  in  $G_v$  such that  $2M + 1 \leq \deg(t) < 2M + 2\lambda + 1$ .

With this set we now describe how to get any odd-degree maximal overlap sequence  $b$  in  $A_w$ . Note that the degrees are larger here than in the generating set of the last proposition for the following reason. Let  $q$  be a maximal overlap sequence in  $G_v$  of odd-degree less than  $\deg(g_v)$ . Then the underlying path of  $q$  is an initial subpath of that of  $g_v$ . Now, if  $b$  is of high-degree, its last relation must be a repetition relation. However, the last relation of  $q$  need not be. We must therefore choose the right-most factor of  $b$ , denoted  $t$  below, to be of sufficiently high degree to end with a repetition relation. We then use Lemma 2.1 and Theorem 2.3 to give us the correct end relation for the product (this is trivial if  $\Lambda$  has only 1 repetition relation):

- If  $\deg(b) \leq \deg(p_w) + 2M + 1$ , then  $b$  is in the chosen generating set.
- If  $\deg(b) > \deg(p_w) + 2M + 1$ , then  $b = p_w g_v^k t$ , for  $k \geq 0$  and where  $t$  is some maximal overlap sequence of  $G_v$  as in (7) above, so that  $\deg(t) = \deg(b) - \deg(p_w) - k \deg(g_v)$ . We illustrate  $p_w g_v^k t$  below and then prove that such a maximal overlap sequence  $b$  may indeed be expressed in this way:



Firstly,  $p_w g_v^k t'$  is non-zero in  $E(\Lambda)$  by Proposition 4.5, where  $t'$  is the even-degree maximal overlap sequence in  $G_v$  of degree  $\deg(t) - 1$ . By maximality of our overlap sequences, and using the remark following Definition 4.4, we have that the last relation of  $t'$  is the same as that of  $b'$ , where  $b'$  is the even-degree maximal overlap sequence in  $A_w$  of degree  $\deg(b) - 1$ . By Proposition 1.2, it is enough to show that the last relation of  $t$  is the same as the last relation of  $b$ . If  $\Lambda$  has only 1 repetition relation then this is immediate, so suppose  $\Lambda$  has more than 1 repetition relation. Then, since  $G_v$  and  $A_w$  are infinite extending sequences, Theorem 2.3 says that  $t(t) \neq t(t')$  and  $t(b) \neq t(b')$ . Moreover, since  $\deg(t) \geq 2M + 1$ , we have that the last relation of  $t$  is a repetition relation. By Lemmas 2.1 and 2.2, we have that the last relation of  $t$  is equal to that of  $b$ . Thus  $p_w g_v^k t$  is non-zero in  $E(\Lambda)$  and is equal to  $b$ .

Since there are only finitely many such  $A_w$ , taking the union of the above sets over each infinite  $A_w$ , along with all other trivial paths, arrows and maximal overlap sequences in finite extending sequences, gives us a finite generating set for  $E(\Lambda)$ .  $\square$

The next two propositions examine the cases where arbitrary powers of a generative multiplication path in the upper half of an extending sequence can also be used to get a finite generating set.

**Proposition 4.11.** *Let  $\Lambda$  be such that  $\ell(r) \leq n$  for all repetition relations  $r$ . Suppose one of the following occurs:*

- (1)  $\lambda \geq 2$ ,  $\Lambda$  has precisely 2 repetitions,  $N = 1$ ,
- (2)  $\lambda \geq 2$ ,  $\Lambda$  has only 1 repetition,  $2N \equiv 1 \pmod{\lambda}$ ,
- (3)  $\lambda = 1$ ,  $\Lambda$  has precisely 2 repetition relations  $r_x$  and  $r_y$ .

*Let  $A_w$  be an infinite extending sequence. Then  $E_w$  has a finite generating set if and only if there is a generative multiplication path  $g_v$  in the lower or upper half of  $A_w$ .*

**Proof.** Suppose first that  $E_w$  has a finite generating set  $\mathcal{S}$ , which we fix. We consider a maximal overlap sequence in  $A_w$  of sufficiently high even-degree such that, in any expression of it as a product of elements of  $\mathcal{S}$ , at least one element of  $\mathcal{S}$  of degree  $\geq 2$  occurs with multiplicity at least 3. Without loss of generality, we may choose a maximal overlap sequence  $a$  in  $A_w$  with  $a = h_0 d h_1 d h_2 d$ , where  $d \in \mathcal{S}$ ,  $\deg(d) \geq 2$ , each  $h_i$  is a product of generators,  $i = 0, 1, 2$ , and  $\deg(h_1) \geq 2M + 1$ .

We have that  $t(h_i) = o(d)$ , for  $i = 0, 1$  and  $2$ . Thus  $o(d)$  is a zero-length connective path. We need to show that  $o(d)$  has a generative multiplication path. Since at least one out of  $dh_1$ ,  $dh_2$ , and  $dh_1 dh_2$  is of even degree, one of the three will be a generative multiplication path or a power of one. Thus  $A_w$  has a generative multiplication path in its lower or upper half.

Conversely, suppose  $\Lambda$  has a zero-length connective path  $e_v$  that has a generative multiplication path  $g_v$ . If  $g_v$  is in the lower half of  $A_w$ , then by the proof of Theorem 4.10 we get a finite generating set for  $E_w$ . Thus suppose  $g_v$  is in the upper half of  $A_w$ . Note that for condition (2), since we have only 1 repetition,  $g_v$  being in the upper half of  $A_w$  is equivalent to  $g_v$  being in the lower half of  $A_w$ . Thus we get our result immediately for condition (2). Define  $s_w$  as the maximal overlap sequence of least odd-degree in  $A_w$ , such that  $\deg(s_w) \geq 2M + 1$  and  $t(s_w) = v$ . Let  $G_v$  be the extending sequence starting at  $v$ , a sequence infinite by Corollary 3.11 and Proposition 4.5. For condition (1) we take as a finite generating set for  $E_w$ :

- (1) The trivial path  $e_w$  and the arrow starting at  $w$ .
- (2) All even-degree maximal overlap sequences in  $A_w$  with degree less than or equal to  $\deg(s_w) + 2M - 1$ .
- (3) All odd-degree maximal overlap sequences in  $A_w$  with degree less than or equal to  $\deg(s_w)$ .

- (4) All even-degree maximal overlap sequences in  $G_v$  with degree less than or equal to that of  $g_v$ .
- (5) All odd-degree maximal overlap sequences  $t$  in  $G_v$ , such that  $2M + 1 \leq \deg(t) < 2M + 2\lambda + 1$ .

For condition (3) we take:

- (1) The trivial path  $e_w$  and the arrow starting at  $w$ .
- (2) All even-degree maximal overlap sequences in  $A_w$  with degree less than or equal to  $\deg(s_w) + 2M - 1$ .
- (3) All odd-degree maximal overlap sequences in  $A_w$  with degree less than or equal to  $\deg(s_w)$ .
- (4) The (degree 2) maximal overlap sequence  $g_v$ ,
- (5) The maximal overlap sequence in  $G_v$  of degree  $2M + 1$ .

It is left to the reader to verify that in each case we have a finite generating set for  $E_w$ .  $\square$

**Proposition 4.12.** *Let  $\Lambda$  be such that  $\ell(r) \leq n$  for all repetition relations  $r$ . Suppose one of the following occurs:*

- (1)  $\lambda \geq 2$ ,  $\Lambda$  has precisely 2 repetitions,  $N = 1$ ,
- (2)  $\lambda \geq 2$ ,  $\Lambda$  has only 1 repetition,  $2N \equiv 1 \pmod{\lambda}$ ,
- (3)  $\lambda = 1$ ,  $\Lambda$  has precisely 2 repetition relations  $r_x$  and  $r_y$ .

*Then  $E(\Lambda)$  is finitely generated as a  $\mathbb{k}$ -algebra if and only if each infinite extending sequence has a generative multiplication path in either its lower or upper half.*

**Proof.** This follows immediately from Proposition 4.11 since  $E(\Lambda)$  is finitely generated as a  $\mathbb{k}$ -algebra if and only if for each infinite extending sequence  $A_w$ , we have a finite generating set for  $E_w$ .  $\square$

The last proposition in this section deals with a special case.

**Proposition 4.13.** *Let  $\Lambda$  have only one order 1 repetition relation,  $r_x$ , with  $\ell(r_x) > n$ , and let  $A_w$  be the single infinite extending sequence. Then  $E(\Lambda)$  is finitely generated as a  $\mathbb{k}$ -algebra if and only if  $w = o(r_x) = t(r_x)$ .*

**Proof.** If  $r_x$  is the only repetition relation, then from Theorem 3.10 we have a single unflagged row in  $\mathcal{T}_\Lambda$ , which corresponds to  $A_w$ . Thus finite generation of  $E(\Lambda)$  is equivalent to  $E_w$  having a finite generating set. If  $w = o(r_x) = t(r_x)$ , then we take as generating set:

- (1) All odd-degree maximal overlap sequences in  $A$  up to degree  $2M + 1$ .
- (2) The (degree 2) maximal overlap sequence  $r_x$ .

Clearly this is a finite generating set for  $E_w$ .

Conversely, suppose  $E_w$  has a finite generating set. As there is only one repetition relation, and by Proposition 4.8, we must have a generative multiplication path in the upper or lower half of  $A_w$ . Since  $r_x$  is the only repetition relation this means we must have  $\mathfrak{o}(r_x) = \mathfrak{t}(r_x)$ . Now,  $r_x$  is a maximal overlap sequence (of degree 2), and by hypothesis there are no flag0s or flag2s in  $\mathcal{T}_\Lambda$ . We may therefore apply Proposition 4.5 to get  $\mathfrak{o}(r_x)$  the start of some infinite extending sequence  $G$ . By hypothesis  $G = A_w$ ; hence  $w = \mathfrak{o}(r_x)$ .  $\square$

The following theorem is our main result and provides the classification of the finite generation of  $E(\Lambda)$  as a  $\mathbb{k}$ -algebra.

We put Theorem 3.10 with Corollary 2.7, and Propositions 4.12 and 4.13 together to form conditions (1)–(6) in Theorem 4.14 below. Any conditions other than those of (1)–(6) are captured by Proposition 4.2. Proposition 4.9 and Theorem 4.10 then yield the stated result for these.

It is remarked here that, for any of our algebras  $\Lambda$ , Theorem 4.10 gives sufficient conditions for  $E(\Lambda)$  to be finitely generated.

**Theorem 4.14.** *Let  $\Lambda = \mathbb{k}\mathcal{Q}/\mathcal{I}$  be a finite-dimensional algebra, with  $\mathcal{Q}$  an oriented cycle and  $\mathcal{I}$  an admissible ideal, with the notation of the previous section. Then  $E(\Lambda)$  is finite-dimensional if and only if one of the following occurs:*

- (1)  $\lambda \geq 2$ , there is only 1 repetition,  $N = 1$  and  $\ell(r) \leq n$  for all repetition relations  $r$ ,
- (2)  $\lambda = 1$ , there is only 1 repetition relation  $r_x$  and  $\ell(r_x) \leq n$ .

*If  $E(\Lambda)$  has infinite dimension and one of the following occurs:*

- (3)  $\lambda \geq 2$ ,  $\Lambda$  has precisely 2 repetitions,  $N = 1$  and  $\ell(r) \leq n$  for all repetition relations  $r$ ,
- (4)  $\lambda \geq 2$ ,  $\Lambda$  has only 1 repetition,  $2N \equiv 1 \pmod{\lambda}$  and  $\ell(r) \leq n$  for all repetition relations  $r$ ,
- (5)  $\lambda = 1$ ,  $\Lambda$  has precisely 2 repetition relations  $r_x$  and  $r_y$ , and  $\ell(r_x), \ell(r_y) \leq n$ , then  $E(\Lambda)$  is finitely generated as a  $\mathbb{k}$ -algebra if and only if each infinite extending sequence has a generative multiplication path in either its lower or upper half.

*If  $E(\Lambda)$  has infinite dimension and the following occurs:*

- (6)  $\lambda = 1$ ,  $\Lambda$  has only 1 repetition relation  $r_x$  and  $\ell(r_x) > n$ , then  $E(\Lambda)$  is finitely generated as a  $\mathbb{k}$ -algebra if and only if  $\mathfrak{o}(A) = \mathfrak{o}(r_x) = \mathfrak{t}(r_x)$ , where  $A$  is the single infinite extending sequence of  $\Lambda$ .

*Otherwise, if  $E(\Lambda)$  has infinite dimension, then  $E(\Lambda)$  is finitely generated as a  $\mathbb{k}$ -algebra if and only if each infinite extending sequence has a generative multiplication path in its lower half.*

We can now use the above theorem to yield an immediate result in some special cases; many of these algebras are considered in the literature. It should be noted that the bound on the size of the smo-tube given after Definition 1.7 does not grow too large next to the size of the algebra. Therefore any reasonably sized examples can easily be checked by hand.

**Corollary 4.15.** *Let  $\Lambda$  have only one relation  $r$  and let  $\ell(r) = kn + c$ , for  $k \geq 0$ ,  $0 \leq c < n$  and  $\ell(r) \geq 2$ .*

- (1) *If  $k = 0$ , then  $E(\Lambda)$  is finite-dimensional.*
- (2) *If  $c = 0$ , then  $E(\Lambda)$  is finitely generated as a  $\mathbb{k}$ -algebra (with  $E(\Lambda)$  finite-dimensional if  $k = 1$ ).*
- (3) *If  $k \neq 0$  and  $c \neq 0$ , then  $E(\Lambda)$  is infinitely generated as a  $\mathbb{k}$ -algebra.*

**Proof.** If  $k = 0$  or if  $k = 1$  and  $c = 0$  we are in condition (2) of the theorem. If  $k \geq 2$  and  $c = 0$  we have condition (6) in Theorem 4.14, with  $\sigma(r) = \tau(r)$ . If  $k \neq 0$  and  $c \neq 0$  we have condition (6) again, but this time  $\sigma(r) \neq \tau(r)$ .  $\square$

The algebras covered in the following corollary are those  $\Lambda$  which are self-injective.

**Corollary 4.16.** *Let  $J$  be the 2-sided ideal of  $\mathbb{k}\mathcal{Q}$  generated by the arrows and let  $\Lambda = \mathbb{k}\mathcal{Q}/J^l$ , so that  $\Lambda$  has  $m = n$  relations, each of length  $l \geq 2$ . Then  $E(\Lambda)$  has infinite dimension but is finitely generated as a  $\mathbb{k}$ -algebra.*

**Proof.** Since the tail of any relation is the head of another, any relation is the start of some infinite extending sequence. This also means each extending sequence has a generative multiplication path in its lower (and upper) half.  $\square$

**Corollary 4.17.** *Let  $\Lambda$  have  $m$  relations, each of length  $kn \geq 2$ , for some fixed  $k \geq 1$ . Then  $E(\Lambda)$  is finitely generated as a  $\mathbb{k}$ -algebra. Moreover, if  $m = 1$  and  $k = 1$ , then  $E(\Lambda)$  is finite-dimensional.*

**Proof.** If  $m = 1$  and  $k = 1$ , then we are in case (2) of the theorem. If  $m \neq 1$  or  $k \neq 1$ , then since  $\tau(r) = \sigma(r)$  for all relations  $r$ , every extending sequence is infinite so we cannot have any flags in  $\mathcal{T}_\Lambda$ . Every relation  $r$  is then the start of an infinite extending sequence that has  $r$  as a (degree 2) generative multiplication path in its lower half.  $\square$

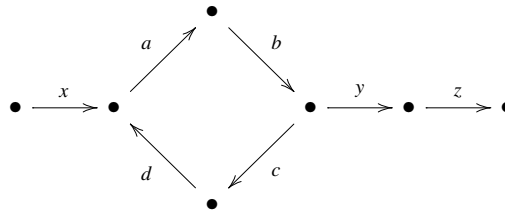
**Example 10.** From its smo-tube, the algebra in Example 1 can be identified as having 2 repetitions of order  $\lambda = 2$ . The repetition shift is  $N = 1$  and the length of all repetition relations is less than or equal to  $n$ . We thus have condition (3), so by Theorem 4.14 we need to find a zero-length connective path  $e_v$  that has a generative multiplication path  $g_v$ . From Example 2 and an inspection of the quiver we see that  $e_{14}$  is a zero-length connective path, with multiplication path  $g_v = r_5 \eta_{25} r_1$ . In this case  $g_v$  is generative. Checking the smo-tube, we find  $g_v$  is in either the upper or lower half of each of the 4 infinite extending sequences. This gives  $E(\Lambda)$  finitely generated as a  $\mathbb{k}$ -algebra.

We close this section with a remark on generalising to monomial path algebras. We also give the counter-example provided by the authors of [4] to their claim of the reverse implication of Proposition 0.3.

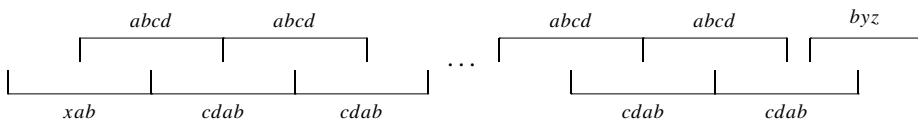
**Remark.** Let  $\Gamma$  be any finite quiver; we form the path algebra  $\mathbb{k}\Gamma$ . Recall that if  $\mathcal{I}$  is an admissible ideal of  $\mathbb{k}\Gamma$  generated by a finite set of paths such that  $B := \mathbb{k}\Gamma/\mathcal{I}$  is finite-dimensional, then we say that  $B$  is a monomial algebra. From Proposition 0.3, to determine if  $E(B)$  is infinitely generated it is enough to find one infinitely generated  $E(\Lambda)$  for any minimal cycle algebra  $\Lambda$  overlying  $B$ . We can now use Theorem 4.14 on each overlying minimal cycle algebra  $\Lambda$  to determine whether or not  $E(\Lambda)$  is infinitely generated. The bound on the size of the smo-tube means that for each cycle-algebra  $\Lambda$  this determination can be quickly made.

We now present an example of a monomial algebra with infinitely generated Ext-algebra, but with all overlying minimal cycle algebras having finitely generated Ext-algebra.

**Example 11** ((E.L. Green, D. Zacharia)). Let  $B$  be the  $\mathbb{k}$ -algebra with quiver



and relations  $abcd, bcda, cdab, dabc, xab, daby$ , and  $byz$ . This algebra has only one overlying minimal cycle algebra and by Theorem 4.14 this has finitely generated Ext-algebra. To show  $E(B)$  is infinitely generated we consider basis elements in  $E(B)$  with underlying path  $x(abcd)^nabyz$ , for  $n \geq 1$ . Such an element takes the form



Since multiplication relies on concatenation of paths, to non-trivially factor an element of the above form we need to split its underlying path in two at a vertex that is both the start of a relation and the end of one. So far we have many choices. However, it is clear that no matter where one chooses the split to be, the right hand path will not be a maximal overlap sequence. Thus maximal overlap sequences of the sort above cannot be non-trivially factored. Since we have infinitely many such maximal overlap sequences,  $E(B)$  is infinitely generated.

## 5. Noetherian Ext-algebras

Now that we have determined precisely when  $E(\Lambda)$  is finitely generated as a  $\mathbb{k}$ -algebra, we determine for which cycle algebras  $\Lambda$  the Ext-algebra is a Noetherian ring. In doing

so we produce a class of examples for which the Ext-algebra is finitely generated but not Noetherian, and a further class that have Noetherian Ext-algebra with  $\Lambda \not\cong \mathbb{k}Q/J^n$  for any  $n \geq 2$ , where  $J$  is the 2-sided ideal of  $\mathbb{k}Q$  generated by the arrows.

We may immediately state the main result of this section.

**Theorem 5.1.** *Let  $\Lambda = \mathbb{k}Q/I$  be a finite-dimensional algebra, with  $Q$  an oriented cycle,  $I$  an admissible ideal. Suppose further that the Ext-algebra  $E(\Lambda)$  has infinite dimension. Then  $E(\Lambda)$  is a Noetherian ring if and only if every connective path of  $E(\Lambda)$  is of zero length.*

Before we can prove this theorem we need the following.

**Remark.** So far in the paper we have talked of connective paths. In fact, just as a maximal overlap sequence is considered to be a left maximal overlap sequence if it is constructed from the left, so a left connective path and a left repetition come from an extending sequence constructed from the left. In the preceding sections, connective paths and repetitions have both been constructed from the left. This left construction of the extending sequences naturally shows the right  $E(\Lambda)$ -module structure of  $E(\Lambda)$ , which we exploit in Theorem 5.1. However, we also need to show how  $E(\Lambda)$  behaves as a left  $E(\Lambda)$ -module: this is done by constructing right maximal overlap sequences. From [1] we know that left and right maximal overlap sequences have the same underlying path. In general however, the left and right repetitions need not be the same and so the left and right connective paths need not be the same. The following proposition gives us conditions under which the left and right repetitions do coincide.

**Proposition 5.2.** *Let  $\Lambda$  be such that  $E(\Lambda)$  has infinite dimension and suppose all left connective paths are of zero length. Then the set of left repetition relations is equal to the set of right repetition relations. In particular, all right connective paths are also of zero length.*

**Proof.** Suppose  $E(\Lambda)$  is of infinite dimension and that all left connective paths are of zero length. Let  $r_{a_1}, r_{a_2}, \dots, r_{a_\lambda}$  be the left repetition relations of  $(a_1, a_2, \dots, a_\lambda)$ , one of the left repetitions of  $E(\Lambda)$ . Then  $r_{a_1}r_{a_2} \cdots r_{a_\lambda}$  is a non-zero path in  $\mathbb{k}Q$ . Since each  $r_{a_i}$  is a degree 2 left maximal overlap sequence, and  $E(\Lambda)$  has infinite dimension, we may use Theorems 1.10 and 3.10 and Proposition 4.5 to conclude that the path  $h := r_{a_1}r_{a_2} \cdots r_{a_\lambda}$  is also a left maximal overlap sequence. Similarly  $r_{a_i}r_{a_{i+1}} \cdots r_{a_\lambda}h^k$  is a left maximal overlap sequence for all  $1 \leq i \leq \lambda$  and  $k \geq 0$ . Hence we can construct a left maximal overlap sequence beginning at the vertex  $\sigma(r_{a_i})$  for all  $1 \leq i \leq \lambda$ , of degree greater than  $2M$ . From [1] we have that, as a path in  $\mathbb{k}Q$ , each left maximal overlap sequence of degree  $l$  is also a right maximal overlap sequence of degree  $l$ . Thus the path  $r_{a_i}r_{a_{i+1}} \cdots r_{a_\lambda}h^k$  is a right maximal overlap sequence for all  $1 \leq i \leq \lambda$  and  $k \geq 0$  and so  $r_{a_i}$  is a right repetition relation for all  $1 \leq i \leq \lambda$ . We thus have that all left repetition relations are also right repetition relations. By an identical argument we get that all right repetition relations are also left repetition relations. It follows immediately that all right connective paths are of zero length.  $\square$



A dual argument yields the following corollary.

**Corollary 5.3.** *All left connective paths are of zero length if and only if all right connective paths are of zero length.*

We can now prove Theorem 5.1.

**Proof of Theorem 5.1.** Assume first that all left connective paths of  $E(\Lambda)$  are of zero length. We will show that  $E(\Lambda)$  is a Noetherian right  $E(\Lambda)$ -module. Let  $r_{b_1}, r_{b_2}, \dots, r_{b_\mu}$  be the left repetition relations of  $\Lambda$ . Since all left connective paths are of zero length, and  $E(\Lambda)$  has infinite dimension, we have by Theorems 1.10 and 3.10 that every left repetition relation is the start of a generative multiplication path. Let  $g_{b_i}$  be the generative multiplication path such that  $\mathfrak{o}(g_{b_i}) = \mathfrak{o}(r_{b_i})$ . Set  $\zeta = g_{b_1} + g_{b_2} + \dots + g_{b_\mu} \in E(\Lambda)$ . Then by Lemma 1.6,  $\zeta$  is a homogeneous element of  $E(\Lambda)$  in degree  $2\lambda$  and has the property that  $\zeta^l = g_{b_1}^l + g_{b_2}^l + \dots + g_{b_\mu}^l$  for all  $l \geq 1$ . We thus have that 1 and  $\zeta$  in  $E(\Lambda)$  generate a graded subalgebra of  $E(\Lambda)$  which is isomorphic to the polynomial ring in one variable, which we denote by  $\mathbb{k}[\zeta]$ .

Consider the usual basis of  $E(\Lambda)$  consisting of trivial paths, arrows and maximal overlap sequences constructed from the left. Since  $\mathbb{k}[\zeta]$  is a subring of  $E(\Lambda)$  we consider  $E(\Lambda)$  as a right  $\mathbb{k}[\zeta]$ -module. Let  $S' = \{A^z: A \text{ is an infinite extending sequence of } E(\Lambda), 2 \leq z \leq 2M + 2\lambda - 1\}$  and let  $S = S' \cup \{\text{trivial paths and arrows of } \Lambda\}$ . We will show that  $S$  is a (finite) generating set for  $E(\Lambda)$  as a right  $\mathbb{k}[\zeta]$ -module.

Let  $A^y$  be a maximal overlap sequence of degree  $y \geq 2M + 2\lambda$ , in some infinite extending sequence  $A$  of  $E(\Lambda)$ . Then  $\mathfrak{t}(A^y) = \mathfrak{t}(r_{b_i})$  for some  $1 \leq i \leq \mu$ . Write  $y - 2M = c(2\lambda) + k'$ , for some  $0 \leq k' < 2\lambda$ ,  $c \geq 1$ , so that  $y = c(2\lambda) + k$ , for some  $2M \leq k \leq 2\lambda + 2M - 1$ ,  $c \geq 1$ . Then  $A^y = A^k g_{b_i}^c = A^k \zeta^c$ , with  $A^k \in S$ . Since  $A^y$  was arbitrary we get all maximal overlap sequences of degree greater or equal to  $2M + 2\lambda$  in this way. Hence  $E(\Lambda)$  is finitely generated as a right  $\mathbb{k}[\zeta]$ -module with generating set  $S$ . As  $\mathbb{k}[\zeta]$  is a Noetherian ring, we get that  $E(\Lambda)$  is a Noetherian right  $\mathbb{k}[\zeta]$ -module. Hence  $E(\Lambda)$  is a Noetherian right  $E(\Lambda)$ -module.

Now we must show that  $E(\Lambda)$  is a Noetherian left  $E(\Lambda)$ -module. By Proposition 5.2 since the left connective paths of  $E(\Lambda)$  are of zero length, so are the right ones. Also the right repetition relations are the same as the left. By a similar argument to that above, it follows that  $E(\Lambda)$  is finitely generated as a left  $\mathbb{k}[\zeta]$ -module. That  $E(\Lambda)$  is a Noetherian left  $E(\Lambda)$ -module then follows. Hence  $E(\Lambda)$  is a Noetherian ring.

Conversely, assume now that there exists a connective path of  $E(\Lambda)$  that has positive length. We can take this to be a left connective path by Corollary 5.3. Suppose this path starts at  $\mathfrak{t}(r_{a_{i-1}})$  and ends at  $\mathfrak{o}(r_{a_i})$ , for  $r_{a_{i-1}}$  and  $r_{a_i}$  repetition relations in some left repetition. The connective path is then denoted  $\mathfrak{c}_{a_i}$ . By taking the basis of  $E(\Lambda)$  of left maximal overlap sequences, we view  $E(\Lambda)$  as a right  $E(\Lambda)$ -module. We now construct a strictly ascending chain of right submodules of  $E(\Lambda)$  that is of infinite length. First consider some special basis elements of  $E(\Lambda)$ , namely those left maximal overlap sequences of degree greater than  $2M$  that end at  $\mathfrak{t}(r_{a_{i-1}})$ . Since  $r_{a_{i-1}}$  is a left repetition relation, there is some extending sequence  $A$  in which there are infinitely many of these maximal overlap se-

quences. Label these left maximal overlap sequences in  $A$  that end at the vertex  $t(r_{a_{i-1}})$  by  $\xi_1, \xi_2, \dots$  in increasing order of degree.

Now let  $q$  be an element from our basis of  $E(\Lambda)$ : then  $q$  corresponds to a vertex, an arrow or a left maximal overlap sequence of degree  $\geq 2$ . Pick  $j \geq 1$ ; then, since  $t(\xi_j) = o(c_{a_i})$  and  $\ell(c_{a_i}) > 0$ , we get that the product  $\xi_j q$  is zero in  $E(\Lambda)$ , for all  $q \neq t(r_{a_{i-1}})$ . Thus  $\deg(\xi_j a) \leq \deg(\xi_j)$ , for all  $a \in E(\Lambda)$ ,  $j \geq 1$ . We now construct our chain of submodules. Let  $I_0 = \{0\}$  and for  $j \geq 1$  let  $I_j = (\xi_1, \xi_2, \dots, \xi_j)E(\Lambda)$ . Then  $I_0 \subset I_1 \subset I_2 \subset \dots$  is clearly an infinite, strictly ascending chain of right  $E(\Lambda)$ -submodules of  $E(\Lambda)$ . Hence  $E(\Lambda)$  is not right Noetherian and therefore not Noetherian.  $\square$

Recalling the definition of a minimal cycle algebra from Definition 0.1, we have the following corollary.

**Corollary 5.4.** *Let  $Z_Q$  be a cycle algebra overlying a minimal cycle algebra  $Z_Q$ . Then  $E(Z_Q)$  is Noetherian if and only if  $E(Z_Q)$  is Noetherian.*

**Proof.** The result is immediate since by definition all the connective paths in  $E(Z_Q)$  are of zero length if and only if all the connective paths in  $E(Z_Q)$  are of zero length.  $\square$

Notice that Theorem 5.1 says nothing about the relations on  $\Lambda$  being of equal length if  $E(\Lambda)$  is Noetherian. The following example shows they need not be. Note that in both examples below,  $E(\Lambda)$  has infinite dimension.

**Example 12.** Let  $Q$  be an oriented cycle with 9 vertices labelled  $1, \dots, 9$ . Let  $\eta_i$  be the arrow which starts at the vertex  $i$ . Let  $\mathcal{I} = \langle r_1, r_2, r_3, r_4, r_5 \rangle$ , where  $r_1 = \eta_1 \eta_2 \eta_3$ ,  $r_2 = \eta_2 \eta_3 \eta_4 \eta_5$ ,  $r_3 = \eta_3 \eta_4 \eta_5 \eta_6$ ,  $r_4 = \eta_4 \eta_5 \eta_6 \eta_7 \eta_8 \eta_9$ ,  $r_5 = \eta_6 \eta_7 \eta_8 \eta_9 \eta_1$ . Let  $\Lambda = \mathbb{k}Q/\mathcal{I}$ . Then the repetition relations of  $\Lambda$  are  $r_1, r_2, r_4$ , and  $r_5$ ; the connective paths are the trivial paths  $e_1, e_2, e_4$ , and  $e_6$ . By Theorem 5.1, since  $E(\Lambda)$  has infinite dimension, we get that  $E(\Lambda)$  is Noetherian (and hence also finitely generated).

**Example 13.** Let  $\Lambda$  be as in Example 12, with the exception that here  $r_4 = \eta_4 \eta_5 \eta_6 \eta_7 \eta_8$ . We have the same left repetition relations as above, but now the left connective paths are  $\eta_9, e_2, e_4$ , and  $e_6$ . Thus from Theorem 5.1,  $E(\Lambda)$  is not Noetherian. The positive length left connective path  $\eta_9$  means that the  $\xi_j$ 's from the proof of Theorem 5.1 arise, each ending at the vertex 9. However, from Theorem 4.14, we get that  $E(\Lambda)$  is finitely generated.

**Remark.** It is clear that using Theorems 4.14 and 5.1, we can extend Examples 12 and 13 in both cases to a large class of examples with the same finiteness conditions on the Ext-algebra.

Lastly we return to our discussion on monomial algebras.

**Proposition 5.5.** *Let  $B$  be a monomial algebra and let the Ext-algebra  $E(B)$  be Noetherian. Then the  $\mathbb{k}$ -algebras  $E(Z_Q)$  are Noetherian for all minimal cycle algebras  $Z_Q$  overlying  $B$ .*

**Proof.** Let  $Z_Q$  be a minimal cycle algebra overlying  $B$  and let  $E(Z_Q)$  be non-Noetherian. Let  $B$  have quiver  $\Gamma$ . We will show that  $E(B)$  is non-Noetherian. By Theorem 5.1, we have a connective path of  $E(Z_Q)$  of positive length, so in particular  $E(Z_Q)$  is not right Noetherian. We thus have the infinite strictly ascending chain of right ideals of  $E(Z_Q)$  constructed in the proof of Theorem 5.1. Recall that an ideal from this chain was written  $I_j^{Z_Q} = (\xi_1^*, \xi_2^*, \dots, \xi_j^*)E(Z_Q)$ . We use the  $*$ -notation to remain consistent with Proposition 0.3. As discussed in the proof of Proposition 0.3, each  $\xi_i^*$  in  $E(Z_Q)$  corresponds to a maximal overlap sequence  $\xi_i$  in  $E(B)$ . We can thus form an infinite ascending chain of right ideals of  $E(B)$ :  $I_0^B \subset I_1^B \subset I_2^B \subset \dots$ , where  $I_0^B = \{0\}$  and  $I_j^B = (\xi_1, \xi_2, \dots, \xi_j)E(B)$ . It remains to show that this chain is strictly ascending. Fix some  $j \geq 0$  and consider the basis element  $\xi_{j+1}$ . To seek a contradiction suppose  $I_j^B = I_{j+1}^B$ . Then since  $\xi_{j+1}$  is one maximal overlap sequence (not a linear combination),  $\xi_{j+1} = \xi_i b$ , for some  $1 \leq i \leq j$  and  $b$  some basis element of  $E(B)$ . Then the underlying path of  $b$  is a terminal subpath of  $\xi_{j+1}$  and so lies along the path in  $\Gamma$  that is covered by  $Q$ . Thus  $b$  corresponds to a basis element  $b^*$  in  $E(Z_Q)$ . This gives us  $\xi_{j+1}^* = \xi_i^* b^*$ , and so  $I_j^{Z_Q} = I_{j+1}^{Z_Q}$ . This is a contradiction and therefore we conclude that  $I_j^B \neq I_{j+1}^B$ . Since  $j$  was arbitrary, we have that our chain of right ideals of  $E(B)$  is strictly ascending. Hence  $E(B)$  is not Noetherian.  $\square$

As a counter-example to the reverse implication, Example 11 gives a monomial algebra with non-Noetherian Ext-algebra that has all its overlying minimal cycle algebras possessing Noetherian Ext-algebras.

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## References

- [1] M. Bardzell, The alternating syzygy behavior of monomial algebras, *J. Algebra* 188 (1) (1997) 69–89.
- [2] E.L. Green, D. Happel, D. Zacharia, Projective resolutions over Artin algebras with zero relations, *Illinois J. Math.* 29 (1) (1985) 180–190.
- [3] E.L. Green, E. Kirkman, J. Kuzmanovich, Finitistic dimensions of finite-dimensional monomial algebras, *J. Algebra* 136 (1) (1991) 37–50.
- [4] E.L. Green, D. Zacharia, The cohomology ring of a monomial algebra, *Manuscripta Math.* 85 (1) (1994) 11–23.