



A variant of Wang's theorem

Kei-ichi Watanabe, Ken-ichi Yoshida *

^a Department of Mathematics, College of Humanities and Sciences, Nihon University, Setagaya-ku, Tokyo 156-8550, Japan

ARTICLE INFO

Article history:

Received 15 October 2010

Available online xxxx

Communicated by Bernd Ulrich

MSC:

primary 13A35

secondary 13B22, 13H10

Keywords:

Gorenstein ring

Cohen–Macaulay ring

Regular ring

Test ideal

Multiplier ideal

Integral closure

Goto number

ABSTRACT

In this paper, we give a new formula of $J : \bar{J}$ for any parameter ideal J in a Gorenstein local ring R of positive characteristic in terms of test ideals: $J : \bar{J} = J + \tau(J^{d-1})$, where $\tau(J^{d-1})$ denotes the J^{d-1} -test ideal of R .

As an application, we give a variant of Wang's theorem. Namely, we prove that if J is a parameter ideal in a Cohen–Macaulay local ring (R, \mathfrak{m}) of dimension $d \geq 2$ with $J \subseteq \mathfrak{m}^s$, then $J : \mathfrak{m}^{(d-1)(s-1)}$ (resp. $J : \mathfrak{m}^{(d-1)(s-1)+1}$) is integral over J (resp. if R is not regular). Moreover, we prove that, after reduction to characteristic $p \gg 0$, a similar assertion holds true for Cohen–Macaulay \mathbb{Q} -Gorenstein normal local domain essentially of finite type over a field of characteristic zero under some extra assumption.

© 2012 Elsevier Inc. All rights reserved.

1. Introduction

Throughout this paper, let R be a commutative Noetherian ring with unit element. Our study of this paper was motivated by the following Wang's theorem.

Theorem 1.1. (See Wang [Wa, Theorems 1.1 and 1.2].) *Let (R, \mathfrak{m}) be a Cohen–Macaulay local ring of dimension $d \geq 2$, with $d \geq 3$ if R is regular. Let $s \geq 2$ be an integer. If J is a parameter ideal with $J \subseteq \mathfrak{m}^s$, then $J : \mathfrak{m}^s \subseteq \mathfrak{m}^s$ and $L^2 = JL$, where $L = J : \mathfrak{m}^s$.*

The above theorem was given by Wang [Wa], which was posed by Polini and Ulrich [PU] as a conjecture in studying the linkage class of a Gorenstein local ring. Recall that two ideals I and L of

* Corresponding author.

E-mail addresses: watanabe@math.chs.nihon-u.ac.jp (K.-i. Watanabe), yoshida@math.chs.nihon-u.ac.jp (K. Yoshida).

height h are said to be (directly) *linked* if there exists a regular sequence a_1, \dots, a_h in $I \cap L$ such that $I = J : L$ and $L = J : I$, where $J = (a_1, \dots, a_h)$. If (R, \mathfrak{m}) is a Gorenstein local ring and J is a parameter ideal contained in an \mathfrak{m} -primary ideal I , then $L = J : I$ and I are linked. In particular, under the notation as in the above theorem, $L = J : \mathfrak{m}^s$ and \mathfrak{m}^s are linked.

But the above Wang's theorem also claims that $L = J : \mathfrak{m}^s$ is integral over J because $L^2 = JL$. It is well known that any parameter ideal J which is not integrally closed in a Cohen–Macaulay local ring R satisfies $L^2 = JL$, where $L = J : \mathfrak{m}$. So it is natural to ask the following question.

Question 1.2. Let (R, \mathfrak{m}) be a Cohen–Macaulay local ring, and let s, t be positive integers. When is $J : \mathfrak{m}^t$ integral over J ? What is

$$\max\{t \in \mathbb{N} \mid J : \mathfrak{m}^t \text{ is integral over } J\}?$$

We recall the following result, which give a motivation of the question above.

Theorem 1.3. (See Corso, Huneke and Vasconcelos [CHV], Corso and Polini [CP], Corso, Polini and Vasconcelos [CPV], Goto [G].) Let (R, \mathfrak{m}) be a Cohen–Macaulay local ring of dimension $d \geq 1$. Let J be a parameter ideal in R and set $I = J : \mathfrak{m}$. Then the following conditions are equivalent:

- (1) $I^2 \neq JI$.
- (2) J is integrally closed.
- (3) R is a regular local ring and $\mu_A(\mathfrak{m}/J) \leq 1$.

On the other hand, the question above has been studied by several people. The properties of quasi-socle ideals, that is, ideals like $J : \mathfrak{m}^s$ have already been studied by Goto and others; see e.g. [GTM, GKM]. After Goto's pioneering works, Heinzer and Swanson [HS] introduced the notion of Goto number. That is, for any parameter ideal J in a Noetherian local ring (R, \mathfrak{m}) ,

$$g(J) = \max\{t \in \mathbb{N} \mid J : \mathfrak{m}^t \text{ is integral over } J\}$$

is called the *Goto number* of J . Recently, the Goto numbers have been studied by many researchers. For instance, Heinzer and Swanson [HS] computed or gave bounds for Goto numbers of parameter ideals in the following cases: (a) local rings of dimension 1 (b) regular local rings of dimension 2 (c) numerical semigroup rings. Moreover, Goto et al. [GKPT] determined $g(J)$ for many parameter ideals J under the assumption that the associated graded ring $G(\mathfrak{m})$ is Gorenstein. Their approaches need a high standard technique in ideal theory.

But, in this paper, we use the so-called characteristic p method involving the theory of tight closure, which enables us to attack this problem in higher dimensional cases. For this we first recall the description of the integral closure \bar{J} for a parameter ideal J in terms of the ideal-adic tight closures introduced by [HY], and we establish a new formula for $J : \bar{J}$ for any parameter ideal J in a (complete) Gorenstein local ring R . Secondly, we give a variant of Wang's theorem as an application of our formula and Skoda-type theorem. Moreover, we have similar results in the equicharacteristic zero case via reduction to modulo $p \gg 0$.

In what follows, let us explain the organization of this paper. Section 2 is devoted to some preliminary results and definitions which will be needed later. For instance, we will recall the notion of test ideals.

In Section 3, we give a description of the integral closure of the ideal generated by a subsystem of parameters in terms of ideal-adic tight closures. In Section 4, we establish a formula describing $J : \bar{J}$ for any parameter ideal J in a complete Gorenstein local ring R , where \bar{J} denotes the integral closure of J . Notice that this theorem is closely related to [HS, Proposition 1.9].

Theorem 1.4. (See Theorem 4.1.) Let (R, \mathfrak{m}) be a complete Gorenstein local ring of characteristic $p > 0$. If J is a parameter ideal of R , then we have

$$J : \bar{J} = J + \tau(J^{d-1}),$$

where \bar{J} denotes the integral closure of J and $\tau(J^{d-1})$ the test ideal of J^{d-1} ; see Section 2 for more details.

In Section 5, we recall the Skoda-type theorem, and give some improvement; see Proposition 5.5, Proposition 5.7.

In Section 6, we prove a variant of Wang's theorem as an application of Theorem 4.1 in the case of positive characteristic.

Theorem 1.5. (See Theorem 6.2.) Let (R, \mathfrak{m}) be a Cohen–Macaulay local ring of characteristic $p > 0$. For any parameter ideal J of R with $J \subseteq \mathfrak{m}^s$, $J : \mathfrak{m}^{(d-1)(s-1)}$ is integral over J . If, in addition, R is not regular, then $J : \mathfrak{m}^{(d-1)(s-1)+1}$ is integral over J .

In other words,

$$g(J) \geq (d-1)(s-1) \quad (\text{resp. } \geq (d-1)(s-1)+1)$$

if R is Cohen–Macaulay (resp. and not regular).

This gives a partial answer to the previous question and this bound is the best possible one if R is regular.

In Section 7, we extend our results in Sections 4, 5, 6 to the case of equicharacteristic zero under some extra conditions.

Theorem 1.6. (See Theorem 7.3.) Let (R, \mathfrak{m}) be a normal Gorenstein local domain essentially of finite type over a field of characteristic zero. For any parameter ideal J of R , we have

$$J : \bar{J} = J + \mathcal{J}(J^{d-1}),$$

where $\mathcal{J}(J^{d-1})$ denotes the multiplier ideal of J^{d-1} .

Theorem 1.7. (See Theorem 7.4.) Let (R, \mathfrak{m}) be a normal local domain of dimension $d \geq 2$ essentially of finite type over a field of characteristic zero. Assume that R is Gorenstein or log-terminal. Then for any parameter ideal $J \subseteq \mathfrak{m}^s$, we have that $J : \mathfrak{m}^{(d-1)(s-1)}$ is integral over J .

We will discuss some related questions in Section 8.

2. Preliminaries

In this section, we give some notations and terminologies which will be needed later.

2.1. Integral closure

Let I be an ideal of a ring R . For an element $z \in R$, z is said to be *integral* over I if there exist an integer $n \geq 1$ and a monic polynomial $\varphi(X) = X^n + a_1 X^{n-1} + \cdots + a_{n-1} X + a_n$ such that

$$\varphi(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0, \quad a_i \in I^i \quad (i = 1, \dots, n).$$

The ideal generated by all integral elements over I , denoted by \bar{I} , is called the *integral closure* of I . Clearly, $I \subseteq \bar{I} \subseteq \sqrt{I}$.

Let I be an ideal of a Noetherian ring R . An ideal J is called a reduction of I if $J \subseteq I$ and $I^{r+1} = JI^r$ holds for some integer $r \geq 0$. In particular, J is called a *minimal reduction* of I if it is a minimal element among all reductions of I . The minimum integer r such that $I^{r+1} = JI^r$ is said to be the *reduction exponent* of I with respect to J . If $J \subseteq I$ are ideals of R , then $I \subseteq \bar{J}$ if and only if J is a reduction of I .

The following two lemmata are useful in the proof of a variant of Wang's theorem.

Lemma 2.1. (See [SH, Proposition 1.6.1].) Let $R \subseteq S$ be an integral extension of rings. For any ideal I of R , we have $\bar{I} = \overline{IS} \cap R$.

Lemma 2.2. (See [SH, Proposition 1.6.2].) Let $R \subseteq S$ be a faithfully flat extension of rings. For any ideal I of R , we have $\bar{I} = \overline{IS} \cap R$.

Let $R = k[x_1, \dots, x_d]$ be a polynomial ring over a field k . Let I be an ideal of R generated by monomials. For each monomial $M = x_1^{n_1} \cdots x_d^{n_d}$, its exponent vector is $(n_1, \dots, n_d) \in \mathbb{N}^d$. The convex hull in \mathbb{R}^d of the set of all exponent vectors of monomials in I is called the *Newton Polyhedron* of I , and denoted by $P(I)$.

The following fact is well-known; see e.g. [SH, Proposition 1.4.6].

Proposition 2.3. Let I be a monomial ideal in $R = k[x_1, \dots, x_d]$. Then \bar{I} is a monomial ideal and

$$x_1^{m_1} \cdots x_d^{m_d} \in \bar{I} \iff (m_1, \dots, m_d) \in P(I).$$

2.2. Test ideal

Let R be a Noetherian ring of characteristic $p > 0$. Let $F : R \rightarrow R$ be the Frobenius map, that is, $F(z) = z^p$ for every $z \in R$. The ring R viewed as an R -module via the e -times iterated Frobenius map $F^e : R \rightarrow R$ is denoted by eR . The ring R is called *F-finite* if 1R is finitely generated as an R -module.

Let M be an R -module. For each integer $e \geq 0$, we define $\mathbb{F}^e(M) = {}^eR \otimes_R M$ and regard it as an R -module by the action from the left. Then we have the induced e -times Frobenius map $F^e : M \rightarrow \mathbb{F}^e(M)$ ($z \mapsto z^{p^e} := 1 \otimes z$). For any R -submodule N of M , $N_M^{[p^e]}$ is defined by the image of the natural map $\mathbb{F}^e(N) \rightarrow \mathbb{F}^e(M)$.

In [HY], Hara and the second author introduced the notion of α -tight closure, which is a generalization of the tight closure due to Hochster and Huneke (see [HH1, HH2, HH3]).

Definition 2.4 (Tight closure). (Cf. [HY, Section 1].) Put $R^\circ = R \setminus \bigcup_{P \in \text{Min}(R)} P$, where $\text{Min}(R)$ denotes the set of all minimal prime divisors of R . Let α be an ideal of R such that $\alpha \cap R^\circ \neq \emptyset$, and let $t \geq 0$ be a real number. Let $N \subseteq M$ be an R -modules. The α^t -tight closure of N in M , denoted by $N_M^{*\alpha^t}$, is the submodule generated by all elements $z \in M$ for which there exists an element $c \in R^\circ$ such that $cz^{p^e} \alpha^{[tp^e]} \subseteq N_M^{[p^e]}$ for all sufficiently large $e \gg 0$.

The *finitistic α^t -tight closure* $N_M^{*\text{fg}, \alpha^t}$ is defined by the union of $N_M^{*\alpha^t}$ for finitely generated R -submodule M' of M . For an ideal I of R , we define $I^{*\alpha^t} = I_R^{*\alpha^t}$.

When $\alpha = R$ and $t = 1$, $N_M^* = N_M^{*R}$ denotes the original tight closure introduced by Hochster and Huneke. An R -submodule N is called *tightly closed* in M if $N_M^* = N$.

If $\mathfrak{b} \subseteq \alpha$ are ideals of R such that $\mathfrak{b} \cap R^\circ \neq \emptyset$, then $N_M^{*\mathfrak{b}^t} \supseteq N_M^{*\alpha^t}$ and equality holds if \mathfrak{b} is a reduction of α .

Definition 2.5 (Test ideal). (Cf. [HY, HT].) Let α be an ideal of R with $\alpha \cap R^\circ \neq \emptyset$. Let $t \geq 0$ be a real number. Let $E := \bigoplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$, the direct sum, taken over all maximal ideal \mathfrak{m} of R , of the injective envelope of the residue fields R/\mathfrak{m} . Then the α^t -test ideal (resp., CS α^t -test ideal) is defined by

$$\tau(\mathfrak{a}^t) = \bigcap_{M \subseteq E} \text{Ann}_R((0)_M^{*\mathfrak{a}^t}) \quad (\text{resp., } \tilde{\tau}(\mathfrak{a}^t) = \text{Ann}_R((0)_E^{*\mathfrak{a}^t}))$$

where M runs through all finitely generated R -submodules of E .

The following properties are easy to check; see also [HY, Proposition 1.11].

Lemma 2.6. Let $\mathfrak{a}, \mathfrak{b}$ be ideals of R , and let $t \geq 0$ be a real number. Then

- (1) $\tau(\mathfrak{a})\mathfrak{b} \subseteq \tau(\mathfrak{a}\mathfrak{b})$.
- (2) If $\mathfrak{b} \subseteq \mathfrak{a}$, then $\tau(\mathfrak{b}^t) \subseteq \tau(\mathfrak{a}^t)$. Equality holds if \mathfrak{b} is a reduction of \mathfrak{a} .
- (3) $\tilde{\tau}(\mathfrak{a}^t) \subseteq \tau(\mathfrak{a}^t)$.

Definition 2.7. (See [HH2].) The ring R for which $I^* = I$ for any ideal I of R is called *weakly F -regular*.

A ring R is weakly F -regular if and only if $\tau(R) = R$, or, equivalently, if $(0)_E^{*\text{fg}} = (0)$. A quasi-unmixed local ring R is called *F -rational* if every (resp. some) parameter ideal of R is tightly closed. Such a ring R is always normal, and Cohen–Macaulay if R is excellent or a homomorphic image of a Cohen–Macaulay local ring.

In general, it is not known whether $(0)_E^{*\mathfrak{a}^t} = (0)_E^{*\text{fg}, \mathfrak{a}^t}$ holds true. But if R is an excellent normal \mathbb{Q} -Gorenstein local domain, then equality holds, and thus $\tau(\mathfrak{a}^t) = \tilde{\tau}(\mathfrak{a}^t)$; see [HY, Theorem 1.13].

In the case of monomial ideals, it is easy to compute test ideals.

Example 2.8. (See Howald-type theorem [Ho], [HY, Theorem 4.8].) Let $t \geq 0$ be a real number. Let $R = k[x_1, \dots, x_d]$ be a polynomial ring over a field of characteristic $p > 0$. Let $\mathfrak{a} \subseteq R$ be a monomial ideal. Then $\tau(\mathfrak{a}^t)$ is a monomial ideal, and

$$x_1^{m_1} \cdots x_d^{m_d} \in \tau(\mathfrak{a}^t) \iff (m_1, \dots, m_d) + (1, \dots, 1) \in \text{Int}(t \cdot P(\mathfrak{a})),$$

where $\text{Int}(t \cdot P(\mathfrak{a}))$ denotes the relative interior of $t \cdot P(\mathfrak{a})$ in \mathbb{R}^d .

3. A description of the integral closure in terms of tight closure

Throughout this section, let (R, \mathfrak{m}, k) be a Noetherian local ring of characteristic $p > 0$ with $d = \dim R \geq 1$. The purpose of this section is to describe the integral closure of a parameter ideal (more generally, an ideal generated by a subsystem of parameters) in terms of ideal-adic tight closures. The key idea can be seen in [HY, Section 2].

Theorem 3.1. Assume that R is quasi-unmixed. Let $m \geq 1$. Let a_1, \dots, a_m be a subsystem of parameters of R , and put $J = (a_1, \dots, a_m)$. Then we have

$$\bar{J} = J^* J^{m-1}.$$

To prove the theorem, we begin with the following lemma.

Lemma 3.2. If J is an ideal generated by m elements, then $\bar{J} \subseteq J^* J^{m-1}$.

Proof. Suppose $z \in \bar{J}$. Then there exists an element $c \in R^\circ$ such that $cz^q \in J^q$ for all sufficiently large $q = p^e$. This implies

$$cz^q (J^{m-1})^q \subseteq J^q J^{(m-1)q} = J^{mq} \subseteq J^{[q]}.$$

Hence $z \in J^* J^{m-1}$. \square

In order to complete the proof of Theorem 3.1, we use the following lemma, which can be proved by a method similar to the proof of [HY, Lemma 2.11 and Corollary 2.12].

Lemma 3.3. *Let $m \geq 1$ be an integer. Then:*

(1) *If a_1, \dots, a_m forms an R -sequence, then for any integers $\ell, r \geq 1$, we have*

$$(a_1^\ell, \dots, a_m^\ell) : (a_1, \dots, a_m)^r = (a_1, \dots, a_m)^{m\ell-r-m+1} + (a_1^\ell, \dots, a_m^\ell).$$

(2) *Suppose that R is an excellent equidimensional local ring of characteristic $p > 0$ and a_1, \dots, a_m is a subsystem of parameters of R . Then*

$$(a_1^\ell, \dots, a_m^\ell) : (a_1, \dots, a_m)^r \subseteq ((a_1, \dots, a_m)^{m\ell-r-m+1} + (a_1^\ell, \dots, a_m^\ell))^*.$$

Proof of Theorem 3.1. By Lemma 3.2, it suffices to show that $J^{*J^{m-1}} \subseteq \bar{J}$. Let $S = \widehat{R}$ be the m -adic completion of R . Note that we clearly have $J^{*J^{m-1}} S \subseteq (JS)^{*(JS)^{m-1}}$. Moreover, we have $\bar{J} = \overline{JS} \cap R$ by Lemma 2.2. So we may assume that R is complete. In particular, R is excellent equidimensional.

Let $z \in J^{*J^{m-1}}$. Then there exists an element $c \in R^\circ$ such that $cz^q J^{(m-1)q} \subseteq J^{[q]}$ for all sufficiently large $q = p^e$. Let a' denote the homomorphic image of $R \rightarrow R_{\text{red}} = R/\sqrt{0}$. Then $c'(z')^q (JR_{\text{red}})^{(m-1)q} \subseteq (JR_{\text{red}})^{[q]}$. That is, $z' \in (JR_{\text{red}})^{*(JR_{\text{red}})^{m-1}}$. If the assertion holds true for R_{red} , then $z' \in \bar{J}R_{\text{red}} = \bar{J}R_{\text{red}}$. This implies that $z \in \bar{J}$. So we may also assume that R is reduced. Then R has a test element c' , that is, $c' \in \tau(R) \cap R^\circ$; see [HH3]. Fix $c'' \in J^{m-1} \cap R^\circ$. Then by Lemma 3.3 we have

$$cz^q \in J^{[q]} : J^{(m-1)q} \subseteq (J^{[q]} + J^{q-m+1})^* = (J^{q-m+1})^*$$

and so

$$c''c'cz^q \in c''J^{q-m+1} \subseteq J^q.$$

Therefore $z \in \bar{J}$, as required. \square

The following assertion is well-known. Our theorem generalizes this in some sense.

Example 3.4. Let R be a Noetherian ring of characteristic $p > 0$. For an element $a \in R^\circ$, we have $\overline{(a)} = (a)^*$.

4. A formula for $J : \bar{J}$ in positive characteristic

Throughout this section, let (R, \mathfrak{m}, k) be a Noetherian local ring of characteristic $p > 0$ with $d = \dim R \geq 1$. The main purpose of this section is to provide a formula of $J : \bar{J}$ for any parameter ideal J in a complete Gorenstein local ring R in terms of test ideals. Namely, we prove the following theorem. A similar formula in the case of characteristic zero will be given in Section 7.

Theorem 4.1. *Assume that R is a complete (or F -finite normal) Gorenstein local ring. Then for any parameter ideal J of R , we have*

$$J : \bar{J} = J + \tau(J^{d-1}).$$

Before proving the theorem, we prove the following proposition. Theorem 4.1 can be deduced from this proposition and Theorem 3.1.

Proposition 4.2. Assume that R is a complete Gorenstein local ring. Let $J \subseteq R$ be a parameter ideal and $\mathfrak{a} \subseteq R$ an ideal with $\mathfrak{a} \cap R^\circ \neq \emptyset$. Then we have

$$J : J^{*\mathfrak{a}} = J + \tau(\mathfrak{a}).$$

Proof. Put $E = E_R(R/\mathfrak{m})$ and $J = (a_1, \dots, a_d)$. Since R is Gorenstein, there exists an injection $R/J \hookrightarrow E = H_{\mathfrak{m}}^d(R) = \varinjlim R/(a_1^\ell, \dots, a_d^\ell)$. Then

$$J^{*\mathfrak{a}}/J = (0)_{R/J}^{*\mathfrak{a}} = (0 :_E J) \cap (0)_E^{*\mathfrak{a}}. \quad (1)$$

As R is complete, $\text{Ann}_E(\tau(\mathfrak{a})) = (0)_E^{*\mathfrak{a}}$ by Matlis duality. Hence the right-hand side is equal to the following module:

$$(0 :_E J) \cap (0 :_E \tau(\mathfrak{a})) = (0 :_E (J + \tau(\mathfrak{a}))) = \text{Hom}_R(R/J + \tau(\mathfrak{a}), E).$$

By taking annihilators of both sides in Eq. (1), we get

$$J : J^{*\mathfrak{a}} = \text{Ann}_R \text{Hom}_R(R/J + \tau(\mathfrak{a}), E) = J + \tau(\mathfrak{a}),$$

as required. \square

Proof of Theorem 4.1. First note that $J + \tau(J^{d-1}) \subseteq J : \bar{J}$ holds true in general. Indeed, we can see that $\tau(J^{d-1})\bar{J} = \tau(\bar{J}^{d-1})\bar{J} \subseteq \tau(\bar{J}^d) = \tau(J^d) \subseteq J$ by [HY, Proposition 1.11, Theorem 2.1]; see also the next section.

So it is enough to show the converse. First suppose that R is an F -finite normal (Gorenstein) local domain. Then $\tau(J^{d-1})\hat{R} = \tau((J\hat{R})^{d-1})$ by [HY, Theorem 1.13] and [HT, Proposition 3.2]. Hence we may assume that R is complete. Then Theorem 3.1 and Proposition 4.1 imply that

$$J : \bar{J} = J : J^{*J^{d-1}} = J + \tau(J^{d-1}),$$

as required. \square

The following example follows from Proposition 4.2.

Example 4.3. (See [Hu, Exercise 2.14].) Suppose that R is a complete Gorenstein local ring. Let J be a parameter ideal generated by test elements. Then $J : J^* = \tau(R)$.

In the case of monomial ideals, our formula immediately follows from a Howald-type theorem (cf. Example 2.8).

Example 4.4. Let $R = k[x_1, \dots, x_d]$ be a polynomial ring over a field of characteristic $p > 0$. Let $J = (x_1^{a_1}, \dots, x_d^{a_d})$. Take a monomial $M = x_1^{m_1} \cdots x_d^{m_d} \in J : \bar{J} \setminus J$. Then $m_i \leq a_i - 1$ for each i . Set $N = x_1^{a_1 - m_1 - 1} \cdots x_d^{a_d - m_d - 1}$. Then N is the “largest” monomial among monomials which are not contained in $J : M$. As $\bar{J} \subseteq J : M$, we have $N \notin \bar{J}$. This yields $(a_1 - m_1 - 1, \dots, a_d - m_d - 1) \notin P(J)$, which means

$$\frac{a_1 - m_1 - 1}{a_1} + \cdots + \frac{a_d - m_d - 1}{a_d} < 1.$$

Hence

$$\frac{m_1 + 1}{a_1} + \cdots + \frac{m_d + 1}{a_d} > d - 1.$$

Therefore Howald-type theorem implies that $M \in \tau(J^{d-1})$, as required.

The next example indicates that our formula follows from “duality” in some sense. For an ideal $I \subseteq R$,

$$R(I) = R[It] = \sum_{n \geq 0} I^n t^n \subseteq R[t] \quad \left(\text{resp. } G(I) = R[It]/IR[It] \cong \bigoplus_{n \geq 0} I^n / I^{n+1} \right)$$

denotes the *Rees algebra* (resp. *the associated graded ring*) of I over R , respectively.

Example 4.5. Assume that R is an excellent Gorenstein normal local domain, and let I be an \mathfrak{m} -primary ideal of R . Suppose that $R(I)$ is F -rational and $G := G(I)$ is Gorenstein. Then $\tau(I^t) = I^{t+G(I)+1}$ for every integer $t \geq 1$ by [HY, Proposition 5.8]. Let $J = (a_1, \dots, a_d)$ be a minimal reduction of I . Then $\tau(J^{d-1}) = \tau(I^{d-1}) = I^{a(G)+d} = I^{a(G(I/J))}$. Since G is Gorenstein, a_1^*, \dots, a_d^* forms a regular sequence in G , where $a_i^* = a_i + I^2 \in G_1$. Then since $G(I/J) \cong G/(a_1^*, \dots, a_d^*)$ is an Artinian Gorenstein ring, its symmetry (e.g. [GI, Proposition 2.4]) implies that

$$(I/J)^{a(G(I/J))} = \frac{J:I}{J}.$$

Hence $J + \tau(J^{d-1}) = J : I$.

5. Skoda-type theorem

Let R be a Noetherian ring of characteristic $p > 0$ with dimension d . Let $\mathfrak{a} \subseteq R$ be an ideal that is generated by r elements. Then $\mathfrak{a}^{(n+r-1)q-r+1} \subseteq (\mathfrak{a}^n)^{[q]}$ for every $q = p^e$ and for every integer $n \geq 0$. This implies that $[(0) : \mathfrak{a}^n]_M \subseteq (0)_M^{*\mathfrak{a}^{n+r-1}}$ for every R -module M . Using this argument, we have the first statement in the following theorem:

Theorem 5.1 (*Skoda-type theorem*). *Let R be a Noetherian ring of characteristic $p > 0$.*

- (1) ([HY, Theorem 2.1]) *If $\mathfrak{a} \subseteq R$ is an ideal generated by r elements, then $\tau(\mathfrak{a}^{n+r-1}) \subseteq \mathfrak{a}^n$ for every $n \geq 0$.*
- (2) ([HT, Theorem 4.1]) *Assume that (R, \mathfrak{m}) is a complete local ring. Then for every integer $t \geq d$, we have $\tau(\mathfrak{m}^t) = \mathfrak{m}^{t-d+1} \tau(\mathfrak{m}^{d-1})$.*

This theorem can be regarded as a prime characteristic analogue of Lipman’s “modified Briançon–Skoda theorem”, which is stated in terms of adjoint ideals by Lipman. After that, his theorem is reformulated in the theory of multiplier ideals.

Corollary 5.2. (See [Li, La].) *Let R be a normal \mathbb{Q} -Gorenstein local ring essentially of finite type over a field of characteristic zero. Let \mathfrak{a} be a nonzero ideal of R . Then $\mathcal{J}(\mathfrak{a}^{n+r-1}) \subseteq \mathfrak{a}^n$ for every $n \geq 0$, where $\mathcal{J}(\mathfrak{b})$ denotes the multiplier ideal of an ideal \mathfrak{b} ; see Section 7 for more details.*

In order to prove a variant of Wang’s theorem using our formula, we need a refinement of Skoda-type theorem. We first state a special case of Skoda-type theorem. In what follows, let (R, \mathfrak{m}, k) be a Noetherian local ring of dimension $d \geq 1$ with infinite residue field.

Corollary 5.3. *For any local ring of R , we have*

$$\tau(\mathfrak{m}^t) \subseteq \mathfrak{m}^{t-d+1}$$

for every integer $t \geq d - 1$. Equality holds if R is excellent regular.

Proof. The first assertion is a special case of Theorem 5.1, and the second assertion follows from [HY, Corollary 2.4] or [HY, Corollary 5.9] if R is excellent. \square

Remark 5.4. The last statement of the corollary can be proved without excellentness, but we cannot find any suitable reference.

In the case of non-regular local rings, a stronger statement holds.

Proposition 5.5. *Assume that R is complete. If R is not regular, then*

$$\tau(\mathfrak{m}^t) \subseteq \mathfrak{m}^{t-d+2}$$

holds true for every integer $t \geq d - 1$.

Proof. By Theorem 5.1 it is enough to show that $\tau(\mathfrak{m}^{d-1}) \subseteq \mathfrak{m}$. If R is weakly F -regular, then it is reduced and equidimensional. Then $\tau(\mathfrak{m}^{d-1}) \subseteq \mathfrak{m}$ by [HY, Theorem 2.15]. Otherwise, $\tau(\mathfrak{m}^{d-1}) \subseteq \tau(R) \subseteq \mathfrak{m}$, as required. \square

Without extra assumptions, this is the best possible result as the next example shows.

Example 5.6. If $R = k[[x_0, x_1, \dots, x_d]]/(x_0^2 + x_1^2 + \dots + x_d^2)$, where $d \geq 1$ and k is a field of characteristic $p > 2$, then $\tau(\mathfrak{m}^t) = \mathfrak{m}^{t-d+2}$ for every integer $t \geq d - 1$.

Proof. It follows from [HY, Proposition 5.8]. \square

In the Gorenstein case, we can improve our result.

Proposition 5.7. *Assume that R is complete and Gorenstein with multiplicity $e(R) \geq 3$. Then*

$$\tau(\mathfrak{m}^t) \subseteq \mathfrak{m}^{t-d+3}$$

for every integer $t \geq d - 1$.

Proof. Put $E = E_R(R/\mathfrak{m}) \cong H_{\mathfrak{m}}^d(R)$. By virtue of [HT, Theorem 4.1], it suffices to show $\tau(\mathfrak{m}^{d-1}) \subseteq \mathfrak{m}^2$. Namely, it is enough to show

$$[(0) : \mathfrak{m}^2]_E \subseteq (0)_E^{*\mathfrak{m}^{d-1}}.$$

We first show the following claim.

Claim. For any $\xi \in [(0) : \mathfrak{m}^2]_E$, we can take a minimal reduction J of \mathfrak{m} such that $\xi \in [(0) : J]_E$.

To show the claim, we may assume that $\xi \notin [(0) : \mathfrak{m}]_E$. Then the image of ξ in

$$[(0) : \mathfrak{m}^2]_E / [(0) : \mathfrak{m}]_E \cong \text{Hom}_R(\mathfrak{m}/\mathfrak{m}^2, E)$$

is nonzero. Notice that the right-hand side is a k -vector space of dimension $v \geq d + 1$, where $v = \dim_k \mathfrak{m}/\mathfrak{m}^2$ denotes the embedding dimension of R . A one-dimensional subspace $k\bar{\xi}$ corresponds to a homomorphic image \mathfrak{m}/L of $\mathfrak{m}/\mathfrak{m}^2$, where L is an ideal with $\mathfrak{m}^2 \subseteq L \subseteq \mathfrak{m}$ and $\dim_k \mathfrak{m}/L = 1$. Then we can regard $\xi \in [(0) : L]_E$. As $\dim_k L/\mathfrak{m}^2 = v - 1 \geq d$, we can choose a minimal reduction J of \mathfrak{m} such that $J + \mathfrak{m}^2 \subseteq L$. Then J is the required ideal.

We now return to the proof. Fix a minimal reduction $J = (a_1, \dots, a_d)$ such that $\xi \in [(0) : J]_E$. Recall that

$$E = H_{\mathfrak{m}}^d(R) = \varprojlim_e R/J^{[p^e]} = \{[b + J^{[q]}] \mid b \in R, q = p^e\},$$

where $[b + J^{[q]}] = [(a_1 \cdots a_d)^{Q-q} b + J^{[Q]}]$ for every $q = p^e < Q = p^f$. Moreover, we can regard the Frobenius map $F : E \rightarrow \mathbb{F}(E) \cong E$ as follows:

$$F([b + J^{[q]}]) = [b^p + J^{[qp]}].$$

Under this notation, we have $[(0) : J]_E = \{[b + J] \mid b \in R\}$. Write $\xi = [b + J]$ for some $b \in R$. Since $\xi \in [(0) : \mathfrak{m}^2]_E$, we get $\mathfrak{m}^2 b \subseteq J$. By the assumption that $e(R) \geq 3$, we have $\mathfrak{m}^2 \not\subseteq J$. Hence $b \in \mathfrak{m}$. If we put $\eta = [1 + J] \in [(0) : J]_E$, then $\xi = b\eta$.

Fix an integer $r \geq 0$ such that $\mathfrak{m}^{r+1} = J\mathfrak{m}^r$. Then for any power $q = p^e$, we have

$$\begin{aligned} J^r F^e(\xi) J^{(d-1)q} &\subseteq \mathfrak{m}^q J^{(d-1)q} J^r F^e(\eta) \\ &\subseteq \mathfrak{m}^r J^{dq} F^e(\eta) \\ &\subseteq \mathfrak{m}^r J^{[q]} F^e(\eta) = 0. \end{aligned}$$

This yields $\xi \in (0)_E^{*J^{d-1}} = (0)_E^{*\mathfrak{m}^{d-1}}$, as required. \square

The following question holds true if $G(\mathfrak{m})$ is Gorenstein and $R(\mathfrak{m})$ is F -rational.

Question 5.8. Let $G = G(\mathfrak{m})$ denote the associated graded ring of R with respect to the maximal ideal \mathfrak{m} , and let $a(G)$ denote the a -invariant of G . Then does $\tau(\mathfrak{m}^t) \subseteq \mathfrak{m}^{t+a(G)+1}$ hold? If so, when does equality hold?

6. A variant of Wang's theorem in positive characteristic

Throughout this section, let (R, \mathfrak{m}, k) be a d -dimensional Noetherian local ring of characteristic $p > 0$. Let \widehat{R} denote the \mathfrak{m} -adic completion of R . In this section, we prove a variant of Wang's theorem as an application of our formula.

Let us recall Wang's theorem [Wa] briefly. Suppose that (R, \mathfrak{m}, k) is a Cohen–Macaulay local ring (of any characteristic) with $d = \dim R \geq 2$, with $d \geq 3$ if R is regular. Let J be a parameter ideal such that $J \subseteq \mathfrak{m}^s$, where $s \geq 2$ is an integer. Then Wang [Wa] proved that $J : \mathfrak{m}^s \subseteq \mathfrak{m}^s$ and $(J : \mathfrak{m}^s)^2 = J(J : \mathfrak{m}^s)$. In particular, $J : \mathfrak{m}^s$ is integral over J . In general, since

$$J \subseteq J : \mathfrak{m} \subseteq J : \mathfrak{m}^2 \subseteq \cdots$$

is an increasing sequence of ideals, if $J : \mathfrak{m}^t$ is integral over J , then so is $J : \mathfrak{m}^{t-1}$. Hence it is natural to ask the following question.

Question 6.1. Let $t \geq 1$ be an integer. When is $J : \mathfrak{m}^t$ integral over J ? Can you determine such a maximal integer t ?

As an answer to the above question, we prove the following theorem.

Theorem 6.2. Assume that R is Cohen–Macaulay and $d = \dim R \geq 2$. Let $s \geq 2$ be an integer. If J is a parameter ideal with $J \subseteq \mathfrak{m}^s$, then $J : \mathfrak{m}^{(d-1)(s-1)}$ is integral over J , that is, $J : \mathfrak{m}^{(d-1)(s-1)} \subseteq \bar{J}$.

Furthermore, if R is not regular, then $J : \mathfrak{m}^{(d-1)(s-1)+1}$ is also integral over J .

Any regular local ring or any homogeneous toric singularity satisfies the assumption of the following corollary.

Corollary 6.3. Under the same notations as in Theorem 6.2, we further assume that $R(\mathfrak{m})$ is normal. Then for any parameter ideal J with $J \subseteq \mathfrak{m}^s$, we have

$$J : \mathfrak{m}^s \subseteq J : \mathfrak{m}^{(d-1)(s-1)+1} \subseteq \mathfrak{m}^s$$

if $d \geq 3$, or if $d \geq 2$ and R is not regular.

Proof. If $d \geq 3$, then $(d-1)(s-1) \geq s$ for every $s \geq 1$. Moreover, if $d = 2$, then $(d-1)(s-1) + 1 = s$ for every $s \geq 1$.

On the other hand, if $R(\mathfrak{m})$ is normal, then $\bar{\mathfrak{m}}^s = \mathfrak{m}^s$ for every $s \geq 1$. \square

In what follows, we prove Theorem 6.2. Let us begin with the following lemma.

Lemma 6.4. We may assume that R is complete with infinite residue field.

Proof. If we put $S = \widehat{R}[X]_{\mathfrak{m}\widehat{R}[X]}$, then $R \rightarrow S$ is a faithfully flat extension, and $\bar{J} = \overline{JS} \cap R$ by Lemma 2.2. If $JS : \mathfrak{m}^t S$ is integral over JS , then we get

$$\begin{aligned} J : \mathfrak{m}^t &= (J : \mathfrak{m}^t)S \cap R \\ &= (JS : \mathfrak{m}^t S) \cap R \\ &\subseteq \overline{JS} \cap R = \bar{J}. \end{aligned}$$

Since S is a complete local ring with infinite residue field, we obtain the required assertion. \square

Proof of Theorem 6.2. By Lemma 6.4, we may assume that R is complete with infinite residue field.

Case 1. R is Gorenstein but not regular.

Now suppose that $J \subseteq \mathfrak{m}^s$. Since R is not regular, we have

$$\tau(J^{d-1}) \subseteq \tau(\mathfrak{m}^{s(d-1)}) \subseteq \mathfrak{m}^{(d-1)(s-1)+1}$$

by Lemma 5.5. Therefore Theorem 4.1 implies

$$J : \mathfrak{m}^{(d-1)(s-1)+1} \subseteq J : [J + \tau(J^{d-1})] = J : [J : \bar{J}] = \bar{J},$$

where the last equality follows from the local duality theorem because R is Gorenstein.

Case 2. R is regular.

By a similar argument as above, we have

$$J : \mathfrak{m}^{(d-1)(s-1)} = J : \tau(\mathfrak{m}^{s(d-1)}) \subseteq J : [J + \tau(J^{d-1})] = J : [J : \bar{J}] = \bar{J},$$

where the first equality follows from Lemma 5.3.

Case 3. R is Cohen–Macaulay but *not* Gorenstein.

Let $S = R \ltimes \omega_R$ be the trivial extension of R , where ω_R denotes the canonical module of R . Put $\mathfrak{n} = \mathfrak{m} \oplus \omega_R$. Then (S, \mathfrak{n}) is a d -dimensional Gorenstein local ring but *not* a hypersurface because \mathfrak{n} is generated by at least $d + 4$ elements. Hence $e(S) \geq 3$. Moreover, $\mathfrak{n}^\ell = \mathfrak{m}^\ell \oplus \mathfrak{m}^{\ell-1}\omega_R$ for every integer $\ell \geq 1$. In particular, $\mathfrak{m}^\ell S \subseteq \mathfrak{n}^\ell \subseteq \mathfrak{m}^{\ell-1}S$. As J is a parameter ideal of R which is contained in \mathfrak{m}^s , JS is also a parameter ideal of S which is contained in \mathfrak{n}^s . Therefore, by Proposition 5.7 and a method similar to Case 1, we have

$$(J : \mathfrak{m}^{(d-1)(s-1)+1})S \subseteq JS : \mathfrak{m}^{(d-1)(s-1)+1}S \subseteq JS : \mathfrak{n}^{(d-1)(s-1)+2} \subseteq \overline{JS}.$$

On the other hand, one can easily see that $\overline{JS} = \bar{J} \oplus K_R$. It follows that

$$J : \mathfrak{m}^{(d-1)(s-1)+1} \subseteq \bar{J},$$

as required. \square

In the case of regular local rings, $t = (d-1)(s-1)$ is the best possible uniform upper bound for which $J : \mathfrak{m}^t$ is integral over J whenever J is a parameter ideal with $J \subseteq \mathfrak{m}^s$. Indeed, we have the following example.

Example 6.5. Let $d, s \geq 2$ be integers. Let $R = k[[x_1, \dots, x_d]]$ be a formal power series ring over a field k . Put $J = (x_1^s, \dots, x_d^s)$. Then J is a parameter ideal and $J \subseteq \mathfrak{m}^s$. Moreover, $J : \mathfrak{m}^{(d-1)(s-1)}$ is integral over J , that is, $J : \mathfrak{m}^{(d-1)(s-1)} \subseteq \mathfrak{m}^s = \bar{J}$. But $J : \mathfrak{m}^{(d-1)(s-1)+1}$ is *not* integral over J because this ideal contains \mathfrak{m}^{s-1} . In particular, the Goto number of J is given by $g(J) = (d-1)(s-1)$.

But in the non-regular case, $t = (d-1)(s-1)$ is *not* best possible.

Example 6.6. Let $R = k[[x_1, \dots, x_d]^r]$ be an r -th Veronese subring over a field k . Put $\mathfrak{m} = (x_1, \dots, x_d)^r R$. Put $J = (x_1^{rs}, \dots, x_d^{rs})$. Then J is a parameter ideal and $J \subseteq \mathfrak{m}^s$. Moreover, $J : \mathfrak{m}^{(d-1)(s-1) + \lceil \frac{(r-1)d}{r} \rceil}$ is integral over J .

7. The case of equicharacteristic zero

In this section, we discuss a formula and a variant of Wang's theorem in the equicharacteristic zero case using modulo p reduction.

7.1. Multiplier ideals

We first recall the notion of \mathbb{Q} -Gorenstein rings. Let R be a Cohen–Macaulay normal local domain. The ring R is called \mathbb{Q} -Gorenstein if there exists a height one ideal ω that is isomorphic to a canonical module ω_R of R and such that $\omega^{(r)}$ is principal for some integer $r \geq 1$. Then the minimum integer $r \geq 1$ such that $\omega^{(r)}$ is principal is said to be the *index* of R .

Lemma 7.1. (See e.g. [TW].) Let (R, \mathfrak{m}) be a Cohen–Macaulay normal domain containing a field of characteristic zero. Suppose that R is \mathbb{Q} -Gorenstein of index r . Let $S = \bigoplus_{i=0}^{r-1} \omega_R^{(i)}$ be a canonical covering of R . Then S is a quasi-Gorenstein normal local domain.

We next recall the notion of multiplier ideals. To define multiplier ideals, we need to assume that R is a normal \mathbb{Q} -Gorenstein domain essentially of finite type over a field k of characteristic zero. Put $Y = \operatorname{Spec} R$. Let $\mathfrak{a} \subseteq \mathcal{O}_Y = R$ be a nonzero ideal sheaf. Let $f : X \rightarrow Y$ be a log resolution of the ideal \mathfrak{a} , that is, a resolution of singularities of Y such that the ideal sheaf $\mathfrak{a}\mathcal{O}_X$ is invertible (say, $\mathfrak{a}\mathcal{O}_X = \mathcal{O}_X(-D)$) for an effective divisor D on X , and that the union $\operatorname{Exc}(f) \cup \operatorname{Supp}(D)$ of the f -exceptional locus and the support of D is a simple normal cross divisor. Then

$$\mathcal{J}(\mathfrak{a}) = H^0(X, \mathcal{O}_X(\lceil K_X - f^*K_Y - D \rceil))$$

is, independent on the choice of a log resolution $f : X \rightarrow Y$ of \mathfrak{a} , is called the *multiplier ideal* of \mathfrak{a} . Notice that for any real number $t \geq 0$, $\mathcal{J}(t \cdot \mathfrak{a})$ can be defined.

7.2. Reduction to characteristic $p \gg 0$

Let R be an algebra essentially of finite type over a field k of characteristic zero, let $\mathfrak{a} \subseteq R$ be an ideal such that $\mathfrak{a} \cap R^\circ \neq \emptyset$. In order to prove a formula or a variant of Wang’s theorem, we want to use modulo p reduction method. Now let us explain the method briefly.

One can choose a finitely generated \mathbb{Z} -subalgebra A contained in k and a subalgebra R_A of R essentially of finite type over A such that the natural map $R_A \otimes_A k \rightarrow R$ is an isomorphism and $\mathfrak{a}_A = \mathfrak{a} \cap R_A$ generates the ideal \mathfrak{a} . For a maximal ideal μ of A , we consider the base change to its residue field $\kappa = \kappa(\mu)$ over A to get a ring $R_\kappa = R_A \otimes_A \kappa$ and an ideal $\mathfrak{a}_\kappa = \mathfrak{a}_A R_\kappa$. Then we refer to such $(\kappa, R_\kappa, \mathfrak{a}_\kappa)$ for maximal ideals μ in a suitable dense open subset of $\operatorname{Spec} A$ as “reduction to characteristic $p \gg 0$ ”. Given a log resolution $f : Y \rightarrow X$, we can reduce this entire setup reduced from characteristic zero to characteristic $p \gg 0$.

We recall the following theorem.

Theorem 7.2. (See [HY, Theorems 3.4, 6.8].) Let R be a normal \mathbb{Q} -Gorenstein local ring essentially of finite type over a field of characteristic zero, and let \mathfrak{a} be a nonzero ideal of R . Then, after reduction to characteristic $p > 0$, we have $\tau(\mathfrak{a}) = \mathcal{J}(\mathfrak{a})$.

7.3. A formula

Note that $\mathcal{J}(J^{d-1}) \subseteq J : \bar{J}$ always holds true by Skoda’s theorem (cf. Corollary 5.2). Thus, the following formula is obtained from Theorem 4.1 and Theorem 7.2.

Theorem 7.3. Let (R, \mathfrak{m}) be a normal Gorenstein local domain essentially of finite type over a field of characteristic zero. Set $d = \dim R \geq 2$. For any parameter ideal J of R , we have

$$J : \bar{J} = J + \mathcal{J}(J^{d-1}).$$

7.4. A variant of Wang’s theorem

We want to prove an analogous result of Theorem 6.2. In the case of Gorenstein local rings, we can use Theorem 7.3 instead of Theorem 4.1. But, since the trivial extension is *not* reduced, we must use another method in the case of Cohen–Macaulay local rings. Namely, we use the so-called “canonical cover trick”.

Theorem 7.4. Let (R, \mathfrak{m}) be a Cohen–Macaulay \mathbb{Q} -Gorenstein normal local domain essentially of finite type over a field of characteristic zero. Moreover, suppose that a canonical cover $S = \bigoplus_{i=0}^{r-1} \omega_R^{(i)}$ is Cohen–Macaulay (and thus Gorenstein). Put $d = \dim R \geq 2$. Let J be a parameter ideal of R with $J \subseteq \mathfrak{m}^s$. Then $J : \mathfrak{m}^{(d-1)(s-1)}$ is integral over J .

Proof. In the case of Gorenstein local rings, we use Theorem 7.3 instead of Theorem 4.1.

Now suppose that R is Cohen–Macaulay but not Gorenstein. Let $S = \bigoplus_{i=0}^{r-1} \omega_R^{(i)}$ be a canonical cover which is Gorenstein. Then S is a module-finite extension of R .

Now suppose that $J \subseteq \mathfrak{m}^s$ is a parameter ideal of R . Then JS is a parameter ideal of S and

$$\begin{aligned} J : \mathfrak{m}^{(d-1)(s-1)} &\subseteq (JS : \mathfrak{m}^{(d-1)(s-1)} S) \cap R \\ &\subseteq (JS : \mathcal{J}(\mathfrak{m}^{s(d-1)} S)) \cap R \\ &\subseteq (JS : [J + \mathcal{J}(J^{d-1} S)]) \cap R \\ &= (JS : (JS : \overline{JS})) \cap R \\ &= \overline{JS} \cap R = \overline{J}, \end{aligned}$$

where the last equality follows from Lemma 2.1. \square

8. Several questions

8.1. A generalization of Theorem 6.2

Recall that one does not need to assume that R contains a field in the original Wang's theorem (see also Theorem 1.1).

Question 8.1. Let (R, \mathfrak{m}, k) be any Cohen–Macaulay local ring of dimension $d \geq 2$. Let $s \geq 1$ be an integer. For any parameter ideal of $J \subseteq \mathfrak{m}^s$, is $J : \mathfrak{m}^{(d-1)(s-1)}$ integral over J ?

Many authors have studied quasi-socle ideals, that is, ideals like $J : \mathfrak{m}^s$. In particular, Goto et al. [GHS] extended Wang's results to the Buchsbaum case.

Question 8.2. Does Theorem 6.2 hold even if R is not Cohen–Macaulay?

It is not difficult to prove the following example by a similar argument as in Theorem 7.4.

Example 8.3. Assume that R is a d -dimensional excellent local domain such that the integral closure \overline{R} of which is regular. Then for any parameter ideal $J \subseteq \mathfrak{m}^s$, we have that $J : \mathfrak{m}^{(d-1)(s-1)}$ is integral over J .

Wang proved that $J : \mathfrak{m}^s$ has reduction exponent one. But our method gives no information about the reduction exponent of $J : \mathfrak{m}^{(d-1)(s-1)}$.

Question 8.4. Let $1 \leq t \leq (d-1)(s-1)$ and put $I = J : \mathfrak{m}^t$.

- (1) What is the reduction exponent of $J : \mathfrak{m}^t$ with respect to J ?
- (2) What is the greatest number t for which $I^2 = JI$?

Example 8.5. (See also Example 6.5.) Let d, s, t be positive integers with $d \geq 2$. Let $R = k[[x_1, \dots, x_d]]$ be a formal power series ring over a field of characteristic $p > 0$. Then $J = (x_1^s, \dots, x_d^s)$ is a parameter ideal which is contained in \mathfrak{m}^s . Set $I = J : \mathfrak{m}^t$.

If $t \leq \lfloor \frac{d(s-1)+1}{2} \rfloor$, then $I^2 = JI$. Moreover, when $(d-2)(s-1) \geq 2$, if we put $t = \lfloor \frac{d(s-1)+1}{2} \rfloor + 1$, then $t \leq (d-1)(s-1)$ and $I^2 \not\subseteq J$ since $x_1^{s-1} \dots x_d^{s-1} \in \mathfrak{m}^{d(s-1)} \subseteq I^2$.

8.2. When does $\tau(J^{d-1}) = J : I$ hold?

Throughout this subsection, let (R, \mathfrak{m}, k) be an F -finite Gorenstein normal local domain of dimension $d \geq 2$ (of characteristic $p > 0$), and let J be a parameter ideal of R and $I = \bar{J}$ denote the integral closure of J . Then Theorem 3.1 implies that

$$J + \tau(J^{d-1}) = J : I.$$

Hence if $\tau(J^{d-1})$ contains J , then we have

$$\tau(I^{d-1}) = \tau(J^{d-1}) = J : I.$$

Indeed, it is always true for 2-dimensional Gorenstein F -rational rings. So it is natural to ask the following question.

Question 8.6. When does $\tau(J^{d-1})$ contain J ?

Although we cannot classify parameter ideals J for which $\tau(J^{d-1})$ contain J yet, we can limit our targets by the following proposition.

Proposition 8.7. Let (R, \mathfrak{m}, k) be an F -finite normal (or a complete) Gorenstein local ring of characteristic $p > 0$ with $d = \dim R \geq 2$. Let J be a parameter ideal of R with $J \subseteq \mathfrak{m}^s$ and $J \not\subseteq \mathfrak{m}^{s+1}$.

If $J \subseteq \tau(J^{d-1})$ holds true, then one of the following is satisfied:

- (1) $d = 2$.
- (2) $d = 3, s = 2$ and R is regular.
- (3) $s = 1$.

Proof. Take an integer $s \geq 2$ for which $J \subseteq \mathfrak{m}^s$ and $J \not\subseteq \mathfrak{m}^{s+1}$ hold. Then $J \subseteq \tau(J^{d-1}) \subseteq \tau((\mathfrak{m}^s)^{d-1}) \subseteq \mathfrak{m}^{(s-1)(d-1)}$ by our assumption. By the choice of s , we get $(s-1)(d-1) \leq s$. This implies that $d \leq 3$.

Now suppose that $d = 3$. Since $2(s-1) \leq s$, we have $s = 2$. If R is not regular, then $J \subseteq \tau(J^2) \subseteq \tau(\mathfrak{m}^4) \subseteq \mathfrak{m}^3$. This contradicts the choice of s . Hence R is regular. \square

On the other hand, in the case where $R(I)$ is F -rational, we have a nice characterization of the above ideals using results in [HY, Section 5].

Proposition 8.8. Suppose that (R, \mathfrak{m}) is an F -finite normal (or a complete) Gorenstein local domain of dimension $d \geq 2$. Let $J \subseteq I$ be as above. Suppose that $R(I)$ is F -rational. Then $\tau(J^{d-1}) = J^{d-1} : I^{d-1}$, and the following conditions are equivalent:

- (1) $J \subseteq \tau(J^{d-1})$.
- (2) $\tau(J^{d-1}) = J : I$.
- (3) I is stable, that is, $I^2 = JI$.

Proof. Let a_1, \dots, a_d be a minimal set of generators of J , and put $J^{[\ell]} = (a_1^\ell, \dots, a_d^\ell)$ for every integer $\ell \geq 1$. Notice that $R(I)$ is F -rational implies that $R(I^{d-1})$ is also F -rational. Hence, by [HY, Theorem 5.1], we have that

$$\tau(J^{d-1}) = J^{[d-1]} : I^{(d-1)^2}.$$

As $R(I)$ is Cohen–Macaulay, we have $I^d = JI^{d-1}$. Hence we get

$$\begin{aligned} \tau(J^{d-1}) &= J^{[d-1]} : J^{(d-2)(d-1)} I^{d-1} \\ &= (J^{[d-1]} : J^{(d-2)(d-1)}) : I^{d-1} \\ &= (J^{[d-1]} + J^{d-1}) : I^{d-1} \\ &= J^{d-1} : I^{d-1}. \end{aligned}$$

The equivalence of (1) and (2) follows from Theorem 4.1. Now suppose (3). If $d = 2$, then it follows from [HY, Theorem 5.1] that $\tau(J) = J : I$ holds true. Hence we get (1). So we may assume that $d \geq 3$. Then $I^{d-1} = J^{d-2}I$. Therefore

$$\tau(J^{d-1}) = J^{d-1} : I^{d-1} = (J^{d-1} : J^{d-2}) : I = J : I,$$

where the last equality follows from the Cohen–Macaulayness of the associated graded ring $\text{gr}_J(R) = \bigoplus_{n \geq 0} J^n / J^{n+1}$.

Conversely, suppose (1). If $d = 2$, then the Cohen–Macaulayness of $R(I)$ implies that I is stable. Hence we may assume that $d \geq 3$. Then since

$$\tau(J^{d-1}) = J : I \subseteq J^2 : I^2 \subseteq \dots \subseteq J^{d-1} : I^{d-1} = \tau(J^{d-1}),$$

we have $\tau(J^{d-1}) = J^2 : I^2$. Hence using the Cohen–Macaulayness of $\text{gr}_J(R)$ and $\text{gr}_I(R)$, we can obtain that

$$I^2 \subseteq (J^2 : J) \cap I^2 = J \cap I^2 = JI,$$

as required. \square

The next example gives ideals which satisfy the condition (1) in Proposition 8.7.

Example 8.9. (See e.g. [HWY, Theorem 3.1].) Let (R, \mathfrak{m}) be a Gorenstein F -rational local ring of dimension 2. Then for any integrally closed \mathfrak{m} -primary ideal I and its minimal reduction J of I , $R(I)$ is F -rational and hence $\tau(J) = J : I$ holds true.

In the case of non- F -rational case, the equality $\tau(J) = J : I$ does not hold in general.

Example 8.10. (See also [HY, Section 5].) Let $R = \mathbb{F}_2[[x, y, z]]/(x^2 + y^3 + z^5)$ and put $I = \mathfrak{m} = (x, y, z)$, $J = (y, z)$. Then $\tau(J) = \tau(\mathfrak{m}) = (x, y, z^2) \neq J : I = \mathfrak{m}$.

The next example gives ideals which satisfy the condition (2) in Proposition 8.7.

Example 8.11. Let $R = k[[x, y, z]]$ be a formal power series ring over a perfect field k of characteristic $p > 0$. Let $J = (x^a, y^b, z^c)R$ and put $I = \overline{J}$. Then $1/a + 1/b + 1/c > 1$ if and only if $\tau(J^2) = J : I$ holds true.

Remark 8.12. Takagi informed us that for any 3-dimensional regular local ring R which is essentially of finite type over a field of characteristic zero, the following three conditions are equivalent (see [La]):

- (1) $\mathcal{J}(J^2) \supseteq J$.
- (2) $\mathcal{J}(J) = R$.
- (3) $R/(f)$ has rational singularity for any general element $f \in J$.

The next example gives ideals which satisfy the condition (3) in Proposition 8.7.

Example 8.13. Let $d \geq 4$ be an integer. Let $R = k[[x_1, x_2, \dots, x_d]]$ be a formal power series ring over a perfect field of characteristic $p > 0$. Let

$$J = (x_1^a, x_2^b, x_3^c, x_4, \dots, x_d)R.$$

If $1/a + 1/b + 1/c > 1$, then $\tau(J^{d-2}) = R$ and hence $\tau(J^{d-1}) \supseteq J$.

Acknowledgments

The first author is partially supported by Grant-in-Aid for Scientific Research 20540050 and Individual Research Expense of College of Humanity and Sciences, Nihon University. The second author is partially supported by Grant-in-Aid for Scientific Research 22540047.

References

- [CHV] A. Corso, C. Huneke, W.V. Vasconcelos, On the integral closure of ideals, *Manuscripta Math.* 95 (1998) 331–347.
- [CP] A. Corso, C. Polini, Links of prime ideals and their Rees algebras, *J. Algebra* 178 (1995) 224–238.
- [CPV] A. Corso, C. Polini, W.V. Vasconcelos, Links of prime ideals, *Math. Proc. Cambridge Philos. Soc.* 115 (1995) 431–436.
- [G] S. Goto, Integral closedness of complete intersection ideals, *J. Algebra* 108 (1987) 151–160.
- [GHS] S. Goto, J. Horiuchi, H. Sakurai, Quasi-socle ideals in Buchsbaum rings, preprint.
- [GKM] S. Goto, S. Kimura, N. Matsuoka, Quasi-socle ideals in Gorenstein numerical semigroup rings, *J. Algebra* 320 (2008) 276–293.
- [GKPT] S. Goto, S. Kimura, T.T. Phuong, H.L. Truong, Quasi-socle ideals and Goto numbers of parameters, *J. Pure Appl. Algebra* 214 (2010) 501–511.
- [GI] S. Goto, S. Iai, Embeddings of certain graded rings into their canonical modules, *J. Algebra* 228 (2000) 377–396.
- [GTM] S. Goto, R. Takahashi, N. Matsuoka, Quasi-socle ideals in a Gorenstein local ring, *J. Pure Appl. Algebra* 212 (2008) 969–980.
- [HT] N. Hara, S. Takagi, On a generalization of test ideals, *Nagoya Math. J.* 175 (2004) 59–74.
- [HWY] N. Hara, K.-i. Watanabe, K. Yoshida, F-rationality of Rees algebras, *J. Algebra* 247 (2002) 153–190.
- [HY] N. Hara, K. Yoshida, A generalization of tight closure and multiplier ideals, *Trans. Amer. Math. Soc.* 355 (2003) 3143–3174.
- [HS] W. Heinzer, I. Swanson, The Goto numbers of parameter ideals, *J. Algebra* 321 (2009) 152–166.
- [HH1] M. Hochster, C. Huneke, Tight closure and strong F-regularity, *Mem. Soc. Math. Fr. (N.S.)* 38 (1989) 119–133.
- [HH2] M. Hochster, C. Huneke, Tight closure, invariant theory and the Briançon–Skoda theorem, *J. Amer. Math. Soc.* 3 (1990) 31–116.
- [HH3] M. Hochster, C. Huneke, F-regularity, test elements, and smooth base change, *Trans. Amer. Math. Soc.* 346 (1994) 1–62.
- [Ho] J.A. Howald, Multiplier ideals of monomial ideals, *Trans. Amer. Math. Soc.* 353 (2001) 2665–2671.
- [Hu] C. Huneke, Tight Closure and Its Applications, *CBMS Reg. Conf. Ser. Math.*, vol. 88, American Mathematical Society, Providence, 1996.
- [La] R. Lazarsfeld, Positivity in Algebraic Geometry, *Ergeb. Math. Grenzgeb.* (3), vol. 48/49, Springer, Berlin, 2004.
- [Li] J. Lipman, Adjoint of ideals in regular local rings, *Math. Res. Lett.* 1 (1994) 739–755, with an Appendix by S.D. Cutkosky.
- [PU] C. Polini, B. Ulrich, Linkage and reduction numbers, *Math. Ann.* 310 (1998) 631–658.
- [SH] I. Swanson, C. Huneke, Integral Closure of Ideals, Rings, and Modules, *London Math. Soc. Lecture Note Ser.*, vol. 336, Cambridge Univ. Press, Cambridge, 2006.
- [TW] M. Tomari, K.-i. Watanabe, Normal \mathbb{Z}_r -graded rings and normal cyclic covers, *Manuscripta Math.* 76 (1992) 325–340.
- [Wa] H.J. Wang, Links of symbolic powers of prime ideals, *Math. Z.* 256 (2007) 749–756.