



Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra



Non-commutative Castelnuovo–Mumford regularity and AS-regular algebras

Z.-C. Dong, Q.-S. Wu*

School of Mathematical Sciences, Fudan University, Shanghai 200433, China

ARTICLE INFO

Article history:

Received 3 August 2008

Available online 1 April 2009

Communicated by J.T. Stafford

Keywords:

Non-commutative graded algebras

Non-commutative projective geometry

Castelnuovo–Mumford regularity

AS-Gorenstein algebras

AS-regular algebras

ABSTRACT

Let A be a connected graded k -algebra with a balanced dualizing complex. We prove that A is a Koszul AS-regular algebra if and only if that the Castelnuovo–Mumford regularity and the Ext-regularity coincide for all finitely generated A -modules. This can be viewed as a non-commutative version of [T. Römer, On the regularity over positively graded algebras, J. Algebra 319 (2008) 1–15, Theorem 1.3]. By using Castelnuovo–Mumford regularity, we prove that any Koszul standard AS-Gorenstein algebra is AS-regular. As a preparation to prove the main result, we also prove that the following statements are equivalent: (1) A is AS-Gorenstein; (2) A has finite left injective dimension; (3) the dualizing complex has finite left projective dimension. This generalizes [I. Mori, Homological properties of balanced Cohen–Macaulay algebras, Trans. Amer. Math. Soc. 355 (2002) 1025–1042, Corollary 5.9].

© 2009 Elsevier Inc. All rights reserved.

1. Introduction

A coherent sheaf \mathcal{M} on \mathbb{P}^n is called m -regular in the sense of Castelnuovo if $H^i(\mathcal{M}(m-i)) = 0$ for all $i \geq 1$. Mumford proved a vanishing theorem [Mu]: if \mathcal{M} is m -regular, then it is \bar{m} -regular for all $\bar{m} > m$. Motivated by Mumford's vanishing theorem, a notion of regularity was introduced by Eisenbud and Goto [EG] for graded modules over polynomial algebras. This notion of regularity is closely related to the regularity of sheaves, also to the existence of linear free resolutions of the truncations of graded modules and to the degrees of generators of the syzygies. If $A = k[x_1, x_2, \dots, x_{n+1}]$, so that $\text{Proj } A = \mathbb{P}^n$, and M is an A -module of the form $\bigoplus_{i \geq 0} H^0(\mathbb{P}^n, \mathcal{O}_X(i))$ for some scheme $X \subset \mathbb{P}^n$, then the regularity of M is the regularity of $X \subset \mathbb{P}^n$ in the sense of Castelnuovo, studied

* Corresponding author.

E-mail addresses: dongzhicheng@msn.com (Z.-C. Dong), qswu@fudan.edu.cn (Q.-S. Wu).

by Mumford [Mu] and many other authors in literature. There are several competing definitions of Castelnuovo–Mumford regularity for a graded module over a commutative or non-commutative connected graded k -algebra A , for example, $\text{CM.reg}_A M$ defined by using local cohomology (Definition 4.1 and Remark 4.2), $\text{Ext.reg}_A M$ defined by Ext-group (Definition 4.4) and $\text{Tor.reg}_A M$ defined by Tor-group (Remark 4.5). The Ext-regularity and Tor-regularity of a finitely generated graded module are always the same. If A is a polynomial algebra with standard grading, Eisenbud and Goto [EG] proved that $\text{CM.reg } M = \text{Tor.reg } M$ for all non-zero finitely generated graded A -modules. Römer [Rö] proved that the converse is true, i.e., if A is a commutative connected graded k -algebra generated in degree 1, such that $\text{CM.reg } M = \text{Tor.reg } M$ holds for all non-zero finitely generated graded A -modules, then A is a polynomial algebra with standard grading (see Theorem 5.1). Castelnuovo–Mumford regularity in the non-commutative setting was first studied by Jørgensen [Jo3,Jo4]. Jørgensen proved a version of Mumford’s vanishing theorem for non-commutative projective schemes [Jo3, Theorem 2.4 and Corollary 3.3]. Under the assumption that A is a Noetherian connected graded k -algebra having a balanced dualizing complex, Jørgensen proved that for any finitely generated non-zero A -module M , $-\text{CM.reg } A \leq \text{Ext.reg } M - \text{CM.reg } M \leq \text{Ext.reg } k$ [Jo4, Theorems 2.5, 2.6]. In the commutative case, this was also proved by Römer [Rö, Theorem 1.2]. For a Noetherian connected graded k -algebra A , by a result of Van den Bergh [VdB, Theorem 6.3], A has a balanced dualizing complex if and only if that A has finite left and right local cohomology dimension (see Definition 2.3) and A satisfies the left and right χ -condition (see Definition 2.4). The χ -condition is easy to check when A is commutative. If A is commutative, then the Krull dimension of A is finite. By a theorem of Grothendieck (see [BH, Theorem 3.5.7]), the local cohomology dimension of A is finite. These conditions are very natural and important in non-commutative projective geometry [AZ]. In this paper, we will prove a non-commutative version of Römer’s result [Rö, Theorem 1.3].

Theorem (Theorem 5.4). *Let A be a Noetherian connected graded k -algebra having a balanced dualizing complex. Then A is Koszul AS-regular if and only if $\text{CM.reg } M = \text{Ext.reg } M$ holds for all non-zero finitely generated graded A -modules.*

Our proof is different from Römer’s in the commutative case. Römer used the following fact that every commutative Noetherian connected graded k -algebra is isomorphic to a graded quotient of some polynomial algebra. However, in the non-commutative case, it is still an unsolved problem whether a Noetherian connected graded k -algebra with a balanced dualizing complex is isomorphic to a graded quotient of an AS-regular (Gorenstein) algebra. Actually we will prove that (iii) \Leftrightarrow (iv) \Leftrightarrow (v) in [Rö, Theorem 4.1] (see also Theorem 5.1) holds in the non-commutative case. Note that (i) \Leftrightarrow (ii) is always true. However, (i) \Leftrightarrow (iii) does not hold in general (see Remark 5.5). The following crucial fact is used in the Römer’s proof: Let A be a commutative Noetherian connected graded k -algebra generated in degree 1. Then A is Koszul if and only if $\text{Ext.reg } k < \infty$. This was conjectured in [AE], and proved by L.L. Avramov and I. Peeva in [AP]. However, this fact is not true in the non-commutative case since $\text{Ext.reg } k < \infty$ holds for those non-Koszul AS-regular algebras (see Proposition 5.6 and Remark 5.5).

Before giving the proof of the main result, we prove the following result by using Castelnuovo–Mumford regularity, which is of independent interest.

Theorem (Theorem 4.10). *Every Koszul standard AS-Gorenstein algebra is AS-regular.*

An AS-Gorenstein algebra of type (d, l) is called standard if $d = l$ (Definition 3.1).

We also generalize a result of Mori [Mo, Corollary 5.9] as a preparation to the proof of the main result.

Theorem (Theorem 3.6). *Let A be a connected graded k -algebra with a balanced dualizing complex. Then the following are equivalent:*

- (1) A is AS-Gorenstein;

- (2) A has finite left injective dimension;
- (3) the balanced dualizing complex has finite left projective dimension.

2. Notations and preliminaries

Throughout this article, k is a fixed field. An \mathbb{N} -graded k -algebra $A = \bigoplus_{i \geq 0} A_i$ is called *connected graded* if $A_0 = k$. We write $\mathfrak{m} = \bigoplus_{i \geq 1} A_i$. When A is connected graded, the graded module ${}_A k \cong A/\mathfrak{m}$ (or k_A) is called the *trivial module*.

The opposite algebra of A is denoted by A^o ; it is the same as A as k -vector spaces, but the product is given by $a \cdot b = ba$. A right graded A -module can be identified with a left graded A^o -module. Unless otherwise stated, we are working with left modules.

A graded module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is called *left-bounded* if $M_i = 0$ for $i \ll 0$; *right-boundedness* and *boundedness* are defined similarly. If each graded piece M_i of M is a finite-dimensional k -vector space, then M is called *locally finite*. For any fixed $n \in \mathbb{Z}$, we use the notations $M_{>n} = \bigoplus_{i=n+1}^{\infty} M_i$, $M_{<n} = \bigoplus_{i < -\infty}^{n-1} M_i$ and $M_{\neq n} = \bigoplus_{i \neq n} M_i$. For any graded A -module M , the n th *shift* $M(n)$ of M , is defined by $M(n)_i = M_{n+i}$.

Let $\text{Gr } A$ denote the category of graded left A -modules and graded homomorphisms of degree 0, and $\text{gr } A$ be the full subcategory of $\text{Gr } A$ consisting of finitely generated graded A -modules. Let $\text{Hom}_{\text{Gr } A}(-, -)$ be the Hom-functor in the category $\text{Gr } A$.

Let A and B be \mathbb{N} -graded k -algebras. The category $\text{Gr } A^o$ is identified with the category of graded right A -modules; the category $\text{Gr } A \otimes_k B^o$ is identified with the category of graded A - B -bimodules. In particular, the category of graded A - A -bimodules is denoted by $\text{Gr } A^e$, where $A^e = A \otimes_k A^o$. The natural restriction functors are denoted by

$$\text{res}_A : \text{Gr } A \otimes_k B^o \rightarrow \text{Gr } A$$

and

$$\text{res}_{B^o} : \text{Gr } A \otimes_k B^o \rightarrow \text{Gr } B^o.$$

We define the graded Hom-functor by

$$\underline{\text{Hom}}_A(M, N) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\text{Gr } A}(M, N(n)).$$

If M is a graded A - B -bimodule and N is a graded A - C -bimodule, then $\underline{\text{Hom}}_A(M, N)$ is a graded B - C -bimodule.

For any $M \in \text{Gr } A \otimes_k B^o$, $M' = \underline{\text{Hom}}_k(M, k)$ is called its *Matlis dual*. By definition, M' becomes a graded B - A -bimodule. Thus there is an exact contravariant functor from $\text{Gr } A \otimes_k B^o$ into $\text{Gr } B \otimes_k A^o$. If M is locally finite, then $M'' \cong M$ as A - B -bimodules.

Let X^\cdot be a cochain complex of graded left A -modules. The i th cohomology module of X^\cdot is denoted by $h^i(X^\cdot)$. Shifting of complexes is denoted by $[]$ so that $(X^\cdot[n])^p = X^{n+p}$. We denote by $Z^n(X^\cdot) = \text{Ker}(d_X^n)$ and $B^n(X^\cdot) = \text{Im}(d_X^{n-1})$. Let $X^{\geq n}$ denote the (brutal) truncated complex

$$0 \rightarrow X^n \rightarrow X^{n+1} \rightarrow \dots,$$

and let $X^{\leq n}$ denote the complex

$$\dots \rightarrow X^{n-1} \rightarrow X^n \rightarrow 0.$$

The homotopy category of $\text{Gr } A$ is denoted by $K(\text{Gr } A)$ and the derived category of $\text{Gr } A$ is denoted by $D(\text{Gr } A)$. For any $X^\cdot \in D(\text{Gr } A)$, we define

$$\sup X^\cdot = \sup\{i \mid h^i(X^\cdot) \neq 0\} \quad \text{and} \quad \inf X^\cdot = \inf\{i \mid h^i(X^\cdot) \neq 0\}.$$

We fix the conventions: $\sup\{\mathbb{Z}\} = \inf\{\emptyset\} = +\infty$, and $\inf\{\mathbb{Z}\} = \sup\{\emptyset\} = -\infty$. When $\sup X^\cdot < \infty$ (resp. $\inf X^\cdot > -\infty$), X^\cdot is said to be *bounded above* (resp. *bounded below*). If X^\cdot is both bounded above and bounded below, then X^\cdot is said to be *bounded*. In this case, $\text{amp } X^\cdot = \sup X^\cdot - \inf X^\cdot$ is called the *amplitude* of X^\cdot . There are various full subcategories $D^*(\text{Gr } A)$ of $D(\text{Gr } A)$, where $*$ = $-$, $+$, b , consisting of bounded above, bounded below and bounded complexes respectively. There are also full subcategories $D_*(\text{Gr } A)$ of $D(\text{Gr } A)$, where $*$ = fg , lf , consisting of complexes with finitely generated cohomologies and locally finite cohomologies respectively. Super- and subscripts are combined freely.

Let A, B , and C be \mathbb{N} -graded k -algebras. For any $X^\cdot \in K(A \otimes_k B^0)$ and any $Y^\cdot \in K(A \otimes_k C^0)$, $\text{Hom}_A(X^\cdot, Y^\cdot)$ is a complex in $K(B \otimes_k C^0)$ where the n th term is

$$\text{Hom}_A^n(X^\cdot, Y^\cdot) = \prod_p \underline{\text{Hom}}_A^n(X^p, Y^{p+n}),$$

and the differential $d_{\text{Hom}_A(X^\cdot, Y^\cdot)}^n$ is

$$(f_p)_p \rightsquigarrow (f_{p+1} \cdot d_{X^\cdot}^p - (-1)^n d_{Y^\cdot}^{p+n} \cdot f_p)_p.$$

This induces a bi- ∂ -functor,

$$\text{Hom}_A(-, -) : K(A \otimes_k B^0) \times K(A \otimes_k C^0) \rightarrow K(B \otimes_k C^0).$$

The right-derived functor of Hom is denoted by $R\underline{\text{Hom}}$.

Similarly, for any $X^\cdot \in K(B \otimes_k A^0)$ and $Y^\cdot \in K(A \otimes_k C^0)$, $X^\cdot \otimes_A Y^\cdot \in K(B \otimes_k C^0)$, where the n th component $(X^\cdot \otimes_A Y^\cdot)^n$ of $X^\cdot \otimes_A Y^\cdot$ is the $B \otimes_k C^0$ -module $\bigoplus_{p+q=n} X^p \otimes_A Y^q$ and the tensor differential $d_{X^\cdot \otimes_A Y^\cdot} = d_{X^\cdot} \otimes \text{id}_Y + \text{id}_X \otimes d_{Y^\cdot}$ (with Koszul sign rule). This induces a bi- ∂ -functor,

$$-\otimes_A - : K(B \otimes_k A^0) \times K(A \otimes_k C^0) \rightarrow K(B \otimes_k C^0).$$

The left derived functor of \otimes is denoted by ${}^L\otimes$.

The Ext and Tor are defined as

$$\underline{\text{Ext}}_A^i(X^\cdot, Y^\cdot) = h^i(R\underline{\text{Hom}}_A(X^\cdot, Y^\cdot)) \quad \text{and} \quad \text{Tor}_i^A(X^\cdot, Y^\cdot) = h^{-i}(X^\cdot {}^L\otimes_A Y^\cdot).$$

Definition 2.1. Let A be a connected graded k -algebra and let $X^\cdot \in D^-(\text{Gr } A)$. A projective resolution of X^\cdot is a complex P^\cdot consisting of graded projective modules such that there is a quasi-isomorphism $f : P^\cdot \xrightarrow{\sim} X^\cdot$. Moreover, if $B^i(P^\cdot) \subseteq \mathfrak{m}P^i$ for each $i \in \mathbb{Z}$, then P^\cdot is called a minimal projective resolution of X^\cdot .

We call

$$\text{pd}_A(X^\cdot) = \inf_{P^\cdot} (-\inf\{i \mid P^i \neq 0\})$$

the projective dimension of X^\cdot , where the infimum is taken over all projective resolutions P^\cdot of X^\cdot .

If A is left Noetherian and $X^\cdot \in D_{\text{fg}}^-(\text{Gr } A)$, then

$$\text{pd}_A(X^\cdot) = \sup\{i \mid \underline{\text{Ext}}_A^i(X^\cdot, M) \neq 0 \text{ for some } M \in \text{gr } A\}.$$

If $P^\cdot \xrightarrow{\sim} X^\cdot$ is a minimal free resolution of X^\cdot , then $\text{pd}_A(X^\cdot) = -\inf\{i \mid P^i \neq 0\}$.

Definition 2.2. Let A be an \mathbb{N} -graded k -algebra and let $X^\cdot \in D^+(\text{Gr } A)$. An injective resolution of X^\cdot is a complex I^\cdot consisting of graded injective modules such that there is a quasi-isomorphism $f : X^\cdot \xrightarrow{\sim} I^\cdot$. Moreover, if $Z^i(I^\cdot)$ is graded essential in I^i for each $i \in \mathbb{Z}$, then I^\cdot is called a minimal injective resolution of X^\cdot .

We call

$$\text{id}_A(X^\cdot) = \inf_I (\sup\{i \mid I^i \neq 0\})$$

the injective dimension of X^\cdot , where the infimum is taken over all injective resolutions I^\cdot of X^\cdot .

It is easy to know that

$$\text{id}_A(X^\cdot) = \sup\{i \mid \underline{\text{Ext}}_A^i(M, X^\cdot) \neq 0 \text{ for some } M \in \text{gr } A\}.$$

Next we collect some definitions and facts in connected graded ring theory we need.

Definition 2.3. (See [AZ, Ye]). Let A be an \mathbb{N} -graded k -algebra and $\mathfrak{m} = A_{\geq 1}$.

(1) For any $M \in \text{Gr } A$, the \mathfrak{m} -torsion submodule of M is defined to be

$$\Gamma_{\mathfrak{m}}(M) = \{x \in M \mid A_{\geq n} \cdot x = 0, \text{ for } n \gg 0\}.$$

If $\Gamma_{\mathfrak{m}}(M) = M$, then M is said to be \mathfrak{m} -torsion.

(2) $\Gamma_{\mathfrak{m}} : \text{Gr } A \rightarrow \text{Gr } A, M \mapsto \Gamma_{\mathfrak{m}}(M)$, is a left exact functor. Its right derived functor $R\Gamma_{\mathfrak{m}}$ is defined on the derived category $D^+(\text{Gr } A)$. The i th local cohomology of $X^\cdot \in D^+(\text{Gr } A)$ is defined to be

$$H_{\mathfrak{m}}^i(X^\cdot) = h^i(R\Gamma_{\mathfrak{m}}(X^\cdot)).$$

(3) The local cohomological dimension of a graded A -module M is defined to be

$$\text{lcd}(M) = \sup\{i \mid H_{\mathfrak{m}}^i(M) \neq 0\}.$$

(4) The cohomological dimension of $\Gamma_{\mathfrak{m}}$ is defined to be

$$\text{cd}(\Gamma_{\mathfrak{m}}) = \sup\{\text{lcd}(M) \mid M \in \text{Gr } A\}.$$

(5) $\text{cd}(\Gamma_{\mathfrak{m}})$ is also called the left local cohomological dimension of the algebra A .

Definition 2.4. (See [AZ, Definition 3.2].) Let A be a Noetherian connected graded k -algebra. Then A is said to satisfy the χ -condition if $\underline{\text{Ext}}_A^i(k, M)$ is right bounded for any $M \in \text{gr } A$ and $i \in \mathbb{Z}$.

The χ -condition arises naturally in non-commutative algebraic geometry. χ can fail for Noetherian connected graded algebras [SZ]. But χ do holds for a large class of algebras. If A is Noetherian PI, then A satisfies the χ -condition [SZ1, Theorem 3.5].

Definition 2.5. Let A be a connected graded k -algebra. For any $X^\cdot \in D^+(\text{Gr } A)$,

$$\text{depth}_A(X^\cdot) = \inf R\underline{\text{Hom}}_A(k, X^\cdot) = \inf\{i \mid \underline{\text{Ext}}_A^i(k, X^\cdot) \neq 0\}.$$

Then $\text{depth}_A(X^\cdot)$ is either an integer or ∞ . The following lemma is well known.

Lemma 2.6. Let A be a left Noetherian connected graded k -algebra. Then for any $M \in \text{Gr } A$,

$$\text{depth}_A(M) = \inf\{i \in \mathbb{Z} \mid H_m^i(M) \neq 0\}.$$

Proof. Let $0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$ be a minimal injective resolution of M . Then $\underline{\text{Ext}}_A^i(k, M) = \underline{\text{Hom}}_A(k, I^i)$.

If $\text{depth}_A(M) = \infty$, then $\underline{\text{Ext}}_A^i(k, M) = 0$ for all $i \in \mathbb{Z}$. It follows from $\Gamma_m(I^i) = E(\underline{\text{Hom}}_A(k, I^i))$ that $H_m^i(M) = 0$ for all $i \in \mathbb{Z}$. Thus the assertion follows.

If $\text{depth}_A(M) = n < \infty$, then $\Gamma_m(I^j) = E(\underline{\text{Hom}}_A(k, I^j)) = 0$ for $j < n$ and $E(H_m^n(M)) = \Gamma_m(I^n) = E(\underline{\text{Hom}}_A(k, I^n)) \neq 0$. Hence $H_m^j(M) = 0$ for $j < n$ and $H_m^n(M) \neq 0$. Thus the assertion follows. \square

Theorem 2.7 (The Auslander–Buchsbaum formula). Let A be a left Noetherian connected graded k -algebra satisfying the χ -condition. Given any $X' \in D_{\text{fg}}^b(\text{Gr } A)$ with $\text{pd}_A(X') < \infty$, one has

$$\text{pd}_A(X') + \text{depth}_A(X') = \text{depth}_A(A).$$

Proof. See [Jo2, Theorem 3.2]. \square

Definition 2.8. (See [Ye, Definition 3.3].) Let A be a Noetherian connected graded k -algebra. A complex $R' \in D^b(\text{Gr } A^e)$ is called a dualizing complex if it satisfies the following conditions:

- (1) $\text{id}_A(R') < \infty$ and $\text{id}_{A^0}(R') < \infty$;
- (2) $\text{res}_A(R') \in D_{\text{fg}}^b(\text{Gr } A)$ and $\text{res}_{A^0}(R') \in D_{\text{fg}}^b(\text{Gr } A^0)$;
- (3) The natural morphisms $A \rightarrow R\underline{\text{Hom}}_A(R', R')$ and $A \rightarrow R\underline{\text{Hom}}_{A^0}(R', R')$ are isomorphisms in $D^b(\text{Gr } A^e)$.

Definition 2.9. (See [Ye, Definition 4.1].) Let A be a Noetherian connected graded k -algebra and $R' \in D^b(\text{Gr } A^e)$ be a dualizing complex over A . If there are isomorphisms $R\Gamma_m(R') \cong R\Gamma_{m^0}(R') \cong A'$ in $D(\text{Gr } A^e)$, then R' is called balanced.

Theorem 2.10. (See [VdB, Theorem 6.3].) Let A be a Noetherian connected graded k -algebra. Then A has a balanced dualizing complex if and only if the following two conditions are satisfied:

- (1) $\text{cd}(\Gamma_m) < \infty$, and $\text{cd}(\Gamma_{m^0}) < \infty$;
- (2) Both A and A^0 satisfy the χ -condition.

If these conditions are satisfied, then the balanced dualizing complex over A is given by $R\Gamma_m(A)'$.

Theorem 2.11 (The local duality theorem). Let A, C be connected graded k -algebras. Assume that A is left Noetherian with $\text{cd}(\Gamma_m) < \infty$. Then for any $X' \in D(\text{Gr } A \otimes_k C^0)$, there is an isomorphism

$$R\Gamma_m(X')' \cong R\underline{\text{Hom}}_A(X', R\Gamma_m(A)')$$

in $D(\text{Gr } C \otimes_k A^0)$.

Proof. See [VdB, Theorem 5.1]. \square

Our basic reference for homological algebra is [We].

3. AS-Gorenstein algebras

As a preparation to prove the main result (Theorem 5.4), we generalize a result of Mori [Mo, Corollary 5.9] in this section. Let us recall some definitions.

Definition 3.1. Let A be a Noetherian connected graded k -algebra. A is called left AS-Gorenstein (AS stands for Artin–Schelter) if

- (1) $\text{id}_A A = d < \infty$;
- (2) $\underline{\text{Ext}}_A^i(k, A) = \begin{cases} 0, & i \neq d, \\ k(l), & i = d, \end{cases}$ for some $l \in \mathbb{Z}$.

Right AS-Gorenstein algebras are defined similarly. An AS-Gorenstein algebra means that it is both left and right AS-Gorenstein.

A left AS-Gorenstein algebra in the Definition 3.1 is sometimes called of type (d, l) . If A is left AS-Gorenstein of type (d, l) and right AS-Gorenstein, say of type (d', l') , then it is well known that $d = d'$ and $l = l'$.

Definition 3.2. Let A be a Noetherian connected graded k -algebra.

- (1) A is called *standard* AS-Gorenstein if A is an AS-Gorenstein algebra with $l = d$.
- (2) A is called AS-regular (Artin–Schelter regular) if A is AS-Gorenstein and A has finite global dimension.

Definition 3.3. Let A be a Noetherian connected graded k -algebra.

- (1) A is called AS-Cohen–Macaulay if $R\Gamma_m(A)$ is concentrated in one degree.
- (2) An A – A -bimodule ω_A is called a balanced dualizing module if $\omega_A[d]$ is a balanced dualizing complex over A for some integer d .
- (3) A is called balanced Cohen–Macaulay if it has a balanced dualizing module.

Lemma 3.4. Let A be a Noetherian connected graded k -algebra with a balanced dualizing complex R^\cdot . If $\text{pd}_A R^\cdot < \infty$, then A is AS-Cohen–Macaulay and balanced Cohen–Macaulay.

Proof. Let $F^\cdot \xrightarrow{\sim} R^\cdot$ be a finitely generated minimal free resolution of $R^\cdot \in D_{\text{fg}}^b(\text{Gr } A)$. Then

$$\text{pd}_A R^\cdot = -\inf\{p \in \mathbb{Z} \mid F^p \neq 0\} \geq -\inf R^\cdot.$$

By the local duality theorem, $R\Gamma_m(k)' \cong \text{RHom}_A(k, R^\cdot)$ (see Theorem 2.11). Hence $\text{depth}_A R^\cdot = 0$. It follows from Auslander–Buchsbaum formula (Theorem 2.7) that $-\inf R^\cdot \leq \text{pd}_A R^\cdot = \text{depth}_A A$. By Lemma 2.6 and $R^\cdot \cong R\Gamma_m(A)'$,

$$\text{depth}_A A = \inf\{j \in \mathbb{Z} \mid H_m^j(A) \neq 0\} = -\sup R^\cdot.$$

Therefore $\sup R^\cdot = \inf R^\cdot$, and so A is AS-Cohen–Macaulay and balanced Cohen–Macaulay. \square

Lemma 3.5. Let A be a Noetherian connected graded k -algebra with a dualizing complex R^\cdot . Then for any $X \in D_{\text{fg}}^b(\text{Gr } A)$, the following holds:

- (1) $\text{id}_{A^0}(\text{RHom}_A(X, R^\cdot)) \leq \text{pd}_A X + \text{id}_{A^0} R^\cdot$;
- (2) $\text{pd}_{A^0}(\text{RHom}_A(X, R^\cdot)) \leq \text{id}_A X - \inf R^\cdot$.

As a consequence, $\text{pd}_A X < \infty$ if and only if $\text{id}_{A^0}(\text{RHom}_A(X, R^\cdot)) < \infty$.

Proof. See [WZ, Lemma 2.1]. \square

The following theorem says that if the balanced dualizing complex has finite projective dimension, then A is AS-Gorenstein. This is a generalization of [Mo, Corollary 5.9], where A is a Noetherian balanced Cohen–Macaulay algebra. For the completeness, we give a proof here.

Theorem 3.6. *Let A be a Noetherian connected graded k -algebra with a balanced dualizing complex R . Then the following are equivalent:*

- (1) A is AS-Gorenstein;
- (2) $\text{id}_A A < \infty$;
- (3) $\text{pd}_A R < \infty$;
- (4) For any $X \in D_{\text{fg}}^b(\text{Gr } A)$, $\text{pd}_A X < \infty$ if and only if $\text{id}_A X < \infty$.

Proof. (1) \Rightarrow (4). Suppose that $M \in \text{gr } A$ with $\text{pd}_A M < \infty$. Then M has a finitely generated free resolution of finite length. Since each term of the resolution has finite injective dimension, so does M . By induction on the amplitude, it is easy to see that $\text{pd}_A X < \infty$ implies $\text{id}_A X < \infty$ for any $X \in D_{\text{fg}}^b(\text{Gr } A)$.

On the other hand, suppose first that $M \in \text{gr } A$ with $\text{id}_A M < \infty$. There is a convergent spectral sequence

$$E_2^{p,q} = \underline{\text{Ext}}_A^p(\underline{\text{Ext}}_{A^0}^{-q}(k, A), M) \Rightarrow \text{Tor}_{-p-q}^A(k, M).$$

Since $\text{id}_A A < \infty$, $E_2^{p,q} = 0$ for $q \ll 0$. Hence $\text{Tor}_n^A(k, M) = 0$ for $n \gg 0$ and so $\text{pd}_A M < \infty$. Now suppose in general that $X \in D_{\text{fg}}^b(\text{Gr } A)$ with $\text{id}_A X < \infty$. We have to show that $\text{pd}_A X < \infty$. Let $F \cong X$ be a finitely generated free resolution of X and $s = \inf X$. It suffices to prove the s th syzygy of F has finite projective dimension. Since $\text{id}_A A < \infty$ and $\text{id}_A X < \infty$, any finitely generated free module has finite injective dimension, and so the s th syzygy of F has finite injective dimension. The claim follows from the module case.

(4) \Rightarrow (3). Since $\text{id}_A R < \infty$, it follows that $\text{pd}_A R < \infty$.

(3) \Rightarrow (2). By Lemma 3.4, $\omega_A = H^{-d}(R)$ is a balanced dualizing module of A , where $\text{pd}_A R = d$ and $\text{pd}_A(\omega_A) < \infty$. Since $\text{depth}_A(\omega_A) = \text{depth}_A A$, it follows from Auslander–Buchsbaum formula (Theorem 2.7) that $\text{pd}_A(\omega_A) = 0$. So ω_A is free. Since $\text{id}_A(\omega_A) < \infty$, it follows that $\text{id}_A A < \infty$.

(2) \Rightarrow (1). Since A has a balanced dualizing complex, A satisfies the χ -condition. Since $\text{id}_A A < \infty$, $c = \sup\{i \mid \underline{\text{Ext}}_A^i(k, A) \neq 0\} < \infty$. It follows from the double Ext spectral sequence

$$E_2^{p,q} = \underline{\text{Ext}}_A^p(\underline{\text{Ext}}_{A^0}^{-q}(k, A), A) \Rightarrow \begin{cases} 0, & p + q \neq 0, \\ k, & p + q = 0, \end{cases}$$

that $d := \text{depth}_{A^0} A < \infty$. Since A^0 satisfies the χ -condition, it follows from the definition of c that $E_2^{c,-d} = E_{\infty}^{c,-d} \neq 0$. So $c = d$, that is,

$$\text{depth}_{A^0} A = \sup\{i \mid \underline{\text{Ext}}_A^i(k, A) \neq 0\}.$$

Since $\text{id}_A A < \infty$, by Lemma 3.5 $\text{pd}_{A^0} R < \infty$. Similar to (3) \Rightarrow (2), we have $\text{id}_{A^0} A < \infty$. Thus by the left–right symmetric version of the above spectral sequence,

$$\text{depth}_A A = \sup\{i \mid \underline{\text{Ext}}_{A^0}^i(k, A) \neq 0\}.$$

Hence

$$\text{depth}_A A = \sup\{i \mid \underline{\text{Ext}}_A^i(k, A) \neq 0\} = \text{depth}_{A^0} A = \sup\{i \mid \underline{\text{Ext}}_{A^0}^i(k, A) \neq 0\}.$$

Therefore A is AS Gorenstein. \square

The idea in (2) \Rightarrow (1) originates from [SZ1, Theorem 3.8] and [Z, Theorem 0.3]. The above result also tells us that if A is a Noetherian connected graded k -algebra with a balanced dualizing complex, then A is left AS-Gorenstein if and only if A is right AS-Gorenstein. This is [Jo5, Corollary 4.6]. Note that the first condition in Theorem 3.6 is two-sided, while the others are one-sided. In particular, when A has a balanced dualizing complex, then A has finite left injective dimension if and only if that A has finite right injective dimension.

4. Castelnuovo–Mumford regularity and AS-Gorenstein algebras

In this section, we first recall the definitions and some facts of Castelnuovo–Mumford regularity and Ext-regularity. Then we prove any Koszul standard AS-Gorenstein algebra is AS-regular (see Definition 3.2). Note that A is a Noetherian connected graded k -algebra as always in this article.

Definition 4.1. (See [Jo4, Definition 2.1].) For any $X \in D(\text{Gr } A)$, the Castelnuovo–Mumford regularity of X is defined to be

$$\text{CM.reg } X = \inf\{p \in \mathbb{Z} \mid H_m^i(X)_{>p-i} = 0, \forall i \in \mathbb{Z}\}.$$

If A has a balanced dualizing complex and $0 \not\cong X \in D_{\text{fg}}^b(\text{Gr } A)$, then $R\Gamma_m(X)' \in D_{\text{fg}}^b(\text{Gr } A^0)$ and $\text{CM.reg } X \neq \pm\infty$ by the local duality theorem (Theorem 2.11) (see [Jo4, Observation 2.3]). Moreover, we have $\text{CM.reg}(A) = \text{CM.reg}(A_A)$ by [VdB, Corollary 4.8].

Remark 4.2. The Castelnuovo–Mumford regularity is called local-regularity by Römer [Rö, Definition 1.1].

Example 4.3. Since ${}_A k$ is m -torsion, we have

$$H_m^i({}_A k) = \begin{cases} 0, & i \neq 0, \\ k, & i = 0. \end{cases}$$

By Definition 4.1, it is easy to see that $\text{CM.reg } {}_A k = 0$.

Definition 4.4. (See [Jo4, Definition 2.2].) For any $X \in D(\text{Gr } A)$, the Ext-regularity of X is defined to be

$$\text{Ext.reg } X = \inf\{p \in \mathbb{Z} \mid \underline{\text{Ext}}_A^i(X, k)_{<-p-i} = 0, \forall i \in \mathbb{Z}\}.$$

Remark 4.5. There is also a notion of regularity defined by Tor-group in literature, which is called the Tor-regularity as in [Rö, 1.1] by Römer. For any $M \in \text{Gr } A$, $\text{Tor.reg}_A(M)$ or $\text{reg}_A^T(M) = \inf\{p \in \mathbb{Z} \mid \text{Tor}_i^A(k_A, M)_{>p+i} = 0, \forall i \geq 0\}$ is called the Tor-regularity of M . If $M \in \text{gr } A$, then as graded k -vector spaces,

$$\underline{\text{Ext}}_A^j(M, k)' \cong \text{Tor}_j^A(k_A, M).$$

Hence $\text{Ext.reg } M = \text{reg}_A^T(M)$, i.e., the Tor-regularity is the same as the Ext-regularity for any $M \in \text{gr } A$.

For all $0 \not\cong X' \in D_{\text{fg}}^b(\text{Gr } A)$, $\text{Ext.reg } X' \neq -\infty$. If $\text{Ext.reg } X \leq p$ and $F' \cong X'$ is a minimal free resolution of X' , then the generators of F^i are concentrated in degree $\leq p - i$. It is possible that $\text{Ext.reg } X' = \infty$. Note that $\text{Ext.reg}({}_A k) = 0$ if and only if A is Koszul, i.e., ${}_A k$ has a linear free resolution. It was conjectured in [AE] and proved in [AP] that a commutative connected graded k -algebra generated in degree 1 is Koszul if and only if that $\text{Ext.reg } k < \infty$.

By using the minimal free resolution, we see that $\text{Ext.reg}({}_A k) = \text{Ext.reg}(k_A)$.

Example 4.6. Since ${}_A A$ is free,

$$\underline{\text{Ext}}_A^i(A, k) = \begin{cases} 0, & i \neq 0, \\ k, & i = 0. \end{cases}$$

By Definition 4.4, $\text{Ext.reg } {}_A A = 0$.

The following result was proved in [Jo4, Theorems 2.5, 2.6], which plays a key role in this article.

Theorem 4.7. *Let A be a Noetherian connected graded k -algebra with a balanced dualizing complex. Given any $X' \in D_{\text{fg}}^b(\text{Gr } A)$ with $X' \not\cong 0$. Then*

$$-\text{CM.reg } A \leq \text{Ext.reg } X' - \text{CM.reg } X' \leq \text{Ext.reg } k.$$

If further, A is Koszul and $\text{CM.reg } A = 0$, then $\text{CM.reg } X' = \text{Ext.reg } X'$ for any $0 \not\cong X' \in D_{\text{fg}}^b(\text{Gr } A)$.

It is an easy fact that any Koszul AS-regular algebra is standard (see 4.10). However, a Koszul AS-Gorenstein algebra is not always standard, e.g. $A = k[x]/(x^2)$ is a Noetherian Koszul connected graded k -algebra which has infinite global dimension. Moreover, $\text{id}_A(A) = 0$ and $\underline{\text{Ext}}_A^0(k, A) \cong k(-1)$. Thus, A is a non-standard AS-Gorenstein algebra with $l = -1$ and $d = 0$. In the final part of this section, we prove that any Koszul standard AS-Gorenstein algebra is AS-regular.

Lemma 4.8. *Let A be an AS-Gorenstein algebra of type (d, l) . Then*

$$\text{CM.reg } A = d - l.$$

Proof. By Definition 3.1,

$$\underline{\text{Ext}}_A^j(k, A) = \begin{cases} 0, & j \neq d, \\ k(l), & j = d. \end{cases}$$

Then

$$H_m^i(A) \cong \begin{cases} 0, & i \neq d, \\ A'(l), & i = d. \end{cases}$$

It follows from Definition 4.1 that $\text{CM.reg } A = d - l$. \square

Corollary 4.9. *Let A be a Koszul standard AS-Gorenstein algebra. Then for any $X' \in D_{\text{fg}}^b(\text{Gr } A)$ with $X' \not\cong 0$,*

$$\text{CM.reg } X = \text{Ext.reg } X.$$

Proof. Direct from Theorem 4.7 and Lemma 4.8 \square

Theorem 4.10. *Let A be a Noetherian connected graded k -algebra. If A is Koszul, then the following statements are equivalent:*

- (1) A is AS-regular;
- (2) A is standard AS-Gorenstein.

Proof. (1) \Rightarrow (2) is well known. We prove (2) \Rightarrow (1). It suffices to prove that $\text{pd}_A k < \infty$. Since A is a Koszul algebra, ${}_A k$ has a minimal free resolution

$$F : \cdots \rightarrow F_i \rightarrow \cdots \rightarrow F_0 \rightarrow k \rightarrow 0,$$

where $F_i = A(-i)^{\beta_i}$. Suppose that $\text{id}_A A = d$. We claim that the d th syzygy $Z_d(F.)$ of $F.$ is 0. Assume on the contrary that $Z_d(F.) \neq 0$. Note that

$$\cdots \rightarrow F_{d+2} \rightarrow F_{d+1} \rightarrow Z_d(F.) \rightarrow 0$$

is a minimal free resolution of $Z_d(F.)$. Then it is easy to see that $\text{Ext.reg } Z_d(F.) = d + 1$. By Corollary 4.9, $\text{CM.reg } Z_d(F.) = d + 1$.

Since $\text{Ext}_A^i(k, A) = 0$ for $i \neq d$, $\text{Ext}_A^i(Z_{d-1}(F.), A) = 0$ for $i \neq 0$. By [Ye, Corollary 4.10] and [Jo1, Theorem 1.2], $A_\alpha(-d)[d]$ is a balanced dualizing complex over A , where α is an automorphism of A as a graded k -algebra. It follows from the local duality theorem (Theorem 2.11) that $H_m^i(Z_{d-1}(F.)) = 0$ for $i \neq d$. Since A is AS-Gorenstein, $H_m^i(A) = 0$ for $i \neq d$. It follows from $\text{lcd}(A) = d$ and the short exact sequence

$$0 \rightarrow Z_d(F.) \rightarrow F_d \rightarrow Z_{d-1}(F.) \rightarrow 0$$

that $H_m^i(Z_d(F.)) = 0$ for $i \neq d$ and the following sequence is exact:

$$0 \rightarrow H_m^d(Z_d(F.)) \rightarrow H_m^d(F_d) \rightarrow H_m^d(Z_{d-1}(F.)) \rightarrow 0. \tag{4.1}$$

Since A is a standard AS-Gorenstein algebra, $\text{CM.reg } A = 0$ by Lemma 4.8. Hence $\text{CM.reg } F_d = d$ by Definition 4.1. Therefore $H_m^d(Z_d(F.))_{>0} = H_m^d(F_d)_{>0} = 0$ by (4.1). This implies that $\text{CM.reg } Z_d(F.) \leq d$, which contradicts to that $\text{CM.reg } Z_d(F.) = d + 1$. Hence $Z_d(F.) = 0$, which means that $\text{pd}_A k < \infty$. \square

5. Castelnuovo–Mumford regularity and AS-regular algebras

If A is a polynomial algebra with standard grading, Eisenbud and Goto [EG] proved that $\text{CM.reg } M = \text{Tor.reg } M$ for all non-zero finitely generated graded A -modules. Recently, Römer proved that the converse is true [Rö, Theorem 4.1]. We copy Römer’s result here for the convenience.

Theorem 5.1. *Let A be a commutative Noetherian connected graded k -algebra generated in degree 1. The following statements are equivalent:*

- (i) For all $M \in \text{gr } A$, $\text{CM.reg}_A(M) - \text{CM.reg}_A(A) = \text{Ext.reg}_A(M)$;
- (ii) For all $M \in \text{gr } A$, $\text{Ext.reg}_A(M) = \text{CM.reg}_A(M) + \text{Ext.reg}_A(k)$;
- (iii) For all $M \in \text{gr } A$, $\text{Ext.reg}_A(M) = \text{CM.reg}_A(M)$;
- (iv) A is Koszul and $\text{CM.reg}_A(A) = 0$;
- (v) $A = k[x_1, \dots, x_n]$ is a polynomial ring with standard grading.

In this section, we prove a non-commutative version of [Rö, Theorem 4.1]. We start from the following two lemmas.

Lemma 5.2. *Let $X^\cdot \in D^b(\text{Gr } A)$ with $h^i(X^\cdot)_{<-i} = 0$ for all $i \in \mathbb{Z}$. Then for all $i \in \mathbb{Z}$, $\underline{\text{Ext}}_A^i(X^\cdot, k)_{>-i} = 0$.*

Proof. If $X^\cdot \cong 0 \in D^b(\text{Gr } A)$, there is nothing to prove. So we assume $X^\cdot \not\cong 0$. We prove the assertion by induction on the amplitude $\text{amp } X^\cdot = \sup X^\cdot - \inf X^\cdot$.

If $\text{amp } X^\cdot = 0$, then $h^s(X^\cdot) \neq 0$ and $X^\cdot \cong h^s(X^\cdot)[-s] \in D^b(\text{Gr } A)$ for $s = \inf X^\cdot$. Since $h^s(X^\cdot)_{<-s} = 0$, $h^s(X^\cdot)[-s]$ has a minimal free resolution F^\cdot , where $(F^i)_{<-i} = 0$ for $i \leq s$ and $F^i = 0$ for $i > s$. Then $\underline{\text{Ext}}_A^{-i}(X^\cdot, k)_{>-i} = \underline{\text{Hom}}_A(F^i, k)_{>-i} = 0$ for $i \leq s$ and $\underline{\text{Ext}}_A^{-i}(X^\cdot, k) = 0$ for $i > s$. Thus the assertion holds for $\text{amp } X^\cdot = 0$.

If $\text{amp } X^\cdot = n > 0$, we may assume that $X^j = 0$ for either $j < s$ or $j > s + n$, where $s = \inf X^\cdot$. Consider the following exact triangle in $D^b(\text{Gr } A)$

$$\text{Ker}(d_{X^\cdot}^s)[-s] \xrightarrow{\lambda} X^\cdot \rightarrow \text{cone}(\lambda) \rightarrow \text{Ker}(d_{X^\cdot}^s)[-s+1]$$

and the induced long exact sequence

$$\dots \rightarrow \underline{\text{Ext}}_A^i(\text{cone}(\lambda), k) \rightarrow \underline{\text{Ext}}_A^i(X^\cdot, k) \rightarrow \underline{\text{Ext}}_A^i(\text{Ker}(d_{X^\cdot}^s)[-s], k) \rightarrow \dots$$

Since $\text{amp}(\text{Ker}(d_{X^\cdot}^s)[-s]) = 0$ and $\text{amp}(\text{cone}(\lambda)) < n$, by induction, it is easy to see that $\underline{\text{Ext}}_A^i(X^\cdot, k)_{>-i} = 0$ for all $i \in \mathbb{Z}$. \square

Let $f : F^\cdot \rightarrow R^\cdot$ be a morphism of bounded below A -module complexes. Set $s = \inf R^\cdot$. Then f naturally induces a morphism between $F^{\geq s}$ and R^\cdot , denoted by \tilde{f} . Define a morphism g from $F^{\leq s-1}$ to $\text{cone}(\tilde{f})$ as follows:

$$\begin{array}{ccccccc} F^{\leq s-1} & \cdots & \longrightarrow & F^{s-2} & \longrightarrow & F^{s-1} & \longrightarrow & 0 & \longrightarrow & \cdots \\ \downarrow g & & & \downarrow & & \downarrow & & \downarrow & & \\ \text{cone}(\tilde{f}) & \cdots & \longrightarrow & R^{s-2} & \longrightarrow & F^s \oplus R^{s-1} & \longrightarrow & F^{s+1} \oplus R^s & \longrightarrow & \cdots \end{array}$$

where

$$g^j = \begin{cases} f^j, & j \leq s-2, \\ 0, & j \geq s, \\ \begin{pmatrix} d_{F^\cdot}^{s-1} \\ f^{s-1} \end{pmatrix}, & j = s-1. \end{cases}$$

Then $g : F^{\leq s-1} \rightarrow \text{cone}(\tilde{f})$ is indeed a morphism of complexes.

Lemma 5.3. *Let f and g be defined as above. If f is a quasi-isomorphism, so is g .*

Proof. It is easy to see that $h^j(F^{\leq s-1}) = 0 = h^j(\text{cone}(\tilde{f}))$ for all $j \neq s-1$, since f is a quasi-isomorphism. We are left to prove that $h^{s-1}(g) : h^{s-1}(F^{\leq s-1}) \rightarrow h^{s-1}(\text{cone}(\tilde{f}))$ is an isomorphism.

For any $x^{s-1} + \text{Im}(d_{F^\cdot}^{s-2}) \in \text{Ker}(h^{s-1}(g))$, there exists $y^{s-2} \in R^{s-2}$ such that $(d_{F^\cdot}^{s-1}(x^{s-1}), f^{s-1}(x^{s-1})) = (0, d_{R^\cdot}^{s-2}(y^{s-2}))$. Hence $x^{s-1} \in \text{Ker}(d_{F^\cdot}^{s-1}) = \text{Im}(d_{F^\cdot}^{s-2})$. It follows that $h^{s-1}(g)$ is injective.

On the other hand, suppose $(x^s, y^{s-1}) \in \text{Ker}(d_{\text{cone}(\tilde{f})}^{s-1})$, that is $(-d_{F^\cdot}^s(x^s), -f^s(x^s) + d_{R^\cdot}^{s-1}(y^{s-1})) = 0$. Since f is a quasi-isomorphism, there exists $x^{s-1} \in F^{s-1}$ such that $d_{F^\cdot}^{s-1}(x^{s-1}) = x^s$. Then

$$\begin{aligned} (x^s, y^{s-1}) - g^{s-1}(x^{s-1}) &= (x^s, y^{s-1}) - (d_{F^\cdot}^{s-1}(x^{s-1}), f^{s-1}(x^{s-1})) \\ &= (0, y^{s-1} - f^{s-1}(x^{s-1})). \end{aligned}$$

Since $d_{R^{\cdot}}^{s-1}(y^{s-1} - f^{s-1}(x^{s-1})) = f^s(x^s) - d_{R^{\cdot}}^{s-1}f^{s-1}(x^{s-1}) = 0$, there exists $x^{s-2} \in R^{s-2}$ such that $d_{R^{\cdot}}^{s-2}(x^{s-2}) = y^{s-1} - f^{s-1}(x^{s-1})$. Thus

$$h^{s-1}(g)(x^{s-1} + \text{Im}(d_{F^{\cdot}}^{s-2})) = (x^s, y^{s-1}) + \text{Im}(d_{\text{cone}(\tilde{f})}^{s-2}).$$

It follows that $h^{s-1}(g)$ is surjective. \square

The following is the main result in this article, which is a generalization of (iii) \Leftrightarrow (iv) \Leftrightarrow (v) in [Rö, Theorem 4.1] (see also Theorem 5.1) to the non-commutative case.

Theorem 5.4. *Let A be a Noetherian connected graded k -algebra with a balanced dualizing complex. Then the following are equivalent:*

- (1) $\text{CM.reg } M = \text{Ext.reg } M$ holds for all $M \in \text{gr } A$.
- (2) A is Koszul and $\text{CM.reg } A = 0$.
- (3) A is a Koszul AS-regular k -algebra.

Proof. (3) \Rightarrow (2). By Theorem 4.10, A is standard. It follows from Lemma 4.8 that $\text{CM.reg } A = 0$.

(2) \Rightarrow (1). Follows from Theorem 4.7.

(1) \Rightarrow (3). Let R^{\cdot} be the balanced dualizing complex over A . Then $R^{\cdot} \cong R\Gamma_m(A)^{\cdot}$ by [VdB, Theorem 6.3]. By assumption, $\text{CM.reg } A = \text{Ext.reg } A = 0$, so $H_m^i(A)_{>-i} = 0$ for any $i \in \mathbb{Z}$ by Definition 4.1. Thus $h^i(R^{\cdot})_{<-i} = 0$ for any $i \in \mathbb{Z}$. By Lemma 5.2, $\underline{\text{Ext}}_A^i(R^{\cdot}, k)_{>-i} = 0$ for any $i \in \mathbb{Z}$.

On the other hand, since R^{\cdot} is a balanced dualizing complex over A , $R\Gamma_m(R^{\cdot}) \cong A^{\cdot}$ by Definition 2.9. Therefore, $\text{CM.reg } R^{\cdot} = 0$ by Definition 4.1. Again by assumption, $\text{Ext.reg } k = \text{CM.reg } k = 0$. It follows from Theorem 4.7 that $\text{Ext.reg } R^{\cdot} = \text{CM.reg } R^{\cdot} = 0$. Thus by Definition 4.4, $\underline{\text{Ext}}_A^i(R^{\cdot}, k)_{<-i} = 0$ for any $i \in \mathbb{Z}$.

Since $R^{\cdot} \in D_{\text{fg}}^b(\text{Gr } A)$, R^{\cdot} has a finitely generated minimal free resolution $F^{\cdot} \xrightarrow{\sim} R^{\cdot}$. Set $s = \inf\{j \mid h^j(R^{\cdot}) \neq 0\}$. Then $s = -\sup\{j \mid H_m^j(A) \neq 0\}$. Since both $F^{\geq s}$ and $F^{\leq s-1}$ are minimal free complexes, and $\underline{\text{Ext}}_A^i(R^{\cdot}, k)_{\neq -i} = 0$ for any $i \in \mathbb{Z}$,

$$\text{Ext.reg } F^{\geq s} = \begin{cases} 0, & F^{\geq s} \text{ is not acyclic,} \\ -\infty, & F^{\geq s} \text{ is acyclic} \end{cases} \tag{5.1}$$

and

$$\text{Ext.reg } F^{\leq s-1} = \begin{cases} 0, & F^{\leq s-1} \text{ is not acyclic,} \\ -\infty, & F^{\leq s-1} \text{ is acyclic.} \end{cases} \tag{5.2}$$

By the choice of s , $F^s \neq 0$ and thus $\underline{\text{Ext}}_A^{-s}(F^{\geq s}, k) = \underline{\text{Hom}}_A(F^s, k) \neq 0$. Hence $F^{\geq s}$ is not acyclic and $\text{Ext.reg } F^{\geq s} = 0$ by (5.1). Let f be the quasi-isomorphism between F^{\cdot} and R^{\cdot} . Then f induces naturally a morphism between $F^{\geq s}$ and R^{\cdot} , denoted by \tilde{f} .

We claim that \tilde{f} is a quasi-isomorphism. If \tilde{f} is not a quasi-isomorphism, then $\text{cone}(\tilde{f}) \not\cong 0$ in $D^b(\text{Gr } A)$. Consider the following exact triangle

$$F^{\geq s} \xrightarrow{\tilde{f}} R^{\cdot} \rightarrow \text{cone}(\tilde{f}) \rightarrow F^{\geq s}[1].$$

Since $R\Gamma_m(R^{\cdot}) \cong A^{\cdot}$ in $D(\text{Gr } A^e)$, we have the following exact sequence and isomorphisms

$$\begin{cases} 0 \rightarrow H_m^{-1}(\text{cone}(\tilde{f})) \rightarrow H_m^0(F^{\geq s}) \rightarrow H_m^0(R^\cdot) \rightarrow H_m^0(\text{cone}(\tilde{f})) \rightarrow H_m^1(F^{\geq s}) \rightarrow 0, \\ H_m^{j-1}(\text{cone}(\tilde{f})) \cong H_m^j(F^{\geq s}), \quad j \neq 0, 1. \end{cases} \tag{5.3}$$

It follows from the proof of Lemma 5.3 that $\text{cone}(\tilde{f}) \cong h^{s-1}(\text{cone}(\tilde{f}))[1-s]$ in $D^b(\text{Gr } A)$. By the local duality theorem, we have the following isomorphisms

$$\begin{aligned} R\Gamma_m(\text{cone}(\tilde{f}))' &\cong R\underline{\text{Hom}}_A(\text{cone}(\tilde{f}), R^\cdot) \\ &\cong R\underline{\text{Hom}}_A(h^{s-1}(\text{cone}(\tilde{f}))[1-s], R^\cdot) \\ &\cong R\underline{\text{Hom}}_A(h^{s-1}(\text{cone}(\tilde{f})), R^\cdot)[s-1]. \end{aligned}$$

By taking the 0th cohomology modules, we have the following isomorphism,

$$H_m^0(\text{cone}(\tilde{f}))' \cong \underline{\text{Ext}}_A^{s-1}(h^{s-1}(\text{cone}(\tilde{f})), R^\cdot).$$

Since $h^j(R^\cdot) = 0$ for $j < s$, we have $H_m^0(\text{cone}(\tilde{f}))' = \underline{\text{Ext}}_A^{s-1}(h^{s-1}(\text{cone}(\tilde{f})), R^\cdot) = 0$. Since $\text{CM.reg } F^{\geq s} = \text{Ext.reg } F^{\geq s} = 0$, $H_m^{j-1}(\text{cone}(\tilde{f}))_{>-j} = 0$ for all $j \neq 1$ by (5.3). Thus $\text{CM.reg}(\text{cone}(\tilde{f})) \leq -1$ by Definition 4.1. However, by Lemma 5.3, $F^{\leq s-1}$ is a minimal free resolution for $\text{cone}(\tilde{f})$. Hence $\text{Ext.reg}(\text{cone}(\tilde{f})) = 0$ by (5.2), which contradicts to $\text{CM.reg}(\text{cone}(\tilde{f})) \leq -1$. Thus we have proved that $\text{cone}(\tilde{f})$ is acyclic.

Since $F^{\leq s-1}$ is a minimal free resolution for $\text{cone}(\tilde{f})$, $F^j = 0$ for $j \leq s-1$, which means that the projective dimension of R^\cdot is finite. By Theorem 3.6, A is AS-Gorenstein. It follows from Lemma 4.8 and Theorem 4.10 that A is AS-regular. \square

Remark 5.5. It is easy to show that (i) \Leftrightarrow (ii) in Theorem 5.1 always holds, even in non-commutative case. By Proposition 5.6, $\text{CM.reg } M - \text{CM.reg } A = \text{Ext.reg } M$ holds for all finitely generated module M with $\text{pd}_A M < \infty$. Since there are lots of non-Koszul AS-regular algebras generated in degree 1 (e.g., see [AS]), (i) \Leftrightarrow (iii) does not hold in general.

Proposition 5.6. *Let A be a Noetherian connected graded k -algebra with a balanced dualizing complex. If $M \in \text{gr } A$ with $\text{pd}_A M < \infty$, then*

$$\text{CM.reg } M - \text{CM.reg } A = \text{Ext.reg } M.$$

Proof. The proof in the commutative case [Rö, Theorem 4.2] works well in the non-commutative case. \square

Acknowledgments

This research is supported by the NSFC (key project 10731070) and supported by the Doctorate Foundation (No. 20060246003), Ministry of Education of China.

References

[AE] L.L. Avramov, D. Eisenbud, Regularity of modules over a Koszul algebra, *J. Algebra* 153 (1992) 85–90.
 [AP] L.L. Avramov, I. Peeva, Finite regularity and Koszul algebras, *Amer. J. Math.* 123 (2001) 275–281.
 [AS] M. Artin, W.F. Schelter, Graded algebras of global dimension 3, *Adv. Math.* 66 (1987) 171–216.
 [AZ] M. Artin, J.J. Zhang, Noncommutative projective schemes, *Adv. Math.* 109 (1994) 228–287.
 [BH] W. Bruns, J. Herzog, *Cohen–Macaulay Rings*, revised ed., Cambridge Stud. Adv. Math., vol. 39, Cambridge University Press, 1998.
 [EG] D. Eisenbud, S. Goto, Linear free resolutions and minimal multiplicity, *J. Algebra* 88 (1984) 89–133.
 [Jo1] P. Jørgensen, Local cohomology for non-commutative graded algebras, *Comm. Algebra* 25 (1997) 575–591.

- [Jo2] P. Jørgensen, Non-commutative graded homological identities, *J. London Math. Soc.* 57 (1998) 336–350.
- [Jo3] P. Jørgensen, Non-commutative Castelnuovo–Mumford regularity, *Math. Proc. Cambridge Philos. Soc.* 125 (1999) 203–221.
- [Jo4] P. Jørgensen, Linear free resolutions over non-commutative algebras, *Compos. Math.* 140 (2004) 1053–1058.
- [Jo5] P. Jørgensen, Gorenstein homomorphism of non-commutative rings, *J. Algebra* 211 (1999) 240–267.
- [Mo] I. Mori, Homological properties of balanced Cohen–Macaulay algebras, *Trans. Amer. Math. Soc.* 355 (2002) 1025–1042.
- [Mu] D. Mumford, *Lectures on Curves on an Algebraic Surface*, *Ann. of Math. Stud.*, vol. 59, Princeton University Press, Princeton, NJ, 1966.
- [Rö] T. Römer, On the regularity over positively graded algebras, *J. Algebra* 319 (2008) 1–15.
- [VdB] M. Van den Bergh, Existence theorems for dualizing complexes over non-commutative graded and filtered rings, *J. Algebra* 195 (1997) 662–679.
- [We] C. Weibel, *An Introduction to Homological Algebra*, Cambridge University Press, Cambridge, 1994.
- [WZ] Q.-S. Wu, J.J. Zhang, Homological identities for non-commutative rings, *J. Algebra* 242 (2001) 516–535.
- [SZ] J.T. Stafford, J.J. Zhang, Examples in non-commutative projective geometry, *Math. Proc. Cambridge Philos. Soc.* 116 (1994) 415–433.
- [SZ] J.T. Stafford, J.J. Zhang, Homological properties of (graded) noetherian PI rings, *J. Algebra* 168 (1994) 988–1026.
- [Ye] A. Yekutieli, Dualizing complexes over non-commutative graded algebras, *J. Algebra* 153 (1992) 41–84.
- [Z] J.J. Zhang, Connected graded Gorenstein algebras with enough normal elements, *J. Algebra* 189 (1997) 390–405.