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On universal Lie nilpotent associative algebras

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ABSTRACT

We study the quotient $Q_i(A)$ of a free algebra A by the ideal $M_i(A)$ generated by the ith commutator of any elements. In particular, we completely describe such quotient for $i=4$ (for $i \leqslant 3$ this was done previously by Feigin and Shoikhet). We also study properties of the ideals $M_i(A)$, e.g. when $M_i(A)M_j(A)$ is contained in $M_{i+j-1}(A)$ (by a result of Gupta and Levin, it is always contained in $M_{i+j-2}(A)$).
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1. Introduction

Let A be an associative unital algebra over a field k. Let us regard it as a Lie algebra with bracket $[a,b]=ab-ba$, and consider the terms of its lower central series $L_i(A)$ defined inductively by $L_1(A)=A$ and $L_{i+1}(A)=[A,L_i(A)]$. Denote by $M_i(A)$ the two-sided ideal in A generated by $L_i(A)$: $M_i(A)=AL_i(A)A$, and let $Q_i(A)=A/M_i(A)$. Thus $Q_i(A)$ is the largest quotient algebra of A which satisfies the higher commutator polynomial identity $[\dots[[a_1,a_2],a_3],\dots,a_i]=0$.

An algebra A is said to be Lie nilpotent of degree i if $M_{i+1}(A)=0$ (i.e. $A=Q_{i+1}(A)$). For example, Lie nilpotent algebras of degree 1 are commutative algebras. Understanding Lie nilpotent algebras of higher degrees is an interesting open problem. Many questions about Lie nilpotent algebras can be reduced to understanding the structure of universal Lie nilpotent algebras, i.e. algebras $Q_{n,i} := Q_i(A_n)$, where A_n is the free associative algebra in n generators, since any finitely generated Lie nilpotent algebra of degree i is a quotient of $Q_{n,i+1}$.

The goal of this paper is to advance our understanding of the algebras $Q_{n,i}$ for $i \geqslant 2$ (in characteristic zero). The structure of these algebras for general i and n is unknown. The only algebras $Q_{n,i}$ whose structure has been known are $Q_{n,2}$, which is easily seen to be isomorphic to the polynomial algebra $k[x_1,\dots,x_n]$, and $Q_{n,3}$, which, according to Feigin and Shoikhet, [FS], is isomorphic to the

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algebra of even polynomial differential forms in n variables, $k[x_1,\dots,x_n]\otimes\bigwedge^{\mathrm{even}}(dx_1,\dots,dx_n)$, with product $a*b=ab+da\wedge db$.

The main result of the paper is an explicit description of the algebra $Q_{n,4}$. We also derive some properties of the algebras $Q_{n,i}$ for $i>4$, and formulate some questions for future study which appear interesting.

2. The associated graded algebra of $Q_{n,i}$ under the Lie filtration

Let A_n be the free algebra over $\mathbb{C}$ in n generators $x_1,\dots,x_n$ ($n\geqslant 2$). The algebra A_n can be viewed as the universal enveloping algebra $U(\ell_n)$ of the free Lie algebra ℓ_n in n generators. Therefore, A_n has an increasing filtration (called the Lie filtration), defined by the condition that ℓ_n sits in degree 1, and the associated graded algebra $\operatorname{gr} A_n$ under this filtration is the commutative algebra $\operatorname{Sym}\ell_n$.

The algebra $Q_{n,i}$ is the quotient of $A_n=U(\ell_n)$ by the ideal $M_{n,i}:=M_i(A_n)$. Hence, $Q_{n,i}$ inherits the Lie filtration from A_n, and one can form the quotient algebra $D_{n,i}=\operatorname{gr} Q_{n,i}=\operatorname{Sym}\ell_n/\operatorname{gr} M_{n,i}$, which is commutative.

Let $\ell_n'=[\ell_n,\ell_n]$. Then we have a natural factorization $\operatorname{Sym}\ell_n=\mathbb{C}[x_1,\dots,x_n]\otimes\operatorname{Sym}\ell_n'$.

Let $\Lambda_{n,i}$ be the image of $\operatorname{Sym}\ell_n'$ in $D_{n,i}$. Then the multiplication map

$$\theta:\mathbb{C}[x_1,\dots,x_n]\otimes\Lambda_{i,n}\to D_{n,i}$$

is surjective.

Theorem 2.1.

(i) *$\Lambda_{n,i}$ is a finite dimensional algebra with a grading by nonnegative integers (defined by setting $\deg x_i=1$), with $\Lambda_{n,i}[0]=k$.*
(ii) *The map θ is an isomorphism.*

Proof. Statement (i) follows from the following theorem of Jennings:

Theorem 2.2. *(See [Jen], Theorem 2.) If A is a finitely generated Lie nilpotent algebra, then $M_2(A)$ is a nilpotent ideal.*

This implies that there exists N such that for any $a_1,\dots,a_N\in M_2(A)$, $a_1a_2\cdots a_N=0$. Taking $A=Q_{n,i}$, we see that for any $a_1,\dots,a_N\in\ell_n'$, we have $a_1a_2\cdots a_N=0$. Since $\Lambda_{n,i}$ is generated by the subspace $\ell_n'[<i]$ of ℓ_n' of degree $<i$, this implies that $\Lambda_{n,i}$ is finite dimensional, proving (i).

To prove (ii), let v_j, $j=1,\dots,d$, be a basis of $\Lambda_{n,i}$, and assume the contrary, i.e. that we have a nontrivial relation in $D_{n,i}$:

$$\sum_{j=1}^d f_j(x_1,\dots,x_n)v_j=0,$$

where $f_j\in\mathbb{C}[x_1,\dots,x_n]$. Pick this relation so that the maximal degree D of f_j is smallest possible. This degree must be positive, since v_j are linearly independent over $\mathbb{C}$. Applying the automorphism g_i^t $(t\in\mathbb{C})$ of A_n acting by $g_i^t(x_i)=x_i+t$, $g_i^t(x_s)=x_s$, $s\neq i$, we get

$$\sum_{j=1}^d f_j(x_1,\dots,x_s+t,\dots,x_n)v_j=0.$$

Differentiating this by t, we get

$$\sum_{j=1}^{d} \partial_s f_j(x_1, \dots, x_n) v_j = 0.$$

This relation must be trivial, since it has smaller degree than D. Thus f_j must be constant, which is a contradiction. □

This shows that to understand the structure of the algebra $Q_{n,i}$, we need to first understand the structure of the commutative finite dimensional algebra $\Lambda_{n,i}$, which gives rise to the following question.

Question 2.3. What is the structure of $\Lambda_{n,i}$ as a $GL(n)$-module?

The answer to Question 2.3 has been known only for $i = 2$, in which case $\Lambda_{n,i} = \mathbb{C}$, and for $i = 3$, in which case it is shown in [FS] that $\Lambda_{n,i} = \bigwedge^{\text{even}}(\xi_1, \dots, \xi_n)$, and hence is the sum of irreducible representations of $GL(n)$ corresponding to the partitions $(1^{2r}, 0, \dots, 0)$, $0 \leqslant 2r \leqslant n$.

In this paper, we answer Question 2.3 for $i = 4$. For $i > 4$, the question remains open.

3. The multiplicative properties of the ideals $M_i(A)$

A step toward understanding of the structure of the algebras $Q_i(A)$ is understanding of the multiplicative properties of the ideals $M_i(A)$. In 1983, Gupta and Levin proved the following result in this direction.

Theorem 3.1. *(See [GL], Theorem 3.2.) For any $m, l \geqslant 2$ and any algebra A, we have*

$$M_m(A) \cdot M_l(A) \subset M_{m+l-2}(A).$$

Corollary 3.2. *$\overline{A} := Q_3(A) \oplus \bigoplus_{i \geqslant 3} M_i(A)/M_{i+1}(A)$ has a structure of a graded algebra, with $Q_3(A)$ sitting in degree zero, and $M_i(A)/M_{i+1}(A)$ in degree $i-2$ for $i \geqslant 3$.*

It is interesting that the result of Theorem 3.1 can sometimes be improved. Namely, let us say that a pair (m, l) of natural numbers is null if for any algebra A

$$M_m(A) \cdot M_l(A) \subset M_{m+l-1}(A)$$

(clearly, this property does not depend on the order of elements in the pair, and any pair $(1, m)$ is null).

Lemma 3.3. *The pair (m, l) is null if and only if the element*

$$\big[\dots[x_1, x_2], \dots, x_m\big] \cdot \big[\dots[x_{m+1}, x_{m+2}], \dots, x_{m+l}\big]$$

is in $M_{m+l-1}(A_{m+l})$.

Proof. By Theorem 3.1, a pair (m, l) is null iff $L_m(A)L_l(A) \subset M_{m+l-1}(A)$ for any A. Clearly, this happens if and only if the statement of Lemma 3.3 holds, as desired. □

Theorem 3.4. *If $l + m \leqslant 7$, then the unordered pair (m, l) is null iff it is not $(2, 2)$ or $(2, 4)$.*

Proof. The property of Lemma 3.3 was checked using the MAGMA program, and it turns out that it holds for $(2,3)$, $(3,3)$, $(2,5)$, $(3,4)$, but not for $(2,2)$ and $(2,4)$.

Actually, it is easy to check by hand that the property of Lemma 3.3 does not hold for $(2,2)$, and here is a computer-free proof that it holds for $(2,3)$.

We need to show that in $Q_{n,4}$, we have

$$[x_i, x_j]\big[x_k, [x_l, x_m]\big] = 0.$$

To do so, define $S(i,j,k,l,m) := [x_i, x_j][x_k, [x_l, x_m]] + [x_k, x_j][x_i, [x_l, x_m]]$. Then in $Q_{n,4}$ we have

$$S(i,j,k,l,m) = 0.$$

Indeed, it suffices to show that in $Q_{n,4}$

$$[x_i, x_j]\big[x_k, [x_l, x_m]\big] + \big[x_i, [x_l, x_m]\big][x_k, x_j] = 0,$$

which follows from the fact that in a free algebra we have

$$[a,b][c,d] + [a,d][c,b] = \big[[ac,b],d\big] + a\big[d,[c,b]\big] - [\big[[a,b],d\big]c,$$

where $a = x_i$, $b = x_j$, $c = x_k$, $d = [x_l, x_m]$.

Now set

$$\begin{aligned}
R(i,j,k,l,m) = {} & -\frac{1}{2}S(x_j, x_k, x_l, x_m, x_i) + \frac{1}{2}S(x_j, x_k, x_m, x_l, x_i) - \frac{1}{2}S(x_j, x_k, x_i, x_l, x_m) \\
& - \frac{1}{2}S(x_j, x_m, x_l, x_k, x_i) + \frac{1}{2}S(x_j, x_m, x_k, x_l, x_i) - \frac{1}{2}S(x_j, x_m, x_i, x_l, x_k) \\
& - S(x_j, x_i, x_k, x_l, x_m) - S(x_j, x_i, x_m, x_l, x_k) + \frac{1}{2}S(x_l, x_k, x_m, x_j, x_i) \\
& - \frac{1}{2}S(x_l, x_k, x_i, x_j, x_m) + \frac{1}{2}S(x_l, x_m, x_k, x_j, x_i) - \frac{1}{2}S(x_l, x_m, x_i, x_j, x_k) \\
& - \frac{1}{2}S(x_k, x_i, x_m, x_j, x_l).
\end{aligned}$$

Then one can show by a direct computation that in A_n we have

$$[x_i, x_j]\big[x_k, [x_l, x_m]\big] = \frac{1}{3}\big(R(i,j,m,l,k) - R(i,j,l,m,k)\big).$$

Therefore, we see that $[x_i, x_j][x_k, [x_l, x_m]] = 0$ in $Q_{n,4}$, as desired. □

Question 3.5. Which pairs of integers $\geqslant 2$ are null?

4. Description of $Q_{n,4}$ by generators and relations

In [FS], Feigin and Shoikhet described the algebra $Q_{n,3}$ by generators and relations. Namely, the proved the following result.

Theorem 4.1. *$Q_{n,3}$ is generated by x_i, $i = 1, \dots, n$, and $y_{ij} = [x_i, x_j]$, $1 \leqslant i, j \leqslant n$, with defining relations*

$$[x_i, y_{jl}] = 0,$$

and the quadratic relation

$$y_{ij} y_{kl} + y_{ik} y_{jl} = 0$$

saying that $y_{ij} y_{kl}$ is antisymmetric in its indices.

Corollary 4.2. *The algebra $\Lambda_{n,3}$ is generated by y_{ij} with defining relations*

$$y_{ij} = -y_{ji}, \qquad y_{ij} y_{kl} + y_{ik} y_{jl} = 0.$$

In this section we would like to give a similar description of the algebras $Q_{n,4}$, $\Lambda_{n,4}$. As we know, the algebra $Q_{n,4}$ is generated by the elements x_i, y_{ij} as above, and also $z_{ijk} = [y_{ij}, x_k]$, $1 \leqslant i, j, k \leqslant n$. Our job is to find what relations to put on x_i, y_{ij}, z_{ijk} to generate the ideal $M_{n,4}$. This is done by the following theorem, which is our main result.

Theorem 4.3.

(i) *The ideal $M_{n,4}$ is generated by the Lie relations*

$$[x_i, z_{jlm}] = 0,$$

the quadratic relations

$$y_{ij} z_{klm} = 0,$$

and the cubic relations

$$y_{ij} y_{kl} y_{mp} + y_{ik} y_{jl} y_{mp} = 0,$$

saying that $y_{ij} y_{kl} y_{mp}$ is antisymmetric in its indices.

(ii) *The algebra $\Lambda_{n,4}$ is generated by y_{ij}, z_{ijk} subject to the linear relations*

$$y_{ij} = -y_{ji}, \qquad z_{ijk} = -z_{jik}, \qquad z_{ijk} + z_{jki} + z_{kij} = 0,$$

and the relations

$$y_{ij} z_{klm} = 0, \qquad z_{ijp} z_{klm} = 0, \qquad y_{ij} y_{kl} y_{mp} + y_{ik} y_{jl} y_{mp} = 0.$$

Proof. Part (ii) follows from (i), so we need to prove (i). The relations $y_{ij} z_{klm} = 0$ follow from the fact that $M_2(A) M_3(A) \subset M_4(A)$ for any algebra A (Theorem 3.4). This fact also implies that $y_{ij} y_{kl} y_{mp}$ is antisymmetric, since by [FS], $y_{ij} y_{kl} + y_{ik} y_{jl} \in M_{n,3}$.

Denote by B_n the quotient of A_n by the relations stated in part (i) of the theorem. We have just shown that there is a natural surjective homomorphism $\eta : B_n \to Q_{n,4}$. We need to show that it is an isomorphism. For this, we need to show that for any $a, b, c, d \in B_n$, $[[[a, b], c], d] = 0$. For this, it suffices to show that $[[a, b], c]$ is a central element in B_n. But $[[a, b], c] = 0$ in $Q_{n,3}$, which implies that $[[a, b], c]$ belongs to the ideal generated by z_{ijk} and $y_{ij} y_{kl} + y_{ik} y_{jl}$. But it is easy to see using the relations of B_n that all elements of this ideal are central in B_n, as desired. $\square$

Let $K_{n,i}$ be the kernel of the projection map $\Lambda_{n,i+1} \to \Lambda_{n,i}$. As a result, we see that $K_{n,3}$ is spanned by elements z_{ijk} and $y_{ij}y_{kl}$ modulo the antisymmetry relation. Therefore, we get

Corollary 4.4. *As a $GL(n)$-module, $K_{n,3}$ is isomorphic to the direct sum of two irreducible modules $F_{2,1,0,\dots,0}$ and $F_{2,2,0,\dots,0}$ corresponding to partitions $(2,1,0,\dots,0)$ and $(2,2,0,\dots,0)$.*

This answers Question 2.3 for $i = 4$.

Proof. Let $V = \mathbb{C}^n$ be the vector representation of $GL(n)$. The span of z_{ijk} is the subrepresentation of $V^{\otimes 3}$ annihilated by $Id + (12)$ and $Id + (123) + (132)$ in $\mathbb{C}[S_3]$, so it corresponds to the partition $(2,1,0,\dots,0)$. The span of $y_{ij}y_{kl}$ is the representation $S^2(\bigwedge^2 V)/\bigwedge^4 V$, so it is the irreducible representation corresponding to the partition $(2,2,0,\dots,0)$. $\square$

5. The W_n-module structure on $M_{n,i}/M_{n,i+1}$

Let $\mathfrak{g}_n = \mathrm{Der}(Q_{n,3})$ be the Lie algebra of derivations of $Q_{n,3}$. Since every derivation of A_n preserves the ideals $M_{n,i}$, we have a natural action of $\mathrm{Der}(A_n)$ on $M_{n,i}/M_{n,i+1}$ and a natural homomorphism $\phi : \mathrm{Der}(A_n) \to \mathfrak{g}_n$. This homomorphism is surjective, since a derivation of A_n is determined by any assignment of the images of the generators x_i.

The following theorem is analogous to results of [FS].

Theorem 5.1. *The action of $\mathrm{Der}(A_n)$ on $M_{n,i}/M_{n,i+1}$ factors through $\mathfrak{g}_n$.*

Proof. Let $D : A_n \to A_n$ be a derivation such that $D(A_n) \subset M_{n,3}$. Our job is to show that $D(M_{n,i}) \subset M_{n,i+1}$ for $i \geqslant 1$. For this, it suffices to show that for any $a_1, \dots, a_i \in A_n$ one has

$$D\big[\dots[a_1, a_2], \dots, a_i\big] \in M_{n,i+1}.$$

For this, it is enough to prove that if $a_1, \dots, a_i \in A_n$, and for some $1 \leqslant k \leqslant i$, $a_k \in M_{n,3}$, then

$$\big[\dots[a_1, a_2], \dots, a_i\big] \in M_{n,i+1}.$$

It is easy to show by induction using the Jacobi identity that we can rewrite $[\dots[a_1, a_2], \dots, a_i]$ as a linear combination of expressions of the form $[\dots[a_k, a_{m_1}], \dots, a_{m_{i-1}}]$, where $(m_1, \dots, m_{i-1})$ is a permutation of $(1, \dots, \hat{k}, \dots, i)$ (k is omitted). Thus we may assume without loss of generality that $k = 1$. In this case, we have to show that for any $b_1, b_2, b_3, b_4 \in A$, one has

$$\big[\dots\big[b_1\big[[b_2, b_3], b_4\big], a_2\big], \dots, a_i\big] \in M_{n,i+1}.$$

This reduces to showing that for any $p, q \geqslant 0$ with $p + q = i - 1$, and any $a_1, \dots, a_p, c_1, \dots, c_q \in A_n$, we have

$$\mathrm{ad}(a_1)\dots\mathrm{ad}(a_p)(b_1) \cdot \mathrm{ad}(c_1)\dots\mathrm{ad}(c_q)\,\mathrm{ad}(b_4)\,\mathrm{ad}(b_3)(b_2) \in M_{n,i+1}.$$

But by Theorem 3.1, we have $M_{n,p+1}M_{n,q+3} \subset M_{n,p+q+2} = M_{n,i+1}$, which implies the desired statement, since the first factor is in $M_{n,p+1}$ and the second one in $M_{n,q+3}$. $\square$

It is pointed out in [FS] that, since $Q_{n,3}$ is the algebra of even differential forms on $\mathbb{C}^n$ with the $*$-product, the Lie algebra W_n of polynomial vector fields on $\mathbb{C}^n$ is naturally a subalgebra of $\mathfrak{g}_n$. Therefore, we get the following corollary.

Corollary 5.2. *There is a natural action of the Lie algebra W_n on the quotients $M_{n,i}/M_{n,i+1}$.*

It is clear from Theorem 2.1 that as W_n-modules, the quotients $M_{n,i}/M_{n,i+1}$ have finite length, and the composition factors are the modules $\mathcal{F}_D$ of tensor fields of type D (where D is a Young diagram) considered in [FS]. In fact, it follows from Theorem 2.1 that if $\tilde{\mathcal{F}}_D$ denotes the space of all polynomial tensor fields of type D (which may be reducible as a W_n-module if D has one column, i.e. in the case of differential forms), and if

$$K_{n,i} = \bigoplus N_D F_D,$$

where $N_D \in \mathbb{Z}_+$ and F_D is the irreducible representation of $GL(n)$ corresponding to D, then in the Grothendieck group of the category of representations of W_n we have

$$M_{n,i}/M_{n,i+1} = \sum N_D \mathcal{F}_D.$$

In particular, Corollary 4.4 implies that in the Grothendieck group,

$$M_{n,3}/M_{n,4} = \mathcal{F}_{2,1,0,\ldots,0} + \mathcal{F}_{2,2,0,\ldots,0}.$$

In fact, we can prove a stronger statement.

Proposition 5.3. *One has an isomorphism of representations*

$$M_{n,3}/M_{n,4} = \mathcal{F}_{2,1,0,\ldots,0} \oplus \mathcal{F}_{2,2,0,\ldots,0}.$$

Proof. Consider the subspace $Y_n := L_3(A_n)/(M_{n,4} \cap L_3(A_n)) \subset M_{n,3}/M_{n,4}$. By [FS], this is a W_n-subrepresentation. It is a proper subrepresentation, because $[x_1, x_2]^2 \in M_{n,3}/M_{n,4}$, but it is not contained in Y_n, as its trace in a matrix representation of A_n can be nonzero. On the other hand, Y_n contains $[x_1, [x_1, x_2]] \neq 0$, so $Y_n \neq 0$, and contains vectors of degree 3. This easily implies that $Y_n = \mathcal{F}_{2,1,0,\ldots,0}$. On the other hand, let Z_n be the subrepresentation generated by the elements $y_{ij} y_{kl} + y_{ik} y_{jl}$. These elements are annihilated by ∂_{x_i}, so they generate a subrepresentation whose lowest degree is 4. Thus, $Z_n = \mathcal{F}_{2,2,0,\ldots,0}$, and $M_{n,3}/M_{n,4} = Y_n \oplus Z_n$, as desired. □

It would be interesting to determine the structure of the representations $M_{n,i}/M_{n,i+1}$ when $i > 3$.

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