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Journal of Algebra

www.elsevier.com/locate/jalgebra



A characterization of cycle-finite generalized double tilted algebras



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ARTICLE INFO

Article history:

Received 16 December 2013

Available online xxxx

Communicated by Changchang Xi

MSC:

16G10

16G60

16G70

Keywords:

Tilted algebra

Generalized double tilted algebra

Cycle-finite algebra

Short chain

Auslander–Reiten quiver

ABSTRACT

We provide a new characterization of cycle-finite generalized double tilted algebras by the existence of a faithful module which is the middle of at most finitely many short chains.

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1. Introduction and the main results

Throughout the paper, by an algebra we mean a basic indecomposable artin algebra over a commutative artin ring K . For an algebra A , we denote by $\text{mod } A$ the category of finitely generated right A -modules, by $\text{ind } A$ the full subcategory of $\text{mod } A$ formed by the indecomposable modules, and by D the standard duality $\text{Hom}_K(-, J)$ on $\text{mod } A$, where

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J is a minimal injective cogenerator in $\text{mod } K$. The Jacobson radical rad_A of $\text{mod } A$ is the (two-sided) ideal generated by all noninvertible homomorphisms between modules in $\text{ind } A$, and the infinite Jacobson radical rad_A^∞ of $\text{mod } A$ is the intersection of all powers rad_A^i , $i \geq 1$, of rad_A . By a result due to M. Auslander [5] (see also [19] for an alternative proof), $\text{rad}_A^\infty = 0$ if and only if A is of finite representation type, that is, $\text{ind } A$ admits only a finite number of pairwise nonisomorphic modules. On the other hand, if A is of infinite representation type, then $(\text{rad}_A^\infty)^2 \neq 0$, by a result proved in [14]. A module M in $\text{mod } A$ is said to be sincere if every simple right A -module occurs as a composition factor of M , and faithful if its annihilator $\text{ann}_A(M) = \{a \in A \mid Ma = 0\}$ vanishes. For a module X in $\text{mod } A$ and its minimal projective presentation $P_1 \xrightarrow{f} P_0 \longrightarrow X \longrightarrow 0$ in $\text{mod } A$, the transpose $\text{Tr } X$ of X is the cokernel of the map $\text{Hom}_A(f, A)$ in $\text{mod } A^{\text{op}}$, where A^{op} is the opposite algebra of A . The homological operator $\tau_A = D \text{Tr}$ on modules in $\text{mod } A$, called the Auslander–Reiten translation, is playing a fundamental role in the modern representation theory of algebras. Finally, following [10,16] a *tilted algebra* is an algebra of the form $\text{End}_H(T)$, where H is a hereditary algebra and T is a (multiplicity-free) tilting module in $\text{mod } H$, that is, $\text{Ext}_H^1(T, T) = 0$ and the number of pairwise nonisomorphic indecomposable direct summands of T is equal to the rank of the Grothendieck group $K_0(H)$ of H .

The tilted algebras have for a long time played a central role in the representation theory of algebras and attracted much attention. In particular, the following handy criterion for an algebra to be a tilted algebra has been established independently by S. Liu [23] and A. Skowroński [37]: an algebra A is a tilted algebra if and only if the Auslander–Reiten quiver Γ_A of A admits a connected component with a faithful section Δ such that $\text{Hom}_A(X, \tau_A Y) = 0$ for all modules X and Y in Δ . This was extended by I. Reiten and A. Skowroński to the double tilted algebras [30] and the generalized double tilted algebras [31], by defining double sections and multisections. The module category $\text{mod } A$ of a generalized double tilted algebra A is determined, up to finitely many indecomposable modules, by the module categories of (left and right) tilted algebras naturally associated with A (see [31, Section 3] for details). We also mention that, for a generalized double tilted algebra A , all but finitely many modules X in $\text{ind } A$ have projective dimension or injective dimension at most one, while A may be of an arbitrary (finite or infinite) global dimension. Note also that the class of generalized double tilted algebras was studied independently by I. Assem, F.U. Coelho, M. Lanzilotta and others under the name of strict lura algebras (see [1,11–13], for some results).

It has been proved by A. Jaworska, P. Malicki and A. Skowroński in [18] that an algebra A is a tilted algebra if and only if there exists a sincere module M in $\text{mod } A$ such that, for any module X in $\text{ind } A$, we have $\text{Hom}_A(X, M) = 0$ or $\text{Hom}_A(M, \tau_A X) = 0$. Recall that, by [6,32], a sequence $X \rightarrow M \rightarrow \tau_A X$ of nonzero homomorphisms in a module category $\text{mod } A$ with X being indecomposable is called a short chain, and M the middle of this short chain. Therefore, an algebra A is a tilted algebra if and only if $\text{mod } A$ admits a sincere module M which is not the middle of a short chain (affirmative

answer for the question raised 20 years ago by I. Reiten, A. Skowroński and S.O. Smalø in [32, Section 3]). We note also that every sincere module M in $\text{mod } A$ which is not the middle of a short chain is a faithful module [32, Corollary 3.2]. Recently A. Skowroński conjectured (oral communication) that an algebra A is a generalized double tilted algebra if and only if $\text{mod } A$ admits a faithful module M which is the middle of at most finitely many short chains. For arbitrary algebra A , the necessity part of this equivalence follows from [31, Section 3] but the sufficiency part seems to be an exciting open problem.

The aim of the paper is to provide an affirmative solution of the above mentioned problem for *cycle-finite* algebras. Recall that, following C.M. Ringel [33], a *cycle* in the module category $\text{mod } A$ of an algebra A is a sequence

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_r} X_r = X_0$$

of nonzero nonisomorphisms in $\text{ind } A$, and such a cycle is said to be *finite* if f_1, \dots, f_r do not belong to rad_A^∞ (see [3,4]). It has been proved in [29] and [39] that the Auslander–Reiten quiver Γ_A of an algebra A admits at most finitely many τ_A -orbits containing indecomposable modules not lying on cycles in $\text{mod } A$ (directing modules in the sense of [33]). Hence, in order to obtain information on nondirecting indecomposable modules in the module category $\text{mod } A$, we may study properties of cycles in $\text{mod } A$ containing these modules. Following I. Assem and A. Skowroński [4], an algebra is said to be *cycle-finite* if all cycles in $\text{mod } A$ are finite. The class of cycle-finite algebras contains the following distinguished classes of algebras: the algebras of finite representation type, the tame tilted algebras [16,33], the tame double tilted algebras [30], the tame generalized double tilted algebras [31], the tubular algebras [33,34], the tame quasitilted algebras [20,43], the tame generalized multicoil algebras [28], and the strongly simply connected algebras of polynomial growth [42]. The representation theory of arbitrary cycle-finite algebras is still only emerging. We refer to [9,24–26,41,44] for some general results on the structure of module categories of cycle-finite algebras.

The following theorem is the main result of the paper.

Theorem 1. *Let A be a cycle-finite algebra. The following statements are equivalent.*

- (i) A is a generalized double tilted algebra.
- (ii) $\text{mod } A$ admits a faithful module M which is the middle of at most finitely many short chains.

We note that the implication (i) \Rightarrow (ii) of the above theorem holds in general (see Lemma 6.1).

In the recent paper [17], A. Jaworska, P. Malicki and A. Skowroński established a complete description of finitely generated modules over artin algebras not being the middle of short chains. The second main result of the paper describes the structure of modules over cycle-finite algebras being the middle of at most finitely many short chains.

Theorem 2. *Let A be a cycle-finite algebra and M be a module in $\text{mod } A$ which is the middle of at most finitely many short chains. Moreover, let B be the quotient algebra $A/\text{ann}_A(M)$, $B = B_1 \times \cdots \times B_m$ a decomposition of B into a product of indecomposable algebras, and $M = M_1 \oplus \cdots \oplus M_m$ the associated decomposition of M in $\text{mod } B$ with M_i a module in $\text{mod } B_i$, for any $i \in \{1, \dots, m\}$. Then the following statements hold.*

- (i) *For any $i \in \{1, \dots, m\}$, B_i is a cycle-finite generalized double tilted algebra.*
- (ii) *For any $i \in \{1, \dots, m\}$, M_i is a faithful module from the additive category $\text{add}(C_i)$ of a connecting component C_i of the Auslander–Reiten quiver Γ_{B_i} of B_i .*

We also mention that, for a generalized double tilted algebra B and a module M from the additive category $\text{add}(C)$ of a connecting component C of Γ_B , M is the middle of at most finitely many short chains in $\text{mod } B$ (see [Lemma 6.1](#) and its proof).

The main results of the paper were presented by the author during the conferences “Advances in Representation Theory of Algebras” (Toruń, September 2013) and “Perspectives of Representation Theory of Algebras” (Nagoya, November 2013).

For basic background on the representation theory applied in the article we refer to [\[2,7,35,36,46\]](#).

2. Preliminaries

In this section we recall basic notation and concepts on algebras and modules needed in the rest part of the paper.

For an algebra A and a complete set e_1, \dots, e_n of pairwise orthogonal primitive idempotents of A with $e_1 + \cdots + e_n = 1_A$, we have the following:

- $P_i = e_i A$, $i \in \{1, \dots, n\}$, is a complete set of pairwise nonisomorphic indecomposable projective modules in $\text{mod } A$;
- $I_i = D(Ae_i)$, $i \in \{1, \dots, n\}$, is a complete set of pairwise nonisomorphic indecomposable injective modules in $\text{mod } A$;
- $S_i = \text{top}(P_i) = e_i A / e_i \text{rad } A = \text{soc}(I_i)$, $i \in \{1, \dots, n\}$, is a complete set of pairwise nonisomorphic simple modules in $\text{mod } A$.

Additionally, for every $i \in \{1, \dots, n\}$, $F_i = \text{End}_A(S_i) \cong e_i A e_i / e_i (\text{rad } A) e_i$ is a division algebra. Further, for all $i, j \in \{1, \dots, n\}$, $e_i (\text{rad } A) e_j / e_i (\text{rad } A)^2 e_j$ has a natural $(F_i - F_j)$ -bimodule structure. Therefore, we may consider the quiver Q_A of A , which is the valued quiver defined in the following way.

- The vertices of Q_A are the indices $1, \dots, n$ of the chosen set e_1, \dots, e_n of primitive idempotents of A .
- For two vertices i and j in Q_A , there is an arrow $i \rightarrow j$ from i to j in Q_A if and only if $e_i (\text{rad } A) e_j / e_i (\text{rad } A)^2 e_j \neq 0$. Moreover, one associates to an arrow $i \rightarrow j$ in Q_A the valuation (d_{ij}, d'_{ij}) , so we have in Q_A the valued arrow

$$i \xrightarrow{(d_{ij}, d'_{ij})} j$$

where the valuations are $d_{ij} = \dim_{F_j} e_i(\text{rad } A)e_j / e_i(\text{rad } A)^2 e_j$ and $d'_{ij} = \dim_{F_i} e_i(\text{rad } A)e_j / e_i(\text{rad } A)^2 e_j$.

We will frequently identify an algebra A with the associated category A^* whose objects are the vertices of the quiver Q_A , $\text{Hom}_{A^*}(i, j) = e_j A e_i$, for any objects i and j of A^* , and the composition of morphisms in A^* is given by the multiplication in A . Moreover, for a module M in $\text{mod } A$, we define *the support* of M to be the full subcategory $\text{supp}(M)$ of A formed by all objects i with $M e_i \neq 0$. In general, for a family $C = (C_i)_{i \in I}$ of components of Γ_A , we denote by $\text{supp}(C)$ the full subcategory of A given by all objects i such that $X e_i \neq 0$ for an indecomposable module X in C , and call it *the support* of C . Further, a module M (respectively, a family of components C of Γ_A) is called *sincere* if $\text{supp}(M) = A$ (respectively, $\text{supp}(C) = A$). Finally, a full subcategory B of A is said to be *convex* if every path in Q_A with source and target in Q_B is contained entirely in Q_B . We note that, for a convex subcategory B of A , there exists a fully faithful embedding of $\text{mod } B$ into $\text{mod } A$ such that $\text{mod } B$ is the full subcategory of $\text{mod } A$ formed by all modules M with $M e_i = 0$, for all vertices i of Q_A which do not belong to Q_B .

Now, we introduce various types of components of the Auslander–Reiten quivers of algebras. Let A be an algebra and C be a component of Γ_A . Then C is called *regular* if C contains neither a projective module nor an injective module, and *semiregular* if C does not contain both a projective module and an injective module. It has been shown in [21] and [47] that a regular component C of Γ_A admits an oriented cycle if and only if C is a stable tube, that is, C is of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, for an integer $r \geq 1$. Moreover, S. Liu proved in [22] that a semiregular component C of Γ_A contains an oriented cycle if and only if C is a semiregular tube, that is, a ray tube or a coray tube. Recall that a ray tube (respectively, a coray tube) is a component of Γ_A obtained from a stable tube by a finite number (possibly zero) of ray (respectively, coray) insertions. Further, a component C of Γ_A is called *postprojective* (respectively, *preinjective*) if C does not contain an oriented cycle and every module in C lies in the τ_A -orbit of a projective (respectively, injective) module. We consider also the concept of a coherent translation quiver. Following [27], a full valued translation subquiver Γ of Γ_A is said to be *coherent* if the following conditions are satisfied:

- (C1) For each projective module P in Γ , there is an infinite sectional path $P = X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_i \rightarrow X_{i+1} \rightarrow \cdots$.
- (C2) For each injective module I in Γ , there is an infinite sectional path $\cdots \rightarrow Y_{j+1} \rightarrow Y_j \rightarrow \cdots \rightarrow Y_2 \rightarrow Y_1 = I$.

Furthermore, a component C of Γ_A is called *almost cyclic* if all but finitely many modules in C lie on oriented cycles in Γ_A . We note that the stable tubes, ray tubes and

coray tubes are semiregular, almost cyclic and coherent. Following A. Skowroński [38], a component \mathcal{C} of Γ_A is said to be *generalized standard* if $\text{rad}_A^\infty(X, Y) = 0$, for all modules X and Y from \mathcal{C} . Recall also that, by [38, Theorem 2.3], every generalized standard component \mathcal{C} of Γ_A is almost periodic, that is, all but finitely many τ_A -orbits in \mathcal{C} are periodic.

For a component \mathcal{C} of Γ_A , we consider the *left stable part* ${}_l\mathcal{C}$ of \mathcal{C} , obtained from \mathcal{C} by deleting in \mathcal{C} the τ_A -orbits containing projective modules, and the *right stable part* ${}_r\mathcal{C}$ of \mathcal{C} , obtained by deleting in \mathcal{C} the τ_A -orbits containing injective modules. The intersection ${}_l\mathcal{C} \cap {}_r\mathcal{C}$ is the *stable part* ${}_s\mathcal{C}$ of \mathcal{C} . Finally, we denote by ${}_c\Gamma_A$ the cyclic part of Γ_A , obtained by removing from Γ_A all acyclic modules and the arrows attached to them. The connected components of ${}_c\Gamma_A$ are called *cyclic components* of Γ_A .

Recall that a component \mathcal{C} of Γ_A is called postprojective (respectively, preinjective) component of Euclidean type if \mathcal{C} is postprojective (respectively, preinjective) and \mathcal{C} contains a section of Euclidean type. The following result on the shape of semiregular components of the Auslander–Reiten quivers of cycle-finite algebras has been proven in [41, Proposition 3.3].

Proposition 2.1. *Let A be a cycle-finite algebra and \mathcal{C} be a semiregular component of Γ_A . Then \mathcal{C} is generalized standard and \mathcal{C} is one of the following forms: a postprojective component of Euclidean type, a preinjective component of Euclidean type, a ray tube, or a coray tube.*

We end this preliminary section with some well-known results on the structure of the module category of triangular matrix algebras. Let A be an algebra of the following form

$$A = \begin{bmatrix} S & M \\ 0 & R \end{bmatrix},$$

where S and R are algebras, $M = {}_S M_R$ is an $(S-R)$ -bimodule, and the multiplication is induced from the multiplication of matrices with respect to the bimodule structure. Recall that the category $\text{mod } A$ can be identified with the category whose objects are triples $Y = (Y_0, Y_1, \varphi)$, with Y_1 a module in $\text{mod } R$, Y_0 a module in $\text{mod } S$, and $\varphi : Y_0 \rightarrow \text{Hom}_R(M, Y_1)$ an S -homomorphism, and a morphism $h : (Y_0, Y_1, \varphi) \rightarrow (X_0, X_1, \psi)$ of such a triples is a pair (h_0, h_1) of homomorphisms $h_0 : Y_0 \rightarrow X_0$ in $\text{mod } S$ and $h_1 : Y_1 \rightarrow X_1$ in $\text{mod } R$ such that $\text{Hom}_R(M, h_1)\varphi = \psi h_0$.

Lemma 2.2. *Let R and S be algebras, M an $(S-R)$ -bimodule, $A = \begin{bmatrix} S & M \\ 0 & R \end{bmatrix}$ the matrix algebra defined by the bimodule ${}_S M_R$, and Y be a module in $\text{mod } A$ represented by a triple (Y_0, Y_1, φ) with $\varphi \neq 0$. Then, for every indecomposable direct summand Z of the R -module Y_1 , we have $\text{Hom}_R(M, Z) \neq 0$.*

Proof. The claim follows immediately from the arguments presented in the proof of [36, Lemma XV.1.8]. \square

We recall also the following well-known lemma (see [36, Corollary XV.1.7] or [45, Lemma 5.6]) on almost split sequences in the module category of a triangular matrix algebras.

Lemma 2.3. *Let R and S be algebras, M an S – R -bimodule and $A = \begin{bmatrix} S & M \\ 0 & R \end{bmatrix}$ be the triangular matrix algebra defined by the bimodule ${}_S M_R$. Then an almost split sequence in $\text{mod } R$*

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

is an almost split sequence in $\text{mod } A$ if and only if $\text{Hom}_R(M, X) = 0$.

3. Modules being the middle of at most finitely many short chains

We investigate here structural properties of algebras A such that $\text{mod } A$ admits a faithful module being the middle of at most finitely many short chains.

First, we prove the following lemma.

Lemma 3.1. *Let A be an algebra such that there exists a faithful module M in $\text{mod } A$ being the middle of at most finitely many short chains. Then the following statements hold.*

- (i) $\text{Hom}_A(M, \mathcal{T}) = 0$, for every ray tube \mathcal{T} of Γ_A containing a projective module;
- (ii) $\text{Hom}_A(\mathcal{T}, M) = 0$, for every coray tube \mathcal{T} of Γ_A containing an injective module.

Proof. We prove only (i). The proof of (ii) is dual. Suppose to the contrary that $\text{Hom}_A(M, X) \neq 0$ for an indecomposable module X in a ray tube \mathcal{T} of Γ_A containing a projective module P . Denote by Σ the sectional path in \mathcal{T} from infinity to P and by Ω the sectional path in \mathcal{T} from X to infinity. Because irreducible homomorphisms attached to arrows of Ω are monomorphisms and $\text{Hom}_A(M, X) \neq 0$, we deduce that $\text{Hom}_A(M, Y) \neq 0$, for every module Y lying on Ω . By [8], for every module Z lying on Σ , we have $\text{Hom}_A(Z, P) \neq 0$. Moreover, since M is a faithful module, there is a composed monomorphism $P \rightarrow A_A \rightarrow M^d$, for some integer $d \geq 1$. Consequently, $\text{Hom}_A(Z, M) \neq 0$ for every module Z lying on Σ . Finally, observe that Σ intersects $\tau_A^{-1}\Omega$ infinitely many times, and hence there are infinitely many pairwise nonisomorphic indecomposable modules Z with $\text{Hom}_A(Z, M) \neq 0$ and $\text{Hom}_A(M, \tau_A Z) \neq 0$, a contradiction, because M is assumed to be the middle of at most finitely many short chains. \square

Now, we prove the following lemma.

Lemma 3.2. *Let A be an algebra, M a module in $\text{mod } A$ which is the middle of at most finitely many short chains, and \mathcal{T} a stable tube of Γ_A . Then, for every two indecomposable direct summands N and N' of M , we have*

$$\text{Hom}_A(N, \mathcal{T}) = 0 \quad \text{or} \quad \text{Hom}_A(\mathcal{T}, N') = 0.$$

Proof. Let N and N' be indecomposable direct summands of M . Suppose to the contrary that $\text{Hom}_A(N, X) \neq 0$ and $\text{Hom}_A(Y, N') \neq 0$, for some indecomposable modules X and Y in \mathcal{T} . Then there are infinite sectional paths in \mathcal{T} of the forms

$$\Sigma: X = X_0 \rightarrow X_1 \rightarrow \cdots \quad \text{and} \quad \Omega: \cdots \rightarrow Y_1 \rightarrow Y_0 = Y.$$

Since irreducible homomorphisms attached to arrows of Σ (respectively, Ω) are monomorphisms (respectively, epimorphisms), we have $\text{Hom}_A(N, X_k) \neq 0$ and $\text{Hom}_A(Y_k, N') \neq 0$, for all $k \geq 0$. Moreover, the paths $\tau_A^{-1}\Sigma$ and Ω intersect infinitely many times. Hence there are infinitely many pairwise nonisomorphic indecomposable modules Z in \mathcal{T} such that $\text{Hom}_A(N, \tau_A Z) \neq 0$ and $\text{Hom}_A(Z, N') \neq 0$. In particular, M is the middle of infinitely many short chains, a contradiction. \square

Corollary 3.3. *Let A be an algebra and M be a faithful module in $\text{mod } A$ being the middle of at most finitely many short chains. Then no indecomposable direct summand of M is lying in a stable tube of Γ_A .*

Finally, we prove the following proposition, which will be of essential use in the proof of the main theorem (see Section 6).

Proposition 3.4. *Let A be a cycle-finite algebra such that there exists a faithful module M in $\text{mod } A$ being the middle of at most finitely many short chains. Then every infinite cyclic component \mathcal{D} of Γ_A is the cyclic part ${}_c\mathcal{C}$ of a stable tube or a semiregular tube \mathcal{C} of Γ_A .*

Proof. Let \mathcal{D} be an infinite cyclic component of Γ_A and \mathcal{C} be the component of Γ_A such that \mathcal{D} is contained in the cyclic part ${}_c\mathcal{C}$ of \mathcal{C} . Because \mathcal{D} is infinite, we deduce from [26, Corollary 2.8] that ${}_l\mathcal{C}$ or ${}_r\mathcal{C}$ contains a connected component Γ containing an oriented cycle and infinitely many modules of \mathcal{D} . Our goal is to prove that \mathcal{C} is a semiregular tube. We shall proceed in three cases.

(1) First, assume that Γ is contained in the stable part ${}_s\mathcal{C}$ of \mathcal{C} . Then, applying the main result of [47], we conclude that Γ is a stable tube. We shall prove that $\Gamma = \mathcal{C}$. Suppose that $\Gamma \neq \mathcal{C}$. Then there is a finite τ_A -orbit $P, \tau_A^{-1}P, \dots, \tau_A^{-r}P = I$ in \mathcal{C} with $r \geq 0$, P a projective module, I an injective module, and such that the following conditions are satisfied:

- an immediate predecessor X of P belongs to Γ ;
- an immediate successor Y of I belongs to Γ .

Observe also that there are infinite sectional paths in Γ of the forms

$$\begin{aligned}\Sigma: \quad & \cdots \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 = X \\ \Omega: \quad & Y = Y_0 \rightarrow Y_1 \rightarrow \cdots \rightarrow Y_n \rightarrow \cdots\end{aligned}$$

Therefore, using the main result of [8], we infer that $\text{Hom}_A(X_n, X) \neq 0$ and $\text{Hom}_A(Y, Y_n) \neq 0$, for all integers $n \geq 0$. Moreover, since M is a faithful module, there is a composed monomorphism $X \rightarrow P \rightarrow A_A \rightarrow M^r$ in $\text{mod } A$ and a composed epimorphism $M^s \rightarrow D(A)_A \rightarrow I \rightarrow Y$ in $\text{mod } A$ with $r, s \geq 0$. Consequently, we conclude that $\text{Hom}_A(X_n, M) \neq 0$ and $\text{Hom}_A(M, Y_n) \neq 0$, for every $n \geq 0$. Furthermore, observe that the path Σ intersects the path $\tau_A^{-1}\Omega$ infinitely many times, hence there are infinitely many pairwise nonisomorphic indecomposable modules Z_k , $k \geq 0$, in Γ such that $\text{Hom}_A(Z_k, M) \neq 0$ and $\text{Hom}_A(M, \tau_A Z_k) \neq 0$, for every $k \geq 0$. But then M is the middle of infinitely many short chains, a contradiction. Thus indeed $\mathcal{C} = \Gamma$ is a stable tube of Γ_A and $\mathcal{D} = {}_c\mathcal{C} = \mathcal{C}$.

(2) Now, assume that Γ is contained in ${}_l\mathcal{C}$ and Γ admits at least one injective module. Then, applying [22, (2.2) and (2.3)], we infer that there is an infinite sectional path

$$\cdots \rightarrow \tau_A^{2r}X_1 \rightarrow \tau_A^rX_s \rightarrow \cdots \rightarrow \tau_A^rX_1 \rightarrow X_s \rightarrow \cdots \rightarrow X_2 \rightarrow X_1,$$

in Γ , where $r > s \geq 1$, X_i is an injective module for some $i \in \{1, \dots, s\}$ and each module in Γ is contained in the τ_A -orbit of one of the modules X_1, \dots, X_s . In particular, there is a nonnegative integer t such that $\tau_A^{-t}X_s = I$ is an injective module in Γ and Γ contains an infinite sectional path of the form

$$\Omega: \quad I = Y_0 \rightarrow Y_1 \rightarrow \cdots \rightarrow Y_n \rightarrow \cdots$$

Moreover, since M is a faithful module, we obtain $\text{Hom}_A(M, Y_n) \neq 0$, for every $n \geq 0$. Therefore, for every module Z_k of the form $Z_k = \tau_A^{-1}Y_k$, $k \geq 1$, we have $\text{Hom}_A(M, \tau_A Z_k) \neq 0$.

Consider now the full valued translation subquiver Γ^* of Γ given by all modules which are targets of infinite sectional paths in Γ . We prove first that Γ^* does not admit an immediate predecessor X of a projective module P in \mathcal{C} . Suppose that this is not the case. Then there exists an infinite sectional path in Γ of the form

$$\Sigma: \quad \cdots \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 = X,$$

where X is a direct summand of the radical $\text{rad } P$ of a projective module P in \mathcal{C} . Again, using [8] and the fact that M is faithful, we deduce that $\text{Hom}_A(X_n, M) \neq 0$, for

all $n \geq 0$. Therefore, because the paths Σ and $\tau_A^{-1}\Omega$ intersect infinitely many times, we conclude that there are infinitely many pairwise nonisomorphic indecomposable modules Z_k in Γ , $k \geq 0$, such that $\text{Hom}_A(Z_k, M) \neq 0$ and $\text{Hom}_A(M, \tau_A Z_k) \neq 0$. This leads to a contradiction with our assumption imposed on M . Hence, indeed no module in Γ^* is an immediate predecessor of a projective module in \mathcal{C} . Moreover, it follows that Γ^* is a left stable full translation subquiver of \mathcal{C} , closed under predecessors. In particular, because Γ^* contains infinitely many modules of \mathcal{D} and \mathcal{D} is a component of ${}_c\Gamma_A$, we conclude from [26, Proposition 2.8] that ${}_c\Gamma^*$ contains all modules of \mathcal{D} , and hence \mathcal{D} is the cyclic part ${}_c\Gamma^*$ of Γ^* .

Finally, observe that Γ^* is a maximal almost cyclic and coherent full translation subquiver of \mathcal{C} . Further, Γ^* does not contain projective modules. Hence, using [27, Theorem A], we infer that Γ^* is a full (valued) translation quiver obtained from a stable tube by an iterated application of admissible operation of type $(ad1^*)$. Thus, because Γ^* does not admit an immediate predecessor of a projective module in \mathcal{C} , we conclude that $\Gamma^* = \mathcal{C}$ is a coray tube (containing at least one injective module). Obviously, then $\mathcal{D} = {}_c\mathcal{C}$.

(3) Applying dual arguments, we prove that, if Γ is a component of ${}_r\mathcal{C}$ containing at least one projective module, then \mathcal{C} is a ray tube and \mathcal{D} is the cyclic part ${}_c\mathcal{C}$ of \mathcal{C} . \square

4. Tilted and generalized double tilted algebras

In this section we recall the structure of the Auslander–Reiten quivers of representation-infinite tilted algebras of Euclidean type and generalized double tilted algebras.

By a tilted algebra of Euclidean type we mean an algebra $B = \text{End}_H(T)$, where H is a hereditary (artin) algebra of one of the Euclidean types $\tilde{A}_{11}, \tilde{A}_{12}, \tilde{A}_m, \tilde{B}_m, \tilde{C}_m, \tilde{BC}_m, \tilde{BD}_m, \tilde{CD}_m, \tilde{D}_m, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{F}_{41}, \tilde{F}_{42}, \tilde{G}_{21}, \tilde{G}_{22}$ (see [15]) and T is a (multiplicity-free) tilting module in $\text{mod } H$. Moreover, if T is contained in the additive category of the postprojective component of Γ_H , then B is called a *tame concealed* algebra. Recall that, for a representation-infinite tilted algebra of Euclidean type, one of the following statements holds.

- (1) B is a domestic tubular (branch) extension of a tame concealed algebra C and

$$\Gamma_B = \mathcal{P}^B \cup \mathcal{T}^B \cup Q^B,$$

where $\mathcal{P}^B = \mathcal{P}^C$ is the postprojective component of Γ_C , \mathcal{T}^B is an infinite family of pairwise orthogonal generalized standard ray tubes, obtained from the family \mathcal{T}^C of stable tubes of Γ_C by ray insertions, Q^B is a preinjective component containing all indecomposable injective B -modules, and \mathcal{T}^B strongly separates \mathcal{P}^B from Q^B ;

- (2) B is a domestic tubular (branch) coextension of a tame concealed algebra C and

$$\Gamma_B = \mathcal{P}^B \cup \mathcal{T}^B \cup Q^B,$$

where \mathcal{P}^B is a postprojective component containing all indecomposable projective B -modules, \mathcal{T}^B is an infinite family of pairwise orthogonal generalized standard coray tubes, obtained from the family \mathcal{T}^C of stable tubes of Γ_C by coray insertions, $Q^B = Q^C$ is the preinjective component of Γ_C , and \mathcal{T}^B strongly separates \mathcal{P}^B from Q^B .

Note that, if B is a tame concealed algebra, then the family \mathcal{T}^B consists only of stable tubes.

For a tilted algebra $B = \text{End}_H(T)$, there are associated torsion pairs $(\mathcal{T}(T), \mathcal{F}(T))$ in $\text{mod } H$ and $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\text{mod } B$, where

- $\mathcal{T}(T) = \{M \in \text{mod } A; \text{Ext}_A^1(T, M) = 0\}$ and $\mathcal{F}(T) = \{N \in \text{mod } A; \text{Hom}_A(T, N) = 0\}$,
- $\mathcal{X}(T) = \{X \in \text{mod } B; X \otimes_B T = 0\}$, and $\mathcal{Y}(T) = \{Y \in \text{mod } B; \text{Tor}_1^B(Y, T) = 0\}$.

Recall also that, by the Brenner–Butler theorem, the functors

$$\text{Hom}_H(T, -) : \text{mod } H \rightarrow \text{mod } B \quad \text{and} \quad \text{Ext}_H^1(T, -) : \text{mod } H \rightarrow \text{mod } B$$

induce equivalences of categories

$$\text{Hom}_H(T, -) : \mathcal{T}(T) \rightarrow \mathcal{Y}(T) \quad \text{and} \quad \text{Ext}_H^1(T, -) : \mathcal{F}(T) \rightarrow \mathcal{X}(T)$$

(see [2, Theorem VI.3.8]). Moreover, it follows from [2, Theorem VI.5.2 and Theorem VI.5.6] that every almost split sequence in $\text{mod } B$ is contained entirely in $\mathcal{Y}(T)$ or entirely in $\mathcal{X}(T)$, or it is a connecting sequence, that is, an almost split sequence in $\text{mod } B$ of the form

$$0 \rightarrow \text{Hom}_H(T, I) \rightarrow E \rightarrow \text{Ext}_H^1(T, P) \rightarrow 0,$$

where $E = \text{Hom}_H(T, I/\text{soc } I) \oplus \text{Ext}_H^1(T, \text{rad } P)$ for an indecomposable projective module P not in $\text{add}(T)$ and an indecomposable injective module I with $\text{soc } I = \text{top } P$.

Finally, we briefly touch upon the concept of generalized double tilted algebras, introduced by I. Reiten and A. Skowroński [31]. Let \mathcal{C} be a component of Γ_A . Following [31, Section 2], a full connected valued subquiver Δ of \mathcal{C} is called a *multisection* if and only if the following conditions are satisfied:

- Δ is almost acyclic;
- Δ is convex in \mathcal{C} ;
- for each τ_A -orbit \mathcal{O} in \mathcal{C} , we have $1 \leq |\Delta \cap \mathcal{O}| < \infty$;
- for all but finitely many τ_A -orbits \mathcal{O} in \mathcal{C} , we have $|\Delta \cap \mathcal{O}| = 1$;
- no proper full valued subquiver of Δ satisfies the conditions (a)–(d).

Note that the concept of a multisection is a generalization of the concept of a section (see [2, Section VIII.1]). Following [31], for a multisection Δ of a component C in Γ_A , we consider the valued subquivers Δ_l , Δ_r and Δ_c of C defined as follows:

- $\Delta_r = (\Delta \setminus \Delta'_l) \cup \tau_A^{-1} \Delta''_l$, where Δ'_l is the full valued subquiver of Δ consisting of all modules $X \in \Delta$ such that there is a nonsectional path $X \rightarrow \cdots \rightarrow P$ with P a projective module, and $\Delta''_l = \{X \in \Delta'_l; \tau_A^{-1} X \notin \Delta'_l\}$;
- $\Delta_l = (\Delta \setminus \Delta'_r) \cup \tau_A \Delta''_r$, where Δ'_r is the full valued subquiver of Δ consisting of all modules $X \in \Delta$ such that there is a nonsectional path $I \rightarrow \cdots \rightarrow X$ with I an injective module, and $\Delta''_r = \{X \in \Delta'_r; \tau_A X \notin \Delta'_r\}$;
- $\Delta_c = \Delta'_l \cap \Delta'_r$.

Moreover, it has been proved in [31, Theorem 2.5] that a component C of Γ_A is almost acyclic (that is, C has at most finitely many modules lying on cycles in C) if and only if C admits a multisection.

Following [31, Section 3], an algebra A is said to be a *generalized double tilted algebra* provided the following three conditions hold.

- (1) Γ_A admits a component C with a faithful multisection Δ .
- (2) There exists a tilted factor algebra A_l of A (not necessarily connected) such that Δ_l is a disjoint union of sections of connecting components of the indecomposable parts of A_l and the category of all predecessors of Δ_l in $\text{ind } A$ coincides with the category of all predecessors of Δ_l in $\text{ind } A_l$.
- (3) There exists a tilted factor algebra A_r of A (not necessarily connected) such that Δ_r is a disjoint union of sections of connecting components of the indecomposable parts of A_r and the category of all successors of Δ_r in $\text{ind } A$ coincides with the category of all successors of Δ_r in $\text{ind } A_r$.

The algebras A_l and A_r are called the *left tilted algebra* and the *right tilted algebra* of A , respectively. We have the following characterization of generalized double tilted algebras established in [31, Theorem 3.1].

Theorem 4.1. *Let A be an algebra. The following conditions are equivalent.*

- (i) A is a generalized double tilted algebra;
- (ii) Γ_A admits a faithful generalized standard almost acyclic component;
- (iii) Γ_A admits a component C with a faithful multisection Δ such that $\text{Hom}_A(X, \tau_A Y) = 0$, for all modules X from Δ_r and Y from Δ_l .

5. Key lemma

In this section we prove an important result needed in the proof of Theorem 1. We will use the following two lemmas proved in [44, Lemmas 6.1 and 6.2].

Lemma 5.1. *Let H be a hereditary algebra of Euclidean type and E be a module on the mouth of a stable tube of Γ_H . Moreover, assume that the valued quiver Q_H of H is a tree (oriented canonically, as in [15]). Then, for every infinite path*

$$\cdots \rightarrow Y_{n+1} \rightarrow Y_n \rightarrow \cdots \rightarrow Y_1 \rightarrow Y_0$$

in the preinjective component Q^H of Γ_H , there is an infinite sequence $n_0 < n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$ of nonnegative integers such that $\text{Hom}_H(E, Y_{n_k}) \neq 0$, for all $k \geq 0$.

Lemma 5.2. *Let H be a hereditary algebra of Euclidean type \tilde{A}_m , $m \geq 2$, with the valued quiver Q_H oriented canonically (as in [15]), and let E be a module lying on the mouth of a stable tube \mathcal{T} of Γ_H . Then, for every module Y from the preinjective component Q^H of Γ_H , there is an infinite sectional path*

$$\cdots \rightarrow Y_{n+1} \rightarrow Y_n \rightarrow \cdots \rightarrow Y_1 \rightarrow Y_0 = Y$$

in Q^H such that the following statements hold.

- (i) *There exists a sequence $n_0 < n_1 < \cdots$ of nonnegative integers such that $\text{Hom}_H(E, Y_{n_k}) \neq 0$, for all $k \geq 0$.*
- (ii) *For every nonnegative integer c , there exists a sequence $n_0^c < n_1^c < \cdots$ of nonnegative integers such that $\text{Hom}_H(E, \tau_H^c Y_{n_k^c}) \neq 0$, for all $k \geq 0$.*

The following lemma will play an essential role in our further considerations.

Lemma 5.3. *Let B be a tilted algebra of Euclidean type such that Γ_B admits an infinite preinjective connecting component. Moreover, assume that there are modules V and R in $\text{ind } B$ satisfying the following conditions:*

- (i) *V lies on the mouth of a stable tube of Γ_B ;*
- (ii) *R belongs to the preinjective component Q^B of Γ_B .*

Then there exist infinitely many pairwise nonisomorphic indecomposable modules Z_k in Q^B , $k \in \mathbb{N}$, such that $\text{Hom}_B(V, \tau_B Z_k) \neq 0$ and $\text{Hom}_B(Z_k, R) \neq 0$, for all $k \in \mathbb{N}$.

Proof. Let H be a hereditary algebra of Euclidean type and T be a tilting module in $\text{mod } H$ such that $B \cong \text{End}_H(T)$. Then T admits a decomposition $T = T^{pp} \oplus T^{rg}$, where T^{pp} (respectively, T^{rg}) is in $\text{add}(\mathcal{P}^H)$ (respectively, in $\text{add}(\mathcal{T}^H)$), the family \mathcal{T}^B of all semiregular tubes of Γ_B has no coray tubes containing injective modules, and the connecting component $\mathcal{C}_T = Q^B$ of Γ_B determined by T contains all indecomposable injective B -modules. Denote by Σ the section in Q^H formed by all indecomposable injective H -modules and by Δ the associated section in Q^B formed by all modules

of the form $\text{Hom}_H(T, I)$ with I in Σ . Because V lies on the mouth of a stable tube of Γ_B , we infer that there is a stable tube of Γ_H without modules from $\text{add}(T)$ and containing a mouth module E such that $V \cong \text{Hom}_H(T, E)$ in $\text{mod } B$. Moreover, we may assume that the valued quiver $Q_H \cong \Delta^{\text{op}}$ of H is oriented canonically (as in [15, 6. Tables]). Indeed, Q^B is an acyclic and generalized standard component of Γ_B with section Δ , hence there is a section Δ' in Q^B with the same number of vertices as Δ and such that Δ' is oriented canonically, Δ and Δ' are of the same Euclidean type, and $\text{Hom}_B(U_0, \tau_B U_1) = 0$, for all modules U_0, U_1 from Δ' . Therefore, by Liu-Skowroński criterion (see [23, 37], or [2, Theorem VIII.5.6]), the direct sum $U = U_B$ of all modules lying on Δ' is a tilting B -module, the algebra $H' = \text{End}_B(U)$ is a hereditary algebra with $Q_{H'} \cong (\Delta')^{\text{op}}$ oriented canonically and $B \cong \text{End}_{H'}(T')$, where $T' = D({}_H U)$ is a tilting module in $\text{mod } H'$.

Obviously, there exists a projective module P in $\mathcal{P}^B \cup \mathcal{T}^B$ such that $\text{Hom}_B(P, R) \neq 0$. Hence, applying [40, Lemma 2.1], we deduce that there is an infinite path in \mathcal{C}_T of the form

$$\cdots \rightarrow Z_{n+1} \rightarrow Z_n \rightarrow \cdots \rightarrow Z_1 \rightarrow Z_0 = R$$

such that $\text{Hom}_B(Z_n, R) \neq 0$, for all $n \geq 0$. Moreover, since \mathcal{C}_T is without projective modules, there is a path in \mathcal{C}_T of the following form

$$\cdots \rightarrow Z'_{n+1} \rightarrow Z'_n \rightarrow \cdots \rightarrow Z'_1 \rightarrow Z'_0,$$

where $Z'_n = \tau_B Z_n$, for all $n \geq 0$. Further, there is an integer $m_0 \geq 0$ such that Z'_n is a predecessor of Δ in \mathcal{C}_T , for every $n \geq m_0$. Hence there exists a path in Q^H of the form

$$\cdots \rightarrow Y_{n+1} \rightarrow Y_n \rightarrow \cdots \rightarrow Y_{m_0+1} \rightarrow Y_{m_0}$$

with $\text{Hom}_H(T, Y_n) = Z'_n$, for all $n \geq m_0$.

Assume first that Q_H is a tree. Then, using Lemma 5.1, we conclude that there is an infinite sequence $m_0 \leq n_0 < n_1 < \cdots$ of integers such that $\text{Hom}_H(E, Y_{n_k}) \neq 0$, for every $k \geq 0$. Consequently, applying the Brenner–Butler theorem (see [2, Theorem VI.3.8]), we obtain that $\text{Hom}_B(V, Z'_{n_k}) = \text{Hom}_B(\text{Hom}_H(T, E), \text{Hom}_H(T, Y_{n_k})) \cong \text{Hom}_H(E, Y_{n_k}) \neq 0$, for all $k \geq 0$, hence the modules Z_{n_k} , for $k \geq 0$, satisfy the required properties.

Now, let Q_H be of Euclidean type \tilde{A}_m , $m \geq 1$. First, we assume that the module R is in the torsion-free part $\mathcal{Y}(T) \cap \mathcal{C}_T$ of \mathcal{C}_T . Then $R \cong \text{Hom}_H(T, Y)$, for some Y in Q^H . Hence, applying Lemma 5.2, we deduce that there is an infinite sectional path in Q^H of the form $\cdots \rightarrow Y_n \rightarrow \cdots \rightarrow Y_0 = Y$ and an infinite sequence $n_0 < n_1 < \cdots$ of nonnegative integers such that $\text{Hom}_H(E, \tau_H Y_{n_k}) \neq 0$. Then, the modules $Z_k = \text{Hom}_H(T, Y_{n_k})$, for $k \geq 0$, satisfy the postulated properties. Finally, let R be a module in the torsion part $\mathcal{X}(T) \cap \mathcal{C}_T$ of \mathcal{C}_T . Then there is a module F in $\mathcal{P}^H \cap \mathcal{F}(T)$ such that $R = \text{Ext}_H^1(T, F)$. Moreover, all irreducible homomorphisms between indecomposable modules in \mathcal{P}^H are

monomorphisms and \mathcal{P}^H is a generalized standard component of Γ_H . Thus a module N belongs to $\mathcal{P}^H \cap \mathcal{F}(T)$ if and only if N is not a successor in \mathcal{P}^H of an indecomposable direct summand of T^{pp} . In particular, it follows that $\mathcal{P}^H \cap \mathcal{F}(T) = \text{ind } H \cap \mathcal{F}(T)$ is a full subcategory of $\text{ind } H$, closed under predecessors in $\text{ind } H$. Therefore, using the dual to Lemma 5.2, we deduce that there is a sectional path in $\mathcal{P}^H \cap \mathcal{F}(T)$ of the form

$$P_j = F_0 \rightarrow F_1 \rightarrow \cdots \rightarrow F_t = F$$

such that P_j is an indecomposable projective module from $\text{mod } H$ but not in $\text{add}(T)$ and $\text{Hom}_H(\tau_H^{-p} F_k, E) \neq 0$, for some integer $p \geq 0$ and every $k \in \{0, \dots, t\}$. Applying Lemma 5.2 again, we conclude that there is a sectional path in \mathcal{Q}^H of the form

$$\cdots \rightarrow Y_{n+1} \rightarrow Y_n \rightarrow \cdots \rightarrow Y_1 \rightarrow Y_0 = I$$

such that the following statements hold:

- the module I is an indecomposable direct summand of the injective H -module $I_j / \text{soc}(I_j)$, where I_j is an indecomposable injective module in $\text{mod } H$ with $\text{soc } I_j = \text{top } P$;
- there is a sequence $n_0 < n_1 < \cdots$ of nonnegative integers such that $\text{Hom}_H(E, \tau_H Y_{n_k}) \neq 0$, for every $k \geq 0$;
- there exists an irreducible homomorphism $\text{Hom}_H(T, I) \rightarrow \text{Ext}_H^1(T, P_j)$ in $\text{mod } B$;
- the modules I_j and Y_1 are nonisomorphic in $\text{mod } H$.

Consequently, \mathcal{Q}^B admits an infinite sectional path of the form

$$\cdots \rightarrow Z_n \rightarrow \cdots \rightarrow Z_0 \rightarrow W_0 \rightarrow \cdots \rightarrow W_t = R,$$

where $Z_n = \text{Hom}_H(T, Y_n)$, for any $n \geq 0$ and $W_k = \text{Ext}_H^1(T, F_k)$, for all $k \in \{0, \dots, t\}$. As above, we have $\text{Hom}_B(V, \tau_B Z_{n_k}) \neq 0$ and $\text{Hom}_B(Z_{n_k}, R) \neq 0$, for all $k \geq 0$, hence the claim follows. \square

6. Proof of Theorem 1

First, we prove the following lemma which provides the proof of the implication (i) \Rightarrow (ii) of Theorem 1. Note that this result does not use the cycle-finiteness assumption.

Lemma 6.1. *Let A be a generalized double tilted algebra. Then there exists a faithful module M in $\text{mod } A$ being the middle of at most finitely many short chains.*

Proof. Observe first that Γ_A admits a faithful component \mathcal{C} with a multisection Δ . Let M be the direct sum of all indecomposable modules from Δ . It follows from [31,

[Lemma 2.7](#)] that M is a faithful module in $\text{mod } A$. We claim that M is the middle of at most finitely many short chains. Assume that there is a short chain of the form $X \rightarrow M \rightarrow \tau_A X$, for a module X in $\text{ind } A$. We will prove that X is in Δ . We have a path Θ in $\text{mod } A$ of the following form

$$\Theta: N \rightarrow \tau_A X \rightarrow Y \rightarrow X \rightarrow N',$$

where N and N' are indecomposable direct summands of M . Let X be a module in \mathcal{C} . Then Y and $\tau_A X$ belong to \mathcal{C} , hence the path Θ admits a refinement to a path of irreducible homomorphisms, because \mathcal{C} is a generalized standard component. But Δ is convex in \mathcal{C} , hence X lies in Δ . Therefore, assume that X is not in \mathcal{C} . Then X is both a predecessor of Δ_l and a successor of Δ_r in $\text{mod } A$. Consequently, applying [\[31, Theorem 3.4\(ii\), \(iii\) and \(vi\)\]](#), we conclude that X is zero module, a contradiction. Concluding, we have proved that, if M is the middle of a short chain $X \rightarrow M \rightarrow \tau_A X$, then the module X is in Δ . Therefore, M is the middle of at most finitely many short chains, because Δ is a finite multisection in \mathcal{C} . \square

The following theorem completes the proof of [Theorem 1](#).

Theorem 6.2. *Let A be a cycle-finite algebra such that $\text{mod } A$ admits a faithful module M being the middle of at most finitely many short chains. Then A is a generalized double tilted algebra.*

Proof. If A is representation-finite, then A is a generalized double tilted algebra with (unique) finite connecting component. We assume that A is of infinite representation type.

(1) Let N be an indecomposable direct summand of M and \mathcal{C} be the component of Γ_A containing N . Then it follows from [Lemma 3.1](#) and [Corollary 3.3](#) that \mathcal{C} is not a semiregular tube. Therefore, by [Proposition 3.4](#), the cyclic part ${}_c\mathcal{C}$ of \mathcal{C} is finite, and hence \mathcal{C} is an almost acyclic component of Γ_A . Moreover, applying [\[31, Theorem 2.5 and Proposition 2.4\]](#), we deduce that \mathcal{C} admits a multisection Δ and there is a disjoint union decomposition

$$\mathcal{C} = \mathcal{C}_l \cup \Delta_c \cup \mathcal{C}_r,$$

where \mathcal{C}_l (respectively, \mathcal{C}_r) is the full valued translation subquiver of \mathcal{C} formed by all predecessors in \mathcal{C} of modules from Δ_l (respectively, formed by all successors in \mathcal{C} of modules from Δ_r). Note also that A is assumed to be indecomposable and of infinite representation type, hence \mathcal{C}_l or \mathcal{C}_r is infinite. We may assume (without loss of generality) that \mathcal{C}_l is infinite.

(2) We claim that A admits a factor tilted algebra $B = B(\mathcal{C}) = B_1 \times \cdots \times B_p$ such that, for every $i \in \{1, \dots, p\}$, B_i is an indecomposable tilted algebra of Euclidean type

and the torsion-free part of a connecting component of Γ_{B_i} is a full valued translation subquiver of \mathcal{C} .

Assume first that \mathcal{C} contains a projective module. We denote by \mathcal{D}_l the full valued translation subquiver of \mathcal{C}_l given by all modules X in the left stable part ${}_l\mathcal{C}_l$ of \mathcal{C}_l such that X is a predecessor of a projective module in \mathcal{C} and every predecessor of X in \mathcal{C} is in ${}_l\mathcal{C}_l$. Then \mathcal{D}_l is a nonempty full valued left stable translation subquiver of \mathcal{C}_l . Moreover, \mathcal{D}_l is acyclic, by [31, Proposition 2.4]. Let $\mathcal{D}_l = \mathcal{D}_l^1 \cup \dots \cup \mathcal{D}_l^p$ be a decomposition of \mathcal{D}_l into a disjoint union of connected full (valued) translation subquivers. Applying [25, Theorem 2.2], we conclude that, for every $i \in \{1, \dots, p\}$, there exists a hereditary algebra H_i of Euclidean type and a tilting module T_i in $\text{mod } H_i$ without nonzero preinjective direct summands such that the tilted algebra $B_i = \text{End}_{H_i}(T_i)$ is an indecomposable factor algebra of A and the torsion-free part $\mathcal{Y}(T_i) \cap \mathcal{C}_{T_i}$ of the connecting component \mathcal{C}_{T_i} of Γ_{B_i} determined by T_i is a full valued translation subquiver of \mathcal{D}_l^i , which is closed under predecessors in \mathcal{C} . Moreover, every module Y in \mathcal{D}_l^i is a B_i -module and belongs to the preinjective connecting component $\mathcal{C}_{T_i} = Q^{B_i}$ of Γ_{B_i} and the product algebra $B = B_1 \times \dots \times B_p$ is a factor algebra of A . Further, for each $i \in \{1, \dots, p\}$, there are a module R_i in \mathcal{D}_l^i and an irreducible monomorphism $R_i \rightarrow P^{(i)}$ with $P^{(i)}$ an indecomposable projective module in \mathcal{C} . Finally, observe that \mathcal{D}_l admits at most finitely many τ_A -orbits.

Now, assume that \mathcal{C} has no projective modules. Then, applying Proposition 2.1, we deduce that \mathcal{C} is a preinjective component of Euclidean type. Moreover, by [25, Theorem 2.2], there exists a hereditary algebra H_1 of Euclidean type and a tilting module T_1 in $\text{mod } H_1$ without nonzero preinjective direct summands such that the tilted algebra $B = B_1 = \text{End}_{H_1}(T_1)$ is an indecomposable factor algebra of A and the torsion-free part $\mathcal{Y}(T_1) \cap \mathcal{C}_{T_1}$ of the connecting component \mathcal{C}_{T_1} of Γ_B is a full valued translation subquiver of \mathcal{C} . In this case $\mathcal{C} = \mathcal{C}_{T_1}$ is a component of Γ_B . In particular, it follows that \mathcal{C} is a generalized standard component of Γ_A .

(3) Consider now the family $\mathcal{T}^{B_i} = (\mathcal{T}_\lambda^{B_i})_{\lambda \in \Lambda_i}$ of all ray tubes of Γ_{B_i} , for $i \in \{1, \dots, p\}$. The disjoint union $\mathcal{T}^{B_1} \cup \dots \cup \mathcal{T}^{B_p}$ is the family $\mathcal{T}^B = (\mathcal{T}_\lambda^B)_{\lambda \in \Lambda}$, $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_p$, of all pairwise orthogonal ray tubes of Γ_B . Moreover, $B = \text{supp}(\mathcal{T}^B)$ is a convex subcategory of A . In fact, for every $i \in \{1, \dots, p\}$, the tilted algebra B_i is a tubular extension of a tame concealed algebra C_i , hence B_i is a convex subcategory of A , by [9, Theorem 1.5].

Take now any two modules X and Y lying in the cyclic part ${}_c\mathcal{T}_\lambda^B$ of a ray tube \mathcal{T}_λ^B , $\lambda \in \Lambda$. Because there is a cycle of irreducible homomorphisms in $\text{mod } B$ passing through X and Y , and A is cycle-finite, we deduce that X and Y lie on a common cycle in Γ_A . Consequently, there is a component \mathcal{T}_λ^A of Γ_A containing all modules of ${}_c\mathcal{T}_\lambda^B$. Then, since ${}_c\mathcal{T}_\lambda^B$ is infinite, ${}_c\mathcal{T}_\lambda^A$ is also infinite, and hence \mathcal{T}_λ^A is a semiregular tube of Γ_A , by Lemma 3.4. Further, if \mathcal{T}_λ^A is a stable tube, then $\mathcal{T}_\lambda^A = \mathcal{T}_\lambda^B$. Moreover, $\mathcal{T}_\lambda^A \neq \mathcal{T}_\mu^A$, for any $\lambda \neq \mu$ in Λ (see [9, Theorem 4.1] and its proof).

We claim that the family $\mathcal{T}^A = (\mathcal{T}_\lambda^A)_{\lambda \in \Lambda}$ of semiregular tubes of Γ_A does not admit a coray tube containing an injective module. Suppose that this is not the case, say $\mathcal{T}_{\lambda_0}^A$,

for some $\lambda_0 \in A_{i_0}$, $i_0 \in \{1, \dots, p\}$, is a coray tube containing an injective module. Then, by [41, Proposition 2.3], $\mathcal{T}_{\lambda_0}^B$ is a stable tube of Γ_B , hence there are a module V lying on the mouth of $\mathcal{T}_{\lambda_0}^B$ and an irreducible epimorphism $I \rightarrow V$ in $\text{mod } A$ with I an injective module in $\text{ind } A$. Therefore, there is an epimorphism $M^r \rightarrow V$ in $\text{mod } A$, for some $r \geq 1$, because M is a faithful module in $\text{mod } A$.

Now, assume that \mathcal{C} contains a projective module. Then, by Lemma 5.3, there are infinitely many pairwise nonisomorphic modules Z_k in $\mathcal{Q}^{B_{i_0}}$, $k \in \mathbb{N}$, such that

$$\text{Hom}_B(V, \tau_B Z_k) \neq 0 \quad \text{and} \quad \text{Hom}_B(Z_k, R_{i_0}) \neq 0,$$

for every $k \in \mathbb{N}$. It follows that $\text{Hom}_A(M, \tau_A Z_k) \neq 0$ and $\text{Hom}_A(Z_k, M) \neq 0$, for all $k \in \mathbb{N}$, because we have an epimorphism $M^r \rightarrow V$ in $\text{mod } A$, a composed monomorphism $R_{i_0} \rightarrow P^{(i_0)} \rightarrow M^s$ in $\text{mod } A$, for some $s \geq 1$, and $\tau_A Z_k = \tau_B Z_k$, for all but finitely many integers $k \geq 0$. Hence M is the middle of infinitely many short chains $Z_k \rightarrow M \rightarrow \tau_A Z_k$, $k \in \mathbb{N}$, a contradiction.

If \mathcal{C} does not contain a projective module, then it follows from arguments presented above that \mathcal{C} is a preinjective component of Euclidean type containing an indecomposable direct summand N of M . Similarly, using Lemma 5.3, we conclude that there are infinitely many pairwise nonisomorphic modules Z_k in $\mathcal{Q}^{B_{i_0}}$, $k \in \mathbb{N}$, such that $\text{Hom}_B(V, \tau_B Z_k) \neq 0$ and $\text{Hom}_B(Z_k, N) \neq 0$, for every $k \in \mathbb{N}$. This also leads to a contradiction with the assumption imposed on M .

Therefore, indeed \mathcal{T}^A has no coray tubes containing injective modules.

Finally, we prove that $\mathcal{T}^A = \mathcal{T}^B$. Because B is a convex subcategory of A , we deduce from [41, Proposition 2.3] that all rays of a nonregular ray tube \mathcal{T}_{λ}^B , $\lambda \in A$, of Γ_B are rays of \mathcal{T}_{λ}^A . Therefore, since \mathcal{T}^A is a family of pairwise orthogonal and generalized standard components, the factor algebra $A' = A/\text{ann}(\mathcal{T}^A)$ is a tubular extension of B and the family \mathcal{T}^A is obtained from \mathcal{T}^B by a finite (possibly zero) number of ray insertions. Suppose that $\mathcal{T}^A \neq \mathcal{T}^B$. Then there is a decomposition $A' = P \oplus Q$ of A' into a direct sum of projective A' -modules with $P = B$, $Q \neq 0$, and $\text{Hom}_{A'}(Q, P) = 0$. Hence A' is isomorphic to an algebra of the following triangular matrix form

$$\begin{bmatrix} F & U' \\ 0 & B \end{bmatrix},$$

where $F = \text{End}_{A'}(Q)$ and $U' = \text{Hom}_{A'}(P, Q)$ is a nonzero $(F-B)$ -bimodule with U'_B in $\text{add}(\mathcal{T}^B)$. Therefore, it follows that there is a module X in \mathcal{D}_l such that $\text{Hom}_B(U', X) \neq 0$ and there is an almost split sequence in $\text{mod } B$ of the form

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

which is also an almost split sequence in $\text{mod } A$. Moreover, because X , Y and Z are A' -modules, the above short exact sequence is an almost split sequence in $\text{mod } A'$. But

this leads to a contradiction, because $\text{Hom}_B(U', X) \neq 0$ (see Lemma 2.3). Consequently, we have proved that $\mathcal{T}^A = \mathcal{T}^B$. In particular, \mathcal{T}^B is a family of ray tubes of Γ_A .

(4) Now we are going to prove that A is a generalized double tilted algebra. Let $\mathcal{P}^{B(C)}$ be the family $\mathcal{P}^{B_1} \cup \dots \cup \mathcal{P}^{B_p}$ of all postprojective components of $\Gamma_{B(C)}$. Consider the factor algebra $A(C) = A/I$ of A , where $I = \text{ann}(\mathcal{T}^{B(C)} \cup \mathcal{C})$. Then $\mathcal{T}^{B(C)} \cup \mathcal{C}$ is a faithful family of components of $\Gamma_{A(C)}$ and all projective modules in $\text{mod } A(C)$ belong to $\mathcal{P}^{B(C)} \cup \mathcal{T}^{B(C)} \cup \mathcal{C}$. In particular, the valued quiver $Q_{A(C)}$ of $A(C)$ can be viewed as a (full) valued subquiver of Q_A .

We claim that $Q_{A(C)} = Q_A$. Suppose, to the contrary, that this is not the case. Then, because Q_A is connected, we infer that there are a vertex i_0 in Q_A not lying in $Q_{A(C)}$ and a vertex j_0 in $Q_{A(C)}$ such that there is either an arrow $i_0 \rightarrow j_0$ in Q_A or an arrow $j_0 \rightarrow i_0$ in Q_A . Suppose first that there is an arrow $j_0 \rightarrow i_0$ in Q_A . Then there is a homomorphism $f_0 : P_{i_0} \rightarrow P_{j_0}$ in $\text{mod } A$, where $P_{j_0} = e_{j_0}A$ and $P_{i_0} = e_{i_0}A$ are indecomposable projective A -modules corresponding to the vertices j_0 and i_0 , respectively, and f_0 is given by an element $a_0 \in e_{j_0}(\text{rad } A)e_{i_0} \setminus e_{j_0}(\text{rad } A)^2e_{i_0}$. Because P_{j_0} and P_{i_0} are nonisomorphic indecomposable projective modules, we infer that f_0 is not an epimorphism, and hence $\text{Im } f_0$ is a submodule of $\text{rad } P_{j_0}$. Then the projectivity of P_{i_0} implies that there exists a commutative diagram in $\text{mod } A$ of the following form

$$\begin{array}{ccc} & P_{i_0} & \\ g_0 \swarrow & \downarrow \bar{f}_0 & \\ P(\text{rad } P_{j_0}) & \xrightarrow{\pi} & \text{rad } P_{j_0} \end{array}$$

with $P(\text{rad } P_{j_0})$ a projective cover of $\text{rad } P_{j_0}$ in $\text{mod } A(C)$ and $f_0 = u\bar{f}_0$, where $u : \text{rad}(P_{j_0}) \rightarrow P_{j_0}$ is the canonical inclusion homomorphism. Observe that, if P_i is a direct summand of $P(\text{rad } P_{j_0})$, then i is in $Q_{A(C)}$. Hence g_0 is a homomorphism in $\text{rad } A$. Moreover, the homomorphism $h_0 = u\pi$ is contained in $\text{rad}_A(P(\text{rad } P_{j_0}), P_{j_0})$. But then $f_0 = h_0g_0$ implies that a_0 is in $e_{j_0}(\text{rad } A)^2e_{i_0}$, a contradiction. Dually, if there is an arrow $i_0 \rightarrow j_0$ in Q_A , then we obtain a similar contradiction using injective modules. Consequently, we get the required equality $Q_{A(C)} = Q_A$.

Therefore, we conclude that all indecomposable projective (respectively, injective) modules in $\text{mod } A$ are projective (respectively, injective) modules in $\text{mod } A(C)$. Hence every indecomposable projective module in $\text{mod } A$ belongs to $\mathcal{P}^{B(C)} \cup \mathcal{T}^{B(C)} \cup \mathcal{C}$. Moreover, $B = B(C)$ is a product of indecomposable tilted algebras of Euclidean type, hence there exists a monomorphism $B \rightarrow L$ in $\text{mod } B$ such that L is a module from $\text{add}(\mathcal{C}_{T_1} \cup \dots \cup \mathcal{C}_{T_p})$. In particular, it follows that there exists a monomorphism $A \rightarrow C$ in $\text{mod } A$ with C a module from $\text{add}(\mathcal{C})$. Hence \mathcal{C} is a faithful component of Γ_A .

Finally, we prove that \mathcal{C} is a generalized standard component. It follows from Proposition 2.1 that, if \mathcal{C} is without projective modules, then \mathcal{C} is a generalized standard component. In that case, A is a tilted algebra, and the claim follows. Therefore assume

that \mathcal{C} contains a projective module. Let $A = P \oplus P'$ be a decomposition of A_A into a direct sum of projective A -modules such that P (respectively, P') is a direct sum of all indecomposable projective modules in $\text{mod } A$ lying in $\mathcal{P}^{B(\mathcal{C})} \cup \mathcal{T}^{B(\mathcal{C})}$ (respectively, in \mathcal{C}). Then $\text{Hom}_A(P', P) = 0$, and hence $A = A(\mathcal{C})$ is isomorphic to an algebra of the following triangular matrix form

$$\begin{bmatrix} D & U \\ 0 & B \end{bmatrix},$$

where $D = \text{End}_A(P')$ and $U = \text{Hom}_A(P, P')$ is a $(D-B)$ -bimodule with U_B lying in \mathcal{C} . Now, we prove that every predecessor Y in $\text{ind } A$ of a module X in \mathcal{D}_l is a predecessor in $\text{ind } B$. Let $X = (X_0, X_1, \psi)$ be a module in \mathcal{D}_l and $Y = (Y_0, Y_1, \varphi)$ be a predecessor of X in $\text{ind } A$. Then X is a module in $\text{mod } B$, hence $X_0 = 0$, $\psi = 0$, and $X_1 = X$. Moreover, there are a homomorphism $h_0 : Y_0 \rightarrow X_0$ in $\text{mod } D$ and a homomorphism $h_1 : Y_1 \rightarrow X_1$ in $\text{mod } B$ such that $h_1 \neq 0$ and $\text{Hom}_B(U, h_1)\varphi = \psi h_0$. It follows that $h_0 = 0$. We claim that $\varphi = 0$. Suppose that this is not the case. Then, using [Lemma 2.2](#), we conclude that, for every indecomposable direct summand Z of Y_1 , we have $\text{Hom}_B(U, Z) \neq 0$. Therefore, Z is a successor in $\text{ind } B$ of an indecomposable direct summand of $\text{rad } P'$, a contradiction, because Z is a predecessor of $X_1 = X$ lying in \mathcal{D}_l . Hence indeed $\varphi = 0$. This implies that $Y \cong Y_0 \oplus Y_1$ in $\text{mod } A$, thus because Y is in $\text{ind } A$, we infer that $Y \cong Y_0$ or $Y \cong Y_1$. In the first case, we have $Y_1 = 0$, hence $h_1 = 0$, a contradiction. Therefore, $Y \cong Y_1$ is a module in $\text{ind } B$, and we are done. Finally, suppose that \mathcal{C} is not a generalized standard component. Then $\text{rad}_A^\infty(Y, X) \neq 0$, for some indecomposable modules X and Y in \mathcal{C} . Applying [\[40, Lemma 2.1\]](#), we deduce that there is an infinite path $\cdots \rightarrow X_n \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 = X$ such that $\text{rad}_A^\infty(Y, X_n) \neq 0$, for all $n \geq 0$. Because \mathcal{C} is almost acyclic with finitely many τ_A -orbits, we infer that X_{n_0} is a module in \mathcal{D}_l , for some integer $n_0 \geq 0$, and consequently, by the above discussion, Y is also a module in \mathcal{D}_l . But this leads to a contradiction, because Y and X_{n_0} are indecomposable B -modules lying in the preinjective component of Γ_B which is generalized standard.

Concluding, we have proved that \mathcal{C} is an almost acyclic, faithful, and generalized standard component of Γ_A . Hence A is a generalized double tilted algebra, by [Theorem 4.1](#). Note also that if \mathcal{C} does not contain projective modules, then A is a tilted algebra of Euclidean type. \square

7. Proof of [Theorem 2](#)

Let A be a cycle-finite algebra, M a module in $\text{mod } A$ which is the middle of at most finitely many short chains, and $B = A/\text{ann}(M)$. Consider the decomposition $B = B_1 \times \cdots \times B_m$ of B and the associated decomposition $M = M_1 \oplus \cdots \oplus M_m$ of M in $\text{mod } B$ as described in the statement of [Theorem 2](#). Then, for every $i \in \{1, \dots, m\}$, M_i is a faithful module in $\text{mod } B_i$. Clearly, B_i is a cycle-finite algebra, for every $i \in \{1, \dots, m\}$

(as a factor algebra of A). Therefore, the statement (i) of [Theorem 2](#) follows from [Theorem 6.2](#) and the following lemma.

Lemma 7.1. *Assume that A is an algebra and I is a two-sided ideal of A . Moreover, let B be the factor algebra A/I of A and M a module in $\text{mod } B$ being the middle of at most finitely many short chains in $\text{mod } A$. Then M is the middle of at most finitely many short chains in $\text{mod } B$.*

Proof. Let $X \rightarrow M \rightarrow \tau_B X$ be a short chain with X in $\text{ind } B$. Then $\text{Hom}_A(X, M) \neq 0$. Moreover, by [\[2, Lemma VIII.5.2\]](#), $\tau_B X$ is isomorphic to a submodule of $\tau_A X$, and hence $\text{Hom}_A(M, \tau_A X) \neq 0$. Therefore, there exists a short chain (in $\text{mod } A$) of the form $X \rightarrow M \rightarrow \tau_A X$. It follows that M is the middle of at most finitely many short chains in $\text{mod } B$. \square

The remaining statement (ii) of [Theorem 2](#) is a direct consequence of the following two propositions.

Proposition 7.2. *Let B be an indecomposable cycle-finite generalized double tilted algebra which is not a tilted algebra. Moreover, assume that M is a faithful module in $\text{mod } B$ being the middle of at most finitely many short chains. Then M is contained in the additive category $\text{add}(C)$ of the (unique) connecting component C of Γ_B .*

Proof. Let N be an indecomposable direct summand of M and C a component of Γ_B containing N . It follows from the proof of [Theorem 6.2](#) that C is an almost acyclic component of Γ_A . Let $C = C_l \cup \Delta_e \cup C_r$ be the decomposition of C described in the proof of [Theorem 6.2](#). It was proved there that, if C_l is infinite and C has no projective modules, then B is a tilted algebra. Therefore, if C_l is infinite, then C contains a projective module. Dually, if C_r is infinite then C contains an injective module. Assume first that C contains a projective module (in particular, C_l is infinite). We claim that C contains also an injective module. Suppose that this is not the case. Then, applying [Proposition 2.1](#), we conclude that C is a postprojective component of Euclidean type. But then, $C = C_r = C_l$ is infinite, hence B is a tilted algebra, a contradiction.

Summing up, we have proved that C contains both a projective module and an injective module. Consequently, every indecomposable direct summand N of M is contained in a nonsemiregular component of Γ_B .

Finally, observe that Γ_B admits exactly one nonsemiregular component (it is the connecting component), therefore all indecomposable direct summands of M are contained in the (unique) connecting component of Γ_A . \square

Proposition 7.3. *Let B be an indecomposable tilted algebra of Euclidean type and M a faithful module in $\text{mod } B$ which is the middle of at most finitely many short chains. Then all indecomposable direct summands of M are contained in one connecting component of Γ_B .*

Proof. First, observe that Γ_B has the following decomposition:

$$\Gamma_B = \mathcal{P}^B \cup \mathcal{T}^B \cup \mathcal{Q}^B,$$

where \mathcal{P}^B is the postprojective component of Γ_B , \mathcal{Q}^B is the preinjective component of Γ_B , and \mathcal{T}^B is a family of all pairwise orthogonal semiregular tubes of Γ_B strongly separating \mathcal{P}^B from \mathcal{Q}^B . Note that either \mathcal{T}^B consists of ray tubes or \mathcal{T}^B consists of coray tubes.

Assume first that \mathcal{T}^B admits at least one ray tube containing a projective module. It follows that \mathcal{Q}^B contains all indecomposable injective modules. Let N be an indecomposable direct summand of M . We claim that N belongs to $\text{add}(\mathcal{Q}^B)$. Applying Lemma 3.1 and Corollary 3.3, we infer that N is not contained in \mathcal{T}^B . Therefore N belongs to $\mathcal{P}^B \cup \mathcal{Q}^B$. Suppose that N lies in \mathcal{P}^B . Then there is an indecomposable injective module I in \mathcal{Q}^B such that $\text{Hom}_B(N, I) \neq 0$. Since \mathcal{T}^B is strongly separating, we conclude that there exists a ray tube \mathcal{T} of \mathcal{T}^B containing a projective module and such that $\text{Hom}_B(N, \mathcal{T}) \neq 0$. But this leads to a contradiction with the statement (i) of Lemma 3.1. Therefore, indeed N belongs to \mathcal{Q}^B . It follows that M is contained in $\text{add}(\mathcal{Q}^B)$.

Dually, we prove that, if \mathcal{T}^B admits at least one coray tube containing an injective module then M belongs to \mathcal{P}^B .

Finally, assume that \mathcal{T}^B admits neither a ray tube with a projective module nor a coray tube with an injective module. Then \mathcal{T}^B consists only of stable tubes. We claim that M belongs either to \mathcal{P}^B or to \mathcal{Q}^B . Suppose to the contrary that there are indecomposable direct summands N and N' of M such that N lies in \mathcal{P}^B and N' lies in \mathcal{Q}^B . As above, because \mathcal{P}^B contains all indecomposable projective modules, \mathcal{Q}^B contains all indecomposable injective modules, and \mathcal{T}^B is a strongly separating family of components of Γ_B , we conclude that there exists a stable tube \mathcal{T} of \mathcal{T}^B such that $\text{Hom}_B(N, \mathcal{T}) \neq 0$ and $\text{Hom}_B(\mathcal{T}, N') \neq 0$. Hence we obtain a contradiction with the statement of Lemma 3.2. \square

Acknowledgment

The author gratefully acknowledges the support from the research grant DEC-2011/02/A/ST1/00216 of the Polish National Science Center.

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