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Apery and micro-invariants of a one-dimensional Cohen–Macaulay local ring and invariants of its tangent cone

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ABSTRACT

Given a one-dimensional equicharacteristic Cohen–Macaulay local ring A, Juan Elias introduced in 2001 the set of micro-invariants of A in terms of the first neighborhood ring. On the other hand, if A is a one-dimensional complete equicharacteristic and residually rational domain, Valentina Barucci and Ralf Fröberg defined in 2006 a new set of invariants in terms of the Apery set of the value semigroup of A. We give a new interpretation for these sets of invariants that allow to extend their definition to any one-dimensional Cohen–Macaulay ring. We compare these two sets of invariants with the one introduced by the authors for the tangent cone of a one-dimensional Cohen–Macaulay local ring and give explicit formulas relating them. We show that, in fact, they coincide if and only if the tangent cone $G(A)$ is Cohen–Macaulay. Some explicit computations will also be given.

1. Introduction

Let $(A, \mathfrak{m})$ be a one-dimensional Cohen–Macaulay local ring with infinite residue field K and set $G(\mathfrak{m}) := \bigoplus_{n\geqslant 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$ for its tangent cone. In recent years, several new families of numerical sets have been defined in order to study its structure and properties. We will denote by e the multiplicity of the ring A and by r its reduction number.

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Given xA a minimal reduction of $\mathfrak{m}$ the ring $F(x) := \bigoplus_{n\geqslant 0} \frac{x^n A}{x^n \mathfrak{m}}$ is isomorphic to a polynomial ring in one variable over $K = A/\mathfrak{m}$ and the equalities $\mathfrak{m}I^n \cap x^n A = x^n \mathfrak{m}$ are satisfied for all n. The authors have observed in [3] that the corresponding Noether normalization

$$F(x) \hookrightarrow G(\mathfrak{m})$$

provides a decomposition of $G(\mathfrak{m})$ as a direct sum of graded cyclic $F(x)$-modules of the form

$$G(\mathfrak{m}) \cong F(x) \bigoplus_{i=1}^{e-1} F(x)(-r_i) \bigoplus_{j=1}^{f} \left(\frac{F(x)}{(x^*)^{t_j} F(x)} \right)(-s_j)$$

for some integers $1 \leqslant s_1 \leqslant \cdots \leqslant s_f$ and $r_1 \leqslant \cdots \leqslant r_{e-1}$ and where x^* denotes the class of x in $\frac{xA}{x\mathfrak{m}} \subseteq F(x)$. In the same paper, the above decomposition is rewritten as

$$G(\mathfrak{m}) \cong \bigoplus_{i=0}^{r} \big(F(x)(-i)\big)^{\alpha_i} \bigoplus_{i=1}^{r-1} \bigoplus_{j=1}^{r-i} \left(\frac{F(x)}{(x^*)^{j} F(x)}(-i) \right)^{\alpha_{i,j}},$$

with $\alpha_0 = 1$, $\alpha_r \neq 0$ and $\sum_{i=1}^{r} \alpha_i = e - 1$.

It turns out that the numbers $\alpha_1, \ldots, \alpha_r$ are independent of the chosen minimal reduction, while the $\alpha_{i,j}$ depend on it. For the purpose of this paper we call $\{\alpha_i, \alpha_{i,j}\}$ the set of invariants of the tangent cone (with respect to x).

Let A' the first neighborhood ring of A and assume that A is equicharacteristic and complete. Then A has a coefficient field K and a transcendental element x such that $W := K[\![x]\!] \subset A$ is a finite extension, xA being a minimal reduction of $\mathfrak{m}$. Juan Elias observed in [6] that A'/A is a torsion finitely generated W-module and that there exist integers $a_1 \leqslant \cdots \leqslant a_{e-1}$ such that

$$\frac{A'}{A} \cong \bigoplus_{j=1}^{e-1} \frac{W}{x^{a_j} W}.$$

In fact, it may be seen that $a_j \leqslant r$ and that the numbers $\{a_1, \ldots, a_{e-1}\}$ are independent of the chosen minimal reduction xA and he defines this set of numbers as the set of micro-invariants of A. By considering $\beta_i = \#\{j; a_j = i\}$ the above decomposition can be rewritten as

$$\frac{A'}{A} \cong \bigoplus_{i=1}^{r} \left(\frac{W}{x^i W} \right)^{\beta_i}.$$

For the purpose of this paper we call $\{\beta_1, \ldots, \beta_r\}$ the set of micro-invariants of A.

Now assume instead that A is a complete equicharacteristic, residually rational local domain with multiplicity e; that is, A is a subring of a formal power series ring $K[\![t]\!]$ with conductor $(A : K[\![t]\!]) \neq 0$. Consider the value semigroup $S := v(A) = \{v(a)\colon\ 0 \neq a \in A\}$ and $\mathrm{Ap}(S) = \{w_0 = 0, w_1, \ldots, w_{e-1}\}$, the Apery set of S with respect to e; that is, the set of the smallest elements in S in each congruence class modulo e. An element x with smallest value $v(x) = e$ generates a minimal reduction of A. A subset $\{g_0 = 1, g_1, \ldots, g_{e-1}\}$ is an Apery basis of A with respect to x if, for each j, $1 \leqslant j \leqslant e - 1$, the following conditions are satisfied:

(1) $v(g_j) = w_j$,
(2) if $g \in \mathfrak{m}^i + xA \setminus \mathfrak{m}^{i+1} + xA$ and $v(g) = v(g_j)$ then $g_j \in \mathfrak{m}^i + xA$.

Fixed an Apery basis $\{g_0 = 1, g_1, \dots, g_{e-1}\}$ with respect to x one may consider, for $1 \leqslant j \leqslant e-1$, the numbers c_j as the largest i such that $g_j \in \mathfrak{m}^i + xA$. Observe that $c_j \leqslant r$. Then, if $\gamma_i = \#\{j;\ c_j = i\}$, we call $\{\gamma_1, \dots, \gamma_r\}$ the set of Apery invariants of A.

The main purpose of this paper is to relate these three families of invariants by giving explicit formulas describing their relations. The formulas are expressed in terms of colon ideals that allow to characterize when the three families coincide: this is precisely when the tangent cone is Cohen–Macaulay. Moreover, we do this completely in general just assuming that the ring A is Cohen–Macaulay. For that, we first extend to any one-dimensional Cohen–Macaulay local ring the definitions of the micro-invariants introduced by Elias and the Apery invariants. Also, some computations are made in general when the reduction number, the embedding dimension or the multiplicity of A are small. In the case of semigroup rings all the computations can be done in terms of usual invariants of the semigroup itself.

2. Background and preliminaries

First, we set up some notation and definitions. Let $(A, \mathfrak{m})$ be a one-dimensional Cohen–Macaulay local ring with infinite residue field, embedding dimension b, reduction number r and multiplicity e.

2.1. Multiplicity, embedding dimension and reduction number

The length of an A-module M will be denoted by $\lambda(M)$ and its minimum number of generators by $\mu(M)$. The *embedding dimension* of A is defined as the number $b = \lambda(\mathfrak{m}/\mathfrak{m}^2) = \mu(\mathfrak{m})$.

An element x in $\mathfrak{m}^s$ is called *superficial of degree s* if $\mathfrak{m}^{n+s} = x\mathfrak{m}^n$ for all large n. Superficial elements generate $\mathfrak{m}$-primary ideals, and hence are regular elements of A. Being the residue field $A/\mathfrak{m}$ infinite, the ring A has superficial elements of degree one, and the ideals they generate are the minimal reductions of $\mathfrak{m}$. Let xA be a minimal reduction of $\mathfrak{m}$. Also, in our situation, the reduction number of $\mathfrak{m}$ with respect to xA, that is, the minimum integer r such that $\mathfrak{m}^{r+1} = x\mathfrak{m}^r$ does not depend of the chosen minimal reduction and it will be called the *reduction number* of A.

We consider $H(n) := \mu(\mathfrak{m}^n) = \lambda(\mathfrak{m}^n/\mathfrak{m}^{n+1})$ the Hilbert function of $\mathfrak{m}$ and $H^1(n) = \sum_{i=0}^{n} H(i)$ its Hilbert–Samuel function. This is of polynomial type of degree 1, and the *multiplicity* of A is defined as the integer e such that $H^1(n) = e(n+1) - \rho$ for all large n.

In the nice book by Judith D. Sally [12] dedicated to the study of the numbers of generators of ideals in local rings, it is proved that $\lambda(I/xI) = \lambda(A/xA) = e$ for any ideal I of A of height 1. Thus, taking $I = \mathfrak{m}^n$ one has

$$e = \lambda\left(\mathfrak{m}^n / x\mathfrak{m}^n\right) = \mu\left(\mathfrak{m}^n\right) + \lambda\left(\mathfrak{m}^{n+1} / x\mathfrak{m}^n\right).$$

Thus, $e = \mu(\mathfrak{m}^n) = \mu(\mathfrak{m}^r)$ for $n \geqslant r$ and $\mu(\mathfrak{m}^n) = e - \lambda(\mathfrak{m}^{n+1}/x\mathfrak{m}^n) < e$ for $n < r$. Also, a result of Paul Eakin and Avinash Sathaye gives the lower bound $n + 1 \leqslant \mu(\mathfrak{m}^n)$ for $n \leqslant r$ (an elementary proof of this bound in the one-dimensional case follows from [3, Proposition 26]). In particular $r \leqslant e-1$ and $b = e - \lambda(\mathfrak{m}^2/x\mathfrak{m}) \leqslant e$.

In order to describe ρ, it is easy to see that for $n \geqslant r$ it is satisfied the equality

$$H^1(n) = \mu\left(\mathfrak{m}^r\right)(n+1) + 1 + \mu(\mathfrak{m}) + \cdots + \mu\left(\mathfrak{m}^{r-1}\right) - r\mu\left(\mathfrak{m}^r\right),$$

thus

$$\rho = r\mu\left(\mathfrak{m}^r\right) - \left(1 + \mu(\mathfrak{m}) + \cdots + \mu\left(\mathfrak{m}^{r-1}\right)\right) = e - 1 + \sum_{i=1}^{r-1} \lambda\left(\mathfrak{m}^{i+1} / x\mathfrak{m}^i\right).$$

2.2. The invariants of the tangent cone

Let xA be a minimal reduction of $\mathfrak{m}$ and $\alpha_i, \alpha_{i,j}$ the numbers defined in the introduction. The α_i's and the $\alpha_{i,j}$'s can be related in terms of lengths of colon ideals. In order to express this fact we first define the numbers $f_{i,j}$ as

$$f_{i,j} := \lambda\left(\frac{\mathfrak{m}^i \cap (\mathfrak{m}^{i+j+1} : x^j)}{\mathfrak{m}^{i+1}}\right).$$

Remark 1. Note that $f_{i,j} = 0$ if $(i,j) \notin \{(k,l) \mid 1 \leqslant k \leqslant r-1 \text{ and } 1 \leqslant l \leqslant r-k\}$, and also $f_{r-1,1} = 0$.

Then, in [3, Proposition 3, Proposition 7] the following result is proved.

Lemma 1. *It holds*:

(1) *for* $1 \leqslant i \leqslant r-1$,

$$\begin{aligned}\alpha_i &= \lambda\big(\mathfrak{m}^i / \big(\mathfrak{m}^i \cap \big(\mathfrak{m}^r : x^{r-i-1}\big) + x\mathfrak{m}^{i-1}\big)\big) \\ &= \lambda\big(\mathfrak{m}^i / \big(\mathfrak{m}^i \cap \big(\mathfrak{m}^r : x^{r-i-1}\big)\big)\big) - \lambda\big(\mathfrak{m}^i / \big(\mathfrak{m}^i \cap \big(\mathfrak{m}^r : x^{r-i}\big)\big)\big) \\ &= \mu\big(\mathfrak{m}^i\big) - f_{i,r-i} - \mu\big(\mathfrak{m}^{i-1}\big) + f_{i-1,r-i+1},\end{aligned}$$

and

$$\alpha_r = \lambda\big(\mathfrak{m}^r / \big(\mathfrak{m}^{r+1} + x\mathfrak{m}^{r-1}\big)\big) = \lambda\big(\mathfrak{m}^r / x\mathfrak{m}^{r-1}\big) = \mu\big(\mathfrak{m}^r\big) - \mu\big(\mathfrak{m}^{r-1}\big);$$

(2) $f_{k,l} = \sum_{(i,j)\in\Lambda} \alpha_{i,j}$

where $\Lambda = \{(i,j){:}\ 1 \leqslant i \leqslant k,\ k-i+1 \leqslant j \leqslant k-i+l\}$;

(3) *the* $f_{i,r-i}$*'s and so, the* α_i*'s are independent of the chosen minimal reduction* xA *of* $\mathfrak{m}$.

Remark 2. Some direct consequences for the tangent cone can be immediately deduced from the above result on the structure of $G(\mathfrak{m})$ as $F(x)$-module.

For instance, the equalities

$$0 = f_{r-1,1} = \sum_{1\leqslant i\leqslant r-1} \alpha_{i,r-i}$$

imply that $\alpha_{i,r-i} = 0$. So the $F(x)$-torsion submodule of $G(\mathfrak{m})$ has the form

$$T\big(G(\mathfrak{m})\big) \cong \bigoplus_{i=1}^{r-1} \bigoplus_{j=1}^{r-i-1} \left(\frac{F(x)}{(x^*)^j F(x)}(-i)\right)^{\alpha_{i,j}},$$

which always vanishes if $r \leqslant 2$. Thus the tangent cone is Cohen–Macaulay for r less or equal to 2, as it is well known.

In the next lemma we resume some characterizations in terms of colon ideals of the Cohen–Macaulay property of the tangent cone that will be used later on.

Given a in A we will denote by a^* the initial form of a. That is, if v is the largest integer n such that $a \in \mathfrak{m}^n$ then a^* is the class of a in $\mathfrak{m}^v/\mathfrak{m}^{v+1} \hookrightarrow G(\mathfrak{m})$. Observe that $(x^i)^* = (x^*)^i$.

Lemma 2. *The following conditions are equivalent*:

(1) $G(\mathfrak{m})$ *is a Cohen–Macaulay ring*;
(2) $(x^*)^i$ *is a regular element in* $G(\mathfrak{m})$ *for some* (*all*) $i \geqslant 1$;
(3) $\mathfrak{m}^n \cap x^i A = x^i \mathfrak{m}^{n-i}$ *for some* (*all*) $i \geqslant 1$ *and all* n;
(4) $(\mathfrak{m}^n : x^i) = \mathfrak{m}^{n-i}$ *for some* (*all*) $i \geqslant 1$ *and all* n;
(5) $\mathfrak{m}^i \cap (\mathfrak{m}^r : x^{r-i-1}) = \mathfrak{m}^{i+1}$ *for* $1 \leqslant i \leqslant r-2$.

Proof. We fix $i \geqslant 1$. The element $(x^i)^*$ is a system of parameters of $G(\mathfrak{m})$. Hence the equivalence between (1) and (2) is clear. Moreover, since x is regular in A, we have by the result of Paolo Valabrega and Giuseppe Valla [13, Corollary 2.7.] that $(x^i)^*$ is a regular element in $G(\mathfrak{m})$ if and only $\mathfrak{m}^n \cap (x^i) = x^i \mathfrak{m}^{n-i}$ for all n. Moreover, by using the regularity of x^i (or x) in A this last equality is equivalent with the equality of (4).

By [3, Proposition 2] the $F(x)$-torsion submodule of $G(\mathfrak{m})$ is

$$T\big(G(\mathfrak{m})\big) = H^0_{F(x)}\big(G(\mathfrak{m})\big) = \big(0 :_{G(\mathfrak{m})} \big(x^*\big)^{r-1}\big) = \bigoplus_{i=1}^{r-1} \big(\mathfrak{m}^i \cap \big(\mathfrak{m}^{r+1} : x^{r-i}\big)\big) / \mathfrak{m}^{i+1}.$$

The tangent cone $G(\mathfrak{m})$ is Cohen–Macaulay if and only if it is a free $F(x)$-module. Since $(\mathfrak{m}^{r+1} : x^{r-i}) = (\mathfrak{m}^r : x^{r-i-1})$ we have now the equivalence of (5) with any of the other assertions. $\square$

Lemma 3. *The following equality holds*

$$\sum_{i=1}^{r} i\alpha_i = \rho + \lambda\big(T\big(G(\mathfrak{m})\big)\big).$$

Proof. By Lemma 1 (1) we have that $\sum_{i=1}^r i\alpha_i = r\mu(\mathfrak{m}^r) - (1 + \mu(\mathfrak{m}) + \cdots + \mu(\mathfrak{m}^{r-1})) + f_{1,r-1} + \cdots + f_{r-2,2} = \rho + \lambda(T(G(\mathfrak{m})))$. $\square$

As a consequence of the above lemma we obtain the following characterization for the Cohen–Macaulay property of the tangent cone.

Corollary 4. $G(\mathfrak{m})$ *is Cohen–Macaulay if and only if* $\sum_{i=1}^r i\alpha_i = \rho$.

2.3. The micro-invariants of the ring

Douglas G. Northcott defined the first neighborhood ring of A as the set of all elements, in the total ring of fractions $Q(A)$ of A, of the form $\frac{b}{a}$, where $b \in \mathfrak{m}^s$ and a is a superficial element of degree s. This is a subring of $Q(A)$ containing A and we will denote it by A'. Let $\overline{A}$ be the integral closure of A in $Q(A)$. We summarize in the following lemma some of the basic facts on A'. For their proof we refer to the works of Eben Matlis [10, Chapter XII] and Joseph Lipman [9, §1], where this ring is studied in a more general context.

Lemma 5. *With the notations above introduced the following it holds*:

(1) $A' = A[\frac{\mathfrak{m}}{x}]$;
(2) $A' = \bigcup_{n \geqslant 0} (\mathfrak{m}^n :_{\overline{A}} \mathfrak{m}^n) = (\mathfrak{m}^r :_{\overline{A}} \mathfrak{m}^r)$;
(3) A' *is a finitely generated* A*-module, and hence is a semi-local, one-dimensional Cohen–Macaulay ring*;
(4) x *is a regular element of* A';
(5) $\mathfrak{m}^n A' = x^n A'$ *for all* n;

(6) $\mathfrak{m}^n = x^n A'$ *for* $n \geqslant r$;
(7) *if* M *is a finitely generated* A*-submodule of* $Q(A)$ *that contains a regular element of* A *then* $\lambda(M/xM) = e$;
(8) $\lambda(A'/\mathfrak{m}^n A') = ne$ *for all* n *and* $\lambda(A'/A) = \rho$.

For any one-dimensional Cohen–Macaulay local ring $(A, \mathfrak{m})$ we define the *micro-invariants* of A as the set of integers

$$\beta_i = \lambda\left(\frac{A + \mathfrak{m}^{i-1}A'}{A + \mathfrak{m}^i A'}\right) - \lambda\left(\frac{A + \mathfrak{m}^i A'}{A + \mathfrak{m}^{i+1}A'}\right)$$

for $i \in \{1, \dots, r\}$, and $\beta_0 = 1$.

Lemma 6. *The following equalities hold*:

(1) $\sum_{i=1}^r \beta_i = e - 1$;
(2) $\sum_{i=1}^r i\beta_i = \rho$.

Proof. For (1) observe that $\beta_r = \lambda((A + \mathfrak{m}^{r-1}A')/(A + \mathfrak{m}^r A'))$ since $A + \mathfrak{m}^r A' = A + \mathfrak{m}^r = A = A + \mathfrak{m}^{r+1} = A + \mathfrak{m}^{r+1}A'$, by Lemma 5 (5) and (6). So

$$\sum_{i=1}^r \beta_i = \lambda\big(A'/A + \mathfrak{m}A'\big) = \lambda\big(A'/\mathfrak{m}A'\big) - \lambda\big(A/\big(A \cap \mathfrak{m}A'\big)\big)$$
$$= \lambda\big(A'/\mathfrak{m}A'\big) - \lambda(A/\mathfrak{m}) = e - 1.$$

On the other hand,

$$\sum_{i=1}^r i\beta_i = \sum_{i=1}^r \lambda\big(\big(A + \mathfrak{m}^{i-1}A'\big)/\big(A + \mathfrak{m}^i A'\big)\big) = \lambda\big(A'/A\big) = \rho$$

by Lemma 5 (8) and so we get (2). $\square$

The following result is an immediate consequence of the above lemma and Corollary 4.

Corollary 7. $G(\mathfrak{m})$ *is Cohen–Macaulay if and only if* $\sum_{i=1}^r i\alpha_i = \sum_{i=1}^r i\beta_i$.

Assume now that A is in addition equicharacteristic and complete. Then A has a coefficient field K, and the extension $W := K[\![x]\!] \subseteq A$ is finite, where W is a discrete valuation ring. Notice that A and A' are finitely generated W-modules without torsion and so W-free modules of rank e, by Lemma 5(7).

Hence A'/A is a W-module of torsion and there exist integers $a_0 \leqslant \cdots \leqslant a_{e-1}$ such that

$$\frac{A'}{A} \cong \bigoplus_{i=0}^{e-1} \frac{W}{x^{a_i}W}.$$

The ideals $x^{a_0}W, \dots, x^{a_{e-1}}W$ are the invariants of A in A'. Elias shows in [6] that $a_0 = 0$ and that these numbers do not depend on W as well. In fact, the following holds, which gives the equivalence of the set of micro-invariants as we have just defined and the one defined by Elias in [6], in the case A is equicharacteristic and complete:

Lemma 8. *(See [6, Proposition 1.4].) For $i \geqslant 1$ it holds $\beta_i = \#\{j;\ a_j = i\} =$*

$$\lambda\big(\big(x^r A + \mathfrak{m}^{r+i-1}\big)/\big(x^r A + \mathfrak{m}^{r+i}\big)\big) - \lambda\big(\big(x^r A + \mathfrak{m}^{r+i}\big)/\big(x^r A + \mathfrak{m}^{r+i+1}\big)\big).$$

Proof. The first equality follows from the definition of the β_i's, the equalities $\mathfrak{m}^i A' = x^i A'$ for all i (Lemma 5(5)) and from the fact that

$$\#\{j;\ a_j = i\} = \lambda\big(\big(A + x^{i-1}A'\big)/\big(A + x^i A'\big)\big) - \lambda\big(\big(A + x^i A'\big)/\big(A + x^{i+1}A'\big)\big).$$

For the second equality one uses that x is a regular element of A' and that $\mathfrak{m}^i = \mathfrak{m}^i A' = x^i A'$ for $i \geqslant r$, by Lemma 5(6). Thus,

$$\begin{aligned}\big(A + \mathfrak{m}^i A'\big)/\big(A + \mathfrak{m}^{i+1}A'\big) &\cong x^r\big(A + \mathfrak{m}^i A'\big)/x^r\big(A + \mathfrak{m}^{i+1}A'\big)\\ &= \big(x^r A + \mathfrak{m}^{r+i}A'\big)/\big(A + \mathfrak{m}^{r+i+1}A'\big)\\ &= \big(x^r A + \mathfrak{m}^{r+i}\big)/\big(x^r A + \mathfrak{m}^{r+i+1}\big). \qquad \square\end{aligned}$$

Observe that, as a consequence of this lemma, the above decomposition of A'/A can also be written as

$$\frac{A'}{A} \cong \bigoplus_{i=1}^{r}\left(\frac{W}{x^i W}\right)^{\beta_i}.$$

2.4. Apery invariants

Let xA be a minimal reduction of $\mathfrak{m}$ and $\overline{\mathfrak{m}} := \mathfrak{m}/xA$ be the maximal ideal of A/xA.
We define the *Apery* invariants of A with respect to x as the set of integers

$$\gamma_i = \dim_K\left(\frac{\overline{\mathfrak{m}}^i}{\overline{\mathfrak{m}}^{i+1}}\right) = \lambda\left(\frac{\mathfrak{m}^i + xA}{\mathfrak{m}^{i+1} + xA}\right)$$

for $i \leqslant r$. That is, the values of the Hilbert–Samuel function of the 0-dimensional local ring A/xA.

Lemma 9. *The following equalities hold*:

(1) $\sum_{i=1}^r \gamma_i = e - 1$;
(2) $\sum_{i=1}^r i\gamma_i = \rho - \sum_{i=1}^{r-1} \lambda(\mathfrak{m}^{i+1} \cap xA/x\mathfrak{m}^i)$.

Proof. By considering the exact sequences

$$0 \longrightarrow \big(\mathfrak{m}^i + xA\big)/\big(\mathfrak{m}^{i+1} + xA\big) \longrightarrow A/\big(\mathfrak{m}^{i+1} + xA\big) \longrightarrow A/\big(\mathfrak{m}^i + xA\big) \longrightarrow 0$$

for $1 \leqslant i \leqslant r$, and taking lengths, the equality $\sum_{i=1}^r \gamma_i = \lambda(A/(\mathfrak{m}^{r+1} + xA)) - \lambda(A/(\mathfrak{m} + xA)) = \lambda(A/xA) - \lambda(A/\mathfrak{m}) = e - 1$ is deduced.

By using the above exact sequence also it is easily deduced that $\sum_{i=1}^r i\gamma_i = re - \sum_{i=1}^r \lambda(A/\mathfrak{m}^i + xA) = e - 1 + (r-1)e - \sum_{i=1}^{r-1} \lambda(A/\mathfrak{m}^{i+1} + xA)$. Then, $\sum_{i=1}^r i\gamma_i = e - 1 + \sum_{i=1}^{r-1} \lambda(\mathfrak{m}^{i+1}/\mathfrak{m}^{i+1} \cap xA)$ follows by taking lengths in the exact sequences

$$0 \longrightarrow \mathfrak{m}^{i+1}/\big(\mathfrak{m}^{i+1} \cap xA\big) \longrightarrow A/xA \longrightarrow A/\big(\mathfrak{m}^{i+1} + xA\big) \longrightarrow 0,$$

for $1 \leqslant i \leqslant r-1$. Now, the equality $e-1=\rho-\sum_{i=1}^{r-1}\lambda(\mathfrak{m}^{i+1}/x\mathfrak{m}^i)$, gives $\sum_{i=1}^{r} i\gamma_i = \rho-\sum_{i=1}^{r-1}\lambda(\mathfrak{m}^{i+1}/x\mathfrak{m}^i)+\sum_{i=1}^{r-1}\lambda(\mathfrak{m}^{i+1}/\mathfrak{m}^{i+1}\cap xA)$. Finally, the exact sequences

$$0 \longrightarrow \mathfrak{m}^i \cap xA/x\mathfrak{m}^i \longrightarrow \mathfrak{m}^{i+1}/x\mathfrak{m}^i \longrightarrow \mathfrak{m}^{i+1}/\big(\mathfrak{m}^i \cap xA\big) \longrightarrow 0,$$

for $1 \leqslant i \leqslant r-1$ transform the last equality into the sentence (2). □

Corollary 10. *It holds*:

$$\sum_{i=1}^{r} i\gamma_i \leqslant \sum_{i=1}^{r} i\beta_i \leqslant \sum_{i=1}^{r} i\alpha_i,$$

and any (*all*) *of the equalities occurs if and only if* $G(\mathfrak{m})$ *is Cohen–Macaulay.*

Proof. Lemma 3, Lemma 6 and Lemma 9 give the inequalities in the corollary. Also, these lemmas, and the characterization of the Cohen–Macaulay property of the tangent cone of A in terms of the Valabrega–Valla conditions (reflected in Lemma 2) and the vanishing of the torsion module $T(G(\mathfrak{m}))$, complete the proof. □

Assume that A is a complete equicharacteristic, residually rational local domain of multiplicity e; that is, A is a subring of the formal power series ring $K[\![t]\!]$ with conductor $(A : K[\![t]\!]) \neq 0$. Let us denote by v the t-adic valuation.

We consider the value semigroup $S := v(A) = \{v(a)\colon\ 0 \neq a \in A\}$. Then x is an element of smallest positive value e. We denote by $\mathrm{Ap}(S) = \{w_0 = 0, w_1, \ldots, w_{e-1}\}$, the Apery set of S with respect to e; that is, the set of the smallest elements in S in each congruence class module e.

We call a subset $\{g_0 = 1, g_1 \ldots, g_{e-1}\}$ of elements of A an Apery basis with respect to x if the following conditions are satisfied for each j, $1 \leqslant j \leqslant e-1$:

1. $v(g_j) = w_j$,
2. $\max\{i \mid g_j \in \mathfrak{m}^i + xA\} = \max\{i \mid w_j \in v(\mathfrak{m}^i + xA)\}$.

We shall denote by $c_j := \max\{i \mid g_j \in \mathfrak{m}^i + xA\}$. Observe that $c_j \leqslant r$. The following observation justifies why we call these invariants, the Apery invariants.

Lemma 11. *For* $i \geqslant 1$, $\gamma_i = \#\{j;\ c_j = i\}$.

Proof. Let $\mathrm{Ap}(S) = \{w_0, w_1, \ldots, w_{e-1}\}$, the Apery set of S and $\{g_0, g_1 \ldots, g_{e-1}\}$ be an Apery basis of A with respect to x.

Fixed i, we consider $\mathfrak{m}^i + xA$. If $i \leqslant c_j$ then $g_j \in \mathfrak{m}^i + xA$ and obviously $w_j \in \mathrm{Ap}(v(\mathfrak{m}^i + xA))$. If $i > c_j$ then, by definition of c_j, $w_j \notin v(\mathfrak{m}^{i+1} + xA))$ and, since $xg_j \in \mathfrak{m}^{i+1} + xA$ with $v(xg_j) = w_j + e$, we have that $w_j + e \in \mathrm{Ap}(v(\mathfrak{m}^i + xA))$. So, applying Lemma 2.1 of [2], $\mathfrak{m}^i + xA$ is a free $k[\![x]\!]$-module of rank e with basis $x^{\epsilon_{i,j}} g_j$ with $\epsilon_{i,j} \in \{0, 1\}$. Thus, $\lambda((\mathfrak{m}^i + xA)/xA) = \#\{j;\ c_j \geqslant i\}$ and $\gamma_i := \lambda((\mathfrak{m}^i + xA)/(\mathfrak{m}^{i+1} + xA)) = \#\{j;\ c_j = i\}$. □

We call a subset $\{f_0 = 1, f_1 \ldots, f_{e-1}\}$ of elements of A a BF-Apery basis if the following conditions are satisfied for each j, $1 \leqslant j \leqslant e-1$:

(1) $v(f_j) = w_j$,
(2′) $\max\{i \mid f_j \in \mathfrak{m}^i\} = \max\{i \mid w_j \in v(\mathfrak{m}^i)\}$.

We shall denote by $b_j := \max\{i \mid f_j \in \mathfrak{m}^i\}$ and we say that A satisfies the BF condition with respect to x if m^i, for all $i \geqslant 0$, is generated freely by elements of type $x^{h_{i,j}} f_j$, $0 \leqslant j \leqslant e-1$, for some exponents $h_{i,j}$.

Note that BF-Apery basis are called Apery basis by Barucci and Fröberg in [2]. In general, as shown by Lance Bryant in his Ph. Dissertation [1], the BF condition is not always satisfied. It is easy to see that under the BF condition with respect to x, then $\gamma_i = \#\{j;\ b_j = i\}$.

3. Comparing invariants

Let $(x) = xA$ be a minimal reduction of $\mathfrak{m}$. In this section we will compare the sets of numbers introduced in the above section; that is

- $\{\alpha_i, \alpha_{i,j}\}$ the invariants of the tangent cone $G(\mathfrak{m})$ with respect to x.
- $\{\beta_1, \dots, \beta_r\}$ the micro-invariants of A.
- $\{\gamma_1, \dots, \gamma_r\}$ the Apery invariants of A with respect to x.

3.1. Micro-invariants of the ring and invariants of its tangent cone

Our first purpose is to measure the difference between β_i and α_i also in terms of lengths of colon ideals. For this, we will begin by writing the β_i's in terms of lengths of specific colon ideals.

Lemma 12. *For $1 \leqslant i \leqslant r-1$, it holds*

$$\beta_i = \lambda\big(\big(\mathfrak{m}^r : x^{r-i}\big)/\big(\mathfrak{m}^r : x^{r-i-1}\big)\big) - \lambda\big(\big(\mathfrak{m}^r : x^{r-i+1}\big)/\big(\mathfrak{m}^r : x^{r-i}\big)\big),$$

and $\beta_r = \mu(\mathfrak{m}^r) - \lambda((\mathfrak{m}^r : x)/\mathfrak{m}^r)$.

Proof. By Lemma 8 we have that

$$\beta_i = \lambda\big(\big(\big(x^r\big) + \mathfrak{m}^{r+i-1}\big)/\big(\big(x^r\big) + \mathfrak{m}^{r+i}\big)\big) - \lambda\big(\big(\big(x^r\big) + \mathfrak{m}^{r+i}\big)/\big(\big(x^r\big) + \mathfrak{m}^{r+i+1}\big)\big).$$

Now, by considering the exact sequence

$$0 \to \big(x^r\big) \cap \mathfrak{m}^{r+i}/\big(x^r\big) \cap \mathfrak{m}^{r+i+1} \to \mathfrak{m}^{r+i}/\mathfrak{m}^{r+i+1} \to \mathfrak{m}^{r+i}/\big(\big(x^r\big) \cap \mathfrak{m}^{r+i} + \mathfrak{m}^{r+i+1}\big) \longrightarrow 0$$

and the isomorphisms

$$\big(\big(x^r\big) + \mathfrak{m}^{r+i}\big)/\big(\big(x^r\big) + \mathfrak{m}^{r+i+1}\big) \cong \mathfrak{m}^{r+i}/\big(\mathfrak{m}^{r+i} \cap \big(x^r\big) + \mathfrak{m}^{r+i+1}\big),$$
$$\mathfrak{m}^{r+i}/\mathfrak{m}^{r+i+1} \cong \mathfrak{m}^r/\mathfrak{m}^{r+1}$$

we obtain the equality

$$\lambda\big(\big(\big(x^r\big) + \mathfrak{m}^{r+i}\big)/\big(\big(x^r\big) + \mathfrak{m}^{r+i+1}\big)\big) = \mu\big(\mathfrak{m}^r\big) - \lambda\big(\big(x^r\big) \cap \mathfrak{m}^{r+i}/\big(\big(x^r\big) \cap \mathfrak{m}^{r+i+1}\big)\big).$$

Also, one can easily prove that $(x^r) \cap \mathfrak{m}^{r+i} = (x^r) \cap x^i\mathfrak{m}^r = x^r(\mathfrak{m}^r : x^{r-i})$. From these considerations it may be deduced that

$$\beta_i = \lambda\big(\big(\mathfrak{m}^r : x^{r-i}\big)/\big(\mathfrak{m}^r : x^{r-i-1}\big)\big) - \lambda\big(\big(\mathfrak{m}^r : x^{r-i+1}\big)/\big(\mathfrak{m}^r : x^{r-i}\big)\big)$$

for $1 \leqslant i \leqslant r-1$ and that $\beta_r = \mu(\mathfrak{m}^r) - \lambda((\mathfrak{m}^r : x)/\mathfrak{m}^r)$. $\square$

In the next proposition we express the difference between the value of the micro-invariant and the invariant for a specific i in terms of lengths of colon ideals.

Proposition 13. *For $1 \leqslant i \leqslant r$ it holds*

$$\beta_i + \lambda\bigl(\bigl(\mathfrak{m}^r : x^{r-i+1}\bigr)/\bigl(\mathfrak{m}^{i-1} + \bigl(\mathfrak{m}^r : x^{r-i}\bigr)\bigr)\bigr) = \alpha_i + \lambda\bigl(\bigl(\mathfrak{m}^r : x^{r-i}\bigr)/\bigl(\mathfrak{m}^i + \bigl(\mathfrak{m}^r : x^{r-i-1}\bigr)\bigr)\bigr).$$

Proof. For $1 \leqslant i \leqslant r-1$, consider the exact sequences

$$\begin{aligned} 0 \to \mathfrak{m}^i/\bigl(\mathfrak{m}^i \cap \bigl(\mathfrak{m}^r : x^{r-i-1}\bigr)\bigr) &\to \bigl(\mathfrak{m}^r : x^{r-i}\bigr)/\bigl(\mathfrak{m}^r : x^{r-i-1}\bigr) \\ &\to \bigl(\mathfrak{m}^r : x^{r-i}\bigr)/\bigl(\mathfrak{m}^i + \bigl(\mathfrak{m}^r : x^{r-i-1}\bigr)\bigr) \to 0 \end{aligned}$$

and

$$0 \to \bigl(\mathfrak{m}^i \cap \bigl(\mathfrak{m}^r : x^{r-i-1}\bigr)\bigr)/\bigl(\mathfrak{m}^{i+1}\bigr) \to \mathfrak{m}^i/\mathfrak{m}^{i+1} \to \mathfrak{m}^i/\bigl(\mathfrak{m}^i \cap \bigl(\mathfrak{m}^r : x^{r-i-1}\bigr)\bigr) \to 0.$$

Taking lengths we get

$$\lambda\bigl(\bigl(\mathfrak{m}^r : x^{r-i}\bigr)/\bigl(\mathfrak{m}^r : x^{r-i-1}\bigr)\bigr) = \lambda\bigl(\bigl(\mathfrak{m}^r : x^{r-i}\bigr)/\bigl(\mathfrak{m}^i + \bigl(\mathfrak{m}^r : x^{r-i-1}\bigr)\bigr)\bigr) + \mu\bigl(\mathfrak{m}^i\bigr) - f_{i,r-i}.$$

Hence, by Lemma 8, Lemma 1 and the above lemma we get, for $1 \leqslant i \leqslant r-1$, that

$$\beta_i - \alpha_i = \lambda\bigl(\bigl(\mathfrak{m}^r : x^{r-i}\bigr)/\bigl(\mathfrak{m}^i + \bigl(\mathfrak{m}^r : x^{r-i-1}\bigr)\bigr)\bigr) - \lambda\bigl(\bigl(\mathfrak{m}^r : x^{r-i+1}\bigr)/\bigl(\mathfrak{m}^{i-1} + \bigl(\mathfrak{m}^r : x^{r-i}\bigr)\bigr)\bigr),$$

and $\alpha_r - \beta_r = \lambda((\mathfrak{m}^r : x)/\mathfrak{m}^r) - \mu(\mathfrak{m}^{r-1}) = \lambda((\mathfrak{m}^r : x)/\mathfrak{m}^{r-1})$. □

3.2. Apery invariants of the ring and invariants of its tangent cone

Put $G := G(\mathfrak{m})$, $F := F(x)$ and $\overline{\mathfrak{m}} := \mathfrak{m}/xA \subseteq A/xA$.

Proposition 14. *For $1 \leqslant i \leqslant r$ it holds*

$$\alpha_i + \sum_{j=1}^{r-i-1} \alpha_{i,j} = \gamma_i + \lambda\bigl(\bigl(\mathfrak{m}^i \cap xA + \mathfrak{m}^{i+1}\bigr)/\bigl(x\mathfrak{m}^{i-1} + \mathfrak{m}^{i+1}\bigr)\bigr).$$

Proof. With the notation just introduced, we have an exact sequence of modules

$$0 \longrightarrow V \longrightarrow G/x^*G \longrightarrow G(\overline{\mathfrak{m}}) \longrightarrow 0,$$

where

$$\begin{aligned} V &= \bigoplus_{n \geqslant 0} \bigl(\mathfrak{m}^n \cap xA + \mathfrak{m}^{n+1}\bigr)/\bigl(x\mathfrak{m}^{n-1} + \mathfrak{m}^{n+1}\bigr), \\ G/x^*G &= \bigoplus_{n \geqslant 0} \mathfrak{m}^n/\bigl(x\mathfrak{m}^{n-1} + \mathfrak{m}^{n+1}\bigr), \quad \text{and} \\ G(\overline{\mathfrak{m}}) &= \bigoplus_{n \geqslant 0} \bigl(\mathfrak{m}^n + xA\bigr)/\bigl(\mathfrak{m}^{n+1} + xA\bigr). \end{aligned}$$

Taking the corresponding Hilbert series (which are polynomials of degree up to r) we get

$$H_{G/x^*G}(z) = H_V(z) + H_{G(\overline{\mathfrak{m}})}(z).$$

By the definition of the γ_i's we have that $H_{G(\overline{\mathfrak{m}})}(z) = \sum_{i=0}^{r} \gamma_i z^i$. On the other hand,

$$G/x^*G \cong \bigoplus_{i=0}^{r} \big(F/x^*F(-i)\big)^{\alpha_i} \bigoplus_{i=1}^{r-1} \bigoplus_{j=1}^{r-i-1} \left(\frac{F}{x^*F}(-i)\right)^{\alpha_{i,j}}$$

and so $H_{G/x^*G}(z) = \sum_{i=0}^{r}(\alpha_i + \sum_{j=1}^{r-i-1} \alpha_{i,j})z^i$. Now, taking coefficients in the above equality between Hilbert series we get the statement. $\square$

Corollary 15. *The following equalities hold*

(1) $\alpha_1 + \sum_{j=1}^{r-2} \alpha_{1,j} = \gamma_1 = \mu(\mathfrak{m}) - 1$;
(2) $\alpha_2 + \sum_{j=1}^{r-3} \alpha_{2,j} = \gamma_2 = \mu(\mathfrak{m}^2) - \mu(\mathfrak{m}) + \alpha_{1,1}$.

3.3. Micro-invariants and Apery invariants of the ring

For short we write

$$\nu_i := \lambda\big(\big(\mathfrak{m}^i \cap xA + \mathfrak{m}^{i+1}\big)/\big(x\mathfrak{m}^{i-1} + \mathfrak{m}^{i+1}\big)\big)$$

and

$$g_i := \lambda\big(\big(\mathfrak{m}^r : x^{r-i}\big)/\big(\mathfrak{m}^i + \big(\mathfrak{m}^r : x^{r-i-1}\big)\big)\big).$$

Then, applying the previous results we obtain the following relation between the micro-invariants of A and the Apery invariants of A with respect to x:

Corollary 16. *For $1 \leqslant i \leqslant r$ it holds*

$$\beta_i + \sum_{j=1}^{r-i-1} \alpha_{i,j} = \gamma_i + \nu_i + g_i - g_{i-1}.$$

4. Cohen–Macaulay tangent cone

Let $(A, \mathfrak{m})$ be an one-dimensional Cohen–Macaulay local ring with infinite residue field K, embedding dimension b, reduction number r and multiplicity e. Let (x) be a minimal reduction of $\mathfrak{m}$.

Let

- $\{\alpha_i, \alpha_{i,j}\}$ the invariants of the tangent con $G(\mathfrak{m})$ with respect to $F(x)$.
- $\{\beta_1, \dots, \beta_r\}$ the micro-invariants of A.
- $\{\gamma_1, \dots, \gamma_r\}$ the Apery invariants of A with respect to x

and, for short, we will write

$$f_i := \lambda\big(\big(\mathfrak{m}^i \cap \big(\mathfrak{m}^r : x^{r-i-1}\big)\big)/\mathfrak{m}^{i+1}\big)$$
$$g_i := \lambda\big(\big(\mathfrak{m}^r : x^{r-i}\big)/\big(\mathfrak{m}^i + \big(\mathfrak{m}^r : x^{r-i-1}\big)\big)\big)$$
$$\nu_i := \lambda\big(\big(\mathfrak{m}^i \cap xA + \mathfrak{m}^{i+1}\big)/\big(x\mathfrak{m}^{i-1} + \mathfrak{m}^{i+1}\big)\big).$$

Theorem 17. *Assume that the tangent cone of A is Cohen–Macaulay, then for* $1 \leqslant i \leqslant r$ *it holds*

$$0 < \alpha_i = \beta_i = \gamma_i = \mu\big(\mathfrak{m}^i\big) - \mu\big(\mathfrak{m}^{i-1}\big).$$

Proof. By Lemma 1, Proposition 13, Proposition 14 and Corollary 16

$$\alpha_i = \mu\big(\mathfrak{m}^i\big) - \mu\big(\mathfrak{m}^{i-1}\big) - f_i + f_{i-1},$$
$$\beta_i - \alpha_i = g_i - g_{i-1},$$
$$\alpha_i + \sum_{j=1}^{r-i-1} \alpha_{i,j} = \gamma_i + \nu_i,$$
$$\beta_i + \sum_{j=1}^{r-i-1} \alpha_{i,j} = \gamma_i + g_i - g_{i-1}.$$

Then, Lemma 2 gives that $f_i = g_i = \nu_i = \alpha_{i,j} = 0$ for all i, j if $G(\mathfrak{m})$ is Cohen–Macaulay and the equalities hold.

Also, [3, Corollary 16] proves that $\alpha_i = \lambda(\mathfrak{m}^i/(\mathfrak{m}^{i+1} + x\mathfrak{m}^{i-1})) > 0$. □

Theorem 18. *The following conditions are equivalent*:

(1) $G(\mathfrak{m})$ *is a Cohen–Macaulay ring*;
(2) $\alpha_i = \beta_i$ *for* $i \leqslant r$;
(3) $\alpha_i = \gamma_i$ *for* $i \leqslant r$;
(4) $\beta_i = \gamma_i$ *for* $i \leqslant r$.

Proof. By Corollary 10 any of the conditions (2), (3) or (4) implies that $G(\mathfrak{m})$ is Cohen–Macaulay. Conversely, if the tangent cone $G(\mathfrak{m})$ is Cohen–Macaulay, by Theorem 17 we have that (2), (3) and (4) hold. □

Proposition 19. *Assume that any of the following equalities hold*:

(1) $\alpha_i = \mu(\mathfrak{m}^i) - \mu(\mathfrak{m}^{i-1})$ *for* $1 \leqslant i \leqslant r$;
(2) $\beta_i = \mu(\mathfrak{m}^i) - \mu(\mathfrak{m}^{i-1})$ *for* $1 \leqslant i \leqslant r$;
(3) $\gamma_i = \mu(\mathfrak{m}^i) - \mu(\mathfrak{m}^{i-1})$ *for* $1 \leqslant i \leqslant r$.

Then the tangent cone of A is Cohen–Macaulay.

Proof. We observe that $\sum_{i=1}^r i(\mu(\mathfrak{m}^i) - \mu(\mathfrak{m}^{i-1})) = \rho$. Then, if the equalities of (1) or (3) occur, applying Lemma 6 and Corollary 10,

We will prove, by induction on i, that $\beta_j = \mu(\mathfrak{m}^j) - \mu(\mathfrak{m}^{j-1})$ for $1 \leqslant j \leqslant i$ implies the equality $(\mathfrak{m}^r : x^{r-i-1}) = \mathfrak{m}^{i+1}$. For $i = 1$, $\beta_1 = \mu(\mathfrak{m}) - 1 - f_1 = \mu(\mathfrak{m}) - 1$ gives $f_1 = 0$, and so, $(\mathfrak{m}^r : x^{r-2}) = \mathfrak{m}^2$. Assume $\beta_j = \mu(\mathfrak{m}^j) - \mu(\mathfrak{m}^{j-1})$ for $1 \leqslant j \leqslant i-1$. Then, by induction, $(\mathfrak{m}^r : x^{r-j-1}) = \mathfrak{m}^{j+1}$ for $1 \leqslant$

$j \leqslant i-1$. In particular, $(\mathfrak{m}^r : x^{r-i}) = \mathfrak{m}^i$ and $(\mathfrak{m}^r : x^{r-i+1}) = \mathfrak{m}^{i-1}$ which produces $f_{i-1} = 0$, $g_i = 0$ and $g_{i-1} = 0$. Hence, $\beta_i = \mu(\mathfrak{m}^i) - \mu(\mathfrak{m}^{i-1}) - f_i = \mu(\mathfrak{m}^i) - \mu(\mathfrak{m}^{i-1})$ implies that

$$f_i = \lambda\big(\big(\mathfrak{m}^i \cap \big(\mathfrak{m}^r : x^{r-i-1}\big)\big)/\mathfrak{m}^{i+1}\big) = \lambda\big(\big(\mathfrak{m}^r : x^{r-i-1}\big)/\mathfrak{m}^i\big) = 0.$$

Thus, $\beta_i = \mu(\mathfrak{m}^i) - \mu(\mathfrak{m}^{i-1})$ for $1 \leqslant i \leqslant r$ implies that $(\mathfrak{m}^r : x^{r-i}) = \mathfrak{m}^{i+1}$ for $1 \leqslant i \leqslant r-1$ and so, $G(\mathfrak{m})$ is Cohen–Macaulay. □

We can summarize the above results in the following way:

Theorem 20. *The following conditions are equivalent*:

(1) $G(\mathfrak{m})$ *is Cohen–Macaulay*;
(2) $G(\mathfrak{m}) \cong K[X] \oplus (K[X](-1))^{\mu(\mathfrak{m})-1} \oplus \cdots \oplus (K[X](-r))^{\mu(\mathfrak{m}^r)-\mu(\mathfrak{m}^{r-1})}$;
(3) $H_{G(\mathfrak{m}/xA)}(z) = 1 + (\mu(\mathfrak{m}) - 1)z + \cdots + (\mu(\mathfrak{m}^r) - \mu(\mathfrak{m}^{r-1}))z^r$.

And in the equicharacteristic and complete case also with

(4) $A'/A \cong (K[\![X]\!]/XK[\![X]\!])^{\mu(\mathfrak{m})-1} \oplus \cdots \oplus (K[\![X]\!]/X^rK[\![X]\!])^{\mu(\mathfrak{m}^r)-\mu(\mathfrak{m}^{r-1})}$.

5. Some computations

Let $(A, \mathfrak{m})$ be an one-dimensional Cohen–Macaulay local ring with infinite residue field K, embedding dimension b, multiplicity e and reduction number r. Let (x) be a minimal reduction of the maximal ideal.

By the previous section, the micro-invariants of A, its Apery numbers, and the invariants of its tangent cone coincide when this last is a Cohen–Macaulay ring. Then, their values are completely determined by the differences of the minimal number of generators of the consecutive powers of the maximal ideal.

Corollary 21. *If* $b = 2$ *then*:

(1) $\alpha_i = \beta_i = \gamma_i = 1$ *for* $1 \leqslant i \leqslant e-1$;
(2) $G(\mathfrak{m}) \cong K[X] \oplus (K[X](-1)) \oplus \cdots \oplus (K[X](-e+1))$;
(3) $H_{G(\mathfrak{m}/xA)}(z) = 1 + z + \cdots + z^{e-1}$;
(4) *in the equicharacteristic complete case*

$$A'/A \cong \big(K[\![X]\!]/XK[\![X]\!]\big) \oplus \cdots \oplus \big(K[\![X]\!]/X^{e-1}K[\![X]\!]\big).$$

Proof. It is known that $b = 2$ implies that $G(\mathfrak{m})$ is Cohen–Macaulay and $\mu(\mathfrak{m}^i) - \mu(\mathfrak{m}^{i-1}) = 1$ for $i = 1, \ldots, r$ (see for example [3, Proposition 26]) and $e = r + 1$. So the result is obtained by applying Theorem 17 and Theorem 20. □

We recall that $e = b + \lambda(\mathfrak{m}^2/x\mathfrak{m})$. So, one says that A has minimal multiplicity when $e = b$ and that A has almost minimal multiplicity if $b = e + 1$.

When the ring has minimal multiplicity, or equivalently has reduction number one, the tangent cone is Cohen–Macaulay and the computation of its invariants, and hence of the micro-invariants and Apery numbers of the ring is direct.

Corollary 22. *Assume A has minimal multiplicity, then*:

(1) $\alpha_1 = \beta_1 = \gamma_1 = e - 1$;

(2) $G(\mathfrak{m}) \cong K[X] \oplus (K[X](-1))^{e-1}$;
(3) $H_{G(\mathfrak{m}/xA)}(z) = 1 + (e-1)z$;
(4) *in the equicharacteristic complete case*

$$A'/A \cong \big(K[\![X]\!]/XK[\![X]\!]\big)^{e-1}.$$

We note that Corollary 21(4) and Corollary 22(4) were already shown in [6, Proposition 4.1].

The case of rings with almost minimal multiplicity will provide examples of micro-invariants and Apery numbers for rings for which their tangent cones are not Cohen–Macaulay. In this case the maximal ideal is a "Sally ideal", which means that $\lambda(\mathfrak{m}^2/x\mathfrak{m}) = 1$. Sally ideals are studied in [11] by M.E. Rossi, [8] by A.V. Jayanthan, T.J. Puthenpurakal and J.K. Verma and [3] by the authors. We collect in a lemma some known results for this case.

Lemma 23. *Assume that A has almost minimal multiplicity e. Then*:

(1) $\mathfrak{m}^2$ *is not contained in* (x);
(2) $\mathfrak{m}^{n+1} \subseteq x\mathfrak{m}^{n-1}$ *for* $n \geqslant 2$;
(3) $\lambda(\mathfrak{m}^{n+1}/x\mathfrak{m}^n) = 1$ *for* $1 \leqslant n \leqslant r-1$;
(4) $\mu(\mathfrak{m}^n) = \mu(\mathfrak{m})$ *for* $1 \leqslant n \leqslant r-1$, *and* $\mu(\mathfrak{m}^n) = \mu(\mathfrak{m}) + 1$ *for* $n \geqslant r$;
(5) $G(\mathfrak{m})$ *is Cohen–Macaulay if and only the reduction number of A is* 2, *if and only if* $\mu(\mathfrak{m}^2) = \mu(\mathfrak{m}) + 1$.

Proof. Observe that A has almost minimal multiplicity if and only if $\lambda(\mathfrak{m}^2/x\mathfrak{m}) = 1$.

If $\mathfrak{m}^2 \subseteq (x)$ then the exact sequence

$$0 \longrightarrow \mathfrak{m}^2/x\mathfrak{m} \longrightarrow (x)/x\mathfrak{m} \longrightarrow (x)/\mathfrak{m}^2 \longrightarrow 0$$

gives, by using the additivity of the length the equality $(x) = \mathfrak{m}^2$ which is not possible since x is part of a minimal set of generators for $\mathfrak{m}$.

In order to prove that $\mathfrak{m}^3 \subseteq x\mathfrak{m}$ we consider the exact sequence

$$0 \longrightarrow \big(\mathfrak{m}^3 + x\mathfrak{m}\big)/x\mathfrak{m} \longrightarrow \mathfrak{m}^2/x\mathfrak{m} \longrightarrow \mathfrak{m}^2/\big(\mathfrak{m}^3 + x\mathfrak{m}\big) \longrightarrow 0.$$

Then, by Nakayama's Lemma and the additivity of the length, one gets the result.

The assertion (3) can be found in the proof of [11, Corollary 1.7] and (5) in [8, Theorem 3.3].

Finally, the equality $b + 1 = e = \lambda(\mathfrak{m}^n/x\mathfrak{m}^n) = \mu(\mathfrak{m}^n) + \lambda(\mathfrak{m}^{n+1}/x\mathfrak{m}^n)$ gives (4) since $\lambda(\mathfrak{m}^{n+1}/x\mathfrak{m}^n) = 0$ for $n \geqslant r$ and $\lambda(\mathfrak{m}^{n+1}/x\mathfrak{m}^n) = 1$ for $n < r$. $\square$

Corollary 24. *Assume that A has almost minimal multiplicity e and reduction number* 2. *Then*:

(1) $\alpha_1 = \beta_1 = \gamma_1 = e - 2$ *and* $\alpha_2 = \beta_2 = \gamma_2 = 1$;
(2) $G(\mathfrak{m}) \cong K[X] \oplus (K[X](-1))^{e-2} \oplus K[X](-2)$;
(3) $H_{G(\mathfrak{m}/xA)}(z) = 1 + (e-2)z + z^2$;
(4) *in the equicharacteristic complete case*

$$A'/A \cong \big(K[\![X]\!]/XK[\![X]\!]\big)^{e-2} \oplus K[\![X]\!]/X^2K[\![X]\!].$$

Corollary 25. *Assume that A has almost minimal multiplicity and reduction number* 3. *Then*

$$(\alpha_1, \alpha_2, \alpha_3) = (e-3, 1, 1),$$

$$(\beta_1, \beta_2, \beta_3) = (e-3, 2, 0),$$

$$(\gamma_1, \gamma_2, \gamma_3) = (e-2, 1, 0).$$

Proof. By Lemma 1 and Lemma 23(4) we get that

$$\alpha_1 = b - 1 - \lambda\big(\big(\mathfrak{m}^3 : x\big)/\mathfrak{m}^2\big),$$
$$\alpha_2 = \lambda\big(\big(\mathfrak{m}^3 : x\big)/\mathfrak{m}^2\big),$$
$$\alpha_3 = 1.$$

Now, again by Lemma 23 (2) and (3) we have $\lambda((\mathfrak{m}^3 : x)/\mathfrak{m}^2) = \lambda((x\mathfrak{m} \cap \mathfrak{m}^3)/x\mathfrak{m}^2) = \lambda(\mathfrak{m}^3/x\mathfrak{m}^2) = 1$ and so the statement for the $\alpha_i's$.

In order to determine the values of the micro-invariants and the Apery numbers we just need to apply respectively Lemma 13 and Lemma 14. □

Corollary 26. *Assume that A has almost minimal multiplicity e and reduction number $r \geqslant 3$. Then*:

(1) $(\gamma_1, \ldots, \gamma_r) = (e-2, 1, 0 \ldots, 0)$;
(2) $\alpha_r = \alpha_{r-1} = 1$;
(3) $\beta_r = 0$.

Proof. By definition, $\gamma_i = \lambda((\mathfrak{m}^i + xA)/(\mathfrak{m}^{i+1} + xA)$ and, since A has almost minimal multiplicity, $\mathfrak{m}^i \subseteq xA$ for $i \geqslant 3$, hence $\gamma_i = 0$ for $i \geqslant 3$. Moreover, $\gamma_1 = \mu(\mathfrak{m}) - 1 = e - 2$ and $\gamma_2 = \lambda((\mathfrak{m}^2 + xA)/(\mathfrak{m}^3 + xA)) = \lambda((\mathfrak{m}^2 + xA)/xA) = \lambda(\mathfrak{m}^2/x\mathfrak{m}) = 1$. So, (1) is proved.

For (2), combining Lemma 1 and Lemma 23(4) one has that $\alpha_r = 1$ and

$$\alpha_{r-1} = \lambda\big(\big(\mathfrak{m}^{r-2} \cap \big(\mathfrak{m}^r : x\big)\big)/\mathfrak{m}^{r-1}\big) = \lambda\big(\big(x\mathfrak{m}^{r-2} \cap \mathfrak{m}^r\big)/x\mathfrak{m}^{r-1}\big).$$

Moreover Lemma 23(2) gives the inclusion $\mathfrak{m}^r \subseteq x\mathfrak{m}^{r-2}$ and so the equalities $\alpha_{r-1} = \lambda(\mathfrak{m}^r/x\mathfrak{m}^{r-1}) = 1$.

Finally, we can obtain (3) by Proposition 13 which provides, in the almost minimal multiplicity case, the equality $\beta_r = \alpha_r - \lambda((\mathfrak{m}^r : x)/\mathfrak{m}^{r-1}) = 1 - \lambda(\mathfrak{m}^r/x\mathfrak{m}^{r-1}) = 0$. □

5.1. Numerical semigroups rings

Let $\mathbb{N}$ be the set of non-negative integers. Recall that a numerical semigroup S is a subset of $\mathbb{N}$ that is closed under addition, contains the zero element and has finite complement in $\mathbb{N}$. A numerical semigroup S is always finitely generated; that is, there exist integers $n_1, \ldots, n_l$ such that $S = \langle n_1, \ldots, n_l \rangle = \{\alpha_1 n_1 + \cdots + \alpha_l n_l;\ \alpha_i \in \mathbb{N}\}$. Moreover, every numerical semigroup has an unique minimal system of generators $n_1, \ldots, n_{b(S)}$. The least integer belonging to S is known as the multiplicity of S and it is denoted by $e(S)$.

A relative ideal of S is a nonempty set I of integers such that $I + S \subset I$ and $d + I \subseteq S$ for some $d \in S$. An ideal of S is then a relative ideal of S contained in S. We denote by M the maximal ideal of S, that is, $M = S \setminus \{0\}$. M is then the ideal generated by a system of generators of S. If I and J are relative ideals of S then $I + J = \{i + j;\ i \in I,\ j \in J\}$ is also a relative ideal of S. If I is a relative ideal, we denote by $\mathrm{Ap}(I)$ the Apery set of I with respect to $e := e(S)$, defined as the set of the smallest elements in I in each residue class module e. Then $\mathrm{Ap}(I) = I \setminus e + I$. Since $I \setminus M + I$ is a minimal set of generators of I we get that in particular the minimal number of generators of I is bounded by e.

Let $V = K[\![t]\!]$ be the formal power series ring over a field K. Given a numerical semigroup $S = \langle n_1, \ldots, n_b \rangle$ minimally generated by $0 < e = e(S) = n_1 < \cdots < n_b = n_{b(S)}$ we consider the ring associated to S defined as $A = K[\![S]\!] = K[\![t^{n_1}, \ldots, t^{n_b}]\!] \subseteq V$. Let $\mathfrak{m} = (t^{n_1}, \ldots, t^{n_b})$ be the maximal ideal of A. Then A is a Cohen–Macaulay local ring of dimension one with multiplicity e and embedding dimension b. These kind of rings are known as numerical semigroup rings. The ideals $(t^{i_1}, \ldots, t^{i_k})$ of A are such that for v, the t-adic valuation, $v((t^{i_1}, \ldots, t^{i_k})) = \{i_1, \ldots, i_k\} + S$. In particular, for the ideals $\mathfrak{m}^n$ one has $v(\mathfrak{m}^n) = nM = M + \overset{n}{\cdots} + M$. Note that the element t^e generates a minimal reduction of $\mathfrak{m}$ and, in terms of semigroups, $(n+1)M \subseteq nM$ for $n \geqslant 0$ (we will set $\mathfrak{m}^0 := A$) and $(n+1)M = e + nM$

for all $n \geqslant r$. Also, for these rings, the first neighborhood ring $A' = K[\![S']\!]$, is a numerical semigroup ring, with $S' = \langle n_1, n_2 - n_1, \ldots, n_b - n_1 \rangle$.

Let $A = K[\![S]\!]$ be a numerical semigroup ring of multiplicity e and reduction number r. Denote by $W = K[\![t^e]\!] \subset K[\![S]\!]$. If we put

$$\mathrm{Ap}(nM) = \{\omega_{n,0}, \ldots, \omega_{n,i}, \ldots, \omega_{n,e-1}\}$$

for $n \geqslant 0$, then

$$\mathfrak{m}^n = Wt^{\omega_{n,0}} \oplus \cdots \oplus Wt^{\omega_{n,i}} \oplus \cdots \oplus Wt^{\omega_{n,e-1}}.$$

The set $\{t^{\omega_{0,0}}, \ldots, t^{\omega_{0,e-1}}\}$ is an Apery basis of $K[\![S]\!]$ (with respect to $x = t^e$ and also a BF-Apery basis) and fixed i, $1 \leqslant i \leqslant e-1$ one has that $\omega_{n+1,i} = \omega_{n,i} + \epsilon \cdot e$ where $\epsilon \in \{0, 1\}$ and $\omega_{n+1,i} = \omega_{n,i} + e$ for $n \geqslant r$. These facts are proved in [4, Lemma 2.1 and Lemma 2.2].

We will show that all the invariants defined in the previous sections can be computed in terms of the information contained in the Apery table:

$\mathrm{Ap}(S)$	$\omega_{0,0}$	$\omega_{0,1}$	$\cdots$	$\omega_{0,i}$	$\cdots$	$\omega_{0,e-1}$
$\mathrm{Ap}(M)$	$\omega_{1,0}$	$\omega_{1,1}$	$\cdots$	$\omega_{1,i}$	$\cdots$	$\omega_{1,e-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathrm{Ap}(nM)$	$\omega_{n,0}$	$\omega_{n,1}$	$\cdots$	$\omega_{n,i}$	$\cdots$	$\omega_{n,e-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathrm{Ap}(rM)$	$\omega_{r,0}$	$\omega_{r,1}$	$\cdots$	$\omega_{r,i}$	$\cdots$	$\omega_{r,e-1}$

Previously we recall the following notation introduced in [4].

Let $E = \{w_0, \ldots, w_m\}$ be a set of integers. We call it a stair if $w_0 \leqslant \cdots \leqslant w_m$. Given a stair, we say that a subset $L = \{w_i, \ldots, w_{i+k}\}$ with $k \geqslant 1$ is a landing of length k if $w_{i-1} < w_i = \cdots = w_{i+k} < w_{i+k+1}$ (where $w_{-1} = -\infty$ and $w_{m+1} = \infty$). In this case, the index i is the beginning of the landing: $s(L)$ and the index $i+k$ is the end of the landing: $e(L)$. A landing L is said to be a true landing if $s(L) \geqslant 1$. Given two landings L and L', we set $L < L'$ if $s(L) < s(L')$. Let $l(E)+1$ be the number of landings and assume that $L_0 < \cdots < L_{l(E)}$ is the set of landings. Then, we define following numbers:

- $s_j(E) = s(L_j)$, $e_j(E) = e(L_j)$, for each $0 \leqslant j \leqslant l(E)$.
- $c_j(E) = s_j - e_{j-1}$, for each $1 \leqslant j \leqslant l(E)$.
- $k_j(E) = e_j - s_j$, for each $1 \leqslant j \leqslant l(E)$.

With this notation, for any $1 \leqslant i \leqslant e-1$, consider the ladder of values $\Omega^i = \{\omega_{n,i}\}_{0 \leqslant n \leqslant r}$, that is, the columns of the Apery table, and define the following integers:

(1) $l_i = l(\Omega^i)$ the number of true landings of the column Ω^i;
(2) $d_i = e_{l_i}(\Omega^i)$ the end of the last true landing;
(3) $b^i_j = e_{j-1}(\Omega^i)$ and $c^i_j = c_j(\Omega^i)$, for $1 \leqslant j \leqslant l_i$.

Then [4, Theorem 2.3] says

$$G(\mathfrak{m}) \cong F\big(t^e\big) \oplus \bigoplus_{i=1}^{e-1} \left(F\big(t^e\big)(-d_i) \bigoplus_{j=1}^{l_i} \frac{F(t^e)}{((t^e)^*)^{c^i_j} F(t^e)} \big(-b^i_j\big) \right).$$

Observe that $\alpha_j = \#\{i;\ d_i = j\}$. Also, if $b_i = \max\{j;\ t^{\omega_{0,i}} \in \mathfrak{m}^j\}$ one has that

(4) $b_i = b_1^i = e_0(\Omega^i)$ is the place where the first landing ends and by the observations made in Section 2.4 we have that $\gamma_j = \#\{i;\ b_i = j\}$.

Moreover, by the definitions, in each column is verified

(5) $d_i = b_i + (c_1^i + k_1^i) + \cdots + (c_{l_i}^i + k_{l_i}^i)$.

Observe also that if $\mathrm{Ap}(S') = \{\omega'_0, \ldots, \omega'_{e-1}\}$, then $\omega_{0,i} - \omega'_i = a_i \cdot e$ for some positive integers and

$$A' = W \oplus W t^{\omega'_1} \oplus \cdots \oplus W t^{\omega'_{e-1}},$$
$$A = W \oplus W\left(t^e\right)^{a_1} \cdot t^{\omega'_1} \oplus \cdots \oplus W\left(t^e\right)^{a_{e-1}} \cdot t^{\omega'_{e-1}}$$

which show that $\{a_1, \ldots, a_{e-1}\}$ are the micro-invariants of A. Moreover, from the equalities $\mathfrak{m}^r = (t^e)^r A'$ and $\omega_{r,i} = \omega_{0,i} + e(c_1^i + \cdots + c_{l_i}^i) + (r - d_i)e$ it is easy to see that

(6) $d_i = a_i + (c_1^i + \cdots + c_{l_i}^i)$

and so, by (5) and (6), also

(7) $a_i = b_i + (k_1^i + \cdots + k_{l_i}^i)$.

Hence, the Cohen–Macaulay property of the tangent cone is equivalent to the no existence of true landings in the columns of the Apery table. Also, each true landing gives a torsion cyclic submodule of the tangent cone and its beginning and ending determine the degree and the order of the correspondent torsion submodule.

Note also that we can read the Hilbert function $H^0(n) = \mu(\mathfrak{m}^n)$ in the Apery table as the number of steps between the nth row and the (n+1)th row.

Suppose that e, the multiplicity of S (equivalently the multiplicity of $k[\![S]\!]$), is given. We recall that then, the embedding dimension b and the reduction number r satisfy $b \leqslant e$ and $r \leqslant e-1$. We will show that, in general, the couple (e, b) does not determine the Apery table of S. However, in the extremal cases $(e, 2)$ and (e, e) the Apery table is completely determined.

Example 1. Suppose that the S has multiplicity e.

- For $b = 2$, we consider $\{\omega_1, \ldots, \omega_{e-1}\}$, with $\omega_1 < \cdots < \omega_{e-1}$ a suitable permutation of $\{\omega_{0,0}, \ldots, \omega_{0,e-1}\}$ the Apery set of S (with this notation $S =< e, \omega_1 >$). In this case the reduction number is $e-1$ and the Apery table is a square box:

0	ω_1	$\cdots$	ω_i	$\cdots$	ω_{e-1}
e	ω_1	$\cdots$	ω_i	$\cdots$	ω_{e-1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ie	$\omega_1 + (i-1)e$	$\cdots$	ω_i	$\cdots$	ω_{e-1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
re	$\omega_1 + (r-1)e$	$\cdots$	$\omega_i + (r-i)e$	$\cdots$	ω_{e-1}

So, for $1 \leqslant i \leqslant e-1$ and observing the columns of the table we have that $a_i = b_i = d_i = i$ and consequently $\alpha_i = \beta_i = \gamma_i = 1$ for $1 \leqslant i \leqslant r$ as we proved in Corollary 21. Moreover $\rho = e(e-1)/2$.

- For $b = e$ the reduction number r is equal to 1, S is minimally generated by the Apery set $\{\omega_{0,0}, \ldots, \omega_{0,e-1}\}$ and the Apery table has two rows:

0	$\omega_{0,1}$	$\cdots$	$\omega_{0,i}$	$\cdots$	$\omega_{0,e-1}$
e	$\omega_{0,1}$	$\cdots$	$\omega_{0,i}$	$\cdots$	$\omega_{0,e-1}$

So, $a_i = b_i = d_i = 1$ for $1 \leqslant i \leqslant e-1$ and $\alpha_1 = \beta_1 = \gamma_1 = e-1$ recovering Corollary 21 for numerical semigroup rings. In this case $\rho = e-1$.

For $3 \leqslant b \leqslant e-1$ there are several possibilities for the reduction number. If $e = 4$ then $b = 3$ and $r = 2$ or $r = 3$. In both cases the Apery table is uniquely determined, see [4, Corollary 4.4 (2)]. For $e = 5$ there is not such uniqueness as shown by the following examples.

The GAP – Groups, Algorithms, Programming – is a system for Computational Discrete Algebra [7]. On the basis of GAP, Manuel Delgado, Pedro A. Garcia-Sánchez and José Morais have developed the NumericalSgps package [5]. Its aim is to make available a computational tool to deal with numerical semigroups. We can determine the values of the diverse families of invariants if we know the Apery sets of the sum ideals nM, where M is the maximal ideal of S. On the other hand, from its definition we have that the Apery set of nM can be calculated as $\mathrm{Ap}(nM) = nM \setminus ((e+S)+nM)$, a computation that can be performed by using the NumericalSgps package. The following examples are just a sample of these computations.

Example 2. We assume in this example that $(e,b) = (5,3)$.

- Set $S = \langle 5, 6, 7\rangle$. The reduction number is 2 and the Apery table is in this case

0	6	7	13	14
5	6	7	13	14
10	11	12	13	14

so, $a_i = b_i = d_i$ for $1 \leqslant i \leqslant 4$, $b_1 = b_2 = 1$ and $b_3 = b_4 = 2$. Also $(\alpha_1, \alpha_2) = (\beta_1, \beta_2) = (\gamma_1, \gamma_2) = (2, 2)$ and $\rho = 6$.

- Set $S = \langle 5, 6, 9\rangle$. The reduction number is 3 and the Apery table in this case is

0	6	12	18	9
5	6	12	18	9
10	11	12	18	14
15	16	17	18	19

so, $a_i = b_i = d_i$, $b_1 = b_4 = 1$, $b_2 = 2$ and $b_3 = 3$. Also $(\alpha_1, \alpha_2, \alpha_3) = (\beta_1, \beta_2, \beta_3) = (\gamma_1, \gamma_2, \gamma_3) = (2, 1, 1)$ and $\rho = 7$.

- Set $S_1 = \langle 5, 6, 13\rangle$, $S_2 = \langle 5, 6, 14\rangle$ and $S_3 = \langle 5, 6, 19\rangle$. The reduction number in these cases is 4.

The Apery table for S_1 is

0	6	12	13	19
5	6	12	13	19
10	11	12	18	19
15	16	17	18	24
20	21	22	23	24

and so, the invariants are

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 1, 1, 1), \quad \alpha_{1,1} = 1, \quad \alpha_{2,1} = 1,$$

$$(\beta_1, \beta_2, \beta_3, \beta_4) = (1, 2, 1, 0),$$

$$(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (2, 2, 0, 0).$$

The Apery table for S_2 is

0	6	12	18	14
5	6	12	18	14
10	11	12	18	19
15	16	17	18	24
20	21	22	23	24

and their invariants are

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 1, 1, 1), \quad \alpha_{1,2} = 1,$$

$$(\beta_1, \beta_2, \beta_3, \beta_4) = (1, 2, 1, 0),$$

$$(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (2, 1, 1, 0).$$

Finally, for S_3 the Apery table is

0	6	12	18	19
5	6	12	18	19
10	11	12	18	24
15	16	17	18	24
20	21	22	23	24

which produces the invariants

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 1, 1, 1), \quad \alpha_{1,1} = 1,$$

$$(\beta_1, \beta_2, \beta_3, \beta_4) = (1, 1, 2, 0),$$

$$(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (2, 1, 1, 0).$$

Observe that (e, b, r) neither determines the Apery table nor any of the families of invariants.

Example 3. Suppose that $(e, b) = (5, 4)$.

- Set $S = \langle 5, 6, 7, 8 \rangle$. In this case $r = 2$, and analyzing its Apery table

0	6	7	8	14
5	6	7	8	14
10	11	12	13	14

we obtain $(\alpha_1, \alpha_2) = (\beta_1, \beta_2) = (\gamma_1, \gamma_2) = (3, 1)$.

- Set $S = \langle 5, 6, 9, 13\rangle$. In this case $r = 3$, the Apery table is

0	6	12	13	9
5	6	12	13	9
10	11	12	18	14
15	16	17	18	19

and

$$(\alpha_1, \alpha_2, \alpha_3) = (2, 1, 1), \quad \alpha_{1,1} = 1,$$
$$(\beta_1, \beta_2, \beta_3) = (2, 2, 0),$$
$$(\gamma_1, \gamma_2, \gamma_3) = (3, 1, 0).$$

- Set $S = \langle 5, 6, 13, 14\rangle$. In this case $r = 4$ and the Apery table

0	6	12	13	14
5	6	12	13	14
10	11	12	18	19
15	16	17	18	24
20	21	22	23	24

gives

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 1, 1, 1), \quad \alpha_{1,1} = 1, \quad \alpha_{1,2} = 1,$$
$$(\beta_1, \beta_2, \beta_3, \beta_4) = (1, 3, 0, 0),$$
$$(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = (3, 1, 0, 0).$$

References

[1] L. Bryant, Filtered numerical semigroups and applications to one-dimensional rings, PhD Dissertation, Purdue University, 2009.
[2] V. Barucci, R. Fröberg, Associated graded rings of one-dimensional analytically irreducible rings, J. Algebra 304 (2006) 349–358.
[3] T. Cortadellas, S. Zarzuela, On the structure of the Fiber Cone, J. Algebra 317 (2007) 759–785.
[4] T. Cortadellas, S. Zarzuela, Tangent cones of numerical semigroups, in: Viviana Ene, Ezra Miller (Eds.), Combinatorial Aspects of Commutative Algebra, in: Contemp. Math., vol. 502, Amer. Math. Soc., 2009, pp. 45–58.
[5] M. Delgado, P.A. Garcia-Sánchez, J. Morais, NumericalSgps – a GAP package, 0.95, 2006, http://www.gap-system.org/Packages/numericalsgps.
[6] J. Elias, On the deep structure of the blowing-up of curve singularities, Math. Proc. Cambridge Philos. Soc. 131 (2001) 227–240.
[7] The GAP Group, GAP – Groups, Algorithms, and Programming – Version 4.4.10, 2007.
[8] A.V. Jayanthan, T.J. Puthenpurakal, J.K. Verma, On fiber cones of m-primary ideals, Canad. J. Math. 59 (1) (2007) 109–126.
[9] J. Lipman, Stable ideals and Arf rings, Amer. J. Math. 93 (1971) 649–685.
[10] E. Matlis, 1-Dimensional Cohen–Macaulay Rings, Lecture Notes in Math., Springer-Verlag, Berlin–Heidelberg–New York, 1973.
[11] M.E. Rossi, A bound on the reduction number of a primary ideal, Proc. Amer. Math. Soc. 128 (5) (2000) 1325–1332.
[12] J.D. Sally, Numbers of Generators of Ideals in Local Rings, Marcel Dekker, Inc., New York–Basel, 1978.
[13] P. Valabrega, G. Valla, Form rings and regular sequences, Nagoya Math. J. 72 (1978) 93–101.