

Affine Lie algebras and product–sum identities

Naihuan Jing ^{a,b,*,1}, Li-meng Xia ^{c,d}

^a Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, USA

^b School of Mathematical Sciences, South China University of Technology, Guangzhou, Guangdong 510640, China

^c Faculty of Science, Jiangsu University, Zhenjiang, 212013 Jiangsu, China

^d Department of Mathematics, Universität Bielefeld, 33619 Bielefeld, Germany

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Abstract

We provide four different decompositions for a special infinite product first studied by I. Schur. Our product–sum decomposition gives two more correspondences between subsets of partitions with parts congruent to 1 or 5 modulo 6. Our method uses a special vertex representation of affine Lie algebra $C_3^{(1)}$ and admissible representations of $A_1^{(1)}$ as well as quintuple product identity.

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1. Introduction

It is well known that there is a close relationship between representations of affine Lie algebras and combinatorics. For example, the Jacobi triple product identity can be obtained as the Weyl–Kac denominator formula for the affine Lie algebra \widehat{sl}_2 [K2]. The famous Rogers–Ramanujan identities can be realized from the character formula of certain level three representations [LW]. Like the Jacobi triple product identity, the quintuple product identity is also equivalent to the Weyl–Kac denominator formula for the affine Lie algebra $A_2^{(2)}$. In this paper we will see that it is also useful in expressing characters of admissible representations of affine Lie algebras.

* Corresponding author.

E-mail addresses: jing@math.ncsu.edu (N. Jing), xialimeng@hotmail.com (L.-m. Xia).

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We start with a simple fact in partition theory. Let a_n be a sequence of natural numbers satisfying some conditions, and assume that

$$\prod_{n=1}^{\infty} \frac{1}{(1 - q^{a_n})} = \sum_{n=0}^{\infty} T(n)q^n.$$

Then for any n , $T(n)$ counts the number of partitions of n into sums of the a_n 's. Thus an identity of involving such infinite products means a relation of different partition classes.

Our purpose in the present work is to provide several product–sum decompositions for a very special infinite product, which is also expressible in terms of theta functions. We will study two related problems. One is to give a product formula for level $-1/2$ admissible modules for affine Lie algebra $A_1^{(1)}$ and another is to give partition-theoretic interpretation of these identities. The product formula for a character is usually better than the summation form in the sense that it often suggests how to explicitly construct the representation. In fact many vertex representations of affine Lie algebras were first suggested by such formulas.

Let us describe our results briefly. We will express the following infinite product

$$\prod_{n=1}^{\infty} \frac{1}{(1 - q^{6n-1})(1 - q^{6n-5})} \quad (1.1)$$

by a sum of two other infinite products in four different ways. The first decomposition, stated below, are proved in section two by considering a principal vertex representation of the affine Lie algebra $C_3^{(1)}$, and the other decomposition is obtained from the character formula of certain admissible representation of the affine Lie algebra $A_1^{(1)}$ and the quintuple product identity. We also derive two more decompositions as direct consequences of them (see the end of the paper).

Theorem 1.1.

$$\begin{aligned} & \prod_{n=1}^{\infty} \frac{1}{(1 - q^{6n-1})(1 - q^{6n-5})} \\ &= q \prod_{n=1}^{\infty} \frac{1}{(1 - q^{10n-4})(1 - q^{10n-6})(1 - q^{15n-6})(1 - q^{15n-9})} \\ & \quad + \prod_{n=1}^{\infty} \frac{1}{(1 - q^{10n-2})(1 - q^{10n-8})(1 - q^{15n-3})(1 - q^{15n-12})} \\ &= \prod_{n=1}^{\infty} \frac{(1 - q^{24n})(1 - q^{24n-4})(1 - q^{24n-20})(1 - q^{48n-16})(1 - q^{48n-32})}{(1 - q^{2n})} \\ & \quad + q \prod_{n=1}^{\infty} \frac{(1 - q^{24n})(1 - q^{24n-8})(1 - q^{24n-16})(1 - q^{48n-8})(1 - q^{48n-40})}{(1 - q^{2n})}. \end{aligned}$$

I. Schur [S] (see also [A]) was probably the first person who studied the partitions described by (1.1). Schur showed that the number of partitions of n into parts congruent to $\pm 1 \pmod{6}$ is

equal to the number of partitions of n into distinct parts congruent to $\pm 1 \pmod{3}$, and is also equal to the number of partitions of n into parts that differ at least 3 with added condition that difference between multiples of 3 is at least 6.

Our results in Theorem 1.1 imply the following partition theorem:

Theorem 1.2. Let $A(n)$ denote the number of partitions of n into parts congruent to $\pm 1 \pmod{6}$. Let $B(n)$ denote the number of partitions of n into

$$2k_1 + \cdots + 2k_i + 3r_1 + \cdots + 3r_j$$

with constraints $k_p - k_{p+1} \geq 2$, $r_p - r_{p+1} \geq 2$. Let $C(n)$ denote the number of partitions of n into

$$2k_1 + \cdots + 2k_i + 3r_1 + \cdots + 3r_j + 1$$

with constraints $k_p - k_{p+1} \geq 2$, $r_p - r_{p+1} \geq 2$ and $k_i > 1$, $r_j > 1$. Let $D(n)$ denote the number of partitions of n into

$$2k_1 + \cdots + 2k_i + 8r_1 + \cdots + 8r_j + 12u_1 + \cdots + 12u_t$$

with constraints that k_p, u_p are odd and $r_p > r_{p+1}$ and $r_p \equiv 1, 2 \pmod{3}$. Let $E(n)$ denote the number of partitions of n into

$$2k_1 + \cdots + 2k_i + 4r_1 + \cdots + 4r_j + 12u_1 + \cdots + 12u_t + 1$$

with constraints that k_p, u_p are odd and $r_p > r_{p+1}$ and $r_p \equiv 1, 5 \pmod{6}$.

Then for any natural number n , we have

$$A(n) = B(n) + C(n) = D(n) + E(n).$$

For example,

$A(15) = 9,$	$B(15) = 6,$	$C(15) = 3,$
$13 + 1 + 1,$	$3 \times 5,$	$2 \times 7 + 1,$
$11 + 1 + 1 + 1 + 1,$	$2 \times 3 + 3 \times 3,$	$2 \times 5 + 2 \times 2 + 1,$
$7 + 7 + 1,$	$2 \times 6 + 3 \times 1,$	$2 \times 4 + 3 \times 2 + 1.$
$7 + 5 + 1 + 1 + 1,$	$2 \times 5 + 2 \times 1 + 3 \times 1,$	
$7 + 1 + 1 + \cdots + 1,$	$2 \times 4 + 2 \times 2 + 3 \times 1,$	
$5 + 5 + 5,$	$3 \times 4 + 3 \times 1,$	
$5 + 5 + 1 + 1 + 1 + 1 + 1,$		
$5 + 1 + \cdots + 1,$		
$1 + 1 + 1 + \cdots + 1,$		

Table 1 lists the values of $A(n)$, $B(n)$, $C(n)$, $D(n)$, $E(n)$ for $n \leq 15$.

Table 1

n	$A(n)$	$B(n)$	$C(n)$	$D(n)$	$E(n)$
1	1	1	0	0	1
2	1	1	0	1	0
3	1	1	0	0	1
4	1	1	0	1	0
5	2	1	1	0	2
6	2	1	1	2	0
7	3	1	2	0	3
8	3	3	0	3	0
9	3	2	1	0	3
10	4	2	2	4	0
11	5	3	2	0	5
12	6	6	0	6	0
13	7	3	4	0	7
14	8	7	1	8	0
15	9	6	3	0	9

Throughout this paper, we will use the following standard notation

$$\varphi(q) = \prod_{n=1}^{\infty} (1 - q^n)$$

and

$$(a, q) = \prod_{n=1}^{\infty} (1 - aq^{n-1}).$$

2. Representations of the affine Lie algebra $C_3^{(1)}$

Let H be a three dimensional complex vector space spanned by $\alpha_1, \alpha_2, \alpha_3$. Let $(,)$ be the non-degenerate symmetric bilinear form such that

$$\begin{aligned} 2(\alpha_1, \alpha_1) &= 2(\alpha_2, \alpha_2) = (\alpha_3, \alpha_3) = 2, \\ 2(\alpha_1, \alpha_2) &= (\alpha_2, \alpha_3) = -1, \quad (\alpha_1, \alpha_3) = 0. \end{aligned}$$

Then $\alpha_1, \alpha_2, \alpha_3$ form a basis of the root system of type C_3 . Let Q be the \mathbb{Z} span of elements $\alpha_1, \alpha_2, \alpha_3$ and $H_S = \mathbb{C}\alpha_1 + \mathbb{C}\alpha_2$. The finite fundamental weights are $\lambda_1 = \alpha_1 + \alpha_2 + \frac{1}{2}\alpha_3$, $\lambda_2 = \alpha_1 + 2\alpha_2 + \alpha_3$, and $\lambda_3 = \alpha_1 + 2\alpha_2 + \frac{3}{2}\alpha_3$.

Let $H(k)$ and $H_S(k + \frac{1}{2})$ be the isomorphic copies of H and H_S , respectively, for each integer k . Then

$$\widehat{H} = \bigoplus_{k \in \mathbb{Z} \setminus \{0\}} H(k) \oplus \bigoplus_{k \in \mathbb{Z}} H_S\left(k + \frac{1}{2}\right) \oplus \mathbb{C}c$$

with bracket relations

$$[a(m), b(n)] = (a, b)m\delta_{m, -n}c, \quad (2.1)$$

$$[a(m), c] = 0 \quad (2.2)$$

is an infinite dimensional Heisenberg Lie algebra. The symmetric algebra $S(\widehat{H}^-)$ generated by its abelian subalgebra

$$\widehat{H}^- = \bigoplus_{k \in \mathbb{Z}_-} H(k) \oplus \bigoplus_{k \in \mathbb{Z}_-} H_S\left(k + \frac{1}{2}\right)$$

has a natural \widehat{H} -module structure by the standard multiplication and differentiation operations.

Let $\mathbb{C}[Q]$ be the group algebra generated by Q . Its basis elements are of the form e^α ($\alpha \in Q$) and product is given by

$$e^\alpha \cdot e^\beta = e^{\alpha+\beta}. \quad (2.3)$$

Then $\mathbb{C}[Q]$ becomes a $H(0) = H$ -module with action:

$$a(0) \cdot e^\beta = (a, \beta)e^\beta. \quad (2.4)$$

Let

$$\widetilde{H} = \widehat{H} \oplus H(0),$$

then

$$V(Q) = S(\widehat{H}^-) \otimes \mathbb{C}[Q]$$

is a \widetilde{H} -module.

In particular, the space $V(Q)$ is a completely reducible $C_3^{(1)}$ -module via vertex operators. As we will not need the explicit formula for the action of the affine Lie algebra, we refer the detailed vertex representation to [XH].

It is easy to see that $V(Q)$ is an integrable module and has weight space decomposition

$$V(Q) = \sum_{\lambda \in P(V(Q))} V_\lambda.$$

In the above module, the element $x = v \otimes e^\alpha$ has the weight

$$\lambda_x = \Lambda_0 + \deg(x)\delta + \alpha,$$

where $\delta = \alpha_0 + 2\alpha_1 + 2\alpha_2 + \alpha_3$ is the canonical imaginary root for type $C_3^{(1)}$, and $\deg(x)$ is the degree of v as a polynomial in $a(-k)$. We also define the degree of λ_x by

$$\deg(\lambda_x) = -\deg(x) \operatorname{ht}(\delta) - \operatorname{ht}(\alpha),$$

where $\text{ht}(\cdot)$ means the height of an element. For example, $\text{ht}(\delta) = 6$. Define

$$V_j = \bigoplus_{\lambda: \deg \lambda = j} V_\lambda,$$

then it is easy to see that V_j is finite dimensional for each j and

$$V(Q) = \bigoplus_{j=0}^{\infty} V_{-j}.$$

The q -character is then defined to be

$$\text{ch}_q V(Q) = \sum_{j=0}^{\infty} \dim(V_{-j}) q^j.$$

The module $V(Q)$ is completely reducible for the Lie algebra $C_3^{(1)}$. In fact

$$V(Q) = L(\Lambda_0) \otimes \Omega_0 \oplus L(\Lambda_2) \otimes \Omega_2,$$

where $L(\Lambda_{2i})$ is the irreducible submodule with the highest vector $v \otimes 1$ and $v \otimes e^{\alpha_1 + 2\alpha_2 + \alpha_3}$ and Ω_{2i} is the corresponding vacuum space, respectively for $i = 0, 1$.

For $\alpha \in Q$ we define the operator

$$\begin{aligned} S(\alpha, z) &= \exp\left(\sum_{n>0} \frac{\alpha(-n + \frac{1}{2})}{n - \frac{1}{2}} z^{2n-1}\right) \exp\left(-\sum_{n>0} \frac{\alpha(n - \frac{1}{2})}{n - \frac{1}{2}} z^{-2n+1}\right) \\ &= \sum_{n \in \frac{1}{2}\mathbb{Z}} S_n(\alpha) z^{-2n}. \end{aligned}$$

Also let

$$\alpha_3 = -\frac{1}{3}\alpha_1 - \frac{2}{3}\alpha_2; \quad y_i = \sum_{j=i}^3 2\alpha_j, \quad i = 1, 2, 3$$

and for $s = 0, 2$,

$$Z^{[s]}(z) = \sum_{i \in \mathbb{Z}} Z_i^{[s]} z^{-i} = \sum_{j=1}^s S(y_j, z) - \sum_{j=s+1}^3 S(y_j, z).$$

Lemma 2.1. Ω_s is generated by $Z_i^{[s]}$ ($i \in \frac{1}{2}\mathbb{Z}$).

Proof. As we know that if $v \otimes 1$ or $v \otimes e^{\alpha_1+2\alpha_2+\alpha_3}$ is a highest weight vector, then $v \otimes 1$ lies in the space generated (acting on 1) by

$$\left\{ S_n(\alpha_1), y_3(m) \mid n \in \mathbb{Z}, m \in \frac{1}{2} + \mathbb{Z} \right\},$$

denoted by $V(\alpha_1)$, for this fact, one can see [JM]. $V(\alpha_1)$ has q -character

$$\prod_{n=1}^{\infty} (1 + q^{6n})(1 + q^{3n}).$$

Now considering the space V' generated by $Z_i^{[s]}$ ($i \in \frac{1}{2}\mathbb{Z}$), $s = 0, 2$. Obviously, it is generated by

$$S(y_1, z) + S(y_2, z), S(y_3, z),$$

notice that

$$S_n(y_1) + S_n(y_2) = \sum_{j \in \mathbb{Z}} S_j(\alpha_1) S_{n-j} \left(-\frac{y_3}{2} \right),$$

and the operator

$$(w - z)^{\frac{2}{3}}(w + z)^{\frac{2}{3}} S(y_3, z) S(y_3, w) - (z - w)^{\frac{2}{3}}(w + z)^{\frac{2}{3}} S(y_3, w) S(y_3, z)$$

generates all $y_3(m)$, $m \in \frac{1}{2} + \mathbb{Z}$, hence V' is generated by

$$\left\{ S_n(\alpha_1), y_3(m) \mid n \in \mathbb{Z}, m \in \frac{1}{2} + \mathbb{Z} \right\},$$

then $V' = V(\alpha_1)$. Thus we have $\Omega_0 \cup \Omega_2 \subset V(\alpha_1)$.

Suppose that $v \otimes 1 \in V(\alpha_1)/(\Omega'_0 \cup \Omega'_2)$ is a highest weight vector. Since $(\alpha_2|y_1) = 0$, so the space S^- is generated by

$$\left\{ \alpha_1(m), \alpha_2(m) \mid m \in \frac{1}{2} + \mathbb{Z} \right\}$$

also is generated by operators

$$\left\{ S_n(\alpha_2), S_n(y_1) \mid m \in \frac{1}{2}\mathbb{Z} \right\},$$

so we can assume that

$$v \otimes 1 = \sum a_{i_1, \dots, i_d; j_1, \dots, j_f} S_{i_1}(\alpha_2) \cdots S_{i_d}(\alpha_2) S_{j_1}(y_1) \cdots S_{j_f}(y_1).$$

By the relation

$$Z_n^{[2]} = S_n(y_1) + 2 \sum_{j \in \frac{1}{2} + \mathbb{Z}} S_j(\alpha_2) S_{n-j}(-2y_1),$$

if $v \otimes 1 \neq 0$, then any summand appearing has the form $S_j(\alpha_2)$ for half integers j . On the other hand, it is easy to see that $S_{\frac{1}{2}}(\alpha_2)v \otimes 1 \neq 0$, which proves the lemma. \square

We remark that these Z -operators are exactly those studied by Lepowsky and Wilson [LW]. It follows from [LW] that

$$\text{ch}_q(\Omega_{2i}) = q^{-i} \prod_{n=1}^{\infty} \frac{1}{(1 - q^{3(5n+i-2)})(1 - q^{3(5n-i-3)})}.$$

Moreover, by the principal Weyl–Kac character formula from [K2], we have

$$\dim_q(L(\Lambda_{2i})) = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{2n-1})(1 - q^{2(5n+i-2)})(1 - q^{2(5n-i-3)})}.$$

So the q -character of $V(Q)$ is

$$\text{ch}_q(V(Q)) = \sum_{i=0,1} q^{-i} \text{ch}_q(\Omega_{2i}) \dim_q(L(\Lambda_{2i})). \quad (2.5)$$

By the construction of the module $V(Q)$, we have

$$\text{ch}_q(V(Q)) = \prod_{n=1}^{\infty} \frac{1}{(1 - q^{3n})^2(1 - q^{6n})} \sum_{\alpha \in Q} q^{3(\alpha, \alpha) - \text{ht}(\alpha)}.$$

Furthermore, we compute the sum inside the character (using standard formulas from [K1])

$$\begin{aligned} \sum_{\alpha \in Q} q^{3(\alpha, \alpha) - \text{ht}(\alpha)} &= \sum_{n_1, n_2, n_3 \in \mathbb{Z}} q^{3(n_1^2 - n_1 n_2 + n_2^2 - 2n_2 n_3 + 2n_3^2) - n_1 - n_2 - n_3} \\ &= \left(\sum_{n \in \mathbb{Z}} q^{3n^2 + n} \right) \left(\sum_{n_1, n_2 \in \mathbb{Z}} q^{6n_1^2 + 5n_1 + 3n_2^2 + 2n_2} + \sum_{n_1, n_2 \in \mathbb{Z}} q^{6n_1^2 + n_1 + 3n_2^2 + 4n_2} \right) \\ &= \left(\sum_{n \in \mathbb{Z}} q^{3n^2 + n} \right) \left(\sum_{n \in \mathbb{Z}} q^{3n^2 + 2n} \right) \left(\sum_{n \in \mathbb{Z}} q^{6n^2 + 5n} + q^{-1} \sum_{n \in \mathbb{Z}} q^{6n^2 + n} \right) \\ &= \prod_{n=1}^{\infty} \frac{(1 - q^{4n})(1 - q^{6n})^2}{(1 - q^{2n})(1 - q^{12n})} \times \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2(1 - q^{12n})(1 - q^{3n})}{(1 - q^n)(1 - q^{4n})(1 - q^{6n})} \\ &\quad \times q^{-1} \prod_{n=1}^{\infty} \frac{(1 - q^{2n})(1 - q^{3n})^2}{(1 - q^n)(1 - q^{6n})} \end{aligned}$$

$$= q^{-1} \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2 (1 - q^{3n})^3}{(1 - q^n)^2},$$

hence

$$\begin{aligned} \text{ch}_q(V(Q)) &= \prod_{n=1}^{\infty} \frac{1}{(1 - q^{3n})^2 (1 - q^{6n})} \times q^{-1} \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2 (1 - q^{3n})^3}{(1 - q^n)^2} \\ &= q^{-1} \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2 (1 - q^{3n})}{(1 - q^n)^2 (1 - q^{6n})} \\ &= q^{-1} \prod_{n=1}^{\infty} \frac{(1 - q^{6n-3})}{(1 - q^{2n-1})^2} \\ &= q^{-1} \prod_{n=1}^{\infty} \frac{1}{(1 - q^{2n-1})(1 - q^{6n-1})(1 - q^{6n-5})}. \end{aligned}$$

This proves the first assertion in Theorem 1.1.

3. Characters of certain highest weight modules of admissible weights

Let \mathfrak{g} be a Kac–Moody Lie algebra and Π , the set of the simple roots, and Δ the set of roots of \mathfrak{g} . Let $\Delta^{\vee\text{re}}$ ($\Delta_+^{\vee\text{re}}$) be the set of real (positive) coroots of \mathfrak{g} . A weight λ is called an *admissible* weight of \mathfrak{g} if it satisfies the following conditions: (1) $\langle \lambda + \rho, \alpha \rangle \notin \mathbb{Z}_+$ for all $\alpha \in \Delta_+^{\vee\text{re}}$; (2) the \mathbb{Q} -span of $\{\alpha \in R: \langle \lambda + \rho, \alpha \rangle \in \mathbb{Z}\} =$ the \mathbb{Q} -span of Π .

Let λ be an admissible weight of level $l = \frac{v}{u}$. Let

$$\Delta_{\lambda+}^{\vee\text{re}} = \{\alpha \in \Delta_+^{\vee\text{re}} \mid \langle \lambda + \rho, \alpha \rangle \in \mathbb{N}\}.$$

A coroot in $\Delta_{\lambda+}^{\vee\text{re}}$ is a simple element in $\Delta_{\lambda+}^{\vee\text{re}}$ if it cannot be written as a sum of two coroots in $\Delta_{\lambda+}^{\vee\text{re}}$. Let Π_{λ}^{\vee} be the set of all such simple elements in $\Delta_{\lambda+}^{\vee\text{re}}$. We define W_{λ} to the special subgroup of the Weyl group

$$W_{\lambda} = \langle r_{\alpha} \mid \alpha \in \Delta_{\lambda+}^{\vee\text{re}} \rangle \leq GL(\mathfrak{h}^*).$$

Clearly $W_{\lambda} = \langle r_{\alpha} \mid \alpha \in \Pi_{\lambda}^{\vee} \rangle$, and it generates the coroots associated with λ : $\Delta_{\lambda}^{\vee\text{re}} := W_{\lambda}(\Pi_{\lambda}^{\vee}) = \Delta_{\lambda+}^{\vee\text{re}} \cup (-\Delta_{\lambda+}^{\vee\text{re}})$. For example, the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ has the following admissible weights of level $\lambda = -\frac{1}{2}$ [W]:

$$-\frac{1}{2}\Lambda_0, -\frac{3}{2}\Lambda_1 + \Lambda_0$$

with $\Pi_{\lambda}^{\vee} = \{c + \alpha_0^{\vee}, \alpha_1^{\vee}\}$.

Let $l = \frac{v}{u}$ (u, v are prime) be the level of the admissible weight λ . When the type of Π_{λ}^{\vee} is of the same type as Π^{\vee} , it is known that the group W_{λ} is the same as the group

$$W_{(u)} = \overline{W} \ltimes M = \{wt_{u\alpha}: w \in \overline{W}, \alpha \in M\},$$

where the lattice $M = \overline{Q}^\vee$ for untwisted types or $A_{2n}^{(2)}$, $M = \overline{Q}$ otherwise, and t_α is the translation

$$t_\alpha(\lambda) := \lambda + (\lambda | \delta)\alpha - \left\{ \frac{1}{2}|\alpha|^2(\lambda | \delta) + (\lambda | \alpha) \right\} \delta, \quad \lambda \in \mathfrak{h}^*.$$

Kac and Wakimoto [KW] proved the following character formula for the admissible module $L(\lambda)$:

$$\text{ch}(L(\lambda)) = \frac{\sum_{w \in W_\lambda} \text{sgn}(w) e^{w(\lambda + \rho) - \rho}}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\text{mult}(\alpha)}}.$$

To prepare for our character identities we first recall the well-known quintuple product identity and then prove a variant identity for our purpose.

Proposition 3.1 (*Quintuple Product Identities*). *For arbitrary s, t we have*

$$\begin{aligned} & \prod_{n=1}^{\infty} (1 - s^n)(1 - s^n t)(1 - s^{n-1} t^{-1})(1 - s^{2n-1} t^2)(1 - s^{2n-1} t^{-2}) \\ &= \sum_{m \in \mathbb{Z}} s^{(3m^2+m)/2} (t^{3m} - t^{-3m-1}), \end{aligned} \quad (3.1)$$

$$\begin{aligned} & \prod_{n=1}^{\infty} (1 - s^n)(1 - s^{n-\frac{1}{2}} t)(1 - s^{n-\frac{1}{2}} t^{-1})(1 - s^{2n} t^2)(1 - s^{2n-2} t^{-2}) \\ &= \sum_{m \in \mathbb{Z}} s^{(3m^2+2m)/2} (t^{3m} - t^{-3m-2}). \end{aligned} \quad (3.2)$$

Proof. The first identity is well known (see e.g. [CS]). We only prove the second one. Replacing t by $x s^{-1/2}$ in (3.1), we have

$$\begin{aligned} \text{RSH (3.1)} &= \sum_{m \in \mathbb{Z}} s^{(3m^2+m)/2} (t^{3m} - t^{-3m-1}) \\ &= \sum_{m \in \mathbb{Z}} s^{(3m^2+m)/2} (x^{3m} s^{-3m/2} - x^{-3m-1} s^{(3m+1)/2}) \\ &= \sum_{m \in \mathbb{Z}} s^{(3m^2-2m)/2} x^{3m} - \sum_{m \in \mathbb{Z}} s^{(3m^2+4m+1)/2} s^{-3m-1} \\ &= \sum_{m \in \mathbb{Z}} s^{(3m^2-2m)/2} x^{3m} - \sum_{m \in \mathbb{Z}} s^{(3m^2-2m)/2} x^{-3m+2} \\ &= \sum_{m \in \mathbb{Z}} s^{(3m^2+2m)/2} (x^{-3m} - x^{3m+2}) \end{aligned}$$

where in the next last step we change m by $m - 1$ and in the last step we changed m to $-m$. Therefore the second quintuple product identity is obtained from the product side of the first quintuple product identity (3.1) by changing t by $t^{-1} s^{-1/2}$. \square

Theorem 3.1. *The characters of level $-1/2$ admissible modules of $\widehat{\mathfrak{sl}}_2$ are given by*

$$\begin{aligned} \text{ch}\left(L\left(-\frac{1}{2}\Lambda_0\right)\right) &= \frac{e^{-\frac{1}{2}\Lambda_0}}{R} \sum_{n \in \mathbb{Z}} e^{-(6n^2+2n)\delta} (e^{3n\alpha_1} - e^{-(3n+1)\alpha_1}) \\ &= \frac{e^{-\frac{\Lambda_0}{2}}}{R} \prod_{n=1}^{\infty} (1 - e^{-4n\delta}) (1 - e^{-4n\delta+\alpha_1}) (1 - e^{-(4n-4)\delta-a_1}) \\ &\quad \cdot (1 - e^{-(8n-4)\delta+2\alpha_1}) (1 - e^{-(8n-4)\delta-2\alpha_1}), \\ \text{ch}\left(L\left(-\frac{3}{2}\Lambda_0 + \Lambda_1\right)\right) &= \frac{e^{-\frac{3}{2}\Lambda_0 + \Lambda_1}}{R} \sum_{n \in \mathbb{Z}} e^{-(6n^2+4n)\delta} (e^{3n\alpha_1} - e^{-(3n+2)\alpha_1}) \\ &= \frac{e^{-\frac{3}{2}\Lambda_0 + \Lambda_1}}{R} \prod_{n=1}^{\infty} (1 - e^{-4n\delta}) (1 - e^{-(4n-2)\delta-\alpha_1}) \\ &\quad \cdot (1 - e^{-(4n-2)\delta+a_1}) (1 - e^{-8n\delta+2\alpha_1}) (1 - e^{-(8n-8)\delta-2\alpha_1}), \end{aligned}$$

where $R = \prod_{n \in \mathbb{N}} (1 - e^{-n\delta}) (1 - e^{-\alpha_1 - (n-1)\delta}) (1 - e^{\alpha_1 - n\delta})$.

Proof. The group $W_\lambda = W_{(2)}$ for the admissible weight of level $-1/2$, and $W_{(2)}$ consists of elements $t_{2n\alpha_1}, r_{\alpha_1} t_{2n\alpha_1}$ for $n \in \mathbb{Z}$. We have

$$\begin{aligned} t_{2n\alpha_1}(\Lambda_0) &= \Lambda_0 + 2n\alpha_1 - 4n^2\delta, \\ t_{2n\alpha_1}(\Lambda_1) &= \Lambda_1 + 2n\alpha_1 - (4n^2 + 2n)\delta, \\ t_{2n\alpha_1}(\rho) &= t_{2n\alpha_1}(\Lambda_0 + \Lambda_1) = \rho + 4n\alpha_1 - (8n^2 + 2n)\delta. \end{aligned}$$

It then follows that

$$\begin{aligned} \text{ch}\left(L\left(-\frac{\Lambda_0}{2}\right)\right) &= \frac{\sum_{w \in W_{(2)}} \text{sgn}(w) e^{w(-\frac{\Lambda_0}{2} + \rho) - \rho}}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\text{mult}(\alpha)}} \\ &= \frac{e^{-\frac{\Lambda_0}{2}}}{R} \sum_{n \in \mathbb{Z}} (e^{3n\alpha_1 - (6n^2+2n)\delta} - e^{-(3n+1)\alpha_1 - (6n^2+2n)\delta}) \\ &= \frac{e^{-\frac{\Lambda_0}{2}}}{R} \sum_{n \in \mathbb{Z}} e^{-(6n^2+2n)\delta} (e^{3n\alpha_1} - e^{-(3n+1)\alpha_1}). \end{aligned}$$

Similarly we have

$$\text{ch}\left(L\left(-\frac{3\Lambda_0}{2} + \Lambda_1\right)\right) = \frac{e^{-\frac{3\Lambda_0}{2} + \Lambda_1}}{R} \sum_{n \in \mathbb{Z}} e^{-(6n^2+4n)\delta} (e^{3n\alpha_1} - e^{-(3n+2)\alpha_1}).$$

Using the quintuple product identity Eq. (3.1), we have immediately that

$$\begin{aligned} \text{ch}\left(L\left(-\frac{\Lambda_0}{2}\right)\right) &= \frac{e^{-\frac{\Lambda_0}{2}}}{R} \prod_{n=1}^{\infty} (1 - e^{-4n\delta})(1 - s^{-4n\delta+\alpha_1})(1 - e^{-(4n-4)\delta-\alpha_1}) \\ &\quad \cdot (1 - e^{-(8n-4)\delta+2\alpha_1})(1 - e^{-(8n-4)\delta-2\alpha_1}). \end{aligned}$$

The product identity for $\text{ch}(L(-\frac{3\Lambda_0}{2} + \Lambda_1))$ follows from a variant of quintuple identity Eq. (3.1). \square

The following summation formula is proved in [JKM] (cf. [L]):

$$\text{ch}\left(L\left(-\frac{1}{2}\Lambda_0\right)\right) + e^{-\frac{1}{2}\delta} \text{ch}\left(L\left(-\frac{3}{2}\Lambda_0 + \Lambda_1\right)\right) = \frac{e^{-\frac{\Lambda_0}{2}}}{\prod_{n=1}^{\infty} (e^{-\frac{\delta}{2}+\frac{\alpha_1}{2}}; e^{-\delta})_{\infty} (e^{-\frac{\delta}{2}-\frac{\alpha_1}{2}}; e^{-\delta})_{\infty}}. \quad (3.3)$$

We can also obtain some interesting q -identities.

Remark 3.2. This result covers that in [L].

Theorem 3.3.

$$\begin{aligned} \frac{1}{\phi(q^2)} \sum_{m \in \mathbb{Z}} q^{36m^2} (1 - q^{24m+4} + q^{12m+1} - q^{36m+9}) &= \frac{1}{(q; q^6)_{\infty} (q^5; q^6)_{\infty}}, \\ \frac{1}{(q; q^6)_{\infty} (q^5; q^6)_{\infty}} &= \frac{(q^{24}; q^{24})_{\infty} (q^4; q^{24})_{\infty} (q^{20}; q^{24})_{\infty} (q^{16}; q^{48})_{\infty} (q^{32}; q^{48})_{\infty}}{(q^2; q^2)_{\infty}} \\ &\quad + \frac{q(q^{24}; q^{24})_{\infty} (q^8; q^{24})_{\infty} (q^{16}; q^{24})_{\infty} (q^8; q^{48})_{\infty} (q^{40}; q^{48})_{\infty}}{(q^2; q^2)_{\infty}}. \quad (3.4) \end{aligned}$$

Proof. The first identity is obtained by specializing $e^{-\delta} = q^6$, $e^{-\alpha_1} = q^4$ in Eq. (3.3), and the second part is obtained from the product identity for the character

$$\begin{aligned} &\frac{1}{(q; q^6)_{\infty} (q^5; q^6)_{\infty}} \\ &= \frac{1}{\phi(q^2)} \sum_{m \in \mathbb{Z}} q^{36m^2} (1 - q^{24m+4}) + \frac{q}{\phi(q^2)} \sum_{m \in \mathbb{Z}} q^{36m^2+12m} (1 - q^{24m+8}) \\ &= \frac{\prod_{n=1}^{\infty} (1 - q^{24n})(1 - q^{24n-4})(1 - q^{24n-20})(1 - q^{48n-16})(1 - q^{48n-32})}{\phi(q^2)} \\ &\quad + \frac{q \prod_{n=1}^{\infty} (1 - q^{24n})(1 - q^{24n-8})(1 - q^{24n-16})(1 - q^{48n-40})(1 - q^{48n-8})}{\phi(q^2)}. \end{aligned}$$

So we have got the second assertion in Theorem 1.1. \square

Proof of Theorem 1.2. The combinatorial interpretation of Rogers–Ramanujan identities says that

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-4})(1-q^{5n-1})} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q, q)_n} = 1 + \sum_{n=1}^{\infty} R(n, 1)q^n,$$

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-3})(1-q^{5n-2})} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q, q)_n} = 1 + \sum_{n=1}^{\infty} R(n, 2)q^n$$

where $R(n, 1)$ is the number of partitions of n into $k_1 + \cdots + k_j$ with constraints $k_i - k_{i+1} \geq 2$, $R(n, 2)$ is the number of partitions of n into $k_1 + \cdots + k_j$ with conditions $k_i - k_{i+1} \geq 2$ and $k_i > 1$. Now the left-hand side of the first identity in Theorem 1.1

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{6n-1})(1-q^{6n-5})} = 1 + \sum_{n=1}^{\infty} A(n)q^n,$$

which counts partitions with parts $6m + 1$ or $6m + 5$. The two summands in the right-hand side can be viewed as product of Rogers–Ramanujan type by changing variables:

$$q \prod_{n=1}^{\infty} \frac{1}{(1-q^{2(5n-2)})(1-q^{2(5n-3)})} \prod_{n=1}^{\infty} \frac{1}{(1-q^{3(5n-2)})(1-q^{3(5n-3)})}$$

and

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{2(5n-1)})(1-q^{2(5n-4)})} \prod_{n=1}^{\infty} \frac{1}{(1-q^{3(5n-1)})(1-q^{3(5n-4)})}.$$

It follows from the Rogers–Ramanujan identities that the partitions are counted exactly by $B(n)$ and $C(n)$, respectively. Therefore we have shown that $A(n) = B(n) + C(n)$ holds for any n .

Next we notice that the numerator in the first summand in the second expression of Theorem 1.1 can be rewritten as:

$$\prod_{n=1}^{\infty} (1-q^{24n})(1-q^{24n-4})(1-q^{24n-20})$$

$$\times (1-q^{24n-8})(1+q^{24n-8})(1-q^{24n-16})(1+q^{24n-16}).$$

Cancellation gives

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{24n-2})(1-q^{24n-6})(1-q^{24n-10})(1-q^{24n-14})(1-q^{24n-18})}$$

$$\times \prod_{n=1}^{\infty} \frac{(1+q^{24n-8})(1+q^{24n-16})}{(1-q^{24n-22})} \prod_{n=1}^{\infty} \frac{1}{(1-q^{24n-12})}.$$

This may be rewritten as

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{2(2n-1)})} \prod_{n=1}^{\infty} (1+q^{8(3n-1)})(1+q^{8(3n-2)}) \prod_{n=1}^{\infty} \frac{1}{(1-q^{12(2n-1)})}.$$

The first factor is the generating function for partitions of an integer into not necessarily distinct odd multiples of 2, the second is the generating function for partitions of an integer into distinct summands of the form $8r$ where r is congruent to 1 or 2 modulo 3. The last factor is the generating function for partitions of an integer into not necessarily distinct odd multiples of 12. This gives the characterization for $D(n)$ asserted above.

The second summand in the second expression of Theorem 1.1 can be similarly rewritten as:

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{2(2n-1)})} \prod_{n=1}^{\infty} (1+q^{4(6n-1)})(1+q^{4(6n-5)}) \prod_{n=1}^{\infty} \frac{1}{(1-q^{12(2n-1)})}.$$

The first factor is the generating function for partitions of an integer into not necessarily distinct odd multiples of 2, the second is the generating function for partitions of an integer into distinct summands of the form $4r$ where r is congruent to 1 or 5 modulo 6. The last factor is the generating function for partitions of an integer into not necessarily distinct odd multiples of 12. This gives the characterization for $E(n)$ asserted above, and the proof is completed. \square

4. More results

Theorem 4.1.

$$\prod_{n=1}^{\infty} \frac{1}{(1+q^{6n-1})(1+q^{6n-5})} = \sum_{n \geq 0} (-1)^n A(n) q^n.$$

Proof. It is clear since for any n , we have

$$1+q^{6n-1} = 1 - (-q)^{6n-1}, \quad 1+q^{6n-5} = 1 - (-q)^{6n-5}. \quad \square$$

Notice that, $A(n) = D(n)$ for even number n and $A(n) = E(n)$ when n is odd. Then we have

Theorem 4.2.

$$\begin{aligned} & 2 \prod_{n=1}^{\infty} \frac{(1-q^{24n})(1-q^{24n-4})(1-q^{24n-20})(1-q^{48n-16})(1-q^{48n-32})}{(1-q^{2n})} \\ &= \prod_{n=1}^{\infty} \frac{1}{(1+q^{6n-1})(1+q^{6n-5})} + \prod_{n=1}^{\infty} \frac{1}{(1-q^{6n-1})(1-q^{6n-5})}, \\ & 2q \prod_{n=1}^{\infty} \frac{(1-q^{24n})(1-q^{24n-8})(1-q^{24n-16})(1-q^{48n-40})(1-q^{48n-8})}{(1-q^{2n})} \\ &= \prod_{n=1}^{\infty} \frac{1}{(1-q^{6n-1})(1-q^{6n-5})} - \prod_{n=1}^{\infty} \frac{1}{(1+q^{6n-1})(1+q^{6n-5})}. \end{aligned}$$

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