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The classification of Leibniz superalgebras of nilindex $n+m$ $(m \neq 0)$ ☆

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ABSTRACT

In this paper we investigate the description of the complex Leibniz superalgebras with nilindex $n+m$, where n and m $(m \neq 0)$ are dimensions of even and odd parts, respectively. In fact, such superalgebras with characteristic sequence equal to $(n_1, \dots, n_k \mid m_1, \dots, m_s)$ (where $n_1 + \dots + n_k = n$, $m_1 + \dots + m_s = m$) for $n_1 \geqslant n-1$ and $(n_1, \dots, n_k \mid m)$ were classified in works by Ayupov et al. (2009) [3], Camacho et al. (2010) [4], Camacho et al. (in press) [5], Camacho et al. (in press) [6]. Here we prove that in the case of $(n_1, \dots, n_k \mid m_1, \dots, m_s)$, where $n_1 \leqslant n-2$ and $m_1 \leqslant m-1$ the Leibniz superalgebras have nilindex less than $n+m$. Thus, we complete the classification of Leibniz superalgebras with nilindex $n+m$.

1. Introduction

During many years the theory of Lie superalgebras has been actively studied by many mathematicians and physicists. A systematic exposition of basic Lie superalgebras theory can be found in [9]. Many works have been devoted to the study of this topic, but unfortunately most of them do not deal with nilpotent Lie superalgebras. In works [2,7,8] the problem of the description of some classes of nilpotent Lie superalgebras has been studied. It is well known that Lie superalgebras are a generaliza-

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tion of Lie algebras. In the same way, the notion of Leibniz algebra, which was introduced in [11], can be generalized to Leibniz superalgebras [1,10]. Some elementary properties of Leibniz superalgebras were obtained in [1].

In the work [8] Lie superalgebras with maximal nilindex were classified. Such superalgebras are two-generated and its nilindex is equal to $n+m$ (where n and m are dimensions of even and odd parts, respectively). In fact, there exists a unique Lie superalgebra of maximal nilindex. This superalgebra is a filiform Lie superalgebra (the characteristic sequence is equal to $(n-1,1 \mid m)$) and we mention about paper [2], where some crucial properties of filiform Lie superalgebras are given.

For nilpotent Leibniz superalgebras the description of the maximal nilindex case (nilpotent Leibniz superalgebras distinguished by the feature of being single-generated) is not difficult and was done in [1].

However, the description of Leibniz superalgebras of nilindex $n+m$ is a very problematic one and it needs to solve many technical tasks. Therefore, they can be studied by applying restrictions on their characteristic sequences. In the present paper we consider Leibniz superalgebras with characteristic sequence $(n_1,\dots,n_k \mid m_1,\dots,m_s)$ ($n_1 \leqslant n-2$ and $m_1 \leqslant m-1$) and nilindex $n+m$. Recall, that such superalgebras for $n_1 \geqslant n-1$ or $m_1 = m$ have been already classified in works [3–6]. Namely, we prove that a Leibniz superalgebra with characteristic sequence equal to $(n_1,\dots,n_k \mid m_1,\dots,m_s)$ ($n_1 \leqslant n-2$ and $m_1 \leqslant m-1$) has nilindex less than $n+m$. Therefore, we complete the classification of Leibniz superalgebras with nilindex $n+m$.

It should be noted that in our study the natural gradation of even part of a Leibniz superalgebra play one of the crucial roles. In fact, we use some properties of naturally graded Lie and Leibniz algebras for obtaining the convenient basis of even part of the superalgebra (so-called adapted basis).

Throughout this work we shall consider spaces and (super)algebras over the field of complex numbers. By asterisks $(*)$ we denote the appropriate coefficients at the basic elements of a superalgebra.

2. Preliminaries

Recall the notion of Leibniz superalgebras.

Definition 2.1. A $\mathbb{Z}_2$-graded vector space $L = L_0 \oplus L_1$ is called a Leibniz superalgebra if it is equipped with a product $[-,-]$ which satisfies the following conditions:

1. $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta \pmod 2}$,
2. $[x,[y,z]] = [[x,y],z] - (-1)^{\alpha\beta}[[x,z],y]$-Leibniz superidentity,

for all $x \in L$, $y \in L_\alpha$, $z \in L_\beta$ and $\alpha, \beta \in \mathbb{Z}_2$.

The vector spaces L_0 and L_1 are said to be even and odd parts of the superalgebra L, respectively. Evidently, even part of the Leibniz superalgebra is a Leibniz algebra.

If the identity

$$[x,y] = -(-1)^{\alpha\beta}[y,x]$$

holds for any $x \in L_\alpha$ and $y \in L_\beta$, then the Leibniz superidentity becomes to the Jacobi superidentity. Thus, Leibniz superalgebras are a simultaneous generalization of Lie superalgebras and Leibniz algebras.

We denote by $Leib_{n,m}$ the set of all Leibniz superalgebras with the dimensions of the even and odd parts, respectively equal to n and m.

For a given Leibniz superalgebra L we define its descending central sequence of two-sided ideals as follows:

$$L^1 = L, \qquad L^{k+1} = \left[L^k, L\right], \quad k \geqslant 1.$$

Definition 2.2. A Leibniz superalgebra L is called nilpotent, if there exists $s \in \mathbb{N}$ such that $L^s = 0$. The minimal number s with this property is called nilindex of the superalgebra L.

Due to coincidence the notions of right nilpotence and nilpotence for Leibniz superalgebras, Definition 2.2 is agreed with the nilpotence in the case of Lie superalgebras.

Definition 2.3. The set

$$\mathcal{R}(L) = \{ z \in L \mid [L, z] = 0 \}$$

is called the right annihilator of a superalgebra L.

Using the Leibniz superidentity, it is easy to check that $\mathcal{R}(L)$ is an ideal of the superalgebra L. Moreover, the elements of the form $[a, b] + (-1)^{\alpha\beta}[b, a]$ ($a \in L_\alpha$, $b \in L_\beta$) belong to $\mathcal{R}(L)$.

The following theorem describes nilpotent Leibniz superalgebras with maximal nilindex.

Theorem 2.1. *(See [1].) Let L be a Leibniz superalgebra of $Leib_{n,m}$ with nilindex equal to $n+m+1$. Then L is isomorphic to one of the following non-isomorphic superalgebras*:

$$[e_i, e_1] = e_{i+1}, \quad 1 \leqslant i \leqslant n-1, \ m = 0; \qquad \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leqslant i \leqslant n+m-1, \\ [e_i, e_2] = 2e_{i+2}, & 1 \leqslant i \leqslant n+m-2 \end{cases}$$

(*omitted products are equal to zero*).

Remark 2.1. From the assertion of Theorem 2.1 it follows that in case of non-trivial odd part L_1 of the superalgebra L there are two possibilities for n and m, namely, $m = n$ if $n+m$ is even and $m = n+1$ if $n+m$ is odd. Moreover, it is clear that the Leibniz superalgebra has the maximal nilindex if and only if it is single-generated.

Let $L = L_0 \oplus L_1$ be a nilpotent Leibniz superalgebra. For an arbitrary element $x \in L_0$, the right multiplication operator $R_x : L \to L$ (given by $R_x(y) = [y, x]$) is a nilpotent endomorphism of the space L_i, where $i \in \{0, 1\}$. Taking into account the property of complex endomorphisms we can consider the Jordan form for R_x. For the operator R_x we denote by $C_i(x)$ ($i \in \{0, 1\}$) the descending sequence of its Jordan blocks dimensions. Consider the lexicographical order on the set $C_i(L_0)$.

Definition 2.4. A sequence

$$C(L) = \Big(\max_{x \in L_0 \setminus L_0^2} C_0(x) \,\Big|\, \max_{\tilde{x} \in L_0 \setminus L_0^2} C_1(\tilde{x}) \Big)$$

is said to be the characteristic sequence of the Leibniz superalgebra L.

Similarly to [7] (Corollary 3.0.1) it can be proved that the characteristic sequence is invariant under isomorphism.

Since Leibniz superalgebras from $Leib_{n,m}$ with nilindex $n+m$ and with characteristic sequence equal to $(n_1, \dots, n_k \mid m_1, \dots, m_s)$ either $n_1 \geqslant n-1$ or $m_1 = m$ were already classified in [3–6], we shall reduce our investigation to the case of the characteristic sequence $(n_1, \dots, n_k \mid m_1, \dots, m_s)$, where $n_1 \leqslant n-2$ and $m_1 \leqslant m-1$.

From Definition 2.4 we have that a Leibniz algebra L_0 has characteristic sequence $(n_1, \dots, n_k)$. Let $l \in \mathbb{N}$ be the nilindex of the Leibniz algebra L_0. Since $n_1 \leqslant n-2$, then we have $l \leqslant n-1$ and the Leibniz algebra L_0 has at least two generators (the elements which belong to the set $L_0 \setminus L_0^2$).

For the completeness of the statement below we recall the classifications of the papers [3–6,8].

$Leib_{1,m}$:

$$\{\, [y_i, x_1] = y_{i+1}, \quad 1 \leqslant i \leqslant m-1.$$

$Leib_{n,1}$:

$$\begin{cases} [x_i, x_1] = x_{i+1}, & 1 \leqslant i \leqslant n-1, \\ [y_1, y_1] = \alpha x_n, & \alpha \in \{0, 1\}. \end{cases}$$

$Leib_{2,2}$:

$$\begin{cases} [y_1, x_1] = y_2, \\ [x_1, y_1] = \frac{1}{2} y_2, \\ [x_2, y_1] = y_2, \\ [y_1, x_2] = 2y_2, \\ [y_1, y_1] = x_2, \end{cases} \qquad \begin{cases} [y_1, x_1] = y_2, \\ [x_2, y_1] = y_2, \\ [y_1, x_2] = 2y_2, \\ [y_1, y_1] = x_2. \end{cases}$$

$Leib_{2,m}$, m is odd:

$$\begin{cases} [x_1, x_1] = x_2, & m \geqslant 3, \\ [y_i, x_1] = y_{i+1}, & 1 \leqslant i \leqslant m-1, \\ [x_1, y_i] = -y_{i+1}, & 1 \leqslant i \leqslant m-1, \\ [y_i, y_{m+1-i}] = (-1)^{i+1} x_2, & 1 \leqslant i \leqslant m-1, \end{cases} \qquad \begin{cases} [y_i, x_1] = -[x_1, y_i] = y_{i+1}, & 1 \leqslant i \leqslant m-1, \\ [y_{m+1-i}, y_i] = (-1)^{i+1} x_2, & 1 \leqslant i \leqslant \frac{m+1}{2}. \end{cases}$$

In order to present the classification of Leibniz superalgebras with characteristic sequence $(n-1, 1 \mid m)$, $n \geqslant 3$ and nilindex $n+m$ we need to introduce the following families of superalgebras:

$\boldsymbol{Leib_{n,n-1}}$:

$L(\alpha_4, \alpha_5, \ldots, \alpha_n, \theta)$:

$$\begin{cases} [x_1, x_1] = x_3, \\ [x_i, x_1] = x_{i+1}, & 2 \leqslant i \leqslant n-1, \\ [y_j, x_1] = y_{j+1}, & 1 \leqslant j \leqslant n-2, \\ [x_1, y_1] = \frac{1}{2} y_2, \\ [x_i, y_1] = \frac{1}{2} y_i, & 2 \leqslant i \leqslant n-1, \\ [y_1, y_1] = x_1, \\ [y_j, y_1] = x_{j+1}, & 2 \leqslant j \leqslant n-1, \\ [x_1, x_2] = \alpha_4 x_4 + \alpha_5 x_5 + \cdots + \alpha_{n-1} x_{n-1} + \theta x_n, \\ [x_j, x_2] = \alpha_4 x_{j+2} + \alpha_5 x_{j+3} + \cdots + \alpha_{n+2-j} x_n, & 2 \leqslant j \leqslant n-2, \\ [y_1, x_2] = \alpha_4 y_3 + \alpha_5 y_4 + \cdots + \alpha_{n-1} y_{n-2} + \theta y_{n-1}, \\ [y_j, x_2] = \alpha_4 y_{j+2} + \alpha_5 y_{j+3} + \cdots + \alpha_{n+1-j} y_{n-1}, & 2 \leqslant j \leqslant n-3. \end{cases}$$

$G(\beta_4, \beta_5, \ldots, \beta_n, \gamma)$:

$$\begin{cases} [x_1,x_1]=x_3, & \\ [x_i,x_1]=x_{i+1}, & 3\leqslant i\leqslant n-1,\\ [y_j,x_1]=y_{j+1}, & 1\leqslant j\leqslant n-2,\\ [x_1,x_2]=\beta_4x_4+\beta_5x_5+\cdots+\beta_nx_n, & \\ [x_2,x_2]=\gamma x_n, & \\ [x_j,x_2]=\beta_4x_{j+2}+\beta_5x_{j+3}+\cdots+\beta_{n+2-j}x_n, & 3\leqslant j\leqslant n-2,\\ [y_1,y_1]=x_1, & \\ [y_j,y_1]=x_{j+1}, & 2\leqslant j\leqslant n-1,\\ [x_1,y_1]=\frac{1}{2}y_2, & \\ [x_i,y_1]=\frac{1}{2}y_i, & 3\leqslant i\leqslant n-1,\\ [y_j,x_2]=\beta_4y_{j+2}+\beta_5y_{j+3}+\cdots+\beta_{n+1-j}y_{n-1}, & 1\leqslant j\leqslant n-3. \end{cases}$$

$\mathit{Leib}_{n,n}$:

$M(\alpha_4,\alpha_5,\ldots,\alpha_n,\theta,\tau)$:

$$\begin{cases} [x_1,x_1]=x_3, & \\ [x_i,x_1]=x_{i+1}, & 2\leqslant i\leqslant n-1,\\ [y_j,x_1]=y_{j+1}, & 1\leqslant j\leqslant n-1,\\ [x_1,y_1]=\frac{1}{2}y_2, & \\ [x_i,y_1]=\frac{1}{2}y_i, & 2\leqslant i\leqslant n,\\ [y_1,y_1]=x_1, & \\ [y_j,y_1]=x_{j+1}, & 2\leqslant j\leqslant n-1,\\ [x_1,x_2]=\alpha_4x_4+\alpha_5x_5+\cdots+\alpha_{n-1}x_{n-1}+\theta x_n, & \\ [x_2,x_2]=\gamma_4x_4, & \\ [x_j,x_2]=\alpha_4x_{j+2}+\alpha_5x_{j+3}+\cdots+\alpha_{n+2-j}x_n, & 3\leqslant j\leqslant n-2,\\ [y_1,x_2]=\alpha_4y_3+\alpha_5y_4+\cdots+\alpha_{n-1}y_{n-2}+\theta y_{n-1}+\tau y_n, & \\ [y_2,x_2]=\alpha_4y_4+\alpha_5y_5+\cdots+\alpha_{n-1}y_{n-1}+\theta y_n, & \\ [y_j,x_2]=\alpha_4y_{j+2}+\alpha_5y_{j+3}+\cdots+\alpha_{n+2-j}y_n, & 3\leqslant j\leqslant n-2. \end{cases}$$

$H(\beta_4,\beta_5,\ldots,\beta_n,\delta,\gamma)$:

$$\begin{cases} [x_1,x_1]=x_3, & \\ [x_i,x_1]=x_{i+1}, & 3\leqslant i\leqslant n-1,\\ [y_j,x_1]=y_{j+1}, & 1\leqslant j\leqslant n-2,\\ [x_1,x_2]=\beta_4x_4+\beta_5x_5+\cdots+\beta_nx_n, & \\ [x_2,x_2]=\gamma x_n, & \\ [x_j,x_2]=\beta_4x_{j+2}+\beta_5x_{j+3}+\cdots+\beta_{n+2-j}x_n, & 3\leqslant j\leqslant n-2,\\ [y_1,y_1]=x_1, & \\ [y_j,y_1]=x_{j+1}, & 2\leqslant j\leqslant n-1,\\ [x_1,y_1]=\frac{1}{2}y_2, & \\ [x_i,y_1]=\frac{1}{2}y_i, & 3\leqslant i\leqslant n-1,\\ [y_1,x_2]=\beta_4y_3+\beta_5y_4+\cdots+\beta_ny_{n-1}+\delta y_n, & \\ [y_j,x_2]=\beta_4y_{j+2}+\beta_5y_{j+3}+\cdots+\beta_{n+2-j}y_n, & 2\leqslant j\leqslant n-2. \end{cases}$$

Analogously, for the Leibniz superalgebras with characteristic sequence $(n \mid m-1, 1)$, $n \geqslant 2$ we introduce the following families of superalgebras:

$\boldsymbol{Leib_{n,n+1}}$:

$E(\gamma, \beta_{[\frac{n+4}{2}]}, \beta_{[\frac{n+4}{2}]+1}, \ldots, \beta_n, \beta)$:

$$\begin{cases} [x_i, x_1] = x_{i+1}, & 1 \leqslant i \leqslant n-1, \\ [y_j, x_1] = y_{j+1}, & 1 \leqslant j \leqslant n-1, \\ [x_i, y_1] = \frac{1}{2} y_{i+1}, & 1 \leqslant i \leqslant n-1, \\ [y_j, y_1] = x_j, & 1 \leqslant j \leqslant n, \\ [y_{n+1}, y_{n+1}] = \gamma x_n, & \\ [x_i, y_{n+1}] = \displaystyle\sum_{k=[\frac{n+4}{2}]}^{n+1-i} \beta_k y_{k-1+i}, & 1 \leqslant i \leqslant \left[\frac{n-1}{2}\right], \\ [y_1, y_{n+1}] = -2 \displaystyle\sum_{k=[\frac{n+4}{2}]}^{n} \beta_k x_{k-1} + \beta x_n, & \\ [y_j, y_{n+1}] = -2 \displaystyle\sum_{k=[\frac{n+4}{2}]}^{n+2-j} \beta_k x_{k-2+j}, & 2 \leqslant j \leqslant \left[\frac{n+1}{2}\right]. \end{cases}$$

$\boldsymbol{Leib_{n,n+2}}$:

$F(\beta_{[\frac{n+5}{2}]}, \beta_{[\frac{n+5}{2}]+1}, \ldots, \beta_{n+1})$:

$$\begin{cases} [x_i, x_1] = x_{i+1}, & 1 \leqslant i \leqslant n-1, \\ [y_j, x_1] = y_{j+1}, & 1 \leqslant j \leqslant n, \\ [x_i, y_1] = \frac{1}{2} y_{i+1}, & 1 \leqslant i \leqslant n, \\ [y_j, y_1] = x_j, & 1 \leqslant j \leqslant n, \\ [x_i, y_{n+2}] = \displaystyle\sum_{k=[\frac{n+5}{2}]}^{n+2-i} \beta_k y_{k-1+i}, & 1 \leqslant i \leqslant \left[\frac{n}{2}\right], \\ [y_j, y_{n+2}] = -2 \displaystyle\sum_{k=[\frac{n+5}{2}]}^{n+2-j} \beta_k x_{k-2+j}, & 1 \leqslant j \leqslant \left[\frac{n}{2}\right]. \end{cases}$$

Let us introduce also the following operators which act on k-dimensional vectors:

$$V^0_{j,k}(\alpha_1, \alpha_2, \ldots, \alpha_k) = \big(0, \ldots, 0, \overset{j}{1}, \delta \sqrt[j]{\delta^{j+1}} S^{j+1}_{m,j} \alpha_{j+1}, \delta \sqrt[j]{\delta^{j+2}} S^{j+2}_{m,j} \alpha_{j+2}, \ldots, \delta \sqrt[j]{\delta^k} S^k_{m,j} \alpha_k\big);$$

$$V^1_{j,k}(\alpha_1, \alpha_2, \ldots, \alpha_k) = \big(0, \ldots, 0, \overset{j}{1}, S^{j+1}_{m,j} \alpha_{j+1}, S^{j+2}_{m,j} \alpha_{j+2}, \ldots, S^k_{m,j} \alpha_k\big);$$

$$V^2_{j,k}(\alpha_1, \alpha_2, \ldots, \alpha_k) = \big(0, \ldots, 0, \overset{j}{1}, S^{2(j+1)+1}_{m,2j+1} \alpha_{j+1}, S^{2(j+2)+1}_{m,2j+1} \alpha_{j+2}, \ldots, S^{2k+1}_{m,2j+1} \alpha_k\big);$$

$$V^0_{k+1,k}(\alpha_1,\alpha_2,\ldots,\alpha_k)=V^1_{k+1,k}(\alpha_1,\alpha_2,\ldots,\alpha_k)=V^2_{k+1,k}(\alpha_1,\alpha_2,\ldots,\alpha_k)=(0,0,\ldots,0);$$

$$W_{s,k}\big(0,0,\ldots,\overset{j-1}{0},\overset{j}{1},S^{j+1}_{m,j}\alpha_{j+1},S^{j+2}_{m,j}\alpha_{j+2},\ldots,S^k_{m,j}\alpha_k,\gamma\big)$$

$$=\big(0,0,\ldots,\overset{j}{1},0,\ldots,\overset{s+j}{1},S^{s+1}_{m,s}\alpha_{s+j+1},S^{s+2}_{m,s}\alpha_{s+j+2},\ldots,S^{k-j}_{m,s}\alpha_k,S^{k+6-2j}_{m,s}\gamma\big);$$

$$W_{k+1-j,k}\big(0,0,\ldots,\overset{j-1}{0},\overset{j}{1},S^{j+1}_{m,j}\alpha_{j+1},S^{j+2}_{m,j}\alpha_{j+2},\ldots,S^k_{m,j}\alpha_k,\gamma\big)$$

$$=(0,0,\ldots,\overset{j}{1},0,\ldots,1);$$

$$W_{k+2-j,k}\big(0,0,\ldots,\overset{j-1}{0},\overset{j}{1},S^{j+1}_{m,j}\alpha_{j+1},S^{j+2}_{m,j}\alpha_{j+2},\ldots,S^k_{m,j}\alpha_k,\gamma\big)=$$

$$=(0,0,\ldots,\overset{j}{1},0,\ldots,0),$$

where $k\in N$, $\delta=\pm1$, $1\leqslant j\leqslant k$, $1\leqslant s\leqslant k-j$, and $S_{m,t}=\cos\frac{2\pi m}{t}+i\sin\frac{2\pi m}{t}$ $(m=0,1,\ldots,t-1)$.

Below we present the complete list of pairwise non-isomorphic Leibniz superalgebras with nilindex $n+m$ and:

characteristic sequence equal to $(n-1,1\mid m)$:

$$L\big(V^1_{j,n-3}(\alpha_4,\alpha_5,\ldots,\alpha_n),S^{n-3}_{m,j}\theta\big),\quad 1\leqslant j\leqslant n-3,$$
$$L(0,0,\ldots,0,1),\quad L(0,0,\ldots,0),\quad G(0,0,\ldots,0,1),\quad G(0,0,\ldots,0),$$
$$G\big(W_{s,n-2}\big(V^1_{j,n-3}(\beta_4,\beta_5,\ldots,\beta_n),\gamma\big)\big),\quad 1\leqslant j\leqslant n-3,\ 1\leqslant s\leqslant n-j,$$
$$M\big(V^1_{j,n-2}(\alpha_4,\alpha_5,\ldots,\alpha_n),S^{n-3}_{m,j}\theta\big),\quad 1\leqslant j\leqslant n-2,$$
$$M(0,0,\ldots,0,1),\quad M(0,0,\ldots,0),\quad H(0,0,\ldots,0,1),\quad H(0,0,\ldots,0),$$
$$H\big(W_{s,n-1}\big(V^1_{j,n-2}(\beta_4,\beta_5,\ldots,\beta_n),\gamma\big)\big),\quad 1\leqslant j\leqslant n-2,\ 1\leqslant s\leqslant n+1-j;$$

characteristic sequence equal to $(n\mid m-1,1)$ if n is odd (i.e. $n=2q-1$):

$$E\big(1,\delta\beta_{q+1},V^0_{j,q-2}(\beta_{q+2},\beta_{q+3},\ldots,\beta_n),0\big),\qquad \beta_{q+1}\neq\pm\frac{1}{2},\quad 1\leqslant j\leqslant q-1,$$
$$E\big(1,\beta_{q+1},V^0_{j,q-1}(\beta_{q+2},\beta_{q+3},\ldots,\beta_n,\beta)\big),\qquad \beta_{q+1}=\pm\frac{1}{2},\quad 1\leqslant j\leqslant q,$$
$$E\big(0,1,V^0_{j,q-2}(\beta_{q+2},\beta_{q+3},\ldots,\beta_n),0\big),\quad 1\leqslant j\leqslant q-1,$$
$$E\big(0,0,W_{s,q-1}\big(V^1_{j,q-1}(\beta_{q+2},\beta_{q+3},\ldots,\beta_n,\beta)\big)\big),\quad 1\leqslant j\leqslant q-1,\ 1\leqslant s\leqslant q-j,$$
$$E(0,0,\ldots,0);$$

if n is even (i.e. $n=2q$):

$$E\big(1,V^2_{j,q-1}(\beta_{q+2},\beta_{q+3},\ldots,\beta_n,),0\big),\quad 1\leqslant j\leqslant q,$$
$$E\big(0,W_{s,q}\big(V^1_{j,q}(\beta_{q+2},\beta_{q+3},\ldots,\beta_n,\beta)\big)\big),\quad 1\leqslant j\leqslant q,\ 1\leqslant s\leqslant q+1-j,$$
$$E(0,0,\ldots,0);$$
$$F\big(W_{s,n+2-[\frac{n+5}{2}]}\big(V^1_{j,n+2-[\frac{n+5}{2}]}(\beta_{[\frac{n+5}{2}]},\beta_{[\frac{n+5}{2}]+1},\ldots,\beta_{n+1})\big)\big),$$

where $1 \leqslant j \leqslant n+2-[\frac{n+5}{2}]$, $1 \leqslant s \leqslant n+3-[\frac{n+5}{2}]-j$,

$$F(0,0,\ldots,0).$$

For a given Leibniz algebra A with nilindex l, we put $gr(A)_i = A^i/A^{i+1}$, $1 \leqslant i \leqslant l-1$ and $gr(A) = gr(A)_1 \oplus gr(A)_2 \oplus \cdots \oplus gr(A)_{l-1}$. Then $[gr(A)_i, gr(A)_j] \subseteq gr(A)_{i+j}$ and we obtain the graded algebra $gr(A)$.

Definition 2.5. The gradation previously constructed is called the natural gradation. If a Leibniz algebra G is isomorphic to $gr(A)$, then we say that the algebra G is naturally graded Leibniz algebra.

3. The main result

Let L be a Leibniz superalgebra with characteristic sequence $(n_1,\ldots,n_k \mid m_1,\ldots,m_s)$, where $n_1 \leqslant n-2$, $m_1 \leqslant m-1$ and nilindex $n+m$. Since the second part of the characteristic sequence of the Leibniz superalgebra L is equal to $(m_1,\ldots,m_s)$ then by Definition 2.4, there exists a nilpotent endomorphism R_x $(x \in L_0 \setminus L_0^2)$ of the space L_1 such that its Jordan form consists of s Jordan blocks. Therefore, we can assume the existence of an adapted basis $\{y_1,y_2,\ldots,y_m\}$ of the subspace L_1, such that

$$\begin{cases} [y_j,x] = y_{j+1}, & j \notin \{m_1, m_1+m_2, \ldots, m_1+m_2+\cdots+m_s\}, \\ [y_j,x] = 0, & j \in \{m_1, m_1+m_2, \ldots, m_1+m_2+\cdots+m_s\}, \end{cases} \tag{1}$$

for some $x \in L_0 \setminus L_0^2$.

Further we shall use a homogeneous basis $\{x_1,\ldots,x_n\}$ with respect to the natural gradation of the Leibniz algebra L_0, which is also agreed with the lower central sequence of L.

The main result of the paper establishes that the nilindex of a Leibniz superalgebra L with characteristic sequence $(n_1,\ldots,n_k \mid m_1,\ldots,m_s)$, $n_1 \leqslant n-2, m_1 \leqslant m-1$ is less than $n+m$.

According to Theorem 2.1 we have the description of single-generated Leibniz superalgebras, which have nilindex $n+m+1$. If the number of generators is greater than two, then the superalgebra has nilindex less than $n+m$. Therefore, we should consider the case of two-generated superalgebras.

The possible cases for the generators are:

1. Both generators lie in L_0, i.e. $\dim(L^2)_0 = n-2$ and $\dim(L^2)_1 = m$.
2. One generator lies in L_0 and another one lies in L_1, i.e. $\dim(L^2)_0 = n-1$ and $\dim(L^2)_1 = m-1$.
3. Both generators lie in L_1, i.e. $\dim(L^2)_0 = n$ and $\dim(L^2)_1 = m-2$.

Moreover, a two-generated superalgebra L has nilindex $n+m$ if and only if $\dim L^k = n+m-k$, for $2 \leqslant k \leqslant n+m$.

Since $m \neq 0$ we omit the case where both generators lie in even part.

3.1. The case of one generator in L_0 and another one in L_1

Since $\dim(L^2)_0 = n-1$ and $\dim(L^2)_1 = m-1$ then there exist some m_j, $0 \leqslant j \leqslant s-1$ (here we assume $m_0 = 0$) such that $y_{m_1+\cdots+m_j+1} \notin L^2$. By a shifting of basic elements we can assume that $m_j = m_0$, i.e. the basic element y_1 can be chosen as a generator of the superalgebra L. Of course, by this shifting the condition from definition of the characteristic sequence $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_s$ can be broken, but further we shall not use the condition.

Let $L = L_0 \oplus L_1$ be a two generated Leibniz superalgebra from $Leib_{n,m}$ with characteristic sequence equal to $(n_1,\ldots,n_k \mid m_1,\ldots,m_s)$ and let $\{x_1,\ldots,x_n,y_1,\ldots,y_m\}$ be a basis of the L.

Lemma 3.1. *Let us assume that one generator of L lies in L_0 and the other one lies in L_1. Then x_1 and y_1 can be chosen as generators of the L. Moreover, we can suppose x_1 instead of the element x in equality* (1)*.*

Proof. As it was mentioned above, y_1 can be chosen as the first generator of L. If $x \in L \setminus L^2$ then the assertion of the lemma is evident. If $x \in L^2$ then there exists some i_0 $(2 \leqslant i_0)$ such that $x_{i_0} \in L \setminus L^2$. Set $x'_1 = Ax + x_{i_0}$ for $A \neq 0$, then x'_1 is a generator of the superalgebra L (since $x'_1 \in L \setminus L^2$). Moreover, making the following transformation on the basis of L_1 as follows

$$\begin{cases} y'_j = y_j, & j \in \{1, m_1+1, \dots, m_1+m_2+\dots+m_{s-1}+1\}, \\ y'_j = [y'_{j-1}, x'_1], & j \notin \{1, m_1+1, \dots, m_1+m_2+\dots+m_{s-1}+1\}, \end{cases}$$

and taking sufficiently large value of the parameter A we preserve the equality (1). Thus, in the basis $\{x'_1, x_2, \dots, x_n, y'_1, y'_2, \dots, y'_m\}$ the elements x'_1 and y'_1 are generators. □

Due to Lemma 3.1 further we shall suppose that $\{x_1, y_1\}$ are generators of the Leibniz superalgebra L. Therefore,

$$L^2 = \{x_2, x_3, \dots, x_n, y_2, y_3, \dots, y_m\}.$$

Let us introduce the notations:

$$[x_i, y_1] = \sum_{j=2}^{m} \alpha_{i,j} y_j, \quad 1 \leqslant i \leqslant n, \qquad [y_i, y_1] = \sum_{j=2}^{n} \beta_{i,j} x_j, \quad 1 \leqslant i \leqslant m. \tag{2}$$

Without loss of generality we can assume that $y_{m_1+\dots+m_i+1} \in L^{t_i} \setminus L^{t_i+1}$, where $t_i < t_j$ for $1 \leqslant i < j \leqslant s-1$.

Since $\dim(L^3)_0 = n-1$, then we have

$$L^3 = \{x_2, x_3, \dots, x_n, y_3, \dots, y_{m_1}, B_1 y_2 + B_2 y_{m_1+1}, y_{m_1+2}, \dots, y_m\},$$

where $(B_1, B_2) \neq (0, 0)$.

Analyzing the way the element x_2 can be obtained, we conclude that there exist i_0 $(2 \leqslant i_0 \leqslant m)$ such that $[y_{i_0}, y_1] = \sum_{j=2}^{n} \beta_{i_0,2} x_j$, $\beta_{i_0,2} \neq 0$. Indeed, due to $C(L_0) = (n_1, n_2, \dots, n_k)$ with $n_1 \leqslant n-2$ and chosen homogeneous basis $\{x_1, x_2, \dots, x_n\}$ with respect to the natural gradation of the Leibniz algebra L_0, we assume that x_2 is the generator of the L_0. Therefore, x_2 cannot be generated by the products $[x_i, x_1]$, $1 \leqslant i \leqslant n$ and hence, it generated by a product $[y_{i_0}, y_1]$ for some i_0.

Let us show that $i_0 \notin \{m_1+1, \dots, m_1+\dots+m_{s-1}+1\}$. It is known that the elements $y_{m_1+m_2+1}, \dots, y_{m_1+\dots+m_{s-1}+1}$ are generated from the products $[x_i, y_1]$ $(2 \leqslant i \leqslant n)$. Due to nilpotency of L we get $i_0 \notin \{m_1+m_2+1, \dots, m_1+\dots+m_{s-1}+1\}$. If y_{m_1+1} is generated by $[x_1, y_1]$, i.e. in the expression $[x_1, y_1] = \sum_{j=2}^{m} \alpha_{1,j} y_j$, $\alpha_{1,m_1+1} \neq 0$ then we consider the product

$$\big[[x_1, y_1], y_1\big] = \left[\sum_{j=2}^{m} \alpha_{1,j} y_j, y_1\right] = \alpha_{1,m_1+1}\beta_{m_1+1,2} x_2 + \sum_{i \geqslant 3} (*) x_i.$$

On the other hand,

$$\big[[x_1, y_1], y_1\big] = \frac{1}{2}\big[x_1, [y_1, y_1]\big] = \frac{1}{2}\left[x_1, \sum_{j=2}^{n} \beta_{1,j} x_j\right] = \sum_{i \geqslant 3} (*) x_i.$$

Comparing the coefficients at the corresponding basic elements we obtain $\alpha_{1,m_1+1}\beta_{m_1+1,2} = 0$, which implies $\beta_{m_1+1,2} = 0$. It means that $i_0 \neq m_1+1$. Therefore, $\beta_{i_0,2} \neq 0$, where $i_0 \notin \{m_1+1, \dots, m_1+\dots+m_{s-1}+1\}$.

Case $y_2 \notin L^3$. Then $B_2 \neq 0$. Let $h \in \mathbb{N}$ be a number such that $x_2 \in L^h \setminus L^{h+1}$, that is,

$$L^h = \{x_2, x_3, \ldots, x_n, y_h, \ldots, y_{m_1}, B_1 y_2 + B_2 y_{m_1+1}, y_{m_1+2}, \ldots, y_m\}, \quad h \geqslant 3,$$
$$L^{h+1} = \{x_3, x_4, \ldots, x_n, y_h, \ldots, y_{m_1}, B_1 y_2 + B_2 y_{m_1+1}, y_{m_1+2}, \ldots, y_m\}.$$

Since the elements $B_1 y_2 + B_2 y_{m_1+1}, y_{m_1+m_2+1}, \ldots, y_{m_1+\cdots+m_{s-1}+1}$ are generated from the multiplications $[x_i, y_1], 2 \leqslant i \leqslant n$ it follows that $h \leqslant m_1 + 1$.

So, x_2 can be obtained only from the product $[y_{h-1}, y_1]$ and thereby $\beta_{h-1,2} \neq 0$. Making the change $x_2' = \sum_{j=2}^n \beta_{h-1,j} x_j$ we can assume that $[y_{h-1}, y_1] = x_2$.

Now let p be a natural number such that $y_h \in L^{h+p} \setminus L^{h+p+1}$. Then for the powers of superalgebra L we have the following

$$L^{h+p} = \{x_{p+2}, x_{p+3}, \ldots, x_n, y_h, \ldots, y_{m_1}, B_1 y_2 + B_2 y_{m_1+1}, y_{m_1+2}, \ldots, y_m\}, \quad p \geqslant 1,$$
$$L^{h+p+1} = \{x_{p+2}, x_{p+3}, \ldots, x_n, y_{h+1}, \ldots, y_{m_1}, B_1 y_2 + B_2 y_{m_1+1}, y_{m_1+2}, \ldots, y_m\}.$$

In the following lemma the useful expression for the products $[y_i, y_j]$ is presented.

Lemma 3.2. *The equality*:

$$[y_i, y_j] = (-1)^{h-1-i} C_{j-1}^{h-1-i} x_{i+j+2-h} + \sum_{t > i+j+2-h} (*) x_t, \tag{3}$$

$1 \leqslant i \leqslant h-1$, $h-i \leqslant j \leqslant \min\{h-1, h-1+p-i\}$, *holds.*

Proof. The proof is deduced by the induction on j at any value of i. □

For the natural number p we have the following

Lemma 3.3. *Under the above conditions* $p = 1$.

Proof. Assume the contrary, i.e. $p > 1$. Then we can suppose

$$[x_i, x_1] = x_{i+1}, \quad 2 \leqslant i \leqslant p, \qquad [x_{p+1}, y_1] = \sum_{j=h}^m \alpha_{p+1,j} y_j, \qquad \alpha_{p+1,h} \neq 0.$$

Using the equality (3) we consider the following chain of equalities

$$\begin{aligned}\big[y_1, [y_{h-1}, x_1]\big] &= \big[[y_1, y_{h-1}], x_1\big] - \big[[y_1, x_1], y_{h-1}\big] = (-1)^{h-2} x_3 + \sum_{t \geqslant 4} (*) x_t \\ &\quad - (-1)^{h-3}(h-2) x_3 + \sum_{t \geqslant 4} (*) x_t = (-1)^h (h-1) x_3 + \sum_{t \geqslant 4} (*) x_t.\end{aligned}$$

If $h \leqslant m_1$, then $[y_1, [y_{h-1}, x_1]] = [y_1, y_h]$. Since $y_h \in L^{h+p}$ and $p > 1$ then in the decomposition of $[y_1, y_h]$ the coefficients at the basic elements x_2 and x_3 are equal to zero. Therefore, from the above equalities we get a contradiction with assumption $p > 1$.

If $h = m_1 + 1$, then $[y_1, [y_{h-1}, x_1]] = 0$ and we also obtain the irregular equality $(-1)^h (h-1) x_3 + \sum_{t \geqslant 4} (*) x_t = 0$. Therefore, the proof of the lemma is completed. □

We summarize our main result of the considered case in the following

Theorem 3.1. *Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra from $Leib_{n,m}$ with characteristic sequence equal to $(n_1, \dots, n_k \mid m_1, \dots, m_s)$, where $n_1 \leqslant n-2$, $m_1 \leqslant m-1$ and let $\dim(L^3)_0 = n-1$ with $y_2 \notin L^3$. Then L has nilindex less than $n+m$.*

Proof. Let us assume the contrary, i.e. the nilindex of the superalgebra L is equal to $n+m$. Then according to Lemma 3.3 we have

$$L^{h+2} = \{x_3, \dots, x_n, y_{h+1}, \dots, y_{m_1}, B_1 y_2 + B_2 y_{m_1+1}, y_{m_1+2}, \dots, y_m\}.$$

Since $y_h \notin L^{h+2}$, it follows that

$$\alpha_{2,h} \neq 0, \qquad \alpha_{i,h} = 0 \quad \text{for } i > 2.$$

Consider the product

$$\big[[y_{h-1}, y_1], y_1\big] = \frac{1}{2}\big[y_{h-1}, [y_1, y_1]\big] = \frac{1}{2}\left[y_{h-1}, \sum_{i=2}^{n} \beta_{1,i} x_i\right].$$

The element y_{h-1} belongs to L^{h-1} and elements $x_2, x_3, \dots, x_n$ lie in L^3. Hence $\frac{1}{2}[y_{h-1}, \sum_{i=2}^{n} \beta_{1,i} x_i] \in L^{h+2}$. Since $y_h \notin L^{h+2}$, we obtain that $[[y_{h-1}, y_1], y_1] = \sum_{j \geqslant h+1} (*) y_j$.

On the other hand,

$$\big[[y_{h-1}, y_1], y_1\big] = [x_2, y_1] = \alpha_{2,h} y_h + \sum_{j=h+1}^{m} \alpha_{2,j} y_j.$$

Comparing the coefficients at the basic elements we obtain $\alpha_{2,h} = 0$, which is a contradiction with the assumption that the superalgebra L has nilindex equal to $n+m$ and therefore the assertion of the theorem is proved. □

Case $y_2 \in L^3$. Then $B_2 = 0$ and the following theorem is true.

Theorem 3.2. *Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra from $Leib_{n,m}$ with characteristic sequence equal to $(n_1, \dots, n_k \mid m_1, \dots, m_s)$, where $n_1 \leqslant n-2$, $m_1 \leqslant m-1$ and let $\dim(L^3)_0 = n-1$ with $y_2 \in L^3$. Then L has nilindex less than $n+m$.*

Proof. We shall prove the assertion of the theorem by the contrary method, i.e. we assume that the nilindex of the superalgebra L equal to $n+m$. The condition $y_2 \in L^3$ implies

$$L^3 = \{x_2, x_3, \dots, x_n, y_2, \dots, y_{m_1}, y_{m_1+2}, \dots, y_m\}.$$

Then $\alpha_{1,m_1+1} \neq 0$ and $\alpha_{i,m_1+1} = 0$ for $i \geqslant 2$. The element y_2 is generated by the products $[x_i, y_1]$, $i \geqslant 2$ which implies $y_2 \in L^4$. Since $[y_{m_1+1}, y_1] = [[x_1, y_1], y_1] = \frac{1}{2}[x_1, [y_1, y_1]] = \frac{1}{2}[x_1, \sum(*) x_i]$ and x_2 is a generator of the Leibniz algebra L_0 then x_2 cannot be generated from the product $[y_{m_1+1}, y_1]$. Thereby x_2 also belongs to L^4.

Consider the equality

$$\big[[x_1, y_1], x_1\big] = \big[x_1, [y_1, x_1]\big] + \big[[x_1, x_1], y_1\big] = [x_1, y_2] - \left[\sum_{i\geqslant 3}(*)x_i, y_1\right].$$

From this it follows that the product $[[x_1, y_1], x_1]$ belongs to L^5 (and therefore belongs to L^4).

On the other hand,

$$\big[[x_1, y_1], x_1\big] = \left[\sum_{j=2}^{m}\alpha_{1,j}y_j, x_1\right] = \alpha_{1,2}y_3 + \cdots + \alpha_{1,m_1-1}y_{m_1} + \alpha_{1,m_1+1}y_{m_1+2} + \cdots + \alpha_{1,m-1}y_m.$$

Since $\alpha_{1,m_1+1} \neq 0$, we obtain that $y_{m_1+2} \in L^4$. Thus, we have $L^4 = \{x_2, x_3, \dots, x_n, y_2, \dots, y_{m_1}, y_{m_1+2}, \dots, y_m\}$, that is, $L^4 = L^3$, which is in contradiction with the nilpotency of the superalgebra L.

Thus, we get a contradiction with the assumption that the superalgebra L has nilindex equal to $n+m$ and therefore the assertion of the theorem is proved. □

From Theorems 3.1 and 3.2 we obtain that a Leibniz superalgebra L satisfying the condition $\dim(L^3)_0 = n-1$ has nilindex less than $n+m$.

The investigation of Leibniz superalgebras satisfying the property $\dim(L^3)_0 = n-2$ shows that the restriction to nilindex depends on the structure of the Leibniz algebra L_0. Below we present some necessary remarks on nilpotent Leibniz algebras.

Let $A = \{z_1, z_2, \dots, z_n\}$ be an n-dimensional nilpotent Leibniz algebra of nilindex l $(l < n)$. Note that the algebra A is not single-generated.

Proposition 3.1. *(See [6].) Let $gr(A)$ be a naturally graded non-Lie Leibniz algebra. Then* $\dim A^3 \leqslant n-4$.

3.2. The case of both generators lie in L_0

The result on nilindex of superalgebras satisfying the condition $\dim(L^3)_0 = n-2$ is established in the following two theorems.

Theorem 3.3. *Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra from $Leib_{n,m}$ with characteristic sequence $(n_1, \dots, n_k \mid m_1, \dots m_s)$, where $n_1 \leqslant n-2$, $m_1 \leqslant m-1$,* $\dim(L^3)_0 = n-2$ *and* $\dim L_0^3 \leqslant n-4$. *Then L has nilindex less than $n+m$.*

Proof. Let us assume the contrary, i.e. the nilindex of the superalgebra L is equal to $n+m$. According to the condition $\dim(L^3)_0 = n-2$ we have

$$L^3 = \{x_3, x_4, \dots, x_n, y_2, y_3, \dots, y_m\}.$$

From the condition $\dim L_0^3 \leqslant n-4$ it follows that there exist at least two basic elements, which do not belong to L_0^3. Without loss of generality, one can assume $x_3, x_4 \notin L_0^3$.

Let h be a natural number such that $x_3 \in L^{h+1} \setminus L^{h+2}$, then we have

$$L^{h+1} = \{x_3, x_4, \dots, x_n, y_h, y_{h+1}, \dots, y_m\}, \quad h \geqslant 2, \qquad \beta_{h-1,3} \neq 0,$$
$$L^{h+2} = \{x_4, \dots, x_n, y_h, y_{h+1}, \dots, y_m\}.$$

Let us suppose $x_3 \notin L_0^2$. Then we have that x_3 cannot be obtained by the products $[x_i, x_1]$, with $2 \leqslant i \leqslant n$. Therefore, it is generated by products $[y_j, y_1]$, $2 \leqslant j \leqslant m$, which implies $h \geqslant 3$ and $\alpha_{2,2} \neq 0$.

If $h = 3$, then $\beta_{2,3} \neq 0$.

Consider the chain of equalities

$$\big[[x_2, y_1], y_1\big] = \left[\sum_{j=2}^{m} \alpha_{2,j} y_j, y_1\right] = \sum_{j=2}^{m} \alpha_{2,j}[y_j, y_1] = \alpha_{2,2}\beta_{2,3}x_3 + \sum_{i \geqslant 4}(*)x_i.$$

On the other hand,

$$\big[[x_2, y_1], y_1\big] = \frac{1}{2}\big[x_2, [y_1, y_1]\big] = \frac{1}{2}\left[x_2, \sum_{i=2}^{n} \beta_{1,i}x_i\right] = \frac{1}{2}\sum_{i=2}^{n} \beta_{1,i}[x_2, x_i] = \sum_{i \geqslant 4}(*)x_i.$$

Comparing the coefficients at the corresponding basic elements, we get a contradiction with $\beta_{2,3} = 0$. Thus, $h \geqslant 4$.

Since $y_2 \in L^3$ and $h \geqslant 4$ we have $y_{h-2} \in L^{h-1}$, which implies $[y_{h-2}, y_2] \in L^{h+2} = \{x_4, \dots, x_n, y_h, y_{h+1}, \dots, y_m\}$. It means that in the decomposition $[y_{h-2}, y_2]$ the coefficient at the basic element x_3 is equal to zero.

On the other hand,

$$\begin{aligned}
[y_{h-2}, y_2] &= \big[y_{h-2}, [y_1, x_1]\big] = \big[[y_{h-2}, y_1], x_1\big] - \big[[y_{h-2}, x_1], y_1\big] \\
&= \left[\sum_{i=2}^{n} \beta_{h-2,i}x_i, x_1\right] - [y_{h-1}, y_1] = -\beta_{h-1,3}x_3 + \sum_{i \geqslant 4}(*)x_i.
\end{aligned}$$

Hence, we get $\beta_{h-1,3} = 0$, which is obtained from the assumption $x_3 \notin L_0^2$.

Therefore, we have $x_3, x_4 \in L_0^2 \setminus L_0^3$. The condition $x_4 \notin L_0^3$ implies that x_4 cannot be obtained by the products $[x_i, x_1]$, with $3 \leqslant i \leqslant n$. Therefore, it is generated by products $[y_j, y_1], h \leqslant j \leqslant m$. Hence, $L^{h+3} = \{x_4, \dots, x_n, y_{h+1}, \dots, y_m\}$ and $y_h \in L^{h+2} \setminus L^{h+3}$, which implies $\alpha_{3,h} \neq 0$.

Let p $(p \geqslant 3)$ be a natural number such that $x_4 \in L^{h+p} \setminus L^{h+p+1}$.

Suppose that $p = 3$. Then $\beta_{h,4} \neq 0$.

Consider the chain of equalities

$$\big[[x_3, y_1], y_1\big] = \left[\sum_{j=h}^{m} \alpha_{3,j} y_j, y_1\right] = \sum_{j=h}^{m} \alpha_{3,j}[y_j, y_1] = \alpha_{3,h}\beta_{h,4}x_4 + \sum_{i \geqslant 5}(*)x_i.$$

On the other hand,

$$\big[[x_3, y_1], y_1\big] = \frac{1}{2}\big[x_3, [y_1, y_1]\big] = \frac{1}{2}\left[x_3, \sum_{i=2}^{n} \beta_{1,i}x_i\right] = \frac{1}{2}\sum_{i=2}^{n} \beta_{1,i}[x_3, x_i] = \sum_{i \geqslant 5}(*)x_i.$$

Comparing the coefficients at the corresponding basic elements in these equations we get $\alpha_{3,h}\beta_{h,4} = 0$, which implies $\beta_{h,4} = 0$. This is a contradiction with the assumption $p = 3$. Therefore, $p \geqslant 4$ and for the powers of the descending lower sequences we have

$$L^{h+p-2} = \{x_4, \dots, x_n, y_{h+p-4}, \dots, y_m\},$$
$$L^{h+p-1} = \{x_4, \dots, x_n, y_{h+p-3}, \dots, y_m\},$$
$$L^{h+p} = \{x_4, \dots, x_n, y_{h+p-2}, \dots, y_m\},$$
$$L^{h+p+1} = \{x_5, \dots, x_n, y_{h+p-2}, \dots, y_m\}.$$

It is easy to see that in the decomposition $[y_{h+p-3}, y_1] = \sum_{i=4}^{n} \beta_{h+p-3,i} x_i$ we have $\beta_{h+p-3,4} \neq 0$. Consider the equalities

$$\begin{aligned}
[y_{h+p-4}, y_2] &= \big[y_{h+p-4}, [y_1, x_1]\big] = \big[[y_{h+p-4}, y_1], x_1\big] - \big[[y_{h+p-4}, x_1], y_1\big] \\
&= \left[\sum_{i=4}^{n} \beta_{h+p-3,i} x_i, x_1\right] - [y_{h+p-3}, y_1] = -\beta_{h+p-3,4} x_4 + \sum_{i \geqslant 5} (*) x_i.
\end{aligned}$$

Since $y_{h+p-4} \in L^{h+p-2}$, $y_2 \in L^3$ and $\beta_{h+p-3,4} \neq 0$, then the element x_4 should lie in L^{h+p+1}, but this fact is in contradiction with $L^{h+p+1} = \{x_5, \dots, x_n, y_{h+p-2}, \dots, y_m\}$. Thus, the superalgebra L has nilindex less than $n+m$. $\square$

From Theorem 3.3 we conclude that a Leibniz superalgebra $L = L_0 \oplus L_1$ with characteristic sequence $(n_1, \dots, n_k \mid m_1, \dots, m_s)$, where $n_1 \leqslant n-2$, $m_1 \leqslant m-1$, and nilindex $n+m$ can appear only if $\dim L_0^3 \geqslant n-3$. Taking into account the condition $n_1 \leqslant n-2$ and the properties of naturally graded subspaces $gr(L_0)_1$, $gr(L_0)_2$ we get $\dim L_0^3 = n-3$.

Let $\dim L_0^3 = n-3$. Then

$$gr(L_0)_1 = \{\bar{x}_1, \bar{x}_2\}, \qquad gr(L_0)_2 = \{\bar{x}_3\}.$$

From Proposition 3.1 the naturally graded Leibniz algebra $gr(L_0)$ is a Lie algebra, i.e. the following multiplication rules hold

$$\begin{cases} [\bar{x}_1, \bar{x}_1] = 0, \\ [\bar{x}_2, \bar{x}_1] = \bar{x}_3, \\ [\bar{x}_1, \bar{x}_2] = -\bar{x}_3, \\ [\bar{x}_2, \bar{x}_2] = 0. \end{cases}$$

Using these products for the corresponding products in the Leibniz algebra L_0 with the basis $\{x_1, x_2, \dots, x_n\}$ we have

$$\begin{cases} [x_1, x_1] = \gamma_{1,4} x_4 + \gamma_{1,5} x_5 + \dots + \gamma_{1,n} x_n, \\ [x_2, x_1] = x_3, \\ [x_1, x_2] = -x_3 + \gamma_{2,4} x_4 + \gamma_{2,5} x_5 + \dots + \gamma_{2,n} x_n, \\ [x_2, x_2] = \gamma_{3,4} x_4 + \gamma_{3,5} x_5 + \dots + \gamma_{3,n} x_n. \end{cases} \tag{4}$$

Theorem 3.4. *Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra from $Leib_{n,m}$ with characteristic sequence $(n_1, \dots, n_k \mid m_1, \dots, m_s)$, where $n_1 \leqslant n-2$, $m_1 \leqslant m-1$, $\dim(L^3)_0 = n-2$ and $\dim L_0^3 = n-3$. Then L has nilindex less than $n+m$.*

Proof. Let us suppose the contrary, i.e. the nilindex of the superalgebra L is equal to $n+m$. Then from the condition $\dim(L^3)_0 = n-2$ we obtain

$$L^2 = \{x_2, x_3, \dots, x_n, y_2, \dots, y_m\},$$
$$L^3 = \{x_3, x_4, \dots, x_n, y_2, \dots, y_m\},$$
$$L^4 \supset \{x_4, \dots, x_n, y_3, \dots, y_{m_1}, B_1 y_2 + B_2 y_{m_1+1}, y_{m_1+2}, \dots, y_m\}, \qquad (B_1, B_2) \neq (0,0).$$

Suppose $x_3 \notin L^4$. Then

$$L^4 = \{x_4, \dots, x_n, y_2, \dots, y_{m_1}, y_{m_1+1}, \dots, y_m\}.$$

Let $B'_1 y_2 + B'_2 y_{m_1+1}$ be an element which earlier disappears in the descending lower sequence for L. Then this element cannot be generated from the products $[x_i, y_1]$, $2 \leqslant i \leqslant n$. Indeed, since $x_3 \notin L^4$, the element cannot be generated from $[x_2, y_1]$. Due to the structure of L_0 the elements x_i $(3 \leqslant i \leqslant n)$ are in L_0^2, i.e. they are generated by the linear combinations of the products of elements from L_0. From the equalities

$$\big[[x_i, x_j], y_1\big] = \big[x_i, [x_j, y_1]\big] + \big[[x_i, y_1], x_j\big] = \left[x_i, \sum_{t=2}^{m} \alpha_{j,t} y_t\right] + \left[\sum_{t=2}^{m} \alpha_{i,t} y_t, x_i\right]$$

we derive that the element $B'_1 y_2 + B'_2 y_{m_1+1}$ cannot be obtained by the products $[x_i, y_1], 3 \leqslant i \leqslant n$. However, it means that $x_3 \in L^4$. Thus, we have

$$L^4 = \{x_3, x_4, \dots, x_n, y_3, \dots, y_{m_1}, B_1 y_2 + B_2 y_{m_1+1}, y_{m_1+2}, \dots, y_m\},$$

where $(B_1, B_2) \neq (0,0)$ and $B_1 B'_2 - B_2 B'_1 \neq 0$.

The simple analysis of the terms L^3 and L^4 in the descending lower sequence implies

$$[x_2, y_1] = \alpha'_{2,2}\big(B'_1 y_2 + B'_2 y_{m_1+1}\big) + \alpha'_{2,m_1+1}(B_1 y_2 + B_2 y_{m_1+1}) + \sum_{\substack{j=3 \\ j \neq m_1+1}}^{m} \alpha_{2,j} y_j, \quad \alpha'_{2,2} \neq 0.$$

Let h be a natural number such that $x_3 \in L^{h+1} \setminus L^{h+2}$, i.e.

$$L^h = \{x_3, x_4, \dots, x_n, y_{h-1}, y_h, \dots, y_{m_1}, B_1 y_2 + B_2 y_{m_1+1}, y_{m_1+2}, \dots, y_m\}, \quad h \geqslant 3,$$
$$L^{h+1} = \{x_3, x_4, \dots, x_n, y_h, y_{h+1}, \dots, y_{m_1}, B_1 y_2 + B_2 y_{m_1+1}, y_{m_1+2}, \dots, y_m\},$$
$$L^{h+2} = \{x_4, \dots, x_n, y_h, y_{h+1}, \dots, y_{m_1}, B_1 y_2 + B_2 y_{m_1+1}, y_{m_1+2}, \dots, y_m\}.$$

If $h = 3$, then $[B'_1 y_2 + B'_2 y_{m_1+1}, y_1] = \beta'_{2,3} x_3 + \sum_{i \geqslant 4} (*) x_4$, $\beta'_{2,3} \neq 0$ and we consider the product

$$\begin{aligned}
\big[[x_2, y_1], y_1\big] &= \left[\alpha'_{2,2}\big(B'_1 y_2 + B'_2 y_{m_1+1}\big) + \alpha'_{2,m_1+1}(B_1 y_2 + B_2 y_{m_1+1}) + \sum_{\substack{j=3 \\ j \neq m_1+1}}^{m} \alpha_{2,j} y_j, y_1\right] \\
&= \alpha'_{2,2}\big[B'_1 y_2 + B'_2 y_{m_1+1}, y_1\big] + \alpha'_{2,m_1+1}[B_1 y_2 + B_2 y_{m_1+1}, y_1] \\
&\quad + \sum_{\substack{j=3 \\ j \neq m_1+1}}^{m} \alpha_{2,j}[y_j, y_1] = \alpha'_{2,2}\beta'_{2,3} x_3 + \sum_{i \geqslant 4} (*) x_4.
\end{aligned}$$

On the other hand, due to (4) we have

$$\big[[x_2, y_1], y_1\big] = \frac{1}{2}\big[x_2, [y_1, y_1]\big] = \frac{1}{2}\left[x_2, \sum_{i=2}^{n} \beta_{1,i} x_i\right] = \sum_{i \geqslant 4} (*) x_i.$$

Comparing the coefficients at the corresponding basic elements we get equality $\alpha'_{2,2}\beta'_{2,3} = 0$, i.e. we have a contradiction with the supposition $h = 3$.

If $h \geqslant 4$, then we obtain $\beta'_{h-1,3} \neq 0$. Consider the chain of equalities

$$\begin{aligned}[][y_{h-2}, y_2] &= \big[y_{h-2}, [y_1, x_1]\big] = \big[[y_{h-2}, y_1], x_1\big] - \big[[y_{h-2}, x_1], y_1\big] \\ &= \left[\sum_{i=3}^{n} \beta_{h-2,i} x_i, x_1\right] - [y_{h-1}, y_1] = -\beta_{h-1,3} x_3 + \sum_{i \geqslant 4} (*) x_i.\end{aligned}$$

Since $y_{h-2} \in L^{h-1}$ and $y_2 \in L^3$ then $x_3 \in L^{h+2} = \{x_4, \dots, x_n, y_{h-1}, \dots, y_m\}$, which is a contradiction with the assumption that the nilindex of L is equal to $n + m$. $\square$

Remark 3.1. In this subsection we used the product $[y_1, x_1] = y_2$. However, it is not difficult to check that the obtained results are also true under the condition $[y_1, x_1] = 0$.

3.3. The case of both generators lie in L_1

Theorem 3.5. *Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra from $Leib_{n,m}$ with characteristic sequence equal to $(n_1, \dots, n_k \mid m_1, \dots, m_s)$, where $n_1 \leqslant n - 2$, $m_1 \leqslant m - 1$, and both generators lying in L_1. Then L has nilindex less than $n + m$.*

Proof. Since both generators of the superalgebra L lie in L_1, they are linear combinations of the elements $\{y_1, y_{m_1+1}, \dots, y_{m_1+\dots+m_{s-1}+1}\}$. Without loss of generality we may assume that y_1 and y_{m_1+1} are generators.

Let $L^{2t} = \{x_i, x_{i+1}, \dots, x_n, y_j, \dots, y_m\}$ for some natural number t and let $z \in L$ be an arbitrary element such that $z \in L^{2t} \setminus L^{2t+1}$. Then z is obtained by the products of even number of generators. Hence $z \in L_0$ and $L^{2t+1} = \{x_{i+1}, \dots, x_n, y_j, \dots, y_m\}$. In a similar way, having $L^{2t+1} = \{x_{i+1}, \dots, x_n, y_j, \dots, y_m\}$ we obtain $L^{2t+2} = \{x_{i+1}, \dots, x_n, y_{j+1}, \dots, y_m\}$.

From the above arguments we conclude that $n = m - 1$ or $n = m - 2$ and

$$L^3 = \{x_2, \dots, x_n, y_2, y_3, \dots, y_{m_1}, y_{m_1+2}, \dots, y_m\}.$$

Applying the above arguments we get that an element of the form $B_1 y_2 + B_2 y_{m_1+2} + B_3 y_{m_1+m_2+1}$ disappears in L^4. Moreover, there exist two elements $B'_1 y_2 + B'_2 y_{m_1+2} + B'_3 y_{m_1+m_2+1}$ and $B''_1 y_2 + B''_2 y_{m_1+2} + B''_3 y_{m_1+m_2+1}$ which belong to L^4, where

$$rank \begin{pmatrix} B_1 & B_2 & B_3 \\ B'_1 & B'_2 & B'_3 \\ B''_1 & B''_2 & B''_3 \end{pmatrix} = 3.$$

Since x_2 does not belong to L^5 then the elements $B'_1 y_2 + B'_2 y_{m_1+2} + B'_3 y_{m_1+m_2+1}$, $B''_1 y_2 + B''_2 y_{m_1+2} + B''_3 y_{m_1+m_2+1}$ lie in L^5. Hence, from the notations

$$[x_1, y_1] = \alpha_{1,2}(B_1 y_2 + B_2 y_{m_1+2} + B_3 y_{m_1+m_2+1}) + \alpha_{1,m_1+2}\big(B'_1 y_2 + B'_2 y_{m_1+2} + B'_3 y_{m_1+m_2+1}\big)$$
$$+ \alpha_{1,m_1+m_2+1}\big(B''_1 y_2 + B''_2 y_{m_1+2} + B''_3 y_{m_1+m_2+1}\big) + \sum_{j=3,\ j\notin\{m_1+2,m_1+m_2+1\}}^{m} \alpha_{1,j} y_j,$$
$$[x_1, y_{m_1+1}] = \delta_{1,2}(B_1 y_2 + B_2 y_{m_1+2} + B_3 y_{m_1+m_2+1}) + \delta_{1,m_1+2}\big(B'_1 y_2 + B'_2 y_{m_1+2} + B'_3 y_{m_1+m_2+1}\big)$$
$$+ \delta_{1,m_1+m_2+1}\big(B''_1 y_2 + B''_2 y_{m_1+2} + B''_3 y_{m_1+m_2+1}\big) + \sum_{j=3,\ j\notin\{m_1+2,m_1+m_2+1\}}^{m} \delta_{1,j} y_j,$$

we have $(\alpha_{1,2}, \delta_{1,2}) \neq (0,0)$.

Similarly, from the notations

$$[B_1 y_2 + B_2 y_{m_1+2} + B_3 y_{m_1+m_2+1}, y_1] = \beta_{2,2}x_2 + \beta_{2,3}x_3 + \cdots + \beta_{2,n}x_n,$$
$$[B_1 y_2 + B_2 y_{m_1+2} + B_3 y_{m_1+m_2+1}, y_{m_1+1}] = \gamma_{2,2}x_2 + \gamma_{2,3}x_3 + \cdots + \gamma_{2,n}x_n,$$

we obtain the condition $(\beta_{2,2}, \gamma_{2,2}) \neq (0,0)$.

Consider the product

$$\begin{aligned}\big[x_1, [y_1, y_1]\big] = 2\big[[x_1, y_1], y_1\big] &= 2\alpha_{1,2}[B_1 y_2 + B_2 y_{m_1+2} + B_3 y_{m_1+m_2+1}, y_1] \\ &+ 2\alpha_{1,m_1+2}\big[B'_1 y_2 + B'_2 y_{m_1+2} + B'_3 y_{m_1+m_2+1}, y_1\big] \\ &+ 2\alpha_{1,m_1+m_2+1}\big[B''_1 y_2 + B''_2 y_{m_1+2} + B''_3 y_{m_1+m_2+1}, y_1\big] \\ &+ 2\sum_{j=3,\ j\notin\{m_1+2,m_1+m_2+1\}}^{m} \alpha_{1,j}[y_j, y_1] = 2\alpha_{1,2}\beta_{2,2}x_2 + \sum_{i\geqslant 3}(*)x_i.\end{aligned}$$

On the other hand,

$$\big[x_1, [y_1, y_1]\big] = [x_1, \beta_{1,1}x_1 + \beta_{1,2}x_2 + \cdots + \beta_{1,n}x_n] = \sum_{i\geqslant 3}(*)x_i.$$

Comparing the coefficients at the basic elements in these equations we obtain $\alpha_{1,2}\beta_{2,2} = 0$.

Analogously, considering the product $[x_1, [y_{m_1+1}, y_{m_1+1}]]$, we obtain $\delta_{1,2}\gamma_{2,2} = 0$.

From these equations and the conditions $(\beta_{2,2}, \gamma_{2,2}) \neq (0,0)$, $(\alpha_{1,2}, \delta_{1,2}) \neq (0,0)$ we easily obtain that the solutions are $\alpha_{1,2}\gamma_{2,2} \neq 0$, $\beta_{2,2} = \delta_{1,2} = 0$ or $\beta_{2,2}\delta_{1,2} \neq 0$, $\alpha_{1,2} = \gamma_{2,2} = 0$.

Consider the following product

$$\big[[x_1, y_1], y_{m_1+1}\big] = \big[x_1, [y_1, y_{m_1+1}]\big] - \big[[x_1, y_{m_1+1}], y_1\big] = -\delta_{1,2}\beta_{2,2}x_2 + \sum_{i\geqslant 3}(*)x_i.$$

On the other hand,

$$\big[[x_1, y_1], y_{m_1+1}\big] = \alpha_{1,2}\gamma_{2,2}x_2 + \sum_{i\geqslant 3}(*)x_i.$$

Comparing the coefficients of the basic elements in these equations we obtain the irregular equation $\alpha_{1,2}\gamma_{2,2} = -\beta_{2,2}\delta_{1,2}$. Hence, we obtain a contradiction with the assumption that the nilindex of the superalgebra is equal to $n+m$. And the theorem is proved. $\square$

Thus, the results of Theorems 3.1–3.5 show that the Leibniz superalgebras with nilindex $n+m$ ($m \neq 0$) are the superalgebras mentioned in Section 2. Hence, the classification of the Leibniz superalgebras with nilindex $n+m$ is completed.

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