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The j-invariant of a plane tropical cubic

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ABSTRACT

In this paper, we prove that for a plane cubic over the field of Puiseux series, the negative of the generic valuation of the j-invariant is equal to the cycle length of the tropicalization of the curve, if there is a cycle at all.

1. Introduction

The j-invariant is an invariant which coincides for two smooth elliptic curves over an algebraically closed field if and only if they are isomorphic. In [1], *equivalences* between abstract tropical curves are defined, and two elliptic abstract tropical curves are equivalent if and only if they have the same cycle length. Thus the cycle length seems to plays the same role in the tropical setting as the j-invariant does in the algebraic setting.

The aim of this paper is to show that for a plane cubic the j-invariant really tropicalizes to the negative of the cycle length.

More precisely, we define plane cubic curves over the field $\mathbb{K}$ of Puiseux series and use the valuation map to tropicalize them. The j-invariant of an elliptic curve over the Puiseux series is a Puiseux series itself. Our main result, Theorem 11 is that if the tropicalization of a smooth cubic curve in $\mathbb{P}^2_{\mathbb{K}}$ has a cycle then the negative of the cycle length is always equal to the generic valuation of the

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j-invariant, and it is equal to the valuation of the j-invariant itself if no terms in the j-invariant cancel—which generically is the case. A corollary of this theorem is that if an elliptic curve has a j-invariant with a non-negative valuation, then its tropicalization does not have a cycle.

After the completion of our work, David Speyer [2, Proposition 9.2] proved a similar result for not necessarily plane genus one tropical curves, which implies our result in the case that the dual subdivision of the Newton polygon is a unimodular triangulation. His proof uses the Tate uniformization of elliptic curves while our approach is combinatorial.

This paper is organized as follows. In Section 2 we recall the definition of the j-invariant of a plane cubic as a rational function in the cubic's coefficients. Its denominator is the discriminant of the cubic. Moreover, we observe that the generic valuation (see Definition 1) of the j-invariant is a piece-wise linear function. In Section 3 we recall basic definitions of tropical geometry and show that the function *cycle length* is piece-wise linear as well. The main theorem is stated in Section 4. As we know already that the two functions we compare are piece-wise linear the proof consists of two main steps: first we compare certain domains of linearity, then we compare the two linear functions on each domain. As domains of linearity we choose cones of the Gröbner fan of the discriminant. The comparison of the two linear functions, *generic valuation of the j-invariant* and *cycle length* on each such cone is done in Section 5. In Section 6 we study the numerator of the j-invariant, which is important to understand the domains of linearity of the *generic valuation of the j-invariant*.

The tropical curves and their Newton subdivisions were partly produced using the procedure `drawTropicalCurve` from the Singular library `tropical.lib` (see [3]) which can be obtained via the URL

http://www.mathematik.uni-kl.de/~keilen/en/tropical.html.

This library contains also a procedure `tropicalJInvariant` which computes the cycle length of a tropical curve as defined in Definition 6. Parts of our proofs and many examples rely on computations performed using `polymake` [4], TOPCOM [5] and Singular [6]. The Singular code that we used for this is contained in the library `jinvariant.lib` (see [7]) and it is available via the URL

http://www.mathematik.uni-kl.de/~keilen/en/jinvariant.html.

More detailed explanations on how to use the code can be found there.

2. The j-invariant and its valuation

In this paper we study plane cubics given by an equation of the form

$$f = \sum_{0 \leqslant i+j \leqslant 3} a_{ij} x^i y^j = 0$$

over the field of Puiseux series

$$\mathbb{K} = \bigcup_{N=1}^{\infty} \mathrm{Quot}\big(\mathbb{C}\,[\![t^{\frac{1}{N}}]\!]\big) = \left\{ \sum_{\nu=m}^{\infty} c_\nu \cdot t^{\frac{\nu}{N}} \;\middle|\; c_\nu \in \mathbb{C},\ N \in \mathbb{Z}_{>0},\ m \in \mathbb{Z} \right\}.$$

The Newton polygon of a general cubic is the triangle Q_c with vertices $(0,0)$, $(0,3)$ and $(3,0)$, and we denote its lattice points by $\mathcal{A}_c := Q_c \cap \mathbb{Z}^2$ (see Fig. 1). In that way we can write the equation as $f = \sum_{(i,j)\in\mathcal{A}_c} a_{ij} x^i y^j$.

We are only interested in the solutions of $f = 0$ in the torus $(\mathbb{K}^*)^2$, but this already determines its closure, say C_f, in the projective plane. Moreover, we are only interested in the case where C_f is smooth, i.e. is an elliptic curve. In this situation the isomorphism class of C_f is determined by a single invariant, the *j-invariant* of the elliptic curve C_f.

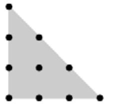


Fig. 1. Q_c and $\mathcal{A}_c$.

The j*-invariant* can be computed from the defining polynomial f as a rational function

$$j(f) = \frac{A}{\Delta}$$

in the coefficients a_{ij} of f where $A, \Delta \in \mathbb{Q}[\underline{a}]$ are homogeneous polynomials of degree 12. We here use the convention $\underline{a} = (a_{ij} \mid (i,j) \in \mathcal{A}_c)$ and if $\omega \in \mathbb{N}^{\mathcal{A}_c}$ is a multi index then $\underline{a}^\omega = \prod_{(i,j)\in\mathcal{A}_c} a_{ij}^{\omega_{ij}}$. The denominator Δ is the *discriminant* of f (see [8]).

The field $\mathbb{K}$ of Puiseux series comes with a *valuation*,

$$\mathrm{val} : \mathbb{K}^* \to \mathbb{Q} : \sum_{\nu=m}^{\infty} c_\nu \cdot t^{\frac{\nu}{N}} \mapsto \min\left\{ \frac{\nu}{N} \;\middle|\; c_\nu \neq 0 \right\},$$

and we may extend the valuation to $\mathbb{K}$ by $\mathrm{val}(0) = \infty$. We sometimes call $\mathrm{val}(k)$ the *tropicalization* of k.

The j-invariant of C_f is a Puiseux series and, unless some unfortunate cancellations occur, its valuation can be read from the polynomials A and Δ. To do this, we introduce the notion of *generic valuation* of the j-invariant.

Definition 1. The generic valuation of a polynomial $0 \neq H = \sum_\omega H_\omega \underline{a}^\omega \in \mathbb{Q}[\underline{a}]$ is the function

$$\mathrm{val}_H : \mathbb{R}^{\mathcal{A}_c} \longrightarrow \mathbb{R} : u \mapsto \mathrm{val}_H(u) = \min\{u \cdot \omega \mid H_\omega \neq 0\},$$

where

$$u \cdot \omega = \sum_{(i,j)\in\mathcal{A}_c} u_{ij} \cdot \omega_{ij}$$

is the standard scalar product of u and ω. The *generic valuation* of the j-invariant is

$$\mathrm{val}_j : \mathbb{R}^{\mathcal{A}_c} \longrightarrow \mathbb{R} : u \mapsto \mathrm{val}_j(u) = \mathrm{val}_A(u) - \mathrm{val}_\Delta(u).$$

Note that the tropical j-invariant is a *tropical rational function* in the sense of [1, Section 2.2] and [9, Definition 3.1].

The following remark is obvious from the definitions, since for u in a top-dimensional cone of the Gröbner fan of $H \in \mathbb{Q}[\underline{a}]$ the minimum in the definition of $\mathrm{val}_H(u)$ is attained by only one fixed term.

Remark 2. The generic valuation of $H \in \mathbb{Q}[\underline{a}]$ is piece-wise linear, and it is linear on a top-dimensional cone of the Gröbner fan of H. Moreover, if $u \in \mathbb{R}^{\mathcal{A}_c}$ is in the interior of a top-dimensional cone of the Gröbner fan of H, then $\mathrm{val}_H(u) = \mathrm{val}(H(b))$ for any $b \in (\mathbb{K}^*)^{\mathcal{A}_c}$ with $\mathrm{val}(b) = u$.

From this it follows that

$$\mathrm{val}_j : \mathbb{R}^{\mathcal{A}_c} \longrightarrow \mathbb{R} : u \mapsto \mathrm{val}_j(u)$$

is linear on intersections $D \cap D'$ of a top-dimensional cone D of the Gröbner fan of the numerator polynomial A and a top-dimensional cone D' of the Gröbner fan of Δ. For u in the open interior of $D \cap D'$, $\mathrm{val}_j(u) = \mathrm{val}(j(f))$ for any $f = \sum_{(i,j)\in\mathcal{A}_c} a_{ij} x^i y^j$ with $\mathrm{val}(a_{ij}) = u_{ij}$.

3. Tropicalizations and the cycle length of a plane tropical cubic

In this section we will study the *tropicalization* of a plane cubic C_f, by which we mean

$$\mathrm{Trop}(C_f) = \overline{\mathrm{val}\big(C_f \cap (\mathbb{K}^*)^2\big)} \subseteq \mathbb{R}^2,$$

i.e. the closure of $\mathrm{val}(C_f \cap (\mathbb{K}^*)^2)$ with respect to the Euclidean topology in $\mathbb{R}^2$. By abuse of notation

$$\mathrm{val} : (\mathbb{K}^*)^2 \longrightarrow \mathbb{Q}^2 : (k_1, k_2) \mapsto \big(\mathrm{val}(k_1), \mathrm{val}(k_2)\big)$$

denotes here the Cartesian product of the valuation map from Section 2.

This definition is not too helpful when it comes down to computing tropical curves. There it is better to consider the *tropical polynomial* associated to f, that is, the piece-wise linear function

$$\mathrm{trop}(f) : \mathbb{R}^2 \longrightarrow \mathbb{R} : (x, y) \mapsto \min\big\{\mathrm{val}(a_{ij}) + i \cdot x + j \cdot y \;\big|\; (i, j) \in \mathcal{A}_c\big\}.$$

Given any *plane cubic tropical polynomial*

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R} : (x, y) \mapsto \min\big\{u_{ij} + i \cdot x + j \cdot y \;\big|\; (i, j) \in \mathcal{A}_c\big\}$$

with $u_{ij} \in \mathbb{R}$ we call the locus of non-differentiability of this piece-wise linear function a *plane cubic tropical curve*, or a *plane tropical cubic* for short.

Sometimes it will be convenient to allow some of the u_{ij} to be ∞, or equivalently the corresponding a_{ij} are allowed to be zero. As long as the point $(1, 1)$ lies in the interior of the convex hull of the (i, j)'s for which u_{ij} is finite, everything makes perfect sense. This allows us to replace Q_c by some subpolygon that has a single interior lattice point at $(1, 1)$.

By *Kapranov's Theorem* (see [10, Theorem 2.1.1]), $\mathrm{Trop}(C_f)$ coincides with the plane tropical cubic defined by $\mathrm{trop}(f)$. In particular, $\mathrm{Trop}(C_f)$ is a piece-wise linear graph.

An important fact is that this graph is *dual* to a subdivision of the *marked Newton polygon* $(Q_c, \mathcal{A}_c)$. For the precise definition of the notions in their full generality and the proofs of the main statements we refer the reader to [8, Chapter 7]. Here we summarize what we need for our special situation.

A *marked polygon* is a 2-dimensional convex lattice polygon Q in $\mathbb{R}^2$ together with a subset $\mathcal{A}$ of the lattice points $Q \cap \mathbb{Z}^2$ containing the vertices of Q. The Newton polygon $(Q_c, \mathcal{A}_c)$ as shown in Fig. 1 is a marked polygon.

A *marked subdivision* of a marked polygon $(Q, \mathcal{A})$ is a collection of marked polygons, $T = \{(Q_1, \mathcal{A}_1), \ldots, (Q_k, \mathcal{A}_k)\}$, such that

- $Q = \bigcup_{i=1}^{k} Q_i$,
- $Q_i \cap Q_j$ is a face (possibly empty) of Q_i and of Q_j for all $i, j = 1, \ldots, k$,
- $\mathcal{A}_i \subset \mathcal{A}$ for $i = 1, \ldots, k$, and
- $\mathcal{A}_i \cap (Q_i \cap Q_j) = \mathcal{A}_j \cap (Q_i \cap Q_j)$ for all $i, j = 1, \ldots, k$.

We do not mandate that $\bigcup_{i=1}^{k} \mathcal{A}_i = \mathcal{A}$. Example 3 shows an example of a marked subdivision of $(Q_c, \mathcal{A}_c)$.

A point in $u \in \mathbb{R}^{\mathcal{A}_c}$ induces a *marked subdivision* of $(Q_c, \mathcal{A}_c)$ by considering the convex hull of

$$\big\{(i, j, u_{ij}) \;\big|\; (i, j) \in \mathcal{A}_c\big\} \subset \mathbb{R}^3$$

in $\mathbb{R}^3$, and projecting the lower faces onto the xy-plane. A lattice point (i, j) will be marked if the point (i, j, u_{ij}) was contained in one of the lower faces. Marked subdivisions of $(Q_c, \mathcal{A}_c)$ obtained in this way are called *regular* or *coherent*. We say two points u and u' in $\mathbb{R}^{\mathcal{A}_c}$ are equivalent if and only if they induce the same regular subdivision of $(Q_c, \mathcal{A}_c)$. This defines an equivalence relation whose equivalence classes are the interiors of cones. The collection of these cones is the *secondary fan* of $\mathcal{A}_c$.

The secondary fan of $\mathcal{A}_c$ is an important object because we will see that the *cycle length* is a function that is linear on each of its top-dimensional cones. Since we have already seen that the valuation of the j-invariant is linear on each cone of the common refinement of the Gröbner fans of A and Δ, our strategy will be to compare the secondary fan with these two Gröbner fans.

Marked subdivisions of $(Q_c, \mathcal{A}_c)$ are dual to plane tropical cubics. Given a point $u \in \mathbb{R}^{\mathcal{A}_c}$ it defines a plane tropical cubic, say $\mathcal{C}_F$, via the plane tropical polynomial

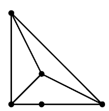
$$F = \min\{u_{ij} + i \cdot x + j \cdot y \mid (i, j) \in \mathcal{A}_c\}$$

and it defines a regular subdivision of $(Q_c, \mathcal{A}_c)$. Each marked polygon of the subdivision is dual to a vertex of $\mathcal{C}_F$, and each edge of a marked polygon is dual to an edge of $\mathcal{C}_F$. Moreover, if the edge, say e, has end points (x_1, y_1) and (x_2, y_2) then the *direction vector* $v(E)$ of the dual edge E in $\mathcal{C}_F$ is defined (up to sign) as

$$v(E) = (y_2 - y_1, x_1 - x_2)$$

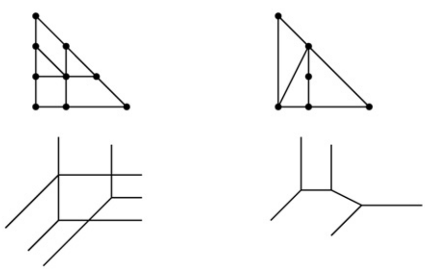
and points in the direction of E. In particular, the edge E is orthogonal to its dual edge e. Finally, the edge E is unbounded if and only if its dual edge e is contained in an edge of Q_c.

Example 3. The marked subdivision below is induced by the plane tropical polynomial $F = \min\{3x, 3y, 0, x, -1 + x + y\}$.



Definition 4. We say that a plane tropical cubic $\mathcal{C}$ *has a cycle* if the interior point $(1, 1)$ is visible as a vertex of a marked polygon in its dual marked subdivision. If this is the case, the *cycle* of $\mathcal{C}$ is the union of those bounded edges of $\mathcal{C}$ which are dual to the edges of marked polygons in the marked subdivision which emanate from $(1, 1)$, and we say that these edges *form the cycle*.

Example 5. In the picture below, the left plane tropical cubic has a cycle while the right one does not, since $(1, 1)$ is visible but it is not a vertex of one of the marked polygons in the subdivision.



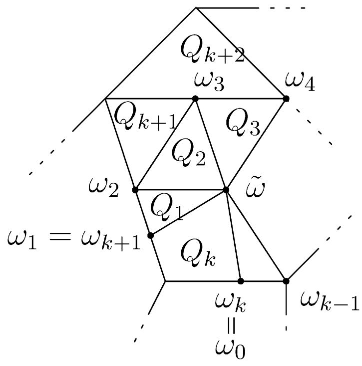


Fig. 2. Marked subdivision determining a cycle.

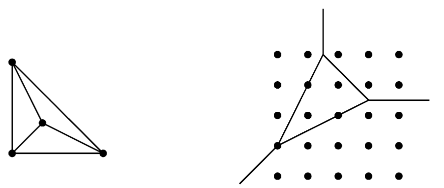
Definition 6. For a bounded edge E of a plane tropical curve with direction vector $v(E)$, we define the *lattice length* $l(E) = \frac{\|E\|}{\|v(E)\|}$ to be the Euclidean length of E divided by the Euclidean length of $v(E)$.

For a plane tropical cubic with cycle, we define its *cycle length* to be the sum of the lattice lengths of the edges which form the cycle. If the plane tropical cubic has no cycle we say its length is zero. This defines a *cycle length* function,

$$\mathrm{cl} : \mathbb{R}^{\mathcal{A}_c} \to \mathbb{R} : u = (u_\omega \mid \omega \in \mathcal{A}_c) \mapsto \mathrm{cl}(u) = \text{“cycle length of } \mathcal{C}_F\text{”}$$

associating to every plane cubic tropical polynomial $F = \min\{u_{ij} + i \cdot x + j \cdot y \mid (i, j) \in \mathcal{A}_c\}$ the cycle length of the corresponding plane tropical cubic $\mathcal{C}_F$.

Example 7. The following picture shows a plane tropical cubic with cycle length $\frac{9}{2}$, since each of the edges of the cycle has lattice length $\frac{3}{2}$.



One can define *plane tropical curves* other than cubics by considering other finite subsets $\mathcal{A} \subset \mathbb{N}^2$ as support. Let Q be the convex hull of $\mathcal{A}$. The duality above works with $(Q_c, \mathcal{A}_c)$ replaced by $(Q, \mathcal{A})$. In this manner we can also generalize Definition 4 to plane tropical curves other than cubics.

Definition 8. Let $\mathcal{C}$ be a plane tropical curve with Newton polygon Q and with dual marked subdivision $\{(Q_i, \mathcal{A}_i) \mid i = 1, \ldots, l\}$. Suppose that $\tilde{\omega} \in \mathrm{Int}(Q) \cap \mathbb{Z}^2$ and that the $(Q_i, \mathcal{A}_i)$ are ordered such that $\tilde{\omega}$ is a vertex of Q_i for $i = 1, \ldots, k$ and it is not contained in Q_i for $i = k+1, \ldots, l$ (see Fig. 2). We then say that $\tilde{\omega}$ *determines a cycle* of $\mathcal{C}$, namely the union of the edges of $\mathcal{C}$ dual to the edges emanating from $\tilde{\omega}$, and we say that these edges *form the cycle* determined by $\tilde{\omega}$. The length of this cycle is defined as in Definition 6.

Lemma 9. *Let $(Q, \mathcal{A})$ be a marked polygon in $\mathbb{R}^2$ with a regular marked subdivision $\{(Q_i, \mathcal{A}_i) \mid i = 1, \ldots, l\}$ and suppose that $\tilde{\omega} \in \mathrm{Int}(Q) \cap \mathbb{Z}^2$ is a vertex of Q_i for $i = 1, \ldots, k$ and it is not contained in Q_i for*

$i = k+1, \ldots, l$. If $u \in \mathbb{R}^{\mathcal{A}}$ is in the cone of the secondary fan corresponding to this subdivision, then $\tilde{\omega}$ determines a cycle in the plane tropical curve $\mathcal{C}$ given by the tropical polynomial

$$\min\{u_{ij} + i \cdot x + j \cdot y \mid (i,j) \in \mathcal{A}\}$$

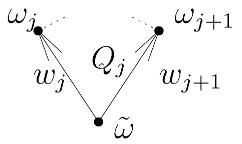
and, using the notation in Fig. 2, its length is

$$\sum_{j=1}^{k} (u_{\tilde{\omega}} - u_{\omega_j}) \cdot \frac{D_{j-1,j} + D_{j,j+1} + D_{j+1,j-1}}{D_{j-1,j} \cdot D_{j,j+1}}$$

where $D_{i,j} = \det(w_i, w_j)$ with $w_i = \omega_i - \tilde{\omega}$ and $w_j = \omega_j - \tilde{\omega}$.

Proof. By definition $\tilde{\omega}$ determines a cycle. It remains to prove the statement on its length.

For this we consider the convex polygon Q_j having ω_{j+1}, $\tilde{\omega}$ and ω_j as neighboring vertices:



The vertex $v_j = (v_{j,1}, v_{j,2})$ of $\mathcal{C}$ dual to Q_j is given by the system of linear equations

$$\omega_j \cdot v_j + u_j = \omega_{j+1} \cdot v_j + u_{j+1} = \tilde{\omega} \cdot v_j + u,$$

where $u_j = u_{\omega_j}$, $u_{j+1} = u_{\omega_{j+1}}$ and $u = u_{\tilde{\omega}}$. This system can be rewritten as

$$\begin{pmatrix} w_j^t \\ w_{j+1}^t \end{pmatrix} \cdot v_j = \begin{pmatrix} u - u_j \\ u - u_{j+1} \end{pmatrix}.$$

Since ω_{j+1}, $\tilde{\omega}$ and ω_j are neighboring vertices of the polygon Q_j the vectors w_j and w_{j+1} are linearly independent and we may apply Cramer's Rule to find

$$v_{j,1} = \frac{\det\begin{pmatrix} u - u_j & w_{j,2} \\ u - u_{j+1} & w_{j+1,2} \end{pmatrix}}{D_{j,j+1}} \quad \text{and} \quad v_{j,2} = \frac{\det\begin{pmatrix} w_{j,1} & u - u_j \\ w_{j+1,1} & u - u_{j+1} \end{pmatrix}}{D_{j,j+1}}. \tag{1}$$

The lattice length of the edge from v_{j-1} to v_j is the real number $\lambda_j \in \mathbb{R}$ such that $(v_j - v_{j-1}) = \lambda_j \cdot w_j^{\perp}$, where $w_j^{\perp} = (-w_{j,2}, w_{j,1})$ is perpendicular to w_j. Thus

$$\lambda_j = \frac{(v_j - v_{j-1}) \cdot w_j^{\perp}}{w_j^{\perp} \cdot w_j^{\perp}} = \frac{(v_j - v_{j-1}) \cdot w_j^{\perp}}{w_j \cdot w_j}. \tag{2}$$

In order to understand the right-hand side of this equation better we need the following observation. The last row of the matrix

$$M = \begin{pmatrix} w_{j-1,1} & w_{j,1} & w_{j+1,1} \\ w_{j-1,2} & w_{j,2} & w_{j+1,2} \\ w_{j-1} \cdot w_j & w_j \cdot w_j & w_{j+1} \cdot w_j \end{pmatrix}$$

is a linear combination of the first two, and thus the determinant of M is zero. Expanding the determinant along the last row we get

$$0 = \det(M) = w_{j-1} \cdot w_j \cdot D_{j,j+1} - w_j \cdot w_j \cdot D_{j-1,j+1} + w_{j+1} \cdot w_j \cdot D_{j-1,j},$$

or equivalently

$$\frac{D_{j+1,j-1}}{D_{j-1,j} \cdot D_{j,j+1}} = -\frac{D_{j-1,j+1}}{D_{j-1,j} \cdot D_{j,j+1}} = -\frac{w_{j-1} \cdot w_j}{w_j \cdot w_j \cdot D_{j-1,j}} - \frac{w_{j+1} \cdot w_j}{w_j \cdot w_j \cdot D_{j,j+1}}.$$

Expanding the right-hand side of (2) using (1) and plugging in this last equality we get

$$\begin{aligned}\lambda_j &= \frac{u - u_{j-1}}{D_{j-1,j}} + \frac{u - u_{j+1}}{D_{j,j+1}} - (u - u_j) \cdot \left(\frac{w_j \cdot w_{j+1}}{w_j \cdot w_j \cdot D_{j,j+1}} + \frac{w_{j-1} \cdot w_j}{w_j \cdot w_j \cdot D_{j-1,j}}\right) \\ &= \frac{u - u_{j-1}}{D_{j-1,j}} + \frac{u - u_{j+1}}{D_{j,j+1}} + \frac{(u - u_j) \cdot D_{j+1,j-1}}{D_{j-1,j} \cdot D_{j,j+1}}.\end{aligned}$$

The lattice length of the cycle of $\mathcal{C}$ is then given by adding the λ_j, i.e. it is

$$\begin{aligned}\lambda_1 + \cdots + \lambda_k &= \sum_{j=1}^{k} \frac{u - u_{j-1}}{D_{j-1,j}} + \frac{u - u_{j+1}}{D_{j,j+1}} + \frac{(u - u_j) \cdot D_{j+1,j-1}}{D_{j-1,j} \cdot D_{j,j+1}} \\ &= \sum_{j=1}^{k} (u - u_j) \cdot \left(\frac{D_{j-1,j} + D_{j,j+1} + D_{j+1,j-1}}{D_{j-1,j} \cdot D_{j,j+1}}\right). \qquad \square\end{aligned}$$

Remark 10. An immediate consequence of Lemma 9 is that the function *cycle length*, cl, from Definition 6 is linear on each cone of the secondary fan of $\mathcal{A}_c$.

4. The main theorem

Theorem 11. *Let $\mathcal{C}_F$ be a plane tropical cubic given by the tropical polynomial*

$$F = \min_{(i,j) \in \mathcal{A}_c} \{u_{ij} + ix + jy\}$$

and assume that $\mathcal{C}_F$ has a cycle. Then the negative of the generic valuation of the j-invariant at $u = (u_{ij} \mid (i,j) \in \mathcal{A}_c)$ is equal to the cycle length of $\mathcal{C}$, i.e.

$$-\mathrm{val}_j(u) = \mathrm{cl}(u).$$

Furthermore, if the marked subdivision dual to $\mathcal{C}$ corresponds to a top-dimensional cone of the secondary fan of $\mathcal{A}_c$ (that is, if it is a triangulation), then

$$\mathrm{val}_j(u) = \mathrm{val}\big(j(f)\big)$$

for any $f = \sum_{(i,j) \in \mathcal{A}_c} a_{ij} x^i y^j$ with coefficients $a_{ij} \in \mathbb{K}$ satisfying $\mathrm{val}(a_{ij}) = u_{ij}$.

There are two main parts of the proof: the first part is to compare certain *domains of linearity* in $\mathbb{R}^{\mathcal{A}_c}$ of the two piece-wise linear functions *cycle length*, cl, and *generic valuation of j*, val_j, and the second part is to compare the two linear functions on each domain.

The proof relies on the results of and the notions introduced in the following two sections.

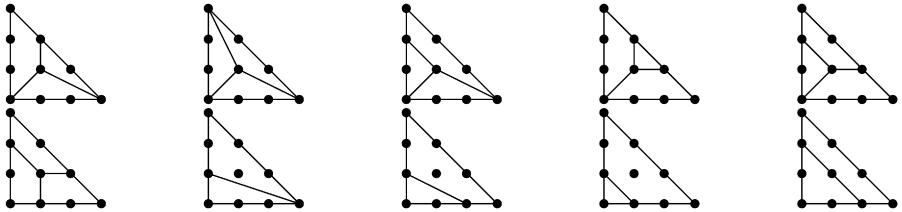


Fig. 3. Classification of the rays of the secondary fan of $\mathcal{A}_c$.

Proof. Note that our claim only involves curves C which have a cycle or, equivalently, where the point $(1,1)$ is a vertex in the dual subdivision of the marked polygon. Therefore we may replace $\mathbb{R}^{\mathcal{A}_c}$ as domain of definition of cl and val_j by the union U of those cones of the secondary fan of $\mathcal{A}_c$ where the corresponding marked subdivision contains $(1,1)$ as a vertex of a marked polygon. The coordinates on U are given by $u = (u_{ij} \mid (i,j) \in \mathcal{A}_c)$ and the canonical basis vector $e_{kl} = (\delta_{ik} \cdot \delta_{jl} \mid (i,j) \in \mathcal{A}_c)$ has a one in position kl and zeros elsewhere.

Lemma 23 below shows that U is contained in a single cone of the Gröbner fan of the numerator A of the j-invariant. That vertex is the one dual to the vertex $12 \cdot e_{11}$ of the Newton polytope of A. Hence the generic valuation of A is linear on U. In fact, it is

$$\mathrm{val}_A : U \to \mathbb{R} : u \mapsto 12 \cdot u_{11}.$$

Thus, if we want to divide U into cones on which val_j is linear, it suffices to consider val_Δ, and we know already that the latter is linear on cones of the Gröbner fan of Δ by Remark 2. Thus so is val_j restricted to U, and by Lemma 16 and Remark 15 the function cl is so as well. Moreover, by definition U is a union of cones of the Gröbner fan of Δ, and each such cone is a union of certain Δ-equivalent cones (see Remark 15) of the secondary fan of $\mathcal{A}_c$.

Hence to prove that the two functions val_j and cl coincide it is enough to compare the linear functions on each cone of the Gröbner fan of Δ contained in U. To do this, we use Theorem 11.3.2 of [8] which enables us to compute the linear function val_Δ on each such cone, D, given a (top-dimensional) marked subdivision T whose corresponding cone in the secondary fan of $\mathcal{A}_c$ is contained in D. In fact, it provides us with a formula to compute the coefficient of u_{ij} for each $(i,j) \in \mathcal{A}_c$. Since we already know that the two functions val_Δ and cl are linear on D, we can for our comparison assume that T is the representative of its class with as few edges as possible. The coefficient of u_{ij} in the linear function cl for the marked subdivision is given by Lemma 9. To compare the two coefficients, there are some cases to distinguish, which is done by Lemma 19. This proves the first part of the theorem.

Finally, for any point u in the interior of a cone of the Gröbner fan of Δ, $\mathrm{val}_j(u) = \mathrm{val}(j(f))$ for any polynomial $f = \sum_{(i,j)\in\mathcal{A}_c} a_{ij}x^i y^j$ with $\mathrm{val}(a_{ij}) = u_{ij}$ by Remark 2. As a point u in the interior of a top-dimensional cone of the secondary fan of $\mathcal{A}_c$ is in the interior of a cone of the Gröbner fan of Δ, the last statement follows as well. □

Remark 12. For our proof we use the cones of the Gröbner fan of Δ as common domains of linearity of cl and val_j. Instead we could have used top-dimensional cones of the secondary fan of $\mathcal{A}_c$. Classifying up to Sym^3-symmetry the rays of $\mathcal{A}_c$ (see Fig. 3) and comparing a generalized cycle length on these rays one gets an alternative proof of Theorem 11.

Corollary 13. *Let $f = \sum_{(i,j)\in\mathcal{A}_c} a_{ij}x^i y^j$ define a smooth elliptic curve over $\mathbb{K}$ such that the valuation of its j-invariant is non-negative. Then its tropicalization does not have a cycle.*

Remark 14. It is obvious that the tropicalization of an elliptic curve over $\mathbb{K}$ depends on the embedding into the projective plane that one chooses. An elliptic curve in reduced Weierstrass form

$$y^2 - x^3 + ax + b = 0$$

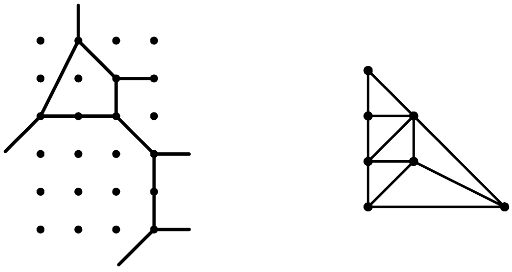


Fig. 4. The tropicalization of C_f and its dual Newton subdivision.

has no xy-term, and thus there is no cycle which could reflect the j-invariant. One might think that the Weierstrass-form is just a bad choice of normal form, and that *generically* things work out better. More precisely, one might expect the following: given a family of embeddings of an elliptic curve, the tropicalization of the *general* member will exhibit a cycle with the right cycle length. This, however, is yet again false as one can verify by the example (see Fig. 4)

$$f = xy + t \cdot \left(1 + y + xy^2 + x^3\right) + t^3 \cdot y^2 + t^7 \cdot y^3 + t^{100} \cdot \left(x + x^2 + x^2 y\right)$$

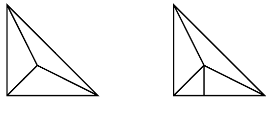
and the 1-dimensional family of coordinate changes given by $(x, y) \mapsto (x + k, y)$ with $k \in \mathbb{K}$. There are infinitely many tropicalizations having a cycle and infinitely many having none.

5. Δ-equivalent marked subdivisions

In this section we want to show that the function *cycle length*, cl, is linear on the union of cones of the secondary fan of $\mathcal{A}_c$ which are Δ-equivalent. Also, we provide the classification of the different cases we need to consider in order to compare the two linear functions val_j and cl on such a union. This is part of our proof of Theorem 11.

Remark 15. The Prime Factorization Theorem, [8, Chapter 10, Theorem 1.2] tells us that the codimension one cones of the Gröbner fan of Δ do not meet the interior of any top-dimensional cone of the secondary fan of $\mathcal{A}_c$. Thus the Gröbner fan of Δ is a coarsening of the secondary fan of $\mathcal{A}_c$. Two cones of the secondary fan of $\mathcal{A}_c$ are called Δ*-equivalent* if they are contained in the same cone of the Gröbner fan of Δ.

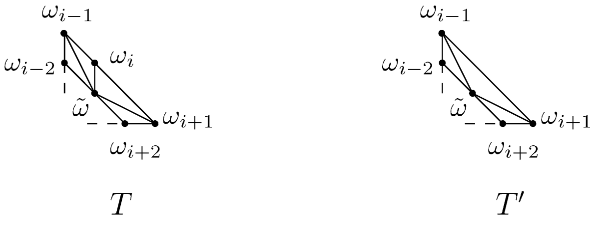
It has been studied how two top-dimensional marked subdivisions whose cones belong to the same Δ-equivalence class can differ. By [8, Chapter 11, Proposition 3.8] they can be obtained from each other by a sequence of modifications along a circuit (see [8, Chapter 7, Section 2C]) such that each intermediate (top-dimensional) marked subdivision belongs to the same equivalence class. Instead of defining in its full generality what a modification along a circuit is, we use [8, Chapter 11, Proposition 3.9] to explain what this means in the case of the marked polygon $(Q_c, \mathcal{A}_c)$: a subdivision T can be obtained from a subdivision T' via a modification along a circuit if there are three points a, b, c in order on *one* edge of Q_c such that T and T' differ by the fact that one contains the triangle $\{a, (1,1), c\}$, while the other one contains the two triangles, $\{a, (1,1), b\}$ and $\{b, (1,1), c\}$ instead. An example is shown in the following picture, the three points are $a = (0,0)$, $b = (1,0)$ and $c = (3,0)$.



Lemma 16. *The function* cl (*see Definition* 6) *is linear on a cone of the Gröbner fan of* Δ.

Proof. Given two Δ-equivalent marked subdivisions T and T' of the secondary fan of $\mathcal{A}_c$, we can use Lemma 9 to determine the function cl on the cone corresponding to each of them. Recall from Remark 10 that the function is linear on each cone of the secondary fan of $\mathcal{A}_c$. We want to show that the these two linear functions coincide.

Without restriction we can assume that T can be obtained from T' by a modification along a circuit, and this circuit consists of three collinear points on an edge of Q_c (see Remark 15).



Recall from Lemma 9 that the coefficients of the linear function cl can be determined using the determinants $D_{i,j} = \det(w_i, w_j)$, where $w_i = \omega_i - \tilde{\omega}$. One easily sees that for T and T' the following two equations hold:

$$D_{i-1,i} + D_{i,i+1} = D_{i-1,i+1}, \quad \text{and} \tag{3}$$

$$D_{i,i+1} = \lambda \cdot D_{i-1,i} \quad \text{for } \lambda \text{ satisfying } \lambda \cdot (w_{i-1} - w_i) = w_i - w_{i+1}. \tag{4}$$

To show that the expressions for cl on the cones for T and T' coincide, we have to show that for T the summand for ω_i equals 0 and the summand for ω_{i-1} equals the summand for ω_{i-1} for T'. The first statement follows immediately from Eq. (3) above. To show the second statement, we subtract the two summands from each other:

$$\frac{D_{i-2,i-1} + D_{i-1,i} + D_{i,i-2}}{D_{i-2,i-1} \cdot D_{i-1,i}} - \frac{D_{i-2,i-1} + D_{i-1,i+1} + D_{i+1,i-2}}{D_{i-2,i-1} \cdot D_{i-1,i+1}}.$$

Multiplied with $(D_{i-1,i} \cdot D_{i-1,i+1})$ this difference is equal to:

$$\begin{aligned}
&D_{i-2,i-1} \cdot D_{i-1,i+1} + D_{i-1,i} \cdot D_{i-1,i+1} + D_{i,i-2} \cdot D_{i-1,i+1} \\
&\quad - D_{i-2,i-1} \cdot D_{i-1,i} - D_{i-1,i+1} \cdot D_{i-1,i} - D_{i+1,i-2} \cdot D_{i-1,i} \\
&= D_{i-2,i-1} \cdot D_{i,i+1} + D_{i,i-2} \cdot D_{i,i+1} + D_{i,i-2} \cdot D_{i-1,i} - D_{i+1,i-2} \cdot D_{i-1,i} \\
&= -\det(w_{i-1} - w_i, w_{i-2}) \cdot D_{i,i+1} + \det(w_i - w_{i+1}, w_{i-2}) \cdot D_{i-1,i} = 0
\end{aligned}$$

where the first equality follows from Eq. (3) above and the last from (4). $\square$

Definition 17. Let us fix a cone C_T of the secondary fan of $\mathcal{A}_c$ corresponding to the marked subdivision T. We then denote by $\eta_T(i, j)$ the coefficient of u_{ij} in the linear function val_Δ on C_T, and by $c_T(i, j)$ we denote the coefficient of u_{ij} in the linear function cl restricted to C_T.

Remark 18. Note that by Remark 2 and Remark 15 $\eta_T(i, j) = \eta_{T'}(i, j)$ for all $(i, j) \in \mathcal{A}_c$ whenever T and T' belong to Δ-equivalent cones of the secondary fan of $\mathcal{A}_c$, and by Lemma 16 also $c_T(i, j) = c_{T'}(i, j)$ for all $(i, j) \in \mathcal{A}_c$ in this situation.

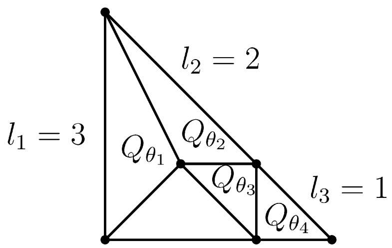
Lemma 19. *Let T be a marked subdivision of $(Q_c, \mathcal{A}_c)$ corresponding to a top-dimensional cone in the secondary fan of $\mathcal{A}_c$ (i.e. a triangulation) such that $(1,1)$ is a vertex of some marked polygon in T (i.e. all dual plane tropical curves have a cycle). Then $c_T(1,1) = \eta_T(1,1) - 12$ and $c_T(i,j) = \eta_T(i,j)$ for all $(i,j) \neq (1,1)$.*

Proof. Due to Remark 18 we may assume for the proof that $T = \{(Q_\theta, \mathcal{A}_\theta) \mid \theta \in \Theta\}$ is the representative of its Δ-equivalence class with as few edges as possible.

Moreover, if two triangulations T and T' can be transformed into each other by an integral unimodular linear isomorphism, i.e. by a linear coordinate change of the projective coordinates (x, y, z) with a matrix in $\mathrm{Gl}_3(\mathbb{Z})$, and the claim holds for T then it obviously also holds for T'. Therefore, we only have to prove the claim up to $\mathrm{Gl}_3(\mathbb{Z})$-symmetry.

We want to use [8, Chapter 11, Theorem 3.2] which explains how $\eta_T(i,j)$ can be computed. For each $(i,j) \in \mathcal{A}_c$ we have to consider all $(Q_\theta, \mathcal{A}_\theta)$ such that $(i,j) \in \mathcal{A}_\theta$. Note that since T by assumption is a triangulation then $(i,j) \in \mathcal{A}_\theta$ implies necessarily that (i,j) is a vertex of Q_θ. We have to distinguish four cases, where in the formulas $\mathrm{vol}(Q_\theta)$ denotes he generalized lattice area (i.e. twice the Euclidean area of Q_θ):

- If (i,j) is a vertex of Q_c, then $\eta_T(i,j) = 1 - l_1 - l_2 + \sum_{(i,j)\in\mathcal{A}_\theta} \mathrm{vol}(Q_\theta)$ where l_1 and l_2 denote the lattice lengths of those edges of some Q_θ adjacent to (i,j) which are contained in edges of Q_c. E.g. if $(i,j) = (0,3)$ in the following triangulation T, then $\eta_T(0,3) = 1 - l_1 - l_2 + \mathrm{vol}(Q_{\theta_1}) + \mathrm{vol}(Q_{\theta_2}) = 1 - 3 - 2 + 3 + 2 = 1$.



- If (i,j) lies on an edge of Q_c, is not a vertex of Q_c, but is a vertex of some $Q_{\theta'}$, then $\eta_T(i,j) = -l_1 - l_2 + \sum_{(i,j)\in\mathcal{A}_\theta} \mathrm{vol}(Q_\theta)$ where again l_1 and l_2 denote the lattice lengths of those edges of some Q_θ adjacent to (i,j) which are contained in edges of Q_c, e.g. if in the previous example $(i,j) = (2,1)$ then $\eta_T(i,j) = -l_2 - l_3 + \mathrm{vol}(Q_{\theta_2}) + \mathrm{vol}(Q_{\theta_3}) + \mathrm{vol}(Q_{\theta_4}) = -2 - 1 + 2 + 1 + 1 = 1$.
- If (i,j) lies on an edge of Q_c, is not a vertex of any Q_θ, then $\eta_T(i,j) = 0$.
- And finally $\eta_T(1,1) = \sum_{(1,1)\in\mathcal{A}_\theta} \mathrm{vol}(Q_\theta)$.

Let Q be the union of all those Q_θ which contain $(1,1)$, and endow the marked polygon $(Q, Q \cap \mathcal{A}_c)$ with the subdivision, T_Q, induced by T. We say that Q meets an edge of Q_c if the intersection of Q with this edge is 1-dimensional (and not only a vertex). Moreover, we say that an edge of Q is multiple if it contains more than two lattice points.

We first want to show that $\eta_T(i,j)$ and $c_T(i,j)$ are as claimed whenever $(i,j) \in Q$. Up to symmetry, we have to distinguish the following cases for Q and T_Q:

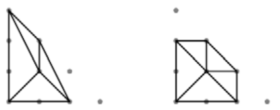
- Assume Q meets all three edges of Q_c and that for all three edges the intersection with Q is multiple. Then Q looks (up to symmetry) like one of the following two pictures:

In the second case, $\eta_T(1,1) = 8$. Using Lemma 9 we can compute $c_T(1,1)$. It is a sum with a summand for each vertex of Q. The summand for $(0,0)$ is

$$\frac{\det\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} + \det\begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix} + \det\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}}{\det\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \cdot \det\begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix}} = -1.$$

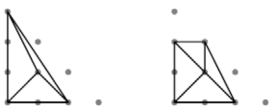
Computing the other 3 summands analogously we get $c_T(1,1) = -4 = \eta_T(1,1) - 12$. In the first case, $\eta_T(1,1) = 9$ and $c_T(1,1) = -3$.

- Assume Q meets two edges of Q_c multiply and one edge non-multiply.



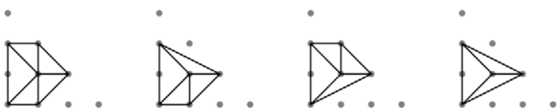
In both cases, $\eta_T(1,1) = 7$ and $c_T(1,1) = -5$.

- Assume Q meets two edges of Q_c multiply and the third edge not at all.



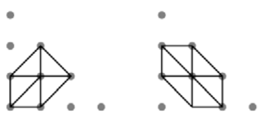
In both cases, $\eta_T(1,1) = 6$ and $c_T(1,1) = -6$.

- Assume Q meets only one edge of Q_c multiply (and the two remaining edges non-multiply, or only one of them and that one non-multiply, or none of them at all).



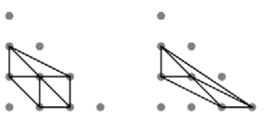
In the first case, $\eta_T(1,1) = 6$ and $c_T(1,1) = -6$, in the second and third case, $\eta_T(1,1) = 5$ and $c_T(1,1) = -7$, and in the last case, $\eta_T(1,1) = 4$ and $c_T(1,1) = -8$.

- Assume Q meets three edges of Q_c, but none of them multiply.



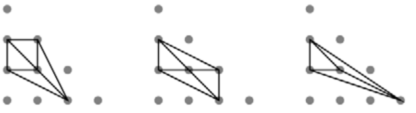
In the first case, $\eta_T(1,1) = 5$ and $c_T(1,1) = -7$, and in the second case, $\eta_T(1,1) = 6$ and $c_T(1,1) = -6$.

- Assume Q meets only two edges of Q_c, and none of them multiply.



In the first case, $\eta_T(1,1) = 5$ and $c_T(1,1) = -7$, and in the second case, $\eta_T(1,1) = 4$ and $c_T(1,1) = -8$.

- Assume Q meets only one edge of Q_c and it does so non-multiply.

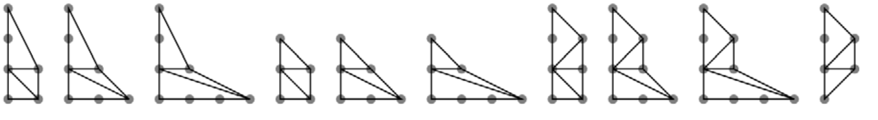


In the first and second case, $\eta_T(1,1) = 4$ and $c_T(1,1) = -8$, and in the third case, $\eta_T(1,1) = 3$ and $c_T(1,1) = -9$.

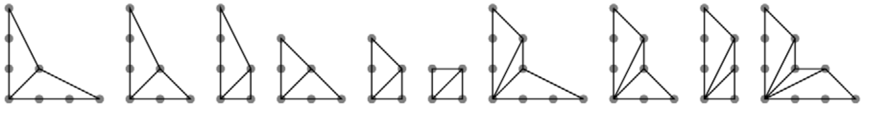
- Assume Q meets no edge of Q_c at all.

Finally, in this case, $\eta_T(1,1) = 3$ and $c_T(1,1) = -9$.

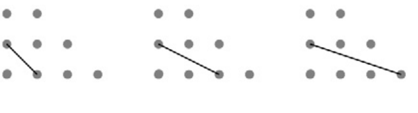
Thus the claim for $(1,1)$ is shown. Now assume $(1,1) \neq (i,j) \in Q$ is not a vertex of Q_c. If (i,j) is also not a vertex of any Q_θ then there is no edge in the subdivision from $(1,1)$ to (i,j) and thus (i,j) does not contribute to the formula for the cycle length, i.e. $c_T(i,j) = 0$. However, the same holds for $\eta_T(i,j)$. We may thus assume that (i,j) is a vertex of some Q_θ, and we may without restriction assume $(i,j) = (0,1)$. The classification of cases we have to consider is very similar to the above, and we will not give the details, leaving the computation of $c_T(i,j)$ and $\eta_T(i,j)$ to the reader. We do not have to consider the whole of Q, but only the triangles which are adjacent to (i,j).



If $(1,1) \neq (i,j) \in Q$ is a vertex of Q_c (without restriction $(i,j) = (0,0)$), the following cases have to be considered:



Finally, we have to consider the case were (i,j) is not part of Q. Obviously, $c_T(i,j) = 0$ in this case and we have to show the same for $\eta_T(i,j)$. Assume first that (i,j) is a vertex of Q_c, without restriction we can assume $(i,j) = (0,0)$. There must be an edge of Q such that $(0,0)$ is on one side of it and $(1,1)$ is on the other. Then (up to symmetry) there are 3 possibilities for that edge.



Since we assumed that T is the representative with as few edges as possible, the triangle formed by that edge of Q and $(0,0)$ can not be additionally subdivided in the second and third picture. In any case, $(0,0)$ is a vertex of only one triangle, which has one edge of integer length 1 and one edge of integer length l where $1 \leqslant l \leqslant 3$. Thus $\eta_T(0,0) = 1 - 1 - l + l = 0$. Now assume that (i,j) is not a vertex of Q_c, without restriction $(i,j) = (1,0)$. Again there must be an edge of Q such that $(1,0)$ is on one side and $(1,1)$ is on the other. Up to symmetry this can only be one of the line segments in the two right pictures above. We assumed that T is the representative of its Δ-equivalence class with as few edges as possible. But that means there is no edge through $(1,0)$ and $(1,0)$ is not a vertex of a triangle in the subdivision. Thus $\eta_T(1,0) = 0$. □

As pointed out by a referee the distinction of cases that we did in the proof for showing $c_T(1,1) = 12 - \eta_T(1,1)$ proves in particular the following corollary, where by definition a tropical curve is *smooth* if and only if the dual Newton subdivision is a unimodular triangulation.

Corollary 20. *Up to symmetry there are precisely* 18 *unimodular triangulations of Q_c, all of which are regular. In particular, there are up to symmetry precisely* 18 *combinatorial types of* smooth *elliptic tropical curves with support set $\mathcal{A}_c$.*

Remark 21. In the proof above the computation that shows that $\eta_T(i,j) = c_T(i,j)$ is different in each of the considered cases. In particular, in the computation of $\eta_T(i,j)$ the part of Q_c which is not part of the cycle is involved while this is not the case for $c_T(i,j)$. Therefore it is unfortunately not possible to replace the consideration of several cases by an argument which holds for all of them at the same time.

However, using `polymake` and SINGULAR one can compute the vertices of the Newton polytope of Δ and for each vertex one can compute the dual cone in the Gröbner fan of Δ and the triangulation of $(Q_c, \mathcal{A}_c)$ with as few edges as possible corresponding to this cone. That way one can verify the above computations for c_T and η_T, since the values for η_T can be read off immediately from the exponents of the vertex of the Newton polytope, while the c_T can be computed with the formula in Lemma 9. These computations have been made using the procedure `displayFan` and the result can be obtained via the URL

http://www.mathematik.uni-kl.de/~keilen/en/jinvariant.html.

The advantage is that the file `discriminant_fan_of_cubic.ps` available via this URL shows the cases not only up to symmetry, but it shows all possible cases.

6. Numerator of the j-invariant

Unfortunately, it is not true that the Gröbner fan of the numerator A of the j-invariant is a coarsening of the secondary fan, as follows from Example 22.

Example 22. We provide an example which shows that the Gröbner fan of A is not a coarsening of the secondary fan for curves of a particular form. The case of the full cubic is more complicated but analogous. It can easily be proved by a computation using `polymake`—this can be done using the procedure `nonrefinementC` in the library `jinvariant.lib` (see [7]).

Let us consider curves of the form

$$y^2 + axy - x^3 - bx^2 - 1 = 0.$$

This corresponds to considering the set $\mathcal{A} = \{(0,2), (1,1), (3,0), (2,0), (0,0)\}$ of lattice points and the corresponding marked polygon. The secondary fan is then 5-dimensional with a 3-dimensional linearity space L. The fixing of the constant coefficient and the coefficients of y^2 and x^3 provides an isomorphism $\mathbb{R}^2 \cong \mathbb{R}^{\mathcal{A}}/L$.

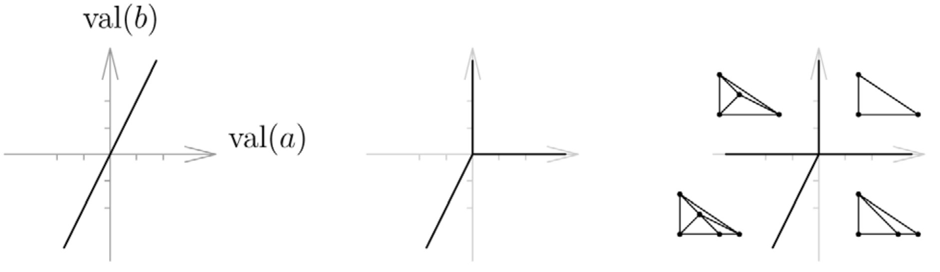
By the usual formulas for the j-invariant, we have

$$A = \left(a^2 + 4b\right)^6 \quad \text{and} \quad \Delta = -\left(a^2 + 4b\right)^3 - 432,$$

so that

$$j = -\frac{(a^2+4b)^6}{(a^2+4b)^3+432}.$$

The following picture shows the tropicalization of the numerator A, the tropicalization of the denominator Δ, and the secondary fan in $\mathbb{R}^{\mathcal{A}}/L$.



Observe that the tropicalization of the denominator is supported on the codimension one skeleton of the secondary fan while that of the numerator intersects a top-dimensional cone of the secondary fan.

However, we are only interested in plane tropical cubics which have a cycle, that is, those that are dual to marked subdivisions for which the interior point can be seen. All these cones of the secondary fan are completely contained in one cone of the Gröbner fan of A. We verified this computationally using `polymake` (see [4]). As usual we use the coordinates u_{ij} with $(i,j) \in \mathcal{A}_c$ on $\mathbb{R}^{\mathcal{A}_c}$ and we denote by $e_{kl} = (\delta_{ik} \cdot \delta_{jl} \mid (i,j) \in \mathcal{A}_c)$ the canonical basis vector in $\mathbb{R}^{\mathcal{A}_c}$ having a one in position kl and zeros elsewhere.

Lemma 23. *Let U be the union of all cones of the secondary fan of $\mathcal{A}_c$ corresponding to marked subdivisions $T = \{(Q_i, \mathcal{A}_i) \mid i = 1, \dots, k\}$ of $(Q_c, \mathcal{A}_c)$ for which $(1,1)$ is a vertex of some Q_i. Then U is contained in a single cone of the Gröbner fan of the A, namely in the cone dual to the vertex $12 \cdot e_{11}$ of the Newton polytope of A.*

Proof. As input for `polymake` we use all exponents of the polynomial $A \in \mathbb{Q}[\underline{a}]$. The convex hull of the set of all exponents is the Newton polytope, say $N(A)$, of A and its vertices are the output of `polymake`. The Newton polytope has 19 vertices. Dual to each vertex is a top-dimensional cone of the Gröbner fan $\mathcal{F}(A)$ of A, because the Gröbner fan is dual to the Newton polytope (see [11, Theorem 2.5 and Proposition 2.8]). The inequalities describing the cone C dual to the vertex V are given by the hyperplanes orthogonal to the edge vectors connecting V with its neighboring vertices in $N(A)$. We compute the neighboring vertices for each vertex using `polymake` and deduce thus inequalities for each of the top-dimensional cones of the Gröbner fan of A.

In order for a marked subdivision $T_u = \{(Q_i, \mathcal{A}_i) \mid i = 1, \dots, k\}$ of $(Q_c, \mathcal{A}_c)$ given by $u \in \mathbb{R}^{\mathcal{A}_c}$ to have the point $(1,1)$ as vertex of some Q_i it is obviously necessary that the u_{ij} satisfy the following inequalities:

$$\begin{aligned}
&3 \cdot u_{01} + 2 \cdot u_{30} + u_{03} > 6 \cdot u_{11}, &\quad &3 \cdot u_{10} + 2 \cdot u_{03} + u_{30} > 6 \cdot u_{11},\\
&3 \cdot u_{12} + u_{30} + 2 \cdot u_{00} > 6 \cdot u_{11}, &\quad &3 \cdot u_{21} + u_{03} + 2 \cdot u_{00} > 6 \cdot u_{11},\\
&2 \cdot u_{30} + 3 \cdot u_{02} + u_{00} > 6 \cdot u_{11}, &\quad &2 \cdot u_{03} + 3 \cdot u_{20} + u_{00} > 6 \cdot u_{11},\\
&u_{12} + u_{30} + u_{00} + u_{02} > 4 \cdot u_{11}, &\quad &u_{21} + u_{03} + u_{00} + u_{20} > 4 \cdot u_{11},
\end{aligned}$$

$$\begin{aligned} u_{01} + u_{10} + u_{03} + u_{30} &> 4 \cdot u_{11}, & 2 \cdot u_{01} + u_{12} + u_{30} &> 4 \cdot u_{11}, \\ 2 \cdot u_{10} + u_{21} + u_{03} &> 4 \cdot u_{11}, & 2 \cdot u_{12} + u_{20} + u_{00} &> 4 \cdot u_{11}, \\ 2 \cdot u_{21} + u_{02} + u_{00} &> 4 \cdot u_{11}, & 2 \cdot u_{02} + u_{10} + u_{30} &> 4 \cdot u_{11}, \\ 2 \cdot u_{20} + u_{01} + u_{03} &> 4 \cdot u_{11}, & u_{20} + u_{01} + u_{12} &> 3 \cdot u_{11}, \\ u_{02} + u_{10} + u_{21} &> 3 \cdot u_{11}, & u_{30} + u_{02} + u_{01} &> 3 \cdot u_{11}, \\ u_{03} + u_{10} + u_{20} &> 3 \cdot u_{11}, & u_{00} + u_{12} + u_{21} &> 3 \cdot u_{11}, \\ u_{00} + u_{30} + u_{03} &> 3 \cdot u_{11}, & u_{21} + u_{01} &> 2 \cdot u_{11}, \\ u_{10} + u_{12} &> 2 \cdot u_{11}, & u_{20} + u_{02} &> 2 \cdot u_{11}. \end{aligned} \tag{5}$$

These inequalities determine a cone in $\mathbb{R}^{\mathcal{A}_c}$ which contains U. A simple computation with `polymake` allows to compute the extreme rays of this cone and to check that they satisfy the inequalities of the single cone of the Gröbner fan of A which is dual to the vertex $12 \cdot e_{11}$ in $N(A)$. This proves the claim. □

Remark 24. The inequalities in (5) precisely determine the cone U as can again be easily tested using `polymake`. The computations were done with the procedure `testInteriorInequalities` in the library `jinvariant.lib` (see [7]).

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