Star configurations in $\mathbb{P}^n$

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A B S T R A C T

Star configurations are certain unions of linear subspaces of projective space. They have appeared in several different contexts: the study of extremal Hilbert functions for fat point schemes in the plane; the study of secant varieties of some classical algebraic varieties; the study of the resurgence of projective schemes. In this paper we study some algebraic properties of the ideals defining star configurations, including getting partial results about Hilbert functions, generators and minimal free resolutions of the ideals and their symbolic powers. We also show that their symbolic powers define arithmetically Cohen–Macaulay subschemes and we obtain results about the primary decompositions of the powers of the ideals. As an application, we compute the resurgence for the ideal of the codimension $n - 1$ star configuration in $\mathbb{P}^n$ in the monomial case (i.e., when the number of hyperplanes is $n + 1$).

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1. Introduction

A *star configuration* of codimension c in $\mathbb{P}^n$ is a certain union of linear subspaces $V_1, \dots, V_i$ each of codimension c. These have arisen as objects of study in numerous research projects lately, including [2,3,5,6,8,13,21,26], but these references make use of only a partial understanding of the properties of star configurations. Thus it is of interest to understand them better.

Here we study powers and symbolic powers of ideals of star configurations in $\mathbb{P}^n$ (over an algebraically closed field of arbitrary characteristic). Since the subspaces V_i are distinct with none

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containing any of the others, and each is a complete intersection, the m-th symbolic power of the ideal I of the star configuration is $I^{(m)} = I(V_1)^m \cap \cdots \cap I(V_i)^m$.

Combinatorially equivalent collections of linear spaces can have very different algebraic properties, as Example 2.5 shows by exhibiting two collections of lines in $\mathbb{P}^3$ with the same intersection posets but where one gives an arithmetically Cohen–Macaulay (ACM) subscheme and the other not. The situation with star configurations (defined below in terms of their intersection posets) is very different. We will show in Proposition 2.9 that every star configuration is ACM, with so-called generic Hilbert function (meaning that the h-vector coincides with the dimension of the appropriate coordinate ring until a prescribed degree and is then zero). We also note that this property does not characterize star configurations, since (at least in codimension two) there exist unmixed configurations of linear varieties with the same Hilbert functions as star configurations, which are themselves not even ACM (see Remark 2.10).

We then show that every symbolic power of the ideal of a star configuration of any codimension defines an ACM subscheme (see Theorem 3.1). This contrasts with what we will see in Example 2.5. We also pose a conjecture for the primary decompositions of powers of ideals of star configurations and in some cases verify the conjecture (see Conjecture 4.1 and Theorem 4.8). As an application we use Theorem 4.8 to determine the resurgence $\rho(I)$ of the ideal I of a positive dimensional star configuration (see Theorem 4.11). The only other exact determination of the resurgence of a positive dimensional subscheme which is not a cone over a 0-dimensional subscheme and for which the resurgence is bigger than 1 is that of [14], using a different method.

2. Preliminaries

We let $R = k[x_0, \ldots, x_n]$ where k is an arbitrary infinite field, and where we regard R as a graded ring with the usual grading (where each variable has degree 1 and nonzero elements of k have degree 0).

Definition 2.1. Let $\mathcal{H} = \{H_1, \ldots, H_s\}$ be a collection of $s \geqslant 1$ distinct hyperplanes in $\mathbb{P}^n$ corresponding to linear forms $L_1, \ldots, L_s$. We assume that the hyperplanes *meet properly*, by which we mean that the intersection of any j of these hyperplanes is either empty or has codimension j. For any $1 \leqslant c \leqslant \min(s, n)$, let $V_c(\mathcal{H}, \mathbb{P}^n)$ be the union of the codimension c linear varieties defined by all the intersections of these hyperplanes, taken c at a time:

$$V_c\big(\mathcal{H}, \mathbb{P}^n\big) = \bigcup_{1 \leqslant i_1 < \cdots < i_c \leqslant s} H_{i_1} \cap \cdots \cap H_{i_c}.$$

(When $\mathbb{P}^n$ or $\mathcal{H}$ is clear from the context, we may write V_c or $V_c(\mathbb{P}^n)$ or $V_c(\mathcal{H})$ in place of $V_c(\mathcal{H}, \mathbb{P}^n)$.) We call V_c *the codimension c skeleton* associated to $\mathcal{H}$ or sometimes simply a *codimension c star configuration*. We denote by $V_c^{(\ell)}$ the subscheme of $\mathbb{P}^n$ defined by the ideal

$$I_{V_c}^{(\ell)} = \bigcap_{1 \leqslant i_1 < \cdots < i_c \leqslant s} (L_{i_1}, \ldots, L_{i_c})^\ell.$$

Note that $I_{V_c}^{(\ell)}$ is the *ℓ-th symbolic power* of I_{V_c}.

Remark 2.2. We are most interested in the case of star configurations in $\mathbb{P}^n$ for which $s \geqslant n+1$. When $1 \leqslant s \leqslant n$, the star is either a linear subvariety of projective space or a projective cone over a star in $\mathbb{P}^{s-1}$. But for some proofs it is convenient to allow $s < n+1$.

We now recall the following definition.

Definition 2.3. Let $Z \subseteq \mathbb{P}^n$ be a closed subscheme whose defining sheaf of ideals is $\mathcal{I}_Z$. If $H^i(\mathbb{P}^n, \mathcal{I}_Z(d)) = 0$ for all $d \in \mathbb{Z}$ and all $0 < i \leqslant \dim Z$, we say the scheme Z is *arithmetically Cohen–Macaulay* or *ACM*. Note that this is equivalent to saying that the graded ring R/I_Z is a Cohen–Macaulay ring [20, Lemma 1.2.3].

Definition 2.4. For a scheme V of codimension c in $\mathbb{P}^n$ (not necessarily ACM), the *h-vector* of V (or, more precisely, of R/I_V) is the $(n - c + 1)$-st difference of the Hilbert function of R/I_V.

Example 2.5. Here we exhibit two subschemes of $\mathbb{P}^3$ consisting of linear subspaces with the same intersection poset, the same degree and arithmetic genus (and hence the same Hilbert polynomial), one being arithmetically Cohen–Macaulay (ACM) and the other not. In both of these cases, CoCoA [7] shows that the symbolic squares of the ideals define subschemes which fail to be ACM. This shows that the ACM property for the reduced curve does not imply it for symbolic powers.

Let Q be the nonsingular quadric surface in $\mathbb{P}^3$. Recall that Q is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and hence has two rulings. Choose any four distinct lines V_1, V_2, V_3, V_4 from one of the rulings and any four distinct lines H_1, H_2, H_3, H_4 from the other. Let p_i be the point where H_i and V_1 meet, $i = 1, 2$, and let q be a point on H_2 not on any of the other lines.

Consider the line L in $\mathbb{P}^3$ through p_1 and q. Note that Q does not contain L. We now have three subschemes: C_1, consisting of the reduced union of $V_1, \ldots, V_4, H_1, H_2$; C_2, consisting of the reduced union of $L, V_2, V_3, V_4, H_1, H_2$; and C_3, consisting of the reduced union of $V_1, V_2, V_3, V_4, H_1, H_2, H_3, H_4$. Note for any i and j that $V_i \cup H_j$ is a hyperplane section of Q. Thus C_3 is the complete intersection of Q with four planes, these four being the planes determined by the pairs of intersecting lines (V_1, H_1), (V_2, H_2), (V_3, H_3), (V_4, H_4).

Note that C_3 is the union of C_1 with the disjoint union $H_3 \cup H_4$. Since, as it is easy to see, ACM subschemes are connected, we see that $H_3 \cup H_4$ is not ACM. Moreover, linked schemes are either both ACM or neither ACM [20]. Since C_1 is linked with $H_3 \cup H_4$, we see that C_1 is not ACM.

Let X be the union of V_2, V_3, V_4, H_1, H_2, so that $X \cup L = C_2$. We see that X is directly linked, by the complete intersection of Q and three planes, to H_3, and thus X is ACM. Also, since L meets Q in the two points $p_1, q \in X \subset Q$, the ideal $I_X + I_L$ defines the reduced scheme $\{p_1\} \cup \{q\}$. The latter is a complete intersection of type $(1, 1, 2)$, and $Q \notin I_L$ (where by abuse of notation, Q also represents the quadratic form defining the quadric surface), so in fact $I_X + I_L$ is saturated. Then from the exact sequence

$$0 \to I_{C_2} \to I_X \oplus I_L \to I_X + I_L \to 0,$$

sheafifying and taking cohomology it follows immediately that the first cohomology of $\mathcal{I}_{C_2}$ is zero in all twists, so C_2 is ACM. Notice that both C_1 and C_2 consist of 6 lines and thus have the same degree, and since the intersection poset of both curves is the same, then both have the same arithmetic genus and hence the same Hilbert polynomial. Checking computationally using CoCoA, we verified that the symbolic square of the ideal of neither curve is ACM.

Recall from [19] the following result.

Proposition 2.6. *Let I_C be a saturated ideal defining a codimension c subscheme $C \subseteq \mathbb{P}^n$. Let $I_S \subset I_C$ be an ideal which defines an ACM subscheme S of codimension $c - 1$. Let F be a form of degree d which is not a zerodivisor on R/I_S. Consider the ideal $I' = F \cdot I_C + I_S$ and let C' be the subscheme it defines. Then I' is saturated, hence equal to $I_{C'}$, and there is an exact sequence*

$$0 \to I_S(-d) \to I_C(-d) \oplus I_S \to I_{C'} \to 0.$$

In particular, since S is an ACM subscheme of codimension one less than C, we see that C′ is an ACM subscheme if and only if C is. Also,

$$\deg C' = \deg C + (\deg F) \cdot (\deg S).$$

Furthermore, as sets on S, we have C′ = C ∪ H_F, where H_F is the hypersurface section cut out on S by F. The Hilbert function $h_{C'}$ of R/$I_{C'}$ is $h_{C'}(t) = h_S(t) − h_S(t − d) + h_C(t − d)$.

Remark 2.7. Under rather mild assumptions, the subscheme C′ obtained in Proposition 2.6 can be linked in two steps to C via Gorenstein ideals, and it was in this context that it was introduced in [19]. We will not use this fact below, but it is worth noting that in the literature this construction is often referred to as *Basic Double G-Linkage*.

As an application we use Proposition 2.6 to obtain the following result. For this we make a definition.

Definition 2.8. Let I be a nonzero homogeneous ideal in the ring R. We define $\alpha(I)$ to be the least degree among degrees of nonzero elements of I.

We recover the fact of [3, Lemma 2.4.2] that the initial degree $\alpha(I_{V_c})$ of I_{V_c} (i.e., the degree of a nonzero homogeneous element of least degree) is $s − c + 1$. We also note that in the case where the hyperplanes are defined by the $s = n + 1$ coordinate variables, V_c was known to be ACM (see [16, Example 2.2(b)]).

Proposition 2.9. *Let $\mathcal{H} = \{H_1, \ldots, H_s\}$ be a collection of distinct hyperplanes in $\mathbb{P}^n$ meeting properly, and let $V_c = V_c(\mathcal{H})$. Then we have the following facts.*

(1) *V_c is ACM.*
(2) *The h-vector of V_c, which has $s − c + 1$ entries, is*

$$\left(1, c, \binom{c+1}{2}, \ldots, \binom{s-1}{c-1}\right) = \left(1, \binom{c}{c-1}, \binom{c+1}{c-1}, \ldots, \binom{s-1}{c-1}\right).$$

Note that the last binomial coefficient occurs in degree $s − c$, and can also be written $\binom{(c-1)+(s-c)}{s-c}$.
(3) *$\deg V_c = \binom{s}{c}$.*
(4) *The initial degree of I_{V_c} is $\alpha(I_{V_c}) = s − c + 1$, and all of its minimal generators occur in this degree and are monomials in the linear forms L_i defining the hyperplanes H_i.*

Proof. Notice that (3) is trivial, and we include it only for completeness. We proceed by induction on c and on $s \geqslant c$. For any c, note that if $s = c$ then V_c is a complete intersection of linear forms, and parts (1) to (4) are trivial. If $c = 1$ and s is arbitrary, V_1 is the union of s hyperplanes, and all four assertions are immediate. Now assume that the assertion is true for codimension $c − 1$ and for up to $s − 1$ hyperplanes. Let $\mathcal{H}' = \{H_1, \ldots, H_{s-1}\}$ and let $\mathcal{H} = \mathcal{H}' \cup \{H_s\}$. By induction, $V_{c-1}(\mathcal{H}')$ and $V_c(\mathcal{H}')$ are both ACM. We now apply Proposition 2.6 to $S = V_{c-1}(\mathcal{H}')$, $C = V_c(\mathcal{H}')$, and $F = L_s$, the defining polynomial of H_s. Since $V_c(\mathcal{H}) = V_c(\mathcal{H}') \cup H_{L_s}$, where H_{L_s} is the hyperplane section of V_{c-1} cut out by H_s, we immediately have (1). Since we have $I_{V_c(\mathcal{H})} = L_s \cdot I_{V_c(\mathcal{H}')} + I_{V_{c-1}(\mathcal{H}')}$, and by induction minimal sets of generators of $I_{V_c(\mathcal{H}')}$ and $I_{V_{c-1}(\mathcal{H}')}$ are monomials in the L_i of degree $s − c$ and $s − c + 1$, respectively, we see $I_{V_c(\mathcal{H})}$ is also generated by monomials in the L_i, and that the generators all have degree $s − c + 1$, which proves (4).

It remains to prove (2). We use the Hilbert function part of Proposition 2.6, still with $S = V_{c-1}(\mathcal{H}')$, $C = V_c(\mathcal{H}')$, and $F = L_s$. The h-vector of $V_c(\mathcal{H}')$ is the $(n − c + 1)$-th difference of $h_{V_c(\mathcal{H}')}$, while the h-vector of $V_{c-1}(\mathcal{H}')$ is the $(n − c + 2)$-th difference of $h_{V_{c-1}(\mathcal{H}')}$. Notice that $d = 1$ in this case (in the

statement of Proposition 2.6), so the portion of the formula coming from $h_s(t) - h_s(t - d)$ amounts to a first difference. The h-vector of $V_c(\mathcal{H}')$ is

$$\left(1, c, \binom{c+1}{c-1}, \ldots, \binom{s-3}{c-1}, \binom{s-2}{c-1}\right),$$

where the last entry is in degree $s - c - 1$, and the h-vector of $V_{c-1}(\mathcal{H}')$ is

$$\left(1, c-1, \binom{c}{c-2}, \ldots, \binom{s-3}{c-2}, \binom{s-2}{c-2}\right),$$

where the last entry is in degree $s - c$. Thus the h-vector of V_c is computed by

$$\left(1, c-1, \binom{c}{c-2}, \binom{c+1}{c-2}, \ldots, \binom{s-3}{c-2}, \binom{s-2}{c-2}\right)$$
$$+ \left(1, c, \binom{c+1}{c-1}, \ldots, \binom{s-3}{c-1}, \binom{s-2}{c-1}\right)$$

from which the desired h-vector of $V_c(\mathcal{H})$ follows. $\square$

Remark 2.10.

(a) The h-vector given in Proposition 2.9(2) is sometimes called a *generic h-vector*, on account of its being the h-vector of a generic finite set of points. Note that the ACM property automatically implies that V_c is a so-called scheme of *maximal rank*, i.e. that the natural restriction map $H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathcal{O}_{V_c}(d))$ has maximal rank for all d. However, even for a scheme of the right degree, having maximal rank does not imply that R/I_{V_c} has generic h-vector. For example, when $s = 4$ and $c = 2$, V_2 has h-vector $(1, 2, 3)$, hence degree 6. However, a general set of six skew lines in $\mathbb{P}^3$ has maximal rank [15] but has h-vector $(1, 2, 3, 4, 0, -4)$.
(b) Notwithstanding the comment in (a), there do exist linear configurations that are not ACM but nevertheless have generic h-vectors. This is an easy consequence of the construction given in [22], starting with a minimal curve consisting of two skew lines. Indeed, here we sketch the argument that *for every codimension two generic h-vector $(1, 2, 3, \ldots)$ of degree at least 6, there is a non-ACM configuration of codimension two linear varieties with the given generic h-vector.* We begin with curves in $\mathbb{P}^3$ and proceed inductively, repeatedly applying Proposition 2.6. Start with a curve C_0 consisting of two skew lines in $\mathbb{P}^3$. Its h-vector is $(1, 2, -1)$. Let S_1 be a union of four planes, such that S_1 contains C_0. Note that I_{S_1} is generated by a form of degree 4 that is the product of four linear forms. Let F_1 be a general linear form. Then $F_1 \cdot I_{C_0} + I_{S_1}$ is the saturated ideal of a union of six lines, C_1, with h-vector $(1, 2, 3)$. For $i \geqslant 2$ (but not $i = 1$) we obtain C_i inductively from C_{i-1} by taking S_i to be a union of $i + 2$ planes containing C_{i-1}, and F_i in each case to be a general linear form (choosing a new F_i each time), and setting $I_{C_i} = F_i \cdot I_{C_{i-1}} + I_{S_i}$. Then C_i has h-vector $(1, 2, 3, \ldots, i + 2)$. We then pass to the codimension two case by taking cones.

Remark 2.11. By Proposition 2.9(1) and (2), the Artinian reduction of the homogeneous coordinate ring of V_c is $k[y_1, \ldots, y_c]/\mathfrak{m}^{s-c+1}$, where $\mathfrak{m} = (y_1, \ldots, y_c)$. Since $\mathfrak{m}^{s-c+1}$ is generated by the maximal (i.e., $r \times r$ for $r = s - c + 1$) minors of the $r \times s$ matrix

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_c & 0 & \cdots & 0 & 0 \\ 0 & y_1 & y_2 & \cdots & y_c & 0 & \cdots & 0 \\ & & & \cdots & & & & \\ 0 & \cdots & 0 & y_1 & y_2 & y_3 & \cdots & y_c \end{pmatrix}$$

and has codimension $(s - c + 1 - r + 1)(s - r + 1) = c$, the graded Betti numbers of the homogeneous coordinate ring of V_c are those given by the Eagon–Northcott resolution of the maximal minors of a generic matrix of size $r \times s$ [17]. Note however, that it is well known that powers of $\mathfrak{m}$ all have linear resolution, consequently the calculation of the graded Betti numbers of the powers is straightforward. In particular, denoting by $\mathbb{E}_{\bullet}^{s,c}$ the minimal free resolution of $I_{V_c(\mathcal{H})}$, we have

$$\operatorname{rk} \mathbb{E}_i^{s,c} = \binom{s}{s - c + i} \cdot \binom{s - c + i - 1}{i - 1}. \tag{2.12}$$

We will need the next result for the proof of Theorem 3.2.

Lemma 2.13. *For each c, we have $I_{V_{c-1}(\mathcal{H})} \subset I_{V_c(\mathcal{H})}^{(2)}$.*

Proof. We have to show the inclusion $V_c(\mathcal{H})^{(2)} \subset V_{c-1}(\mathcal{H})$ of schemes. Since both sides are unmixed, it is enough to do this locally. That is, we show that every component of $V_c(\mathcal{H})^{(2)}$ lies in $V_{c-1}(\mathcal{H})$. To do this, it is enough to look only at the components of V_{c-1} that contain the component of $V_c(\mathcal{H})$ in question. Now, V_{c-1} is a union of codimension $c - 1$ linear spaces and V_c is its singular locus. In particular, each component of V_c is the intersection of c of the hyperplanes H_i, so there pass c components of V_{c-1} through each component of V_c (take away one H_i at a time). It thus is enough to set $\mathcal{H} = \{H_1, \ldots, H_c\}$ and prove it in this case. Now $I_{V_c(\mathcal{H})} = \langle L_1, \ldots, L_c \rangle$ and $I_{V_c(\mathcal{H})}^{(2)} = I_{V_c(\mathcal{H})}^2$.

On the other hand, let $\mathcal{H}' = \{H_1, \ldots, H_{c-1}\}$ and consider the codimension $c - 1$ complete intersection $I_{V_{c-1}(\mathcal{H}')} = \langle L_1, \ldots, L_{c-1} \rangle$. Thanks to Proposition 2.6, we have

$$I_{V_{c-1}(\mathcal{H})} = L_c \cdot I_{V_{c-1}(\mathcal{H}')} + I_{V_{c-2}(\mathcal{H}')}.$$

We can thus use induction on c (the low values are easy to check), and assume that $I_{V_{c-2}(\mathcal{H}')}$ is generated by degree two products of $L_1, \ldots, L_{c-1}$, and since $I_{V_{c-1}(\mathcal{H}')}$ is just the complete intersection of the linear forms $L_1, \ldots, L_{c-1}$, we have that $I_{V_{c-1}(\mathcal{H})}$ is generated by degree two products of $L_1, \ldots, L_c$. This implies the asserted inclusion and completes the proof. □

Remark 2.14. S. Tohǎneanu has pointed out to us a connection between our Proposition 2.9 and his work. In [24] and [25], he has considered ideals generated by products of linear forms, which define zero-dimensional schemes in $\mathbb{P}^{k-1}$. His assumption about the linear forms is weaker than our assumption that the hyperplanes meet properly (see Definition 2.1 above), and he finds a similar connection to the Eagon–Northcott resolution. We are grateful to him for pointing out the connection. It is of interest to note that Tohǎneanu's work is motivated by applications to coding theory. Applying ideals generated by products of linear forms was first introduced by DeBoer and Pellikaan [10].

Motivated by his comment to us, we realized that we can also weaken our hypothesis on meeting properly, for the purpose of obtaining a generalization of Proposition 2.9. Indeed, our key ingredient was Proposition 2.6. In order to be able to apply this result to obtain a codimension c scheme, we only need that any c of the hyperplanes meet in codimension c. For example, taking $n = 3$, $s = 4$ and $c = 3$, a choice of four hyperplanes meeting properly gives a star configuration consisting of four non-degenerate points in $\mathbb{P}^3$, while a choice of four hyperplanes all containing a given point P gives a "fat" point of multiplicity two (hence degree 4) supported on P. However, for the main results below we continue to require that the hyperplanes meet properly.

3. Symbolic powers of ideals of star configurations

Given the ideal I of a reduced ACM subscheme consisting of a union of linear spaces of projective space, it's natural to ask whether the symbolic powers of I also define ACM subschemes. They clearly do if the linear subspaces are points, but otherwise it is not always the case, as Example 2.5 shows.

Thus Theorem 3.1, showing that all of the symbolic powers of any ideal I_{V_c} of a star configuration V_c of any codimension c do define ACM subschemes, is all the more interesting.

Theorem 3.1. *Let $\mathcal{H} = \{H_1, \ldots, H_s\}$ be hyperplanes in $\mathbb{P}^n$ and let $V_c := V_c(\mathcal{H})$ for any c. Every symbolic power of I_{V_c} is ACM. Furthermore, if L_i is a linear form defining the hyperplane H_i, then each symbolic power of I_{V_c} is generated by monomials in the L_i.*

Proof. Say $s \leqslant n$. If $c = s$, then V_c is a linear subvariety and a complete intersection so the result is true. If $c < s \leqslant n$, choose coordinates such that the hyperplanes H_i are defined by coordinate variables $x_1, \ldots, x_s$, and extend to a full set of coordinates $x_0, x_1, \ldots, x_n$. Let Λ be the linear subvariety defined by $x_0 = x_{s+1} = \cdots = x_n = 0$. Then $\Lambda \cap V_c$ is a codimension c star in $\Lambda \cong \mathbb{P}^{s-1}$, and V_c is a projective cone over $\Lambda \cap V_c$. In addition to the canonical surjection $k[\mathbb{P}^n] \to k[\Lambda]$, we have a non-canonical inclusion $k[\Lambda] = k[x_1, \ldots, x_s] \subseteq k[x_0, \ldots, x_n] = k[\mathbb{P}^n]$, with respect to which we have $I_{V_c}^{(m)} = I_{\Lambda \cap V_c}^{(m)} k[\mathbb{P}^n]$ since primary decompositions extend [1, Exercise 4.7(iv)]. Thus $k[\mathbb{P}^n]/I_{V_c}^{(m)}$ is a polynomial ring over $k[\Lambda]/I_{\Lambda \cap V_c}^{(m)}$, so the result for $V_c \subset \mathbb{P}^n$ follows if and only if it follows for $\Lambda \cap V_c \subset \Lambda$. Thus we may assume that $s \geqslant n + 1$.

Now fix the codimension, c, so V_c is the union of $\binom{s}{c}$ linear varieties. First assume that $s = n + 1$, so without loss of generality we may assume that $L_i = x_i$ for each i (modulo s, so $L_s = x_0$).

We claim that I_{V_c} is the Stanley–Reisner ideal of a simplicial complex, Δ, of dimension $n - c$ that is the complete simplicial complex of dimension $n - c$ on $n + 1$ vertices. To construct this simplicial complex, take for the $n + 1$ vertices the $n + 1$ coordinate points in $\mathbb{P}^n$. For convenience of notation, we will label these points by $p_0, \ldots, p_n$, and without loss of generality we will assume that the vertex labelled p_i is the common intersection point of the hyperplanes defined by $x_0, \ldots, \hat{x}_i, \ldots, x_n$.

The component of V_c cut out by the hyperplanes $x_{i_1} = 0, \ldots, x_{i_c} = 0$ has dimension $n - c$. The vertices that it does *not* contain are precisely $x_{i_1}, \ldots, x_{i_c}$; that is, this component corresponds to the face of Δ which is the linear span of the vertices with the complementary labels. There are $n + 1 - c$ such vertices, so Δ has dimension $n - c$. By construction, it is the complete simplicial complex of dimension $n - c$ on these vertices. Thus by construction, the Stanley–Reisner ideal corresponding to this simplicial complex is the ideal of V_c. This completes the proof of our claim.

Recall that a simplicial complex Δ is said to be *pure* if all of its facets have the same dimension. It is said to be a *matroid* if, for every subset W of the vertices (in our case $\{p_0, \ldots, p_n\}$), the restriction $\Delta_W = \{F \in \Delta \mid F \subset W\}$ is a pure simplicial complex. In our setting, simplicial complex Δ is clearly a matroid, since the restriction is again complete.

If $c = n$, the result clearly follows since any zero-dimensional scheme is ACM. Thus we may assume that $c < n$, i.e. that our star configuration has dimension at least one. We now recall a key fact from [23] and [27]:

Let Δ be a simplicial complex and let I_Δ be its Stanley–Reisner ideal. Then $I_\Delta^{(\ell)}$ is Cohen–Macaulay for every $\ell \geqslant 1$ if and only if Δ is a matroid.

It follows from these results that $I_{V_c}^{(\ell)}$ is Cohen–Macaulay for every ℓ, i.e. that the corresponding schemes are ACM.

Now assume that $s > n + 1$. We still have $\mathcal{H} = \{H_1, \ldots, H_s\}$, hyperplanes in $\mathbb{P}^n$ where H_i is the vanishing locus of a linear form L_i. Without loss of generality we may assume that $L_s = x_0, L_1 = x_1, \ldots, L_n = x_n$. We still denote by V_c the codimension c star configuration in $\mathbb{P}^n$ defined by $\mathcal{H}$. Let $N = s - 1$ and consider the star configuration $W_c \subset \mathbb{P}^N$ defined as in our first case above, with the variables $x_0, \ldots, x_N$.

Consider the linear forms $M_{n+1} = x_{n+1} - L_{n+1}, \ldots, M_N = x_N - L_N$. It is clear that for an *ACM* subscheme V of $\mathbb{P}^N$ meeting each of the corresponding hyperplanes, successively, in codimension 1, the *saturated* ideal of I_V is obtained by replacing x_i by L_i, for all $i = n + 1, \ldots, N$, since the ACM property and the assumption about the codimension guarantee that $M_{n+1}, \ldots, M_N$ are a regular sequence. In particular, for any $i \geqslant n + 1$, x_i is replaced by L_i. Thus the star configuration W_c and the schemes $W_c^{(\ell)}$ defined by its symbolic powers in $\mathbb{P}^N$ yield V_c and the schemes $V_c^{(\ell)}$ as the result of a sequence

of hyperplane sections. Since the codimension is preserved, these hyperplane sections are all proper. Since we have shown that $W_c^{(\ell)}$ are all ACM, the claimed result follows from the fact that the ACM property is preserved under proper hyperplane sections (see for instance [20]). From what we have done, the claim about the ideals is also immediate. It is also clear that $\alpha(I_{W_c^{(\ell)}}) = \alpha(I_{V_c^{(\ell)}})$; we will use this in Corollary 4.6. □

Theorem 3.1 makes no assertion about the Hilbert function or the minimal free resolution (apart from its length) of the symbolic powers of the ideal of a star configuration. In Theorem 3.2, only in the case of the symbolic square, we give a different proof of the fact that we obtain an ACM scheme, which allows us to describe the h-vector (equivalently, the Hilbert function) and the graded Betti numbers. For the proof of the theorem, we will give an explicit construction of the symbolic square of I_{V_c} for any c, in a way that makes it clear that it is ACM. Rather than squaring I_{V_c}, throwing away higher codimensional primary components, and trying to verify that the result is ACM, we take a more direct approach. We construct an ideal for which it is easy to see that it is ACM, and then we show that this ideal is actually the symbolic square.

We will use Proposition 2.6 with $C = V_c$ and $S = V_{c-1}$. We will construct an ideal $I_{C'}$ with a special choice of F, so this gives right away that C' is an ACM subscheme, since C is. Furthermore, we can get the minimal free resolution of $I_{C'}$ from that of I_C and I_S by studying a suitable mapping cone. We will then see that C' is precisely the symbolic square of C in this case.

Theorem 3.2. *Let $\mathcal{H} = \{H_1, \ldots, H_s\}$ and let $V_i := V_i(\mathcal{H})$ for all i. Then*

(1) *The h-vector of $V_c^{(2)}$ is as follows*

$$\Delta^{n-c+1} h_{R/I_{V_c^{(2)}}}(t) = \begin{cases} \binom{t+c-1}{c-1} & \text{if } t \leqslant s - c, \\ \binom{s}{c-1} & \text{if } s - c + 1 \leqslant t \leqslant 2s - 2c + 1, \\ 0 & \text{if } t > 2s - 2c + 1. \end{cases}$$

(2) *The minimal free resolution of $I_{V_c}^{(2)}$ has the form*

$$0 \to \mathbb{F}_c \to \cdots \to \mathbb{F}_1 \to \mathbb{I}_{V_c}^{(2)} \to 0$$

where

$$\mathbb{F}_i = \mathbb{E}_i^{s,c}(-1 + c - s) \oplus \mathbb{E}_{i-1}^{s,c-1}(-1 + c - s) \oplus \mathbb{E}_i^{s,c-1}$$

using the notation of Remark 2.11. In particular,

$$\mathbb{F}_i = R(-2s + 2c - 1 - i)^{M_i} \oplus R(-s + c - 1 - i)^{N_i}$$

where

$$M_i = \begin{cases} \binom{s}{s-c+1} & \text{if } i = 1; \\ \binom{s}{s-c+i} \cdot \binom{s-c+i-1}{i-1} + \binom{s}{s-c+i} \cdot \binom{s-c+i-1}{i-2} & \text{if } 2 \leqslant i \leqslant c \end{cases}$$

and

$$N_i = \begin{cases} \binom{s}{s-c+1+i} \cdot \binom{s-c+i}{i-1} & \text{if } 1 \leqslant i \leqslant c - 1; \\ 0 & \text{if } i = c. \end{cases}$$

Proof. By Proposition 2.9(4) applied to V_{c-1}, $I_{V_{c-1}}$ is entirely generated in degree $s - c + 2$, while I_{V_c} is entirely generated in degree $s - c + 1$. Let $F \in I_{V_c}$ be a general element of degree $s - c + 1$. Then F does not vanish on any component of V_{c-1}, i.e. it is a non-zerodivisor on $R/I_{V_{c-1}}$.

As mentioned above, V_{c-1} is a union of codimension $c - 1$ linear spaces and V_c is its singular locus. In particular, each component of V_c is the intersection of c of the hyperplanes H_i, so there pass c components of V_{c-1} through each component of V_c (take away one H_i at a time). Since $F \in I_{V_c}$ and F does not vanish on any component of V_{c-1}, the subscheme of V_{c-1} cut out by F thus has multiplicity at least c locally along each component of V_c. This accounts for a subscheme of degree at least $c \cdot \binom{s}{c}$. On the other hand, a quick calculation shows

$$(\deg F) \cdot (\deg V_{c-1}) = (s - c + 1) \cdot \binom{s}{c-1} = c \cdot \binom{s}{c}.$$

We conclude that F cuts out a subscheme supported on $V_c \subset V_{c-1}$ with multiplicity exactly c along each component of V_c. Consequently, thanks to Proposition 2.6, the subscheme defined by the ideal $F \cdot I_{V_c} + I_{V_{c-1}}$ is supported on V_c and has degree $c + 1$ along each component.

This is the same degree and support as the scheme defined by the symbolic square of I_{V_c}, and both $I_{V_c}^{(2)}$ and $F \cdot I_{V_c} + I_{V_{c-1}}$ are unmixed (in particular, saturated). To show equality, then, we just have to show one inclusion. We will show

$$F \cdot I_{V_c} + I_{V_{c-1}} \subseteq I_{V_c}^{(2)}. \tag{3.3}$$

First, any element of $F \cdot I_{V_c}$ is an element of $I_{V_c}^{(2)}$ since $F \in I_{V_c}$. Furthermore, by Lemma 2.13 we have $I_{V_{c-1}} \subset I_{V_c}^{(2)}$, so the inclusion follows, and the ideals are equal. We thus have a new proof that $V_c^{(2)}$ is ACM.

Now we can write the Hilbert function, a minimal generating set and minimal free resolution using Proposition 2.6. Indeed, observe that the claimed h-vector is actually

$$\Delta^{n-c+1} h_{R/I_{V_c}^{(2)}}(t) = \begin{cases} \Delta^{n-c+1} h_{R/I_{V_{c-1}}}(t) = \binom{t+c-1}{c-1} & \text{if } t \leqslant s - c, \\ \Delta^{n-c+1} h_{R/I_{V_{c-1}}}(s - c + 1) = \binom{s}{c-1} & \text{if } s - c + 1 \leqslant t \leqslant 2s - 2c + 1, \\ 0 & \text{if } t > 2s - 2c + 1. \end{cases}$$

The first two lines are immediate since (3.3) shows that $I_{V_{c-1}}$ and $I_{V_c}^{(2)}$ agree through degree $(s - c + 1) + (s - c) = 2s - 2c + 1$ (since Proposition 2.9 gives the initial degree of I_{V_c} as $s - c + 1$). The third line comes from the fact that

$$\Delta^{n-c+1} h_{R/I_{V_c}^{(2)}}(t) = \left[\Delta^{n-c+1} h_{R/I_{V_{c-1}}}(t) - \Delta^{n-c+1} h_{R/I_{V_{c-1}}}\left(t - (s - c + 1)\right) \right]$$
$$+ \Delta^{n-c+1} h_{R/I_{V_c}}\left(t - (s - c + 1)\right).$$

Now, thanks to Proposition 2.9, the third term is zero in degree $(s - c + 1) + (s - c + 1) = 2s - 2c + 2$. As for the first and second terms, they agree in degrees $\geqslant (s - c + 1) + (s - c + 1)$, so their difference is zero in this range.

It remains to find the minimal free resolution of $I_{V_c}^{(2)}$. From Proposition 2.6 and the above calculations, we have the short exact sequence

$$0 \to I_{V_{c-1}}(-1 + c - s) \to I_{V_c}(-1 + c - s) \oplus I_{V_{c-1}} \to I_{V_c}^{(2)} \to 0.$$

The minimal free resolutions of $I_{V_{c-1}}$ and of I_{V_c} are described in Remark 2.11, and in particular Eq. (2.12). A mapping cone then gives a free resolution of $I_{V_c}^{(2)}$, and since the resolutions of $I_{V_{c-1}}$

and of I_{V_c} are linear, it is immediate that there is no splitting, so this is in fact a minimal free resolution. $\square$

Example 3.4. Let $n = 4$, $s = 7$ and $c = 3$. The h-vectors of R/I_{V_2} and R/I_{V_3} are

$$(1, 2, 3, 4, 5, 6) \quad \text{and} \quad (1, 3, 6, 10, 15),$$

respectively. Let $F \in (I_{V_3})_5$. The h-vector of $R/(F, I_{V_2})$ is

$$(1, 3, 6, 10, 15, 20, 18, 15, 11, 6)$$

so using Proposition 2.6 and Proposition 3.2, we can compute the h-vector of $R/I_{V_3}^{(2)}$ as follows:

1	3	6	10	15	20	18	15	11	6
					1	3	6	10	15
1	3	6	10	15	21	21	21	21	21

The minimal free resolution of $I_{V_3}^{(2)}$, by Theorem 3.2, is as follows (for explicit details, see our arXiv posting of this article):

$$0 \to R(-12)^{21} \to \begin{array}{c} R(-7)^6 \\ \oplus \\ R(-11)^{42} \end{array} \to \begin{array}{c} R(-6)^7 \\ \oplus \\ R(-10)^{21} \end{array} \to I_{V_3}^{(2)} \to 0.$$

We now will consider the case of codimension 2. In preparation for stating our results, we define some matrices. Consider a set $\mathcal{H}$ of $s > n$ hyperplanes $H_i \subset \mathbb{P}^n$ meeting properly, so $V_2(\mathcal{H}, \mathbb{P}^n)$ is the union of the $\binom{s}{2}$ codimension 2 linear spaces of the form $H_i \cap H_j$ for $i \neq j$. Let h_i be the linear form defining H_i. Let $P = h_1 \cdots h_s$, and let $P_i = P/h_i$. Let $A_{m,n}$ be the $m \times n$ 0-matrix and $\delta(d_1, \ldots, d_r)$ the $r \times r$ diagonal matrix with diagonal entries d_i. Furthermore, consider the $1 \times s$ matrix B, the $s \times s$ matrices C and E and the $s \times (s-1)$ matrix D, defined as follows:

$$B = (-P_1 - P_2 - P_3 \cdots - P_s),$$

$$C = \delta(h_1, \ldots, h_s),$$

$$D = \begin{pmatrix} -h_1 & 0 & 0 & \cdots & 0 & 0 \\ h_2 & -h_2 & 0 & \cdots & 0 & 0 \\ 0 & h_3 & -h_3 & \cdots & 0 & 0 \\ & & \cdots & & & \\ 0 & 0 & 0 & \cdots & h_{s-1} & -h_{s-1} \\ 0 & 0 & 0 & \cdots & 0 & h_s \end{pmatrix}, \quad \text{and}$$

$$E = -\delta(P_1, \ldots, P_s).$$

Finally,

- when $m = 2r$ is even, let Δ_m be the $(sr + 1) \times sr$ matrix

$$\Delta_m = \begin{pmatrix} B & A_{1,s} & A_{1,s} & A_{1,s} & \cdots & A_{1,s} & A_{1,s} \\ C & E & A_{s,s} & A_{s,s} & \cdots & A_{s,s} & A_{s,s} \\ A_{s,s} & C & E & A_{s,s} & \cdots & A_{s,s} & A_{s,s} \\ & & & \cdots & & & \\ A_{s,s} & A_{s,s} & A_{s,s} & A_{s,s} & \cdots & C & E \\ A_{s,s} & A_{s,s} & A_{s,s} & A_{s,s} & \cdots & A_{s,s} & C \end{pmatrix};$$

- when $m = 2r + 1$ is odd, let $\triangle_m$ be the $s(r+1) \times s(r+1) - 1$ matrix:

$$
\begin{pmatrix}
D & E & A_{s,s} & A_{s,s} & A_{s,s} & \cdots & A_{s,s} & A_{s,s} \\
A_{s,s-1} & C & E & A_{s,s} & A_{s,s} & \cdots & A_{s,s} & A_{s,s} \\
A_{s,s-1} & A_{s,s} & C & E & A_{s,s} & \cdots & A_{s,s} & A_{s,s} \\
& & & \cdots & & & & \\
A_{s,s-1} & A_{s,s} & A_{s,s} & A_{s,s} & A_{s,s} & \cdots & C & E \\
A_{s,s-1} & A_{s,s} & A_{s,s} & A_{s,s} & A_{s,s} & \cdots & A_{s,s} & C
\end{pmatrix}.
$$

Lemma 3.5. *The maximal minors of* $\triangle_m$ *are* $\{P^r\}$, $\{P^{r-1}P_i^2\}_{i=1}^s$, $\{P^{r-2}P_i^4\}_{i=1}^s, \ldots, \{P_i^{2r}\}_{i=1}^s$ *if m is even and* $\{P^r P_i\}_{i=1}^s$, $\{P^{r-1}P_i^3\}_{i=1}^s, \ldots, \{P_i^{2r+1}\}_{i=1}^s$ *if m is odd.*

Proof. The matrix $\triangle_m$ is close to being upper triangular, so the maximal minors are easy to compute with in some cases a few row and column swaps. We leave the details to the reader. □

Theorem 3.6. *Let* $\mathcal{H}$ *be a set of* $s > n$ *hyperplanes* $H_i \subset \mathbb{P}^n$ *meeting properly, where* h_i *is the linear form defining* H_i. *Let* $I = I_{V_2(\mathcal{H}, \mathbb{P}^n)}$, *the ideal of the codimension 2 skeleton* $V_2(\mathcal{H}, \mathbb{P}^n)$. *The Hilbert–Burch matrix for* $I^{(m)}$ *is* $\triangle_m$ *and the generators for* $I^{(m)}$ *are as given in Lemma* 3.5.

Proof. Let J be the ideal generated by the elements listed in Lemma 3.5. It is easy to see that they have no common divisor and the zero-locus is $V_2(\mathcal{H}, \mathbb{P}^n)$. Thus by the Hilbert–Burch Theorem, J defines an ACM subscheme and the primary decomposition of J consists of ideals primary for the ideals of the components of $V_2(\mathcal{H}, \mathbb{P}^n)$. The prime ideals corresponding to irreducible components of $V_2(\mathcal{H}, \mathbb{P}^n)$ are precisely the ideals of the form (h_i, h_j), $i \neq j$. If one localizes by inverting all h_l with $l \notin \{i, j\}$, it is easy to check by an explicit examination of the generators given in Lemma 3.5 that the localization J' of the ideal J equals the localization of $(h_i, h_j)^m$. Thus J and $I^{(m)}$ have the same primary decompositions, so $J = I^{(m)}$, which concludes the proof. □

In the case of the codimension 2 skeleton, we now give yet another proof that the symbolic powers are Cohen–Macaulay, with an eye, again, to proving more than can be concluded from Theorem 3.1. In fact, we will show that ideals which are "almost" symbolic powers are also Cohen–Macaulay.

Corollary 3.7. *Let* $\mathcal{H}$ *be a set of* $s > n$ *hyperplanes* $H_i \subset \mathbb{P}^n$ *meeting properly, where* h_i *is a linear form defining* H_i. *Let* $I = I_{V_2(\mathcal{H}, \mathbb{P}^n)}$, *the ideal of the codimension 2 skeleton* $V_2(\mathcal{H}, \mathbb{P}^n)$. *For* $1 \leqslant k \leqslant s$ *and* $\ell \geqslant 1$ *arbitrary, the schemes* W_k *defined by the saturated ideals*

$$
I_{W_k} = \bigcap_{1 \leqslant i < j \leqslant k} (L_i, L_j)^{\ell+2} \cap \bigcap_{1 \leqslant i \leqslant k < j \leqslant s} (L_i, L_j)^{\ell+1} \cap \bigcap_{k < i < j \leqslant s} (L_i, L_j)^{\ell}
$$

are all ACM.

Proof. This will be a byproduct of a new proof of the Cohen–Macaulayness of the symbolic powers. This proof is inspired by a construction used in [21]. That paper studied *tetrahedral curves*, i.e. subschemes of $\mathbb{P}^3$ defined by the intersection of powers of the ideals of the six components of V_2. The specialization of the current theorem to $V_2 \subset \mathbb{P}^3$ was proved in that paper as a special case.

Note that we have already shown this result for $V_2^{(1)}$ and $V_2^{(2)}$. The idea of our proof, which worked also in [21], is that we can apply an inductive argument, passing from $I_{V_2}^{(\ell)}$ to $I_{V_2}^{(\ell+2)}$ by a sequence of applications of Proposition 2.6, thus ensuring that each resulting scheme along the way is ACM. In particular, $V_2^{(\ell+2)}$ is ACM, and we have our result.

Recall that we have hyperplanes $H_1, \ldots, H_s$ defined by linear forms $L_1, \ldots, L_s$. We begin with the ideal $I_{V_2}^{(\ell)}$. Clearly we have $L_2^{\ell+1} \cdots L_s^{\ell+1} \in I_{V_2}^{(\ell)}$. We first claim that we have an equality of saturated ideals

$$L_1 \cdot I_{V_2}^{(\ell)} + \left(L_2^{\ell+1} \cdots L_s^{\ell+1}\right) = \bigcap_{2 \leqslant j \leqslant s} (L_1, L_j)^{\ell+1} \cap \bigcap_{2 \leqslant i < j \leqslant s} (L_i, L_j)^{\ell}.$$

To see this, note first that both ideals are automatically saturated and unmixed (the first comes from Proposition 2.6 and the second is an intersection of saturated, unmixed ideals of the same height). Hence as before, we check that they define schemes of the same degree and that there is an inclusion of one into the other. The first ideal defines a scheme of degree

$$\left[\deg\left(V_2^{(\ell)}\right)\right] + \deg(L_1) \cdot \left[\deg\left(L_2^{\ell+1} \cdots L_s^{\ell+1}\right)\right] = \left[\binom{s}{2} \cdot \binom{\ell+1}{2}\right] + (1) \cdot \left[(s-1)(\ell+1)\right]$$

thanks to Proposition 2.6. The ideal on the right defines a scheme of degree

$$(s-1) \cdot \binom{\ell+2}{2} + \left[\binom{s}{2} - (s-1)\right] \cdot \binom{\ell+1}{2}.$$

We leave it to the reader to verify that these degrees are equal. Since the inclusion $\subseteq$ is clear, the claim is established. Note that by induction we may assume that $V_2^{(\ell)}$ is ACM, so the scheme defined by this new ideal is also ACM, thanks to the construction of Proposition 2.6.

We now show that we can construct a sequence of ACM schemes

$$V_2^{(\ell)} \subset W_1 \subset W_2 \subset \cdots \subset W_s = V_2^{(\ell+2)}$$

by sequentially applying Proposition 2.6. What we have shown so far is that the scheme W_1 defined by the ideal

$$L_1 \cdot I_{V_2}^{(\ell)} + \left(L_2^{\ell+1} \cdots L_s^{\ell+1}\right) = \bigcap_{2 \leqslant j \leqslant s} (L_1, L_j)^{\ell+1} \cap \bigcap_{2 \leqslant i < j \leqslant s} (L_i, L_j)^{\ell}$$

is ACM and contains $V_2^{(\ell)}$. Notice that I_{W_1} contains the element $L_1^{\ell+2} L_3^{\ell+1} \cdots L_s^{\ell+1}$.

We now turn to the inductive step. Suppose we have constructed the ACM scheme W_k defined by the saturated ideal

$$I_{W_k} = \left(\bigcap_{1 \leqslant i < j \leqslant k} (L_i, L_j)^{\ell+2}\right) \cap \left(\bigcap_{1 \leqslant i \leqslant k < j \leqslant s} (L_i, L_j)^{\ell+1}\right) \cap \left(\bigcap_{k < i < j \leqslant s} (L_i, L_j)^{\ell}\right)$$

and that this ideal contains the element $L_1^{\ell+2} \cdots L_k^{\ell+2} L_{k+2}^{\ell+1} \cdots L_s^{\ell+1}$. Notice that

$$\deg W_k = \binom{k}{2}\binom{\ell+3}{2} + (k)(s-k)\binom{\ell+2}{2} + \binom{s-k}{2}\binom{\ell+1}{2}.$$

We produce the ACM scheme W_{k+1} via the ideal

$$L_{k+1} \cdot I_{W_k} + \left(L_1^{\ell+2} \cdots L_k^{\ell+2} L_{k+2}^{\ell+1} \cdots L_s^{\ell+1}\right).$$

Notice that thanks to Proposition 2.6, its degree is

$$\deg W_k + (k)(\ell + 2) + (s - k - 1)(\ell + 1).$$

To prove that this ideal is equal to

$$I_{W_{k+1}} = \bigcap_{1 \leqslant i < j \leqslant k+1} (L_i, L_j)^{\ell+2} \cap \bigcap_{1 \leqslant i \leqslant k+1 < j \leqslant s} (L_i, L_j)^{\ell+1} \cap \bigcap_{k+1 < i < j \leqslant s} (L_i, L_j)^{\ell}$$

is an elementary computation along exactly the same lines as above (although showing that the degrees are equal is very tedious). It is not hard to check that this ideal contains the element $L_1^{\ell+2} \cdots L_{k+1}^{\ell+2} L_{k+3}^{\ell+1} \cdots L_s^{\ell+1}$. Thus the inductive step works, and after s steps we obtain $W_s = V_2^{(\ell+2)}$. (The referee has pointed out to us that an alternative approach to showing the equality of these ideals is to observe that the ideals locally coincide, thus avoiding the calculation of the degrees; this can be applied as well to the equality at the beginning of the proof.) □

We remark that in the case $n = 3$, $s = 4$ (the tetrahedral curve case), the study of when the ideals defined by

$$(x_1, x_2)^{\alpha_1} \cap (x_1, x_3)^{\alpha_2} \cap (x_1, x_4)^{\alpha_3} \cap (x_2, x_3)^{\alpha_4} \cap (x_2, x_4)^{\alpha_5} \cap (x_3, x_4)^{\alpha_6}$$

define ACM subschemes of $\mathbb{P}^3$ was begun in [21] and completed in [11]. Corollary 3.7 gives a partial extension to the codimension two case in $\mathbb{P}^n$.

4. Primary decompositions of powers of ideals of star configurations and applications

In this section we consider an important special case: star configurations defined by monomial ideals. Such a star configuration arises from the set of $s = N + 1$ coordinate hyperplanes in $\mathbb{P}^N$. As motivation, we note that given any codimension c star configuration $V_c(\mathcal{H}, \mathbb{P}^n)$ defined by a set $\mathcal{H} = \{H_1, \ldots, H_s\}$ of $s > n$ hyperplanes in $\mathbb{P}^n$, we have $V_c(\mathcal{H}, \mathbb{P}^n) = V_c(\mathcal{H}', \mathbb{P}^N) \cap L$ for an appropriate n-dimensional linear subspace $L \subset \mathbb{P}^N$, where $N + 1 = s$ and $\mathcal{H}' = \{H_0', \ldots, H_N'\}$ are the coordinate hyperplanes for $\mathbb{P}^N$. (In particular, define $\phi : k[\mathbb{P}^N] \to k[\mathbb{P}^n]$ by $\phi : x_i \mapsto L_{i+1}$ for $0 \leqslant i \leqslant N$, where x_i is the i-th coordinate variable and L_i is the linear form which defines H_i. Then L is defined by the kernel of ϕ.) In fact, by Theorem 3.1, we also have $\phi(I_{V_c(\mathcal{H}', \mathbb{P}^N)}^{(m)}) = I_{V_c(\mathcal{H}, \mathbb{P}^n)}^{(m)}$ for all $m \geqslant 1$.

We now make a conjecture on the primary decomposition of $I_{V_c(\mathcal{H}, \mathbb{P}^n)}^l$, which we will verify in the monomial case (i.e., for $I_{V_c(\mathcal{H}', \mathbb{P}^N)}^l$; see Theorem 4.8).

Conjecture 4.1. *Let $s > n$ and let $\mathcal{H} = \{H_1, \ldots, H_s\}$ be hyperplanes $H_i \subset \mathbb{P}^n$ meeting properly, defined by linear forms L_i. Let M be the irrelevant ideal in $k[\mathbb{P}^n]$ and M' the irrelevant ideal in $k[\mathbb{P}^N]$, where $N + 1 = s$ with $k[\mathbb{P}^N] = k[x_0, \ldots, x_N]$ so $M' = (x_0, \ldots, x_N)$, and let $\mathcal{H}'$ be the $N + 1$ coordinate hyperplanes in $\mathbb{P}^N$. Define $\phi : k[\mathbb{P}^N] \to k[\mathbb{P}^n]$ by $\phi : x_i \mapsto L_{i+1}$. Then*

$$I_{V_c(\mathcal{H}, \mathbb{P}^n)}^l = I_{V_c(\mathcal{H}, \mathbb{P}^n)}^{(l)} \cap I_{V_{c+1}(\mathcal{H}, \mathbb{P}^n)}^{(2l)} \cap \cdots \cap I_{V_n(\mathcal{H}, \mathbb{P}^n)}^{((n-c+1)l)} \cap M^{(N-c+2)l}.$$

Remark 4.2. The point of Conjecture 4.1 is to give primary decompositions of $I_{V_c(\mathcal{H}, \mathbb{P}^n)}^l$ in terms of intersections of symbolic powers. Of course, the symbolic powers are not primary, but they are by definition intersections of primary ideals; for example, $I_{V_c(\mathcal{H}, \mathbb{P}^n)}^{(l)}$ is the intersection of the l-th powers

of the ideals defining the various linear components of $V_c(\mathcal{H}, \mathbb{P}^n)$. The equality in Conjecture 4.1 is known to be at least a containment. What is true is:

$$
\begin{aligned}
I^l_{V_c(\mathcal{H}, \mathbb{P}^n)} &= \phi(I_{V_c(\mathcal{H'}, \mathbb{P}^N)})^l = \phi\big(I^l_{V_c(\mathcal{H'}, \mathbb{P}^N)}\big) \\
&= \phi\big(I^{(l)}_{V_c(\mathcal{H'}, \mathbb{P}^N)} \cap I^{(2l)}_{V_{c+1}(\mathcal{H'}, \mathbb{P}^N)} \cap \cdots \cap I^{((N-c+1)l)}_{V_N(\mathcal{H'}, \mathbb{P}^N)} \cap (M')^{(N-c+2)l}\big) \\
&\subseteq \phi\big(I^{(l)}_{V_c(\mathcal{H'}, \mathbb{P}^N)}\big) \cap \phi\big(I^{(2l)}_{V_{c+1}(\mathcal{H'}, \mathbb{P}^N)}\big) \cap \cdots \cap \phi\big(I^{((N-c+1)l)}_{V_N(\mathcal{H'}, \mathbb{P}^N)}\big) \cap \phi\big((M')^{(N-c+2)l}\big) \\
&= I^{(l)}_{V_c(\mathcal{H}, \mathbb{P}^n)} \cap \cdots \cap I^{((n-c+1)l)}_{V_n(\mathcal{H}, \mathbb{P}^n)} \\
&\qquad \cap \big(\phi\big(I^{((n-c+2)l)}_{V_{n+1}(\mathcal{H'}, \mathbb{P}^N)}\big) \cap \cdots \cap \phi\big(I^{((N-c+1)l)}_{V_N(\mathcal{H'}, \mathbb{P}^N)}\big) \cap \phi\big((M')^{(N-c+2)l}\big)\big) \\
&\subseteq I^{(l)}_{V_c(\mathcal{H}, \mathbb{P}^n)} \cap I^{(2l)}_{V_{c+1}(\mathcal{H}, \mathbb{P}^n)} \cap \cdots \cap I^{((n-c+1)l)}_{V_n(\mathcal{H}, \mathbb{P}^n)} \cap M^{(N-c+2)l}. \qquad (4.3)
\end{aligned}
$$

The first equality follows from Proposition 2.9(4), the second since ϕ is a homomorphism and the third by Theorem 4.8 below. The third line (i.e., the first inclusion) holds since the image of an intersection is always contained in the intersection of the images (for any mapping), and the fourth line holds since $\phi(I^{((i+1)l)}_{V_{c+i}(\mathcal{H'}, \mathbb{P}^N)}) = I^{((i+1)l)}_{V_{c+i}(\mathcal{H}, \mathbb{P}^n)}$ for each i by Theorem 3.1. The fifth line holds since $\phi(M') = M$ and since some of the terms in the intersection have been deleted. Thus the conjecture is that the two inclusions are equalities.

The conjecture that the first inclusion is an equality says that ϕ commutes with the intersections. Having equality would give a primary decomposition of $I^l_{V_c(\mathcal{H}, \mathbb{P}^n)}$. Note that the tail end of this conjectured primary decomposition, namely

$$
\phi\big(I^{((n-c+2)l)}_{V_{n+1}(\mathcal{H'}, \mathbb{P}^N)}\big) \cap \phi\big(I^{((n-c+3)l)}_{V_{n+2}(\mathcal{H'}, \mathbb{P}^N)}\big) \cap \cdots \cap \phi\big(I^{((N-c+1)l)}_{V_N(\mathcal{H'}, \mathbb{P}^N)}\big) \cap \phi\big((M')^{(N-c+2)l}\big),
$$

is primary for the irrelevant ideal, M. The last line of the conjecture simply asserts that this irrelevant component, which is not itself in general a pure power of M, can nonetheless be replaced by a pure power of M.

Finally, note that the primary decompositions proposed here need not be irredundant. For example, when $l = 1$, the last line of (4.3) is contained in (hence equal to) $I_{V_c(\mathcal{H}, \mathbb{P}^n)}$, hence Conjecture 4.1 holds for $l = 1$, but obviously the primary decomposition it gives is not minimal.

Remark 4.4. Here we note some cases where Conjecture 4.1 is known to hold. Conjecture 4.1 holds when $l = 1$, as noted at the end of Remark 4.2. It is easy to see that Conjecture 4.1 holds when $c = 1$, since $I_{V_c(\mathcal{H}, \mathbb{P}^n)}$ is principal and $V_c(\mathcal{H}, \mathbb{P}^n)$ is a complete intersection. Conjecture 4.1 holds when $c = n$, since $(I^l_{V_c(\mathcal{H}, \mathbb{P}^n)})_t = (I^{(l)}_{V_c(\mathcal{H}, \mathbb{P}^n)})_t$ for $t \geqslant \alpha(I^l_{V_c(\mathcal{H}, \mathbb{P}^n)}) = l(s - c + 1)$ by [3, Lemma 2.3.3(c), Lemma 2.4.2], and hence $I^l_{V_n(\mathcal{H}, \mathbb{P}^n)} = I^{(l)}_{V_n(\mathcal{H}, \mathbb{P}^n)} \cap M^{(N-c+2)l}$. And Conjecture 4.1 holds when $n = N = s - 1$, by Theorem 4.8.

So now we begin a study of monomial star configurations $V^{(l)}_c(\mathcal{H'}, \mathbb{P}^N)$, where $\mathcal{H'}$ consists of the $N + 1$ coordinate hyperplanes. Consider $k[\mathbb{P}^N] = k[x_0, \ldots, x_N]$. Let $p_0, \ldots, p_N$ be the coordinate vertices, where $I_{p_i} = (\{x_j : j \neq i\})$. More generally, let $\Lambda = \langle p_{i_1}, \ldots, p_{i_r} \rangle$ be the linear subspace spanned by the given points p_{i_j}; then

$$
I_\Lambda = (\{x_j : j \notin \{i_1, \ldots, i_r\}\}).
$$

Given any monomial $\mu = x_0^{m_0} \cdots x_N^{m_N}$, we can define its Λ-degree as $\deg_\Lambda(\mu) = \sum_{j \notin \{i_1, \ldots, i_r\}} m_j = \deg(\mu) - \sum_{j \in \{i_1, \ldots, i_r\}} m_j$. Note that $\deg_\Lambda(\mu)$ is just the order of vanishing of μ on Λ (i.e., the largest power of I_Λ containing μ). Let $V_c = V_c(\mathcal{H'}, \mathbb{P}^N)$ and let $I = I_{V_c}$ be its ideal. It now follows from the definition of symbolic power that $I^{(l)}$ is generated by all monomials μ such that $\deg_\Lambda(\mu) \geqslant l$ for all irreducible components Λ of V_c.

In the next result we determine $\alpha(I^{(l)})$. This is a special case extension of the result $\alpha(I) = N - c + 2$ given in Proposition 2.9(4). We will use this extension in Corollary 4.6 to extend the determination of α given in Proposition 2.9(4) to symbolic powers in general.

Proposition 4.5. *Let I be the ideal of $V_c = V_c(\mathcal{H}', \mathbb{P}^N)$ where $\mathcal{H}'$ consists of the $N + 1$ coordinate hyperplanes, and let $l \geqslant 1$. Define q and r by writing $l = qc + r$ for $1 \leqslant r \leqslant c$. Then $\alpha(I^{(l)}) = (q + 1)(N + 1) - c + r$.*

Proof. Let $\mu = (x_0 \cdots x_N)^q x_0 \cdots x_{N-c+r}$. Every component Λ of V_c is the span of exactly $N - c + 1$ of the coordinate vertices p_i. Thus $\deg_\Lambda(x_0 \cdots x_N) = N + 1 - (N - c + 1) = c$ and $\deg_\Lambda(x_0 \cdots x_{N-c+r}) \geqslant (N - c + r + 1) - (N - c + 1) = r$, so $\deg_\Lambda(\mu) \geqslant qc + r = l$. Thus $\mu \in I^{(l)}$, so $\alpha(I^{(l)}) \leqslant \deg(\mu) = (q + 1)(N + 1) - c + r$.

To show that $\alpha(I^{(l)}) \geqslant (q + 1)(N + 1) - c + r$, it is enough to show for each monomial of degree $(q + 1)(N + 1) - c + r - 1$ that there is a component Λ of V_c on which the monomial has order of vanishing less than l. So let $\mu = x_0^{m_0} \cdots x_N^{m_N}$ be any monomial such that $\deg(\mu) = (q + 1)(N + 1) - c + r - 1 = q(N + 1) + (N - c + r)$. For some permutation $i_0, \ldots, i_N$ of the indices $0, \ldots, N$ we have $m_{i_0} \geqslant m_{i_1} \geqslant \cdots \geqslant m_{i_N}$. Let $\Lambda = \langle p_{i_0}, \ldots, p_{i_{N-c}} \rangle$. The order of vanishing of μ on Λ is

$$\deg_\Lambda(\mu) = m_{i_{N-c+1}} + \cdots + m_{i_N} = \deg(\mu) - (m_{i_0} + \cdots + m_{i_{N-c}}).$$

This is largest when $m_{i_0} + \cdots + m_{i_{N-c}}$ is least. We can replace μ with $\mu' = x_0^{m'_0} \cdots x_N^{m'_N}$ of the same degree such that still $m'_{i_0} \geqslant \cdots \geqslant m'_{i_N}$ but such that the exponents are as close to constant as possible (i.e., such that $m'_{i_0} - m'_{i_N} \leqslant 1$). Doing this increases the smaller exponents at the expense of the larger exponents, so we have $\deg_\Lambda(\mu) \leqslant \deg_\Lambda(\mu')$. Since $\deg(\mu') = q(N + 1) + (N - c + r)$ we see that $m_{i_j} = q + 1$ for $j = 0, \ldots, N - c + r - 1$, while $m_{i_j} = q$ for $j = N - c + r, \ldots, N$. Thus $\deg_\Lambda(\mu) \leqslant \deg_\Lambda(\mu') = \deg(\mu') - (m_{i_0} + \cdots + m_{i_{N-c}}) = q(N + 1) + (N - c + r) - (N - c + 1)(q + 1) = qc + r - 1 < l$. $\quad\square$

More generally we have:

Corollary 4.6. *Now let $V_c(\mathbb{P}^n)$ be the codimension c skeleton for a star configuration on $s > n$ hyperplanes in $\mathbb{P}^n$ and let I be its ideal. Define q and r by writing $l = qc + r$ for $1 \leqslant r \leqslant c$. Then $\alpha(I^{(l)}) = (q + 1)s - c + r$.*

Proof. This follows from Theorem 3.1 (see also the last sentence of the proof of Theorem 3.1) and Proposition 4.5. $\quad\square$

Proposition 4.7. *Let I be the ideal of $V_c = V_c(\mathcal{H}', \mathbb{P}^N)$ where $\mathcal{H}'$ consists of the $N + 1$ coordinate hyperplanes, and let $l \geqslant 1$. Then $I^{(l)}$ is generated in degree at most $l(N - c + 2)$; more precisely, in any minimal set of homogeneous generators of $I^{(l)}$, the degree $\omega(I^{(l)})$ of a generator of maximum degree is $\omega(I^{(l)}) = l(N - c + 2) = \alpha(I^l)$.*

Proof. First we note that $\alpha(I^l) = l\alpha(I) = l(N - c + 2)$. The ideal $I^{(l)}$ is generated by all monomials $\mu = x_0^{m_0} \cdots x_N^{m_N}$ such that the c smallest exponents sum to l. The maximum degree of such a monomial which is not divisible by another such monomial is $l(N - c + 2)$; take for example $\mu = (x_{c-1} \cdots x_N)^l$, and note μ is not divisible by any other monomial in this generating set. $\quad\square$

We now prove Conjecture 4.1 in the monomial case. (See [12] for irreducible decompositions of monomial ideals and for results on their associated primes.)

Theorem 4.8. *Let I be the ideal of $V_c = V_c(\mathcal{H}', \mathbb{P}^N)$ where $\mathcal{H}'$ consists of the $N + 1$ coordinate hyperplanes and M' is the irrelevant ideal, and let $l \geqslant 1$. Then*

$$I^l = I_{V_c}^{(l)} \cap I_{V_{c+1}}^{(2l)} \cap \cdots \cap I_{V_N}^{((N-c+1)l)} \cap (M')^{(N-c+2)l}.$$

Proof. It is enough to show both the forward containment $I^l \subseteq I^{(l)}_{V_c} \cap I^{(2l)}_{V_{c+1}} \cap \cdots \cap I^{((N-c+1)l)}_{V_N} \cap$ $(M')^{(N-c+2)l}$ and the reverse containment $I^l \supseteq I^{(l)}_{V_c} \cap I^{(2l)}_{V_{c+1}} \cap \cdots \cap I^{((N-c+1)l)}_{V_N} \cap (M')^{(N-c+2)l}$. Moreover, if we show the forward containment for $l = 1$, then we clearly have equality for $l = 1$, so it follows for $l > 1$ that

$$I^l = \left(I_{V_c} \cap \cdots \cap I^{(N-c+1)}_{V_N} \cap (M')^{N-c+2}\right)^l \subseteq I^{(l)}_{V_c} \cap I^{(2l)}_{V_{c+1}} \cap \cdots \cap I^{((N-c+1)l)}_{V_N} \cap (M')^{(N-c+2)l};$$

i.e., the forward containment for $l = 1$ implies the reverse containment for $l = 1$ and it implies the forward containments for all $l > 1$.

So now we verify the forward containment for $l = 1$. As noted in the proof of Proposition 4.7, I is generated by all monomials $\mu = x_0^{m_0} \cdots x_N^{m_N}$ such that the c smallest exponents sum to $l = 1$. We also know that I is generated by monomials of degree $N - c + 2$. Thus exactly $c - 1$ of the exponents m_i must be 0, so the other $N - c + 2$ must be equal to 1. I.e., I is generated by the square-free monomials of degree $N - c + 2$, so each μ is of the form $\mu = x_{i_0} \cdots x_{i_{N-c+1}}$ for some indices $0 \leqslant i_0 < \cdots < i_{N-c+1} \leqslant N$. Thus it is enough to show for every square-free monomial μ of degree $N - c + 2$ that $\mu \in I^{(i+1)}_{V_{c+i}}$ for $i = 0, \ldots, N - c + 1$ and that $\mu \in (M')^{N-c+2}$. Clearly we have $\mu \in (M')^{N-c+2}$, so consider $\mu \in I^{(i+1)}_{V_{c+i}}$. We must check that $\deg_\Lambda(\mu) \geqslant i + 1$ for each component Λ of V_{c+i}. But Λ is spanned by exactly $N - c - i + 1$ coordinate vertices, hence $\deg_\Lambda(\mu) \geqslant \deg(\mu) - (N - c - i + 1) = i + 1$, as needed.

We now verify the reverse inclusion. Let $\mu = x_0^{m_0} \cdots x_N^{m_N}$ and assume

$$\mu \in I^{(l)}_{V_c} \cap I^{(2l)}_{V_{c+1}} \cap \cdots \cap I^{((N-c+1)l)}_{V_N} \cap (M')^{(N-c+2)l}. \tag{**}$$

We will show that $\mu \in I^l$. For simplicity we demonstrate the argument only in case $m_0 \geqslant m_1 \geqslant \cdots \geqslant m_N$; up to a permutation of the indices, the general argument is the same. Our proof will be by induction on l, the case $l = 1$ having been established above.

If $m_{N-c+1} \geqslant l$, then $(x_0 \cdots x_{N-c+1})^l$ divides μ, but $x_0 \cdots x_{N-c+1} \in I$, so $\mu \in I^l$. Now assume $m_{N-c+1} < l$. In any case we have $m_{N-c+1} > 0$, since if $m_{N-c+1} = 0$, then μ is not divisible by any square-free monomial of degree $N - c + 2$ and hence $\mu \notin I$, but by assumption (**), $\mu \in I^{(l)} \subseteq I$. In particular, μ is divisible by $x_0 \cdots x_{N-c+1}$; let $\mu' = \mu/(x_0 \cdots x_{N-c+1})$. If we check that

$$\mu' \in I^{(l-1)}_{V_c} \cap I^{(2(l-1))}_{V_{c+1}} \cap \cdots \cap I^{((N-c+1)(l-1))}_{V_N} \cap (M')^{(N-c+2)(l-1)};$$

then $\mu' \in I^{l-1}$ by induction, so $\mu = \mu' x_0 \cdots x_{N-c+1} \in I^l$, as claimed. We will use the following function. Given distinct elements $j_1, \ldots, j_r \in \{0, \ldots, N\}$ and $0 \leqslant t \leqslant N$, let $\nu_{j_1,\ldots,j_r}(t) = |\{0, \ldots, t\} \cap \{j_1, \ldots, j_r\}|$. Thus, for example, $\nu_j(t)$ is 1 if $0 \leqslant j \leqslant t \leqslant N$ and $\nu_j(t)$ is 0 if $0 \leqslant t < j \leqslant N$.

We first check that $\mu' \in (M')^{(N-c+2)(l-1)}$. Since $\mu \in (M')^{(N-c+2)l}$, we have $\deg(\mu) \geqslant (N - c + 2)l$, so $\deg(\mu') \geqslant (N - c + 2)l - (N - c + 2) = (N - c + 2)(l - 1)$, hence $\mu' \in (M')^{(N-c+2)(l-1)}$.

Now we check that $\mu' \in I^{((N-c+1)(l-1))}_{V_N}$. It suffices to check that $\deg_{\langle p_i \rangle}(\mu') \geqslant (N - c + 1)(l - 1)$ for each i, where $p_0, \ldots, p_N$ are the coordinate vertices. For all i we have

$$\deg_{\langle p_i \rangle}(\mu') = \deg_{\langle p_i \rangle}(\mu) - \left(N - c + 2 - \nu_i(N - c + 1)\right).$$

If $i \leqslant N - c + 1$, then $\nu_i(N - c + 1) = 1$ so using $\deg_{\langle p_i \rangle}(\mu) \geqslant (N - c + 1)l$ (which we have since $\mu \in I^{((N-c+1)l)}_{V_N}$) we obtain

$$\deg_{\langle p_i \rangle}(\mu) - \left(N - c + 2 - \nu_i(N - c + 1)\right) \geqslant (N - c + 1)l - (N - c + 1) = (N - c + 1)(l - 1).$$

If $i > N - c + 1$, then $\nu_i(N - c + 1) = 0$ and

$$\deg_{\langle p_i \rangle}(\mu') = \deg_{\langle p_i \rangle}(\mu) - \big(N - c + 2 - \nu_i(N - c + 1)\big)$$

$$= \deg(\mu) - m_i - (N - c + 2)$$

$$\geqslant (N - c + 2)l - m_i - (N - c + 2)$$

$$\geqslant (N - c + 2)l - (l - 1) - (N - c + 2)$$

$$= (N - c + 1)(l - 1),$$

where the fourth line uses the assumption that $m_{N-c+1} < l$.

Now we check that $\mu' \in I_{V_{N-1}}^{((N-c)(l-1))}$. Let p_{i_1} and p_{i_2} be arbitrary distinct coordinate vertices, and assume $i_1 < i_2$. It suffices to check that $\deg_{\langle p_{i_1}, p_{i_2} \rangle}(\mu') \geqslant (N - c)(l - 1)$. For all i we have

$$\deg_{\langle p_{i_1}, p_{i_2} \rangle}(\mu') = \deg_{\langle p_{i_1}, p_{i_2} \rangle}(\mu) - \big(N - c + 2 - \nu_{i_1, i_2}(N - c + 1)\big).$$

If $i_2 \leqslant N - c + 1$, then $\nu_{i_1, i_2}(N - c + 1) = 2$ so using $\deg_{\langle p_{i_1}, p_{i_2} \rangle}(\mu) \geqslant (N - c)l$ we have

$$\deg_{\langle p_{i_1}, p_{i_2} \rangle}(\mu) - \big(N - c + 2 - \nu_{i_1, i_2}(N - c + 1)\big) \geqslant (N - c)l - (N - c) = (N - c)(l - 1).$$

If $i_1 \leqslant N - c + 1 < i_2$, then $\nu_{i_1, i_2}(N - c + 1) = 1$ so using $\deg_{\langle p_{i_1}, p_{i_2} \rangle}(\mu) = \deg_{\langle p_{i_1} \rangle}(\mu) - m_{i_2} \geqslant (N - c + 1)l - m_{i_2} \geqslant (N - c + 1)l - (l - 1)$ gives

$$\deg_{\langle p_{i_1}, p_{i_2} \rangle}(\mu) - \big(N - c + 2 - \nu_{i_1, i_2}(N - c + 1)\big) \geqslant (N - c + 1)l - (l - 1) - (N - c + 1)$$

$$= (N - c)(l - 1).$$

If $N - c + 1 < i_1$, then $\nu_{i_1, i_2}(N - c + 1) = 0$ so using $\deg_{\langle p_{i_1}, p_{i_2} \rangle}(\mu) = \deg(\mu) - m_{i_1} - m_{i_2} \geqslant (N - c + 2)l - 2(l - 1) = (N - c)l + 2$ gives

$$\deg_{\langle p_{i_1}, p_{i_2} \rangle}(\mu) - \big(N - c + 2 - \nu_{i_1, i_2}(N - c + 1)\big) \geqslant (N - c)l + 2 - (N - c + 2) = (N - c)(l - 1).$$

Now we must check that $\mu' \in I_{V_{N-2}}^{((N-c-1)(l-1))}$, and then $\mu' \in I_{V_{N-3}}^{((N-c-2)(l-1))}$, etc., but the argument follows the same pattern of checking cases depending on how many of the indices of $\langle p_{i_1}, \ldots, p_{i_r} \rangle$ are less than or equal to $N - c + 1$, and each case is verified in the same way as indicated above. $\square$

We can partially extend this to the non-monomial case. Given a homogeneous ideal J in a polynomial ring, we denote the saturation of J by $\mathrm{sat}(J)$, meaning the intersection of the primary components of J excluding the component primary to the irrelevant ideal (if there is one).

Corollary 4.9. *Let $I \subset k[\mathbb{P}^n] = R$ be the ideal of $V_c = V_c(\mathcal{H}, \mathbb{P}^n)$ where $\mathcal{H}$ consists of $s > n$ hyperplanes $H_1, \ldots, H_s$ meeting properly where M is the irrelevant ideal, and let $l \geqslant 1$. Then*

$$\mathrm{sat}(I^l) = I_{V_c}^{(l)} \cap I_{V_{c+1}}^{(2l)} \cap \cdots \cap I_{V_n}^{((n-c+1)l)}.$$

Proof. Since $I^l \subseteq I_{V_c}^{(l)} \cap I_{V_{c+1}}^{(2l)} \cap \cdots \cap I_{V_n}^{((n-c+1)l)}$ by Remark 4.2 but the latter is saturated, we at least have $\mathrm{sat}(I^l) \subseteq I_{V_c}^{(l)} \cap I_{V_{c+1}}^{(2l)} \cap \cdots \cap I_{V_n}^{((n-c+1)l)}$. Since I^l is homogeneous, the associated primes and their primary components are homogeneous also [28, Theorem 9, p. 153]. Thus to show equality it suffices to show equality after localizing for every prime ideal of the form I_p for $p \in V_c$. But after such a localization, every hyperplane H_i not passing through p becomes a unit and hence $I^l R_{I_p}$ is generated

by monomials in the linear forms L_j for all H_j passing through p. Say that these H_j are $H_{j_0}, \ldots, H_{j_r}$ and pick any other $n - r$ of the hyperplanes H_i to obtain $H_{j_0}, \ldots, H_{j_n}$. After a change of coordinates we may assume $H_{j_i} = x_i$ for $i = 0, \ldots, n$. Let $\mathcal{H}' = \{x_0, \ldots, x_n\}$, let $V_i' = V_i(\mathcal{H}', \mathbb{P}^n)$ for all i and let $J = I_{V_c'}$. Clearly $p \in V_c' \subseteq V_c$ and $I^l R_{I_p} = J^l R_{I_p}$. We know the primary decomposition of J^l and hence of $J^l R_{I_p}$ from Theorem 4.8; i.e., we have

$$
\begin{aligned}
I^l R_{I_p} &= J^l R_{I_p} \\
&= \left(I_{V_c'}^{(l)} \cap I_{V_{c+1}'}^{(2l)} \cap \cdots \cap I_{V_n'}^{((n-c+1)l)} \cap M^{(n-c+2)l} \right) R_{I_p} \\
&= I_{V_c'}^{(l)} R_{I_p} \cap I_{V_{c+1}'}^{(2l)} R_{I_p} \cap \cdots \cap I_{V_n'}^{((n-c+1)l)} R_{I_p} \\
&= I_{V_c}^{(l)} R_{I_p} \cap I_{V_{c+1}}^{(2l)} R_{I_p} \cap \cdots \cap I_{V_n}^{((n-c+1)l)} R_{I_p} \\
&= \left(I_{V_c}^{(l)} \cap I_{V_{c+1}}^{(2l)} \cap \cdots \cap I_{V_n}^{((n-c+1)l)} \right) R_{I_p}.
\end{aligned}
$$

Since this holds for all $p \in V_c$, we have

$$
\mathrm{sat}(I^l) = I_{V_c}^{(l)} \cap I_{V_{c+1}}^{(2l)} \cap \cdots \cap I_{V_n}^{((n-c+1)l)},
$$

as claimed. $\quad\square$

As an application we apply our results to compute the resurgence for certain subschemes. We first recall the definition of the resurgence [3]. The point of the resurgence is to provide an asymptotic measure of how far symbolic powers deviate from ordinary powers of the same ideal. This is not interesting in the case of an ideal I if $I = (0)$ or $I = (1)$, so we do not define the resurgence in those cases.

Definition 4.10. Let $(0) \neq I \subsetneq k[\mathbb{P}^n]$ be a homogeneous ideal. The *resurgence* of I, denoted $\rho(I)$ is

$$
\sup \left\{ \frac{m}{r} : I^{(m)} \not\subseteq I^r \right\}.
$$

We always have $1 \leqslant \rho(I) \leqslant n$. (Since $\alpha(I^{(m)}) \leqslant \alpha(I^m) = m\alpha(I)$ and $r\alpha(I) = \alpha(I^r)$, we see that $m < r$ implies $\alpha(I^{(m)}) < \alpha(I^r)$ and hence $I^{(m)} \not\subseteq I^r$. It follows that $1 \leqslant \rho(I)$, and by applying the main result of [18] we know that $\rho(I) \leqslant n$.) However, it is in general quite difficult to compute the resurgence exactly, and only a few cases have been done, so finding methods to provide exact determinations in additional cases is of substantial interest. When I is the ideal of a complete intersection, $I^{(m)} = I^m$ for all $m \geqslant 0$ [28, Lemma 5, Appendix 6] so $\rho(I) = 1$. Moreover, $\rho(I) = \rho(I')$ if $I \subset k[\mathbb{P}^n] \subseteq k[\mathbb{P}^N]$ with $I' = Ik[\mathbb{P}^N]$ [3, Proposition 2.5.1(a)], so if the resurgence is known for a subscheme it is known for projective cones over the subscheme. Some exact values of $\rho(I)$ are known when I defines a 0-dimensional subscheme [3,4,9]. For example, if $\mathcal{H}$ consists of $s > N$ hyperplanes in $\mathbb{P}^N$ meeting properly, then [3] shows that

$$
\frac{c(s - c + 1)}{s} \leqslant \rho(I_{V_c(\mathcal{H}, \mathbb{P}^N)})
$$

with equality in case $c = N$. Thus when $s = N + 1$ we have $\frac{c(N-c+2)}{N+1} \leqslant \rho(I_{V_c(\mathcal{H}, \mathbb{P}^N)})$ with $\rho(I_{V_N(\mathcal{H}, \mathbb{P}^N)}) = 2N/(N+1)$ when $c = N$. We will show equality also holds when $c = N - 1$, giving $\rho(I_{V_N(\mathcal{H}, \mathbb{P}^N)}) = 3(N-1)/(N+1)$. The only exact determinations up to now for subschemes which are not complete intersections nor are 0-dimensional nor are cones over such and for which the resurgence is bigger than 1 are for certain smooth unions of lines in projective space [14].

Theorem 4.11. *Let $N \geqslant 3$ and let $I \subset k[\mathbb{P}^N]$ be the ideal of $V_{N-1} = V_{N-1}(\mathcal{H}', \mathbb{P}^N)$ where $\mathcal{H}'$ consists of the $N + 1$ coordinate hyperplanes, which we denote as $H_1, \ldots, H_{N+1}$. Then*

$$\rho(I) = \frac{3(N-1)}{N+1}.$$

Moreover, given $m, r \geqslant 1$, we have $I^{(m)} \not\subseteq I^r$ if and only if

$$\frac{m}{r} < \left(3 - \frac{2N-4}{(N-1)r}\right)\frac{N-1}{N+1}.$$

Proof. Assume $k[\mathbb{P}^N] = k[x_0, \ldots, x_N]$. Let M be the irrelevant ideal and let $J = I_{V_N}$. By Theorem 4.8

$$I^r = I_{V_{N-1}}^{(r)} \cap I_{V_N}^{(2r)} \cap M^{3r} = I^{(r)} \cap J^{(2r)} \cap M^{3r}.$$

Thus $I^{(m)}$ fails to be contained in I^r if and only if either

$$I^{(m)} \not\subseteq I^{(r)}, \qquad I^{(m)} \not\subseteq J^{(2r)} \quad \text{or} \quad I^{(m)} \not\subseteq M^{3r},$$

so

$$\rho(I) = \max\bigl(\sup\{m/r\colon I^{(m)} \not\subseteq I^{(r)}\}, \sup\{m/r\colon I^{(m)} \not\subseteq J^{(2r)}\}, \sup\{m/r\colon I^{(m)} \not\subseteq M^{3r}\}\bigr).$$

Since $I^{(m)} \not\subseteq I^{(r)}$ if and only if $m < r$, we have $\sup\{m/r\colon I^{(m)} \not\subseteq I^{(r)}\} \leqslant 1$.

Next, $I^{(m)} \not\subseteq J^{(2r)}$ if and only if there is a monomial $x_0^{m_0} \cdots x_N^{m_N}$ in $I^{(m)}$ but not in $J^{(2r)}$. After a permutation of the indices if need be, this condition is equivalent to there being exponents $m_0 \geqslant \cdots \geqslant m_N$ such that $m_2 + \cdots + m_N \geqslant m$ but $m_1 + \cdots + m_N < 2r$. Let $q = \lfloor m/(N-1) \rfloor$ and $\sigma = m - (N-1)q$ so $m = (N-1)q + \sigma$ and $0 \leqslant \sigma < N-1$. Let $m_0' = m_1' = m_2'$, and if $\sigma = 0$, let $m_2' = \cdots = m_N' = q$, while if $\sigma > 0$, let $m_2' = \cdots = m_{\sigma+1}' = q+1$, and $m_{\sigma+2}' = \cdots = m_N' = q$. Note that $m_2 \geqslant \lceil (m_2 + \cdots + m_N)/(N-1) \rceil \geqslant \lceil m/(N-1) \rceil$ so $m_2 \geqslant m_2'$, hence $m_1 \geqslant m_2 \geqslant m_2' = m_1'$. Then $m_0' \geqslant \cdots \geqslant m_N'$ with $m_2' + \cdots + m_N' = (N-1)q + \sigma = m$ and $m_1' + \cdots + m_N' = m_1' + m_2' + \cdots + m_N' = m_2' + m \leqslant m_1 + m_2 + \cdots + m_N < 2r$. Thus $\mu' = x_0^{m_0'} \cdots x_N^{m_N'} \in I^{(m)} \setminus J^{(2r)}$, and we have $m_0' - m_N' \leqslant 1$; in particular, each m_i' is either $\lceil m/(N-1) \rceil$ or $\lfloor m/(N-1) \rfloor$ (and necessarily $m_2' = \lceil m/(N-1) \rceil$ and $m_N' = \lfloor m/(N-1) \rfloor$). The condition that $m_1' + \cdots + m_N' < 2r$ can now be stated as $m + \lceil m/(N-1) \rceil < 2r$, or equivalently as $mN/(N-1) = m + m/(N-1) \leqslant 2r - 1$; i.e., $I^{(m)} \not\subseteq J^{(2r)}$ if and only if

$$\frac{m}{r} \leqslant (N-1)\frac{2 - \frac{1}{r}}{N}.$$

Thus $\sup\{m/r\colon I^{(m)} \not\subseteq J^{(2r)}\} \leqslant 2(N-1)/N$.

Finally, $I^{(m)} \not\subseteq M^{3r}$ if and only if $\alpha(I^{(m)}) < 3r$. By Proposition 4.5, $\alpha(I^{(m)}) = (q+1)(N+1) - (N-1) + r$, where $m = q(N-1) + \sigma$ for $1 \leqslant \sigma \leqslant N-1$. Note that

$$(q+1)(N+1) - (N-1) + \sigma = m + 2q + 2 = m + 2(m-\sigma)/(N-1) + 2$$

$$= m + 2 + 2\left\lfloor \frac{m-1}{N-1} \right\rfloor.$$

Thus $m + 2 + 2\lfloor\frac{m-1}{N-1}\rfloor = \alpha(I^{(m)}) < 3r$ holds if and only if $m + 2 + 2\frac{m-1}{N-1} < 3r$, which simplifies to

$$\frac{m}{r} < \left(3 - \frac{2N-4}{(N-1)r}\right)\frac{N-1}{N+1}.$$

The supremum of the right hand side over all values of $r \geqslant 1$ is $3\frac{N-1}{N+1}$. Since $3\frac{N-1}{N+1}$ is greater than either 1 or $2(N-1)/N$, we see that $\rho(I) \leqslant 3\frac{N-1}{N+1}$. To show that we actually have equality, let $m = 3(N-1)^2 t$ and let $r = (N^2 - 1)t + N - 1$. Then $m + 2 + 2\frac{m-1}{N-1} < 3r$ holds (it simplifies to $(3N-8)N + 7 > 0$), and

$$\frac{m}{r} = \frac{3(N-1)^2 t}{(N^2-1)t + N - 1} = \frac{3(N-1)}{N + 1 + \frac{1}{t}}$$

has supremum $3(N-1)/(N+1)$, taken over all $t \geqslant 1$.

We now have

$$\rho(I) = \max\left(\frac{3(N-1)}{N+1}, \frac{2(N-1)}{N}, 1\right) = \frac{3(N-1)}{N+1}.$$

We close by proving that $I^{(m)} \not\subseteq I^r$ if and only if

$$\frac{m}{r} < \left(3 - \frac{2N-4}{(N-1)r}\right)\frac{N-1}{N+1}.$$

From our work above we have $I^{(m)} \not\subseteq I^r$ if and only if either

$$\frac{m}{r} < 1 \quad \text{or} \quad \frac{m}{r} \leqslant \left(2 - \frac{1}{r}\right)\frac{N-1}{N} \quad \text{or} \quad \frac{m}{r} < \left(3 - \frac{2N-4}{(N-1)r}\right)\frac{N-1}{N+1}.$$

But $1 \leqslant (2 - \frac{1}{r})\frac{N-1}{N} < (3 - \frac{2N-4}{(N-1)r})\frac{N-1}{N+1}$ for $r \geqslant 2$, so the three inequalities are subsumed by the last one when $r \geqslant 2$, while when $r = 1$ it is enough to note that $(3 - \frac{2N-4}{(N-1)})\frac{N-1}{N+1} = 1$. $\quad\square$

One of the things our results suggest is that the nice properties of star configurations generally may derive from the nice behavior coming from stars configurations whose ideals are monomial ideals. As we have seen, a codimension c star $V_c(\mathbb{P}^n)$ coming from s hyperplanes in $\mathbb{P}^n$ is, as a point set, the intersection with an appropriate linear space $L \subset \mathbb{P}^N$ of dimension n of the codimension c star $V_c(\mathbb{P}^N)$ coming from the $N + 1$ coordinate hyperplanes in $\mathbb{P}^N$, where $N + 1 = s$. Thus it is reasonable to ask the following question.

Question 4.12. If $\mathcal{H}$ is a set of $s > n$ hyperplanes in $\mathbb{P}^n$ meeting properly and $\mathcal{H}'$ is the set of coordinate hyperplanes in $\mathbb{P}^N$ for $N = s - 1$, is it true that $\rho(I_{V_c(\mathcal{H},\mathbb{P}^n)}) = \rho(I_{V_c(\mathcal{H}',\mathbb{P}^N)})$?

We do not know the answer, but we at least have $\rho(I_{V_c(\mathcal{H},\mathbb{P}^n)}) \leqslant \rho(I_{V_c(\mathcal{H}',\mathbb{P}^N)})$. (This is because if $(I_{V_c(\mathcal{H}',\mathbb{P}^N)})^{(m)} \subseteq (I_{V_c(\mathcal{H}',\mathbb{P}^N)})^r$, then $(I_{V_c(\mathcal{H},\mathbb{P}^n)})^{(m)} \subseteq (I_{V_c(\mathcal{H},\mathbb{P}^n)})^r$, since by Theorem 3.1 and its proof we have $(I_{V_c(\mathcal{H},\mathbb{P}^n)})^{(m)} = ((I_{V_c(\mathcal{H}',\mathbb{P}^N)})^{(m)} + J)/J$ and $(I_{V_c(\mathcal{H},\mathbb{P}^n)})^r = ((I_{V_c(\mathcal{H}',\mathbb{P}^N)})^r + J)/J$, where J is an ideal generated by linear forms, these forms being the ones defining the linear space whose intersection with $V_c(\mathcal{H}',\mathbb{P}^N)$ gives $V_c(\mathcal{H},\mathbb{P}^n)$.) In addition, Theorem 4.11 shows that Question 4.12 is true when $c = n = N - 1$ and $s = n + 2$: using $\rho(I_{V_n(\mathbb{P}^n)}) = n(s - n + 1)/s$ [3], we have $\rho(I_{V_n(\mathbb{P}^n)}) = n(s - n + 1)/s = 3(N-1)/(N+1) = \rho(I_{V_{N-1}(\mathbb{P}^N)})$.

Acknowledgments

Geramita thanks the NSERC for research support, while Harbourne and Migliore thank the National Security Agency which sponsored this research under Grant/Cooperative agreement H98230-11-1-0139 for Harbourne and H98230-12-1-0204 for Migliore, and thus the United States Government is authorized to reproduce and distribute reprints notwithstanding any copyright notice. Migliore was also partially supported by a grant from the Simons Foundation (#208579). We also thank S. Tohǎneanu for bringing his work to our attention (discussed in Remark 2.14) and the referee for useful comments and suggestions.

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