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The group fixed by a family of endomorphisms of a surface group

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ABSTRACT

For a closed surface S with $\chi(S) < 0$, we show that the fixed subgroup of a family $\mathcal{B}$ of endomorphisms of $\pi_1(S)$ has rank $\operatorname{Fix}\mathcal{B} \leq \operatorname{rank}\pi_1(S)$. In particular, if $\mathcal{B}$ contains a non-epimorphic endomorphism, then $\operatorname{rank}\operatorname{Fix}\mathcal{B} \leq \frac{1}{2}\operatorname{rank}\pi_1(S)$. We also show that geometric subgroups of $\pi_1(S)$ are inert, and hence the fixed subgroup of a family of epimorphisms of $\pi_1(S)$ is also inert.

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1. Introduction

For a finitely generated group G, we denote the rank (i.e., the minimal number of the generators) of G by $\operatorname{rank} G$. There are lots of research on the intersection of subgroups of a finitely generated group G in the literature. For example, when G is a free group,

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H. Neumann (see [18] and [19]) conjectured that for any two finitely generated subgroups A and B of G,

$$\operatorname{rank}(A \cap B) - 1 \leq (\operatorname{rank} A - 1)(\operatorname{rank} B - 1).$$

This conjecture was proved independently by I. Mineyev [17] and J. Friedman [7].

Before this celebrated result was proved, it had been shown that for some special subgroups of free groups, one could say more about their intersection. Denote the set of endomorphisms of G by $\operatorname{End}(G)$. For a family $\mathcal{B}$ of endomorphisms of G, namely, $\mathcal{B} \subseteq \operatorname{End}(G)$, the subgroup fixed by $\mathcal{B}$ is

$$\operatorname{Fix}(\mathcal{B}) := \{ g \in G \mid \phi(g) = g, \forall \phi \in \mathcal{B} \}.$$

It is called the *fixed subgroup* of $\mathcal{B}$. We abbreviate $\operatorname{Fix}(\mathcal{B})$ to $\operatorname{Fix}\mathcal{B}$, and $\operatorname{Fix}(\{\phi\})$ to $\operatorname{Fix}\phi$ for any single endomorphism $\phi : G \to G$ in the context. It is obvious that $\operatorname{Fix}\mathcal{B} = \bigcap_{\phi\in\mathcal{B}} \operatorname{Fix}\phi$.

In [2], M. Bestvina and M. Handel proved Scott's conjecture that for any automorphism ϕ of a finitely generated free group G,

$$\operatorname{rank}\operatorname{Fix}\phi \leq \operatorname{rank} G.$$

In the book [6], W. Dicks and E. Ventura generalized the Bestvina–Handel result on the fixed subgroup of a single automorphism to a family of injective endomorphisms. They proved the following theorem.

Theorem 1.1. *(See [6, Corollary IV.5.8].) Let G be a finitely generated free group, and $\mathcal{B}$ a family of injective endomorphisms of G. Then*

$$\operatorname{rank}\operatorname{Fix}\mathcal{B} \leq \operatorname{rank} G.$$

They also showed that $\operatorname{Fix}\mathcal{B}$ is inert in G (see [6, Theorem IV.5.7]). A subgroup A is *inert* in G if for every subgroup $B \leq G$,

$$\operatorname{rank}(A \cap B) \leq \operatorname{rank} B.$$

In the paper [1], G. Bergman proved that the Dicks–Ventura result (Theorem 1.1) also holds for any family of endomorphisms but kept the following question open: Is the fixed subgroup of a family of endomorphisms of a finitely generated free group inert?

When G is a *surface group*, namely, G is isomorphic to the fundamental group of some connected closed surface S with Euler characteristic $\chi(S) < 0$, T. Soma estimated the rank of the intersection of any two subgroups A and B of G in terms of ranks of A and B. In [22], he showed the following enhanced version of the result of [21]:

$$\operatorname{rank}(A \cap B) - 1 \leq 1161(\operatorname{rank} A - 1)(\operatorname{rank} B - 1).$$

In fact, since Hanna Neumann's Conjecture was proved, by [16, Section 8], T. Soma's result should be improved to be:

$$\operatorname{rank}(A \cap B) - 1 \leq (\operatorname{rank} A - 1)(\operatorname{rank} B - 1).$$

When G is a one-relator group, D. Collins [4,5] studied the intersection of Magnus subgroups that we will describe in Section 4.

In this paper, we consider the intersection of fixed subgroups of endomorphisms of a surface group and prove a theorem similar to what B. Bergman did on free groups. We also consider the intersection of any subgroup with a geometric subgroup. The definition below is similar to, but not the same as the one given by P. Scott [20].

A connected subsurface (i.e., two dimensional submanifold) F of a connected surface S is called *incompressible* if the natural homomorphism $\pi_1(F) \to \pi_1(S)$ induced by the inclusion $F \hookrightarrow S$ is injective. If F is incompressible in S, then we can think of $\pi_1(F)$ as a subgroup of $\pi_1(S)$. Subgroups which arise in this way are called *geometric*.

For the fixed subgroup of a single endomorphism of a surface group, B. Jiang, S. Wang and Q. Zhang [12] showed that

Theorem 1.2. *(See [12, Theorem 1.2].) Let G be a surface group, and ϕ an endomorphism of G. Then*

(1) $\operatorname{rank} \operatorname{Fix} \phi \leq \operatorname{rank} G$ *if ϕ is epimorphic, with equality if and only if $\phi = id$;*
(2) $\operatorname{rank} \operatorname{Fix} \phi \leq \frac{1}{2} \operatorname{rank} G$ *if ϕ is not epimorphic.*

We generalize this result to any family of endomorphisms. The main result of this paper is the following.

Theorem 1.3. *Let G be a surface group, and $\mathcal{B}$ a family of endomorphisms of G. Then*

(1) $\operatorname{rank} \operatorname{Fix} \mathcal{B} \leq \operatorname{rank} G$, *with equality if and only if $\mathcal{B} = \{id\}$;*
(2) $\operatorname{rank} \operatorname{Fix} \mathcal{B} \leq \frac{1}{2} \operatorname{rank} G$, *if $\mathcal{B}$ contains a non-epimorphic endomorphism.*

For the geometric subgroups of a surface group, we prove that

Theorem 1.4. *If A is a geometric subgroup of a surface group G, then A is inert in G. Namely, for any subgroup B of G, we have*

$$\operatorname{rank}(A \cap B) \leq \operatorname{rank} B.$$

As a corollary, we have

Corollary 1.5. *The fixed subgroup of any family of epimorphisms of a surface group G is inert in G.*

The paper is organized as follows. In Section 2, we give some definitions and background knowledge for fixed point theory on surfaces and prove a strong version of Theorem 1.2. In Section 3, we study the inertia of geometric subgroups of surface groups, and give the proofs of Theorem 1.4 and Corollary 1.5. The technology used in this section is the covering theory. In Section 4, we discuss retracts and equalizers of a surface group. These special subgroups play a key role in the proof of our main result which we do in Section 5. At last, we give some examples and questions in Section 6.

2. The fixed subgroup of a single endomorphism

For the fixed subgroup of any single endomorphism of a surface group, we have the following theorem that is not stated but can be obtained from the paper [12].

Theorem 2.1. *Let G be a surface group, ϕ a non-identity epimorphism. If $\operatorname{Fix}\phi$ is not cyclic, then $\operatorname{Fix}\phi$ is a geometric free subgroup of G with*

$$\operatorname{rank}\operatorname{Fix}\phi < \operatorname{rank} G.$$

To prove Theorem 2.1, we need to introduce some facts on fixed points and fixed subgroups of a selfmap of a space.

For a selfmap $f : X \to X$ of a connected compact polyhedron X, the fixed point set

$$\operatorname{Fix} f := \bigl\{x \in X \bigm| f(x) = x\bigr\}$$

splits into a disjoint union of *fixed point classes*: two fixed points are in the same class if and only if they can be joined by a *Nielsen path*, which is a path homotopic (rel. its endpoints) to its f-image. For each fixed point class $\mathbf{F}$, a homotopy invariant *index* $\operatorname{ind}(f, \mathbf{F}) \in \mathbb{Z}$ is defined. A fixed point class is *essential* if its index is non-zero, otherwise, called *inessential* (see [10] for an introduction).

Although there are several approaches to define fixed point classes, we state the one using paths and introduce another homotopy invariant: the *rank* of a fixed point class $\mathbf{F}$ (see [12, §2]).

Definition 2.2. By an *f-route* we mean a homotopy class (rel. endpoints) of path $w : I \to X$ from a point $x \in X$ to $f(x)$. For brevity we shall often say the path w (in place of the path class $[w]$) is an f-route at $x = w(0)$. An f-route w gives rise to an endomorphism

$$f_w : \pi_1(X, x) \to \pi_1(X, x), \quad [a] \mapsto \bigl[w(f \circ a)\overline{w}\bigr]$$

where a is any loop based at x, and $\overline{w}$ denotes the reverse of w.

Two f-routes $[w]$ and $[w']$ are *conjugate* if there is a path $q : I \to X$ from $x = w(0)$ to $x' = w'(0)$ such that $[w'] = [\bar{q}w(f \circ q)]$, that is, w' and $\bar{q}w(f \circ q)$ are homotopic rel. endpoints.

Note that a constant f-route w corresponds to a fixed point $x = w(0) = w(1)$ of f, and the endomorphism f_w becomes the natural homomorphism induced by f,

$$f_* : \pi_1(X, x) \to \pi_1(X, x), \quad [a] \mapsto [f \circ a],$$

where a is any loop based at x. Two constant f-routes are conjugate if and only if the corresponding fixed points can be joint by a Nielsen path. This gives the following definition.

Definition 2.3. With an f-route w (more precisely, with its conjugacy class) we associate a *fixed point class* $\mathbf{F}_w$ of f, which consists of the fixed points that correspond to constant f-routes conjugate to w. Thus fixed point class are associated bijectively with conjugacy classes of f-routes. A fixed point class $\mathbf{F}_w$ can be empty if there is no constant f-route conjugate to w. Empty fixed point classes are inessential and distinguished by their associated route conjugacy classes.

Definition 2.4. For any f-route w, the fixed subgroup of the endomorphism f_w is the subgroup

$$\mathrm{Fix}(f_w) = \{\gamma \in \pi_1\big(X, w(0)\big) \mid f_w(\gamma) = \gamma\}.$$

Hence, we have the *rank* of $\mathbf{F}_w$ defined as

$$\mathrm{rank}(f, \mathbf{F}_w) := \mathrm{rank}\,\mathrm{Fix}(f_w),$$

it is well defined because conjugate f-routes have isomorphic fixed subgroups. Moreover, $\mathrm{rank}(f, \mathbf{F}_w)$ of a fixed point class is also a homotopy invariant.

According to Nielsen–Thurston's canonical classification theorem of surface homeomorphisms, any homeomorphism of a compact connected surface with negative Euler characteristic is isotopic to either a periodic, pseudo-Anosov or reducible map f (see W. Thurston [23]). Moreover, B. Jiang and J. Guo [11] stated that such f has a standard form so we call it a *standard map*. A standard map has fine-tuned local behavior and nice properties.

By the complete list of possible types of fixed point classes of a standard map given in [11, Lemma 3.6], we have

Lemma 2.5. *Every fixed point class of a standard map of a closed surface with negative Euler characteristic is an incompressible compact connected subsurface (possibly a point or a circle).*

Theorem 2.6. *(See [12, Theorem 3.2].) Let $f : S \to S$ be a standard map of a connected closed surface S with $\chi(S) < 0$. Then for any empty fixed point class $\mathbf{F}$, we have*

$$\operatorname{rank}(f, \mathbf{F}) \leq 1.$$

Now we give the proof of Theorem 2.1.

Proof of Theorem 2.1. Let G be a surface group, namely, $G = \pi_1(S, x)$ for some closed surface S with $\chi(S) < 0$. Then it is obvious that $\operatorname{rank} \operatorname{Fix} \phi < \operatorname{rank} G$ by Theorem 1.2. Now we show that $\operatorname{Fix} \phi$ is a geometric free subgroup.

Note that S is a $K(G, 1)$ space, then the endomorphism $\phi : G \to G$ is induced by a selfmap $g : (S, x) \to (S, x)$ (see [8, Proposition 1B.9]). Namely,

$$\phi = g_* : \pi_1(S, x) \to \pi_1(S, x), \quad [a] \mapsto [g \circ a],$$

where a is any loop based at $x \in \operatorname{Fix} g$.

Since ϕ is epimorphic, it is an automorphism because G is Hopfian. Hence, g can be homotopic to a homeomorphism, even to a standard map $f : (S, x) \to (S, f(x))$, via a homotopy $H = \{h_t\}_{t \in I}$. Then

$$\phi = f_w : \pi_1(S, x) \to \pi_1(S, x), \quad [a] \mapsto \big[w(f \circ a)\bar{w}\big],$$

where $w = \{h_t(x)\}_{t \in I}$. Therefore, $\operatorname{Fix} \phi = \operatorname{Fix}(f_w)$.

By assumption, $\operatorname{Fix}(f_w)$ is not cyclic. Then the fixed point class $\mathbf{F}_w$ corresponding to w is not empty according to Theorem 2.6. Thus there is a fixed point $* \in \mathbf{F}_w \subseteq \operatorname{Fix} f$ that is conjugate to w, namely, the loop $\bar{q}w(f \circ q)$ is homotopic to the point $*$, where q is a path from $x = w(0)$ to $*$. We have the following commutative diagram

$$\begin{tikzcd}
\pi_1(S, x) \arrow[r, "f_w"] \arrow[d, "q_\sharp"', "\cong"] & \pi_1(S, x) \arrow[d, "q_\sharp", "\cong"'] \\
\pi_1(S, *) \arrow[r, "f_*"] & \pi_1(S, *)
\end{tikzcd}$$

where $q_\sharp : [a] \mapsto [\bar{q}aq]$ is an isomorphism. This isomorphism carries the problem to the group $\pi_1(S, *)$. Thus, we may assume that $G = \pi_1(S, *)$, and

$$\phi = f_* : \pi_1(S, *) \to \pi_1(S, *), \quad [a] \mapsto [f \circ a], \tag{2.1}$$

where a is any loop based at $*$.

Recall that f is a standard map, then each Nielsen path of f can be deformed (rel. endpoints) into $\operatorname{Fix} f$ by [12, Proof of Corollary T] or [11, Lemmas 1.2, 2.2 and 3.4]. Hence every fixed point class is connected, and

$$\mathrm{Fix}(f_*) = \pi_1(\mathbf{F}_w, *) \leq \pi_1(S, *),$$

the last inequality holds because the fixed point class $\mathbf{F}_w$ is an incompressible subsurface according to Lemma 2.5. Therefore, the fixed subgroup $\mathrm{Fix}(f_*)$ is geometric. Clearly, $\mathbf{F}_w$ is a compact subsurface with nonempty boundary because $\phi \neq id$, hence $\mathrm{Fix}(f_*)$ is free. Therefore, $\mathrm{Fix}\,\phi$ is a geometric free subgroup of G by Eq. (2.1). □

At the end of this section, we give a lemma used frequently in this paper.

Lemma 2.7. *Let H be a proper subgroup of a surface group G with $\mathrm{rank}\, H \leq \mathrm{rank}\, G$. Then H is a free group. Furthermore, if $\phi : G \to G$ is an endomorphism but non-epimorphic, then $\phi(G)$ is a free group with*

$$\mathrm{rank}\,\phi(G) \leq \frac{1}{2}\,\mathrm{rank}\, G.$$

Proof. Let $G = \pi_1(S)$, where S is a closed surface with $\chi(S) < 0$. By covering theory of surfaces, the proper subgroup H is either free or $H \cong \pi_1(\widetilde{S})$ for some closed surface $\widetilde{S}$ with $\chi(\widetilde{S})/\chi(S) = |G : H| > 1$. But the latter implies $\mathrm{rank}\,\pi_1(\widetilde{S}) > \mathrm{rank}\,\pi_1(S)$ which contradicts to $\mathrm{rank}\, H \leq \mathrm{rank}\, G$. Therefore, H is a free group.

If $\phi : G \to G$ is an endomorphism but non-epimorphic, then $\phi(G) < G$ and $\mathrm{rank}\,\phi(G) \leq \mathrm{rank}\, G$. Thus $\phi(G)$ is a free group. Moreover, we have

$$\mathrm{rank}\,\phi(G) \leq \mathrm{Ir}(G),$$

where $\mathrm{Ir}(G)$ denotes the *inner rank* of G defined as the maximal rank of free homomorphic images of G. It is known that when G is a surface group, then $\mathrm{Ir}(G) = [\frac{1}{2}\,\mathrm{rank}\, G]$, the greatest integer not more than $\frac{1}{2}\,\mathrm{rank}\, G$. (See Lyndon and Schupp [14, p. 52] where it is attributed to Zieschang [24].) Therefore, we have $\mathrm{rank}\,\phi(G) \leq \frac{1}{2}\,\mathrm{rank}\, G$. □

3. The inertia of geometric subgroups of surface groups

In this section, we study the inertia of geometric subgroups of surface groups firstly, and then give the proofs of Theorem 1.4 and Corollary 1.5.

3.1. The inertia of geometric subgroups of surface groups

For the intersection of a subgroup with a geometric subgroup in the fundamental group of a surface, we have

Theorem 3.1. *Let S be a connected surface (may has punctures) with $\pi_1(S)$ finitely generated. If the subgroup $A \leq \pi_1(S)$ is geometric, and B is any subgroup of $\pi_1(S)$, then*

$$\mathrm{rank}(A \cap B) \leq \mathrm{rank}\, B.$$

Before giving the proof of Theorem 3.1, we need the lemma below. For brevity, a subsurface means it is connected unless it is specially stated otherwise.

Lemma 3.2. *Suppose S is a connected surface with $\pi_1(S)$ finitely generated, and $F \subseteq S$ is an incompressible subsurface. Then* $\operatorname{rank}\pi_1(F) \leq \operatorname{rank}\pi_1(S)$. *In particular, if S is closed, then $\pi_1(F)$ is either $\pi_1(S)$ itself or a free group with*

$$\operatorname{rank}\pi_1(F) < \operatorname{rank}\pi_1(S).$$

Proof. If F is closed, then S must be closed and $\pi_1(F) = \pi_1(S)$. Now we suppose F is neither closed nor a disk, then $\pi_1(F)$ is a free group. There are two cases:

Case (1). Both F and S are compact. Via a slightly push of ∂F into $\operatorname{int} S$, we can assume that $\partial F \cap \partial S = \emptyset$. Then each component of $\partial(S - \operatorname{int} F)$ is a circle which is either contained in ∂F or ∂S. Recall that F is incompressible and not a disk, then no component of $S - \operatorname{int} F$ is a disk. In fact, if there is a disk D, then $\partial D \subseteq \partial F$ which contradicts to that the natural homomorphism $\pi_1(F) \to \pi_1(S)$ is injective. Thus the Euler characteristic $\chi(S - \operatorname{int} F) \leq 0$ and

$$\chi(S) = \chi(F) + \chi(S - \operatorname{int} F) \leq \chi(F).$$

Therefore, $\operatorname{rank}\pi_1(F) \leq \operatorname{rank}\pi_1(S)$ when S is not closed, and $\operatorname{rank}\pi_1(F) < \operatorname{rank}\pi_1(S)$ when S is closed.

Case (2). At least one of F and S is not compact. If F is not compact, pick a core C_F of F, which is a compact subsurface of F such that each component of $F - C_F$ is an open annulus; if F is compact, set $C_F = F$. Moreover, we can choose a compact core C_S of S such that $C_F \subseteq C_S$ (see [20, Lemma 1.5]). Thus $\pi_1(C_F) = \pi_1(F)$, $\pi_1(C_S) = \pi_1(S)$ and the natural homomorphism $\pi_1(C_F) \to \pi_1(C_S)$ is also injective. The conclusion holds by Case (1). □

Proof of Theorem 3.1. If B is infinitely generated (i.e., $\operatorname{rank} B = \infty$) or $A \cap B = \{1\}$, it is trivial. Thus we assume B is finitely generated and $A \cap B \neq \{1\}$ below.

Since A is geometric, there is an incompressible subsurface $F \subseteq S$ such that $A = \pi_1(F, *) \leq \pi_1(S, *)$ for some base point $* \in F$. We have two maps: the inclusion $i : (F, *) \hookrightarrow (S, *)$, and the covering $k : (K, *_K) \to (S, *)$ associated to B, namely, $k_*(\pi_1(K, *_K)) = B$. Consider the commutative diagram:

$$\begin{tikzcd}
(F_0, *_0) \subseteq (\tilde{F}, *_0) \arrow[r, "i'"] \arrow[d, "p'"'] \arrow[dr, "p"] & (K, *_K) \arrow[d, "k"] \\
(F, *) \arrow[r, hook, "i"] & (S, *)
\end{tikzcd}$$

where

$$\tilde{F} = \{(x, y) \in F \times K \mid i(x) = k(y)\},$$

$p : \tilde{F} \to S$ is the pull back map such that

$$p\big((x, y)\big) = i(x) = k(y),$$

and F_0 is the component of $\tilde{F}$ containing the base point $*_0 = (*, *_K)$. Since $i : F \hookrightarrow S$ is an inclusion, $\tilde{F}$ can be identified as $k^{-1}(F)$, and $p' : \tilde{F} \to F$ can be identified as the covering $k|_{k^{-1}(F)} : k^{-1}(F) \to F$. Thus F_0 is a connected subsurface of K, and by the commutative diagram, $i'_* : \pi_1(F_0) \to \pi_1(K)$ is injective since $p' : F_0 \to F$ is a covering. Therefore, F_0 is incompressible in K. By Lemma 3.2, we have

$$\operatorname{rank} \pi_1(F_0) \le \operatorname{rank} \pi_1(K) = \operatorname{rank} B. \tag{3.1}$$

Moreover, $p_* : \pi_1(F_0) \to \pi_1(S)$ is also injective according to the commutative diagram, and we have

$$p_*\big(\pi_1(F_0)\big) \le i_*\big(\pi_1(F)\big) \cap k_*\big(\pi_1(K)\big) = A \cap B.$$

Now we claim that

$$p_*\big(\pi_1(F_0)\big) = A \cap B. \tag{3.2}$$

To prove the claim, it suffices to prove $A \cap B \le p_*(\pi_1(F_0))$. In fact, any nontrivial element $a \in A \cap B$ can be represented by a loop $\alpha \subset F \subseteq S$ containing the base point $*$. Since $[\alpha] = a \in A \cap B$, there is a lifting loop $\tilde{\alpha} \subset K$ containing the base point $*_K$. Therefore there is a loop $\tilde{\alpha}_0 \subset F_0$ containing the base point $*_0$ and $p(\tilde{\alpha}_0) = \alpha$, which implies $A \cap B \le p_*(\pi_1(F_0))$. Thus the claim holds.

Therefor we have

$$\operatorname{rank}(A \cap B) = \operatorname{rank} p_*\big(\pi_1(F_0)\big) = \operatorname{rank} \pi_1(F_0) \le \operatorname{rank} B,$$

where the first equality holds by Eq. (3.2), the second equality holds because p_* is injective, and the last inequality holds by Eq. (3.1). $\square$

3.2. *Proofs of Theorem 1.4 and Corollary 1.5*

It is obvious that Theorem 1.4 follows from Theorem 3.1. Now we give the proof of Corollary 1.5.

Proof of Corollary 1.5. Let G be a surface group, namely, $G \cong \pi_1(S)$ for a closed surface S with $\chi(S) < 0$, and $\mathcal{B}$ a family of epimorphisms of G. Suppose $K \le G$ is any subgroup of G.

If $\mathcal{B} = \{id\}$, or some $\beta \in \mathcal{B}$ has $\operatorname{Fix}\beta$ cyclic, then $\operatorname{rank}(K \cap \operatorname{Fix}\mathcal{B}) \leq \operatorname{rank} K$ is obvious.

Now we suppose $\mathcal{B} \neq \{id\}$ and $\operatorname{Fix}\beta$ is not cyclic for all $\beta \in \mathcal{B}$. Then by Theorem 2.1, for any non-identity epimorphism $\beta \in \mathcal{B}$, $\operatorname{Fix}\beta$ is a geometric free subgroup with $\operatorname{rank}\operatorname{Fix}\beta < \operatorname{rank} G$. Hence without loss of generality, we assume $id \notin \mathcal{B}$ in the following.

Note that G is finitely generated, then $\operatorname{End}(G)$, the set of all endomorphisms of G, is countable and hence $\mathcal{B} \subseteq \operatorname{End}(G)$ is also countable. Set $\mathcal{B} = \{\beta_1, \beta_2, \ldots\}$ and $\mathcal{B}_i = \{\beta_1, \ldots, \beta_i\}, i = 1, 2, \ldots$. (If $\mathcal{B}$ has a finite cardinality n, set $\beta_j = \beta_n$ and $\mathcal{B}_j = \mathcal{B}_n$ for all $j > n$.) Then we have a descending chain of fixed subgroups

$$\operatorname{Fix}\beta_1 = \operatorname{Fix}\mathcal{B}_1 \geq \operatorname{Fix}\mathcal{B}_2 \geq \ldots \geq \operatorname{Fix}\mathcal{B}_i \geq \ldots$$

whose terms are all free groups. Furthermore, we have a descending chain of free groups

$$K \cap \operatorname{Fix}\beta_1 = K \cap \operatorname{Fix}\mathcal{B}_1 \geq K \cap \operatorname{Fix}\mathcal{B}_2 \geq \ldots \geq K \cap \operatorname{Fix}\mathcal{B}_i \geq \ldots.$$

Note that $K \cap \operatorname{Fix}\mathcal{B}_{i+1} = K \cap \operatorname{Fix}\mathcal{B}_i \cap \operatorname{Fix}\beta_{i+1}$ and $\operatorname{Fix}\beta_{i+1}$ is geometric in G for all $i \geq 1$, by Theorem 1.4, we have $\operatorname{rank}(K \cap \operatorname{Fix}\mathcal{B}_{i+1}) \leq \operatorname{rank}(K \cap \operatorname{Fix}\mathcal{B}_i)$. Thus

$$\begin{aligned}&\operatorname{rank}(K \cap \operatorname{Fix}\beta_1)\\ &\quad = \operatorname{rank}(K \cap \operatorname{Fix}\mathcal{B}_1) \geq \operatorname{rank}(K \cap \operatorname{Fix}\mathcal{B}_2) \geq \ldots \geq \operatorname{rank}(K \cap \operatorname{Fix}\mathcal{B}_i) \geq \ldots.\end{aligned}$$

Note that

$$K \cap \operatorname{Fix}\mathcal{B} = K \cap \left(\bigcap_{i=1}^{\infty} \operatorname{Fix}\mathcal{B}_i\right) = \bigcap_{i=1}^{\infty}(K \cap \operatorname{Fix}\mathcal{B}_i),$$

we have

$$\operatorname{rank}(K \cap \operatorname{Fix}\mathcal{B}) = \operatorname{rank}\left(\bigcap_{i=1}^{\infty}(K \cap \operatorname{Fix}\mathcal{B}_i)\right) \leq \operatorname{rank}(K \cap \operatorname{Fix}\beta_1) \leq \operatorname{rank} K$$

where the first inequality is according to [15, Exercise 33, p. 118] that if the intersection of a descending chain of free groups has rank $\geq n$, then almost all terms of the chain have rank $\geq n$, and the second inequality holds since $\operatorname{Fix}\beta_1$ is geometric in G. It implies that $\operatorname{Fix}\mathcal{B}$ is inert. $\square$

4. Equalizers and retracts

In this section, we study the equalizers and retracts of surface groups.

4.1. Introduction to equalizers and retracts

Suppose G and H are two groups, $\phi : G \to H$ is an epimorphism. A *section* of ϕ is a homomorphism $\sigma : H \to G$ such that

$$\phi \circ \sigma = id : H \to H.$$

Then for any family $\mathcal{B}$ of sections of ϕ, the *equalizer* of $\mathcal{B}$

$$\mathrm{Eq}(\mathcal{B}) := \{h \in H \mid \sigma_1(h) = \sigma_2(h), \forall \sigma_1, \sigma_2 \in \mathcal{B}\}$$

is a subgroup of H.

Suppose H is a subgroup of a group G. If there is a homomorphism $\pi : G \to G$ such that $\pi(G) \le H$ and

$$\pi|_H = id : H \to H,$$

we say that π is a *retraction*, and H is a *retract* of G. We have $\operatorname{rank} H \le \operatorname{rank} G$ obviously. Moreover, if H is a proper subgroup, it is called a *proper retract*. Note that if a retract H of G is contained in a subgroup $K \le G$, then H is also a retract of K. Hence $\operatorname{rank} H \le \operatorname{rank} K$.

The following is a relation between equalizers and retracts.

Lemma 4.1. *Let G, H be two groups and $\phi : G \to H$ an epimorphism. If $\mathcal{B}$ is a family of sections of ϕ, then for any section $\sigma \in \mathcal{B}$, $\sigma(H)$ is a retract of G, and*

$$\sigma|_{\mathrm{Eq}(\mathcal{B})} : \mathrm{Eq}(\mathcal{B}) \to \bigcap_{\alpha \in \mathcal{B}} \alpha(H)$$

is an isomorphism.

Proof. For any $\sigma \in \mathcal{B}$, $\sigma : H \to G$ is a section of $\phi : G \to H$ implies

$$\phi \circ \sigma = id : H \to H,$$

and hence $\sigma \circ \phi : G \to \sigma(H)$ is an epimorphism such that

$$(\sigma \circ \phi)\big(\sigma(h)\big) = \sigma\big(\phi\big(\sigma(h)\big)\big) = \sigma(h)$$

for any $h \in H$. Therefore, $\sigma(H)$ is a retract of G.

Clearly σ is injective and $\sigma(\mathrm{Eq}(\mathcal{B})) \le \bigcap_{\alpha \in \mathcal{B}} \alpha(H)$. To prove $\sigma|_{\mathrm{Eq}(\mathcal{B})}$ is an isomorphism, it suffices to show

$$\sigma\big(\mathrm{Eq}(\mathcal{B})\big) = \bigcap_{\alpha \in \mathcal{B}} \alpha(H).$$

In fact, for any $g \in \bigcap_{\alpha \in \mathcal{B}} \alpha(H)$ and any $\alpha \in \mathcal{B}$, there exists $h_\alpha \in H$ such that $g = \alpha(h_\alpha)$. Hence $\phi(g) = \phi(\alpha(h_\alpha)) = h_\alpha$, which implies $\phi(g) \in \mathrm{Eq}(\mathcal{B})$ and hence $g = \sigma(\phi(g)) \in \sigma(\mathrm{Eq}(\mathcal{B}))$. $\square$

For equalizers and retracts of finitely generated free groups, G. Bergman gave the following results (see [1, Corollary 12 and Lemma 18]).

Proposition 4.2 *(Bergman).* (1) *Any intersection of retracts of a finitely generated free group is also a retract;*

(2) *If $\phi : G \to H$ is an epimorphism of free groups with H finitely generated, then the equalizer of any family of sections of ϕ is a free factor in H.*

4.2. Some facts on surface groups

In this subsection, we introduce some facts on surface groups.

Let S be a closed surface of genus g. It is well known that $\pi_1(S)$ has a standard presentation:

$$\pi_1(S) = \left\langle a_1, b_1, \ldots, a_g, b_g \,\middle|\, \prod_{i=1}^{g} [a_i, b_i] \right\rangle, \quad \text{or} \quad \pi_1(S) = \left\langle a_1, a_2, \ldots, a_g \,\middle|\, \prod_{i=1}^{g} a_i^2 \right\rangle$$

according to whether S is orientable or not.

For any generating set $X = \{x_1, \ldots, x_n\} \subset \pi_1(S)$, let $F_X = \langle y_1, ..., y_n \rangle$ be a free group with one generator for each element of X and denote the natural map $F_X \to \pi_1(S)$ by ϕ_X. Two generating sets X and X' of the same cardinality are *Nielsen equivalent* if there is an isomorphism $\epsilon : F_{X'} \to F_X$ such that the following diagram commutes.

$$\begin{tikzcd} F_{X'} \arrow[rr, "\epsilon"] \arrow[dr, "\phi_{X'}"'] & & F_X \arrow[dl, "\phi_X"] \\ & \pi_1(S) & \end{tikzcd}$$

In the paper [25], H. Zieschang showed that for any generating set $X \subset \pi_1(S)$ with cardinality $|X| = \operatorname{rank} \pi_1(S)$ for a closed orientable surface S of genus not 3 is Nielsen equivalent to the standard generating set. In [13], L. Louder generalized the result to any closed surface of any genus whether it is orientable or not.

For standard generating set X' of $\pi_1(S)$, the kernel of $\phi_{X'}$ is the normal closure of a word w in $F_{X'}$. Thus for any generating set X of $\pi_1(S)$ with $|X| = \operatorname{rank} \pi_1(S)$, the kernel of ϕ_X is the normal closure of a word $r = \epsilon(w)$ in F_X. Moreover, we can let r be cyclically reduced. Hence we have

Lemma 4.3. *Let S be a closed surface, and $n = \operatorname{rank} \pi_1(S)$. If $X = \{x_1, x_2, \ldots, x_n\}$ is any generating set of $\pi_1(S)$, then $\pi_1(S)$ has a new one-relator presentation*

$$\pi_1(S) = \langle x_1, x_2, \dots, x_n | r \rangle,$$

where r is a cyclically reduced word in the free group on the generating set X.

Let $G = \langle X|r \rangle$ be a one-relator group where r is a cyclically reduced word in the free group on the generating set X. A subset $Y \subset X$ is called a *Magnus subset* if Y omits a generator which appears in the relator r. A subgroup H of G is called a *Magnus subgroup* if $H = \langle Y \rangle$ for some Magnus subset Y of X, and hence by the Magnus Freiheitssatz [15, Theorem 4.10], H is free of rank $|Y|$. There were many studies ([3–5,9], etc.) on intersections of Magnus subgroups. In particular, D. Collins showed that

Theorem 4.4. *(See [4, Theorem 2].) The intersection $\langle Y \rangle \cap \langle Z \rangle$ of two Magnus subgroups of the one-relator group G is either $\langle Y \cap Z \rangle$ or the free product of $\langle Y \cap Z \rangle$ with an infinite cyclic group and thus of rank $|Y \cap Z| + 1$.*

4.3. Equalizers and retracts on surface groups

Now we consider equalizers and retracts of surface groups, which will play a key role in the proof of Theorem 1.3.

Lemma 4.5. *Let G be a surface group. If K is any proper retract of G, then K is a free group with rank*

$$\operatorname{rank} K \le \frac{1}{2} \operatorname{rank} G.$$

Furthermore, if H_1, H_2 are two proper retracts of G, and $H = \langle H_1, H_2 \rangle \le G$, the subgroup generated by H_1 and H_2, then

(1) *If $H < G$, then H is a free group, $H_1 \cap H_2$ is a retract of both H_1 and H_2, and*

$$\operatorname{rank}(H_1 \cap H_2) \le \min\{\operatorname{rank} H_1, \operatorname{rank} H_2\}.$$

(2) *If $H = G$, then $H_1 \cap H_2$ is cyclic (possibly trivial).*

Proof. Since K is a proper retract of the surface group G, there is an endomorphism $\beta : G \to G$ such that $\beta(G) = K < G$ and $\beta|_K = id$. By Lemma 2.7, K is a free group with rank $K \le \frac{1}{2} \operatorname{rank} G$.

Furthermore, since H_1 and H_2 are two proper retracts of G and $H = \langle H_1, H_2 \rangle$, we have

$$\operatorname{rank} H \le \operatorname{rank} H_1 + \operatorname{rank} H_2 \le \operatorname{rank} G. \tag{4.1}$$

There are two cases.

Case (1). $H < G$. Then H is a free group by Lemma 2.7. Note that H_1 and H_2 are both retracts of the free group H, then $H_1 \cap H_2$ is also a retract of H according to Proposition 4.2. It implies

$$\operatorname{rank}(H_1 \cap H_2) \leq \min\{\operatorname{rank} H_1, \operatorname{rank} H_2\}.$$

Case (2). $H = G$. Let X_i be a generating set of H_i, $i = 1, 2$. Then $X_1 \cup X_2$ is a generating set of G, moreover $|X_1 \cup X_2| = \operatorname{rank} G$ and $X_1 \cap X_2 = \emptyset$ by Eq. (4.1). Thus by Lemma 4.3, G has a one-relator presentation

$$G = \langle X_1 \cup X_2 | r\rangle$$

where r is a cyclic reduced word in the free group on the generating set $X_1 \cup X_2$. It implies that both X_1 and X_2 are Magnus subset and hence H_1 and H_2 are both Magnus subgroup of G. Therefore, the intersection $H_1 \cap H_2$ is a cyclic (possibly trivial) subgroup of G according to Theorem 4.4. □

Proposition 4.6. *Let G be a surface group and $\mathcal{R}$ a family of retracts of G. Then*

$$\operatorname{rank}\left(\bigcap_{H \in \mathcal{R}} H\right) \leq \min\{\operatorname{rank} H \mid H \in \mathcal{R}\} \leq \begin{cases} \operatorname{rank} G, & \mathcal{R} = \{G\} \\ \frac{1}{2} \operatorname{rank} G, & \mathcal{R} \neq \{G\} \end{cases}$$

Proof. For any proper retract $H \in \mathcal{R}$, H is a free group with rank

$$\operatorname{rank} H \leq \frac{1}{2} \operatorname{rank} G < \operatorname{rank} G$$

according to Lemma 4.5. Therefore, it suffices to assume that $\mathcal{R}$ consists of proper retracts in the following. There are two cases.

Case (1). There exist two retracts $H, H' \in \mathcal{R}$ such that $G = \langle H, H'\rangle$, then $H \cap H'$ is cyclic by Lemma 4.5. Note that $\bigcap_{H \in \mathcal{R}} H$ is a subgroup of the cyclic group $H \cap H'$, we have $\bigcap_{H \in \mathcal{R}} H$ is also cyclic, which implies

$$\operatorname{rank}\left(\bigcap_{H \in \mathcal{R}} H\right) \leq 1 \leq \min\{\operatorname{rank} H \mid H \in \mathcal{R}\}.$$

Case (2). For any two retracts H, H', $\langle H, H'\rangle < G$. Let $H_0 \in \mathcal{R}$ be the retract which has the minimal rank in $\mathcal{R}$, namely,

$$\operatorname{rank} H_0 = \min\{\operatorname{rank} H \mid H \in \mathcal{R}\}.$$

By Lemma 4.5, $\{H_0 \cap H \mid H \in \mathcal{R}\}$ is a family of retracts of the free group H_0. Therefore

$$\bigcap_{H\in\mathcal{R}} H = H_0 \cap \left(\bigcap_{H\in\mathcal{R}} H\right) = \bigcap_{H\in\mathcal{R}} (H_0 \cap H)$$

is a retract of H_0 according to Proposition 4.2. Hence

$$\operatorname{rank}\left(\bigcap_{H\in\mathcal{R}} H\right) \leq \operatorname{rank} H_0 = \min\{\operatorname{rank} H \mid H \in \mathcal{R}\}.$$

The proof is finished. □

Proposition 4.7. *Let G be a surface group and F a finitely generated free group. If $\phi : G \to F$ is an epimorphism, and $\mathcal{B}$ is a family of sections of ϕ, then*

$$\operatorname{rank} \operatorname{Eq}(\mathcal{B}) \leq \operatorname{rank} F \leq \frac{1}{2} \operatorname{rank} G.$$

Proof. For any section $\sigma \in \mathcal{B}$, $\sigma(F) < G$ since $\sigma(F)$ is isomorphic to the free group F. By Lemma 4.1, we have an isomorphism

$$\sigma|_{\operatorname{Eq}(\mathcal{B})} : \operatorname{Eq}(\mathcal{B}) \to \bigcap_{\alpha\in\mathcal{B}} \alpha(F)$$

where $\{\alpha(F) \mid \alpha \in \mathcal{B}\}$ is a family of proper retracts of G. Therefore, the conclusion holds according to Proposition 4.6. □

5. Proof of Theorem 1.3

The aim of this section is to prove Theorem 1.3.

Proof of Theorem 1.3. Suppose G is a surface group and $\mathcal{B}$ is a family of endomorphisms of G. There are two cases.

Case (1). $\mathcal{B}$ consists of epimorphisms. Then $\operatorname{Fix} \mathcal{B}$ is inert in G by Corollary 1.5, and we have

$$\operatorname{rank} \operatorname{Fix} \mathcal{B} = \operatorname{rank}(G \cap \operatorname{Fix} \mathcal{B}) \leq \operatorname{rank} G,$$

and when $\mathcal{B} = \{id\}$, the equality holds obviously. Moreover, if $\mathcal{B} \neq \{id\}$, then there is a non-identity epimorphism $\beta \in \mathcal{B}$ with $\operatorname{rank} \operatorname{Fix} \beta < \operatorname{rank} G$ according to Theorem 2.1, and we have

$$\operatorname{rank} \operatorname{Fix} \mathcal{B} = \operatorname{rank}(\operatorname{Fix} \beta \cap \operatorname{Fix} \mathcal{B}) \leq \operatorname{rank} \beta < \operatorname{rank} G.$$

Case (2). $\mathcal{B}$ contains a non-epimorphic endomorphism.

The proof of this case is partly inspired by G. Bergman's paper [1]. Since $\operatorname{Fix}\alpha \cap \operatorname{Fix}\beta \subseteq \operatorname{Fix}(\alpha\beta)$ for any $\alpha, \beta \in \mathcal{B}$, we may assume that $\mathcal{B}$ is closed under composition and contains the identity endomorphism. Recall that $\mathcal{B}$ contains a non-epimorphic endomorphism, we can choose $\beta \in \mathcal{B}$ such that $\beta(G)$ is a free group with

$$\operatorname{rank}\bigl(\beta(G)\bigr) = \min\bigl\{\operatorname{rank}\bigl(\gamma(G)\bigr) \bigm| \gamma \in \mathcal{B}\bigr\} \le \frac{1}{2}\operatorname{rank} G$$

according to Lemma 2.7. Thus all elements of $\mathcal{B}$ act injective on $\beta(G)$. Indeed, if there is $\gamma \in \mathcal{B}$ acts not injective on $\beta(G)$, then $\operatorname{rank}(\gamma\beta(G)) < \operatorname{rank}(\beta(G))$ contradicts to the minimality of $\operatorname{rank}(\beta(G))$. Let $\beta\mathcal{B} = \{\beta\gamma \mid \gamma \in \mathcal{B}\}$. Note that $\beta\gamma(\beta(G)) \le \beta(G)$, thus we have a family $\beta\mathcal{B}|_{\beta(G)}$ of injective endomorphisms of the free group $\beta(G)$,

$$\beta\gamma|_{\beta(G)} : \beta(G) \to \beta(G).$$

Since $\operatorname{Fix}(\beta\mathcal{B}) = \operatorname{Fix}(\beta\mathcal{B}|_{\beta(G)}) \le \beta(G)$, for brevity, we omit the restriction if no confusion is possible. Therefore, by Theorem 1.1, we have

$$\operatorname{rank}\operatorname{Fix}(\beta\mathcal{B}) \le \operatorname{rank}\bigl(\beta(G)\bigr) \le \frac{1}{2}\operatorname{rank} G. \tag{5.1}$$

Clearly, $\operatorname{Fix}\mathcal{B}$ is a subgroup of the free group $\operatorname{Fix}(\beta\mathcal{B})$. Now we claim that

Claim 5.1. $\operatorname{rank}\operatorname{Fix}\mathcal{B} \le \operatorname{rank}\operatorname{Fix}(\beta\mathcal{B})$.

Proof. Let

$$E = \beta^{-1}\bigl(\operatorname{Fix}(\beta\mathcal{B})\bigr) \le G,$$

then there is an epimorphism

$$\beta : E \to \operatorname{Fix}(\beta\mathcal{B}),$$

and a family of sections $\mathcal{B}|_{\operatorname{Fix}(\beta\mathcal{B})}$ of β

$$\gamma|_{\operatorname{Fix}(\beta\mathcal{B})} : \operatorname{Fix}(\beta\mathcal{B}) \to E, \quad \forall\gamma \in \mathcal{B}.$$

Note that $\operatorname{Fix}\mathcal{B} \le \operatorname{Fix}(\beta\mathcal{B})$ and $\mathcal{B}$ contains the identity (and hence $\mathcal{B}|_{\operatorname{Fix}(\beta\mathcal{B})}$ contains the identity), then

$$\operatorname{Fix}\mathcal{B} = \operatorname{Fix}(\mathcal{B}|_{\operatorname{Fix}(\beta\mathcal{B})}) = \operatorname{Eq}(\mathcal{B}|_{\operatorname{Fix}(\beta\mathcal{B})}). \tag{5.2}$$

Recall that E is a subgroup of the surface group G, then E is either free or isomorphic to a surface group. If E is a free group, then by Proposition 4.2, $\operatorname{Eq}(\mathcal{B}|_{\operatorname{Fix}(\beta\mathcal{B})})$ is a free factor of $\operatorname{Fix}(\beta\mathcal{B})$, and hence $\operatorname{rank}\operatorname{Eq}(\mathcal{B}|_{\operatorname{Fix}(\beta\mathcal{B})}) \le \operatorname{rank}\operatorname{Fix}(\beta\mathcal{B})$; if E is a surface group, then we also have $\operatorname{rank}\operatorname{Eq}(\mathcal{B}|_{\operatorname{Fix}(\beta\mathcal{B})}) \le \operatorname{rank}\operatorname{Fix}(\beta\mathcal{B})$ according to Proposition 4.7. Thus by Eq. (5.2), Claim 5.1 holds.

Therefore, by Eq. (5.1) and Claim 5.1, we have

$$\operatorname{rank}\operatorname{Fix}\mathcal{B} \le \frac{1}{2}\operatorname{rank} G$$

and the proof is finished. □

6. Examples and questions

In this section, we give some examples and questions on surface groups.

The example below shows that the fundamental group of a torus also satisfies the conclusion of Theorem 1.3.

Example 6.1. Let $G = \langle a, b | a^{-1}b^{-1}ab\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, and ϕ an endomorphism of G. It is well known that any subgroup of G is also abelian with rank ≤ 2. Thus

$$\operatorname{rank}\operatorname{Fix}\phi \le \operatorname{rank} G.$$

Now we claim that $\operatorname{Fix}\phi$ is a cyclic group (possibly trivial) when $\phi \ne id$.

Indeed, pick a basis $a = (1, 0)$ and $b = (0, 1)$ of G. Then $G = \{(u, v) \mid u, v \in \mathbb{Z}\}$, and ϕ can be presented as a 2×2 matrix A with integral entries

$$\phi(x) = xA, \quad \forall x = (u, v) \in G.$$

If $\operatorname{rank}\operatorname{Fix}\phi = 2$, then there are two non-parallel vectors $x_1, x_2 \in \operatorname{Fix}\phi$ such that $x_1A = x_1$ and $x_2A = x_2$. For any $x \in G$, suppose $x = kx_1 + lx_2$, $k, l \in \mathbb{Q}$, it implies

$$xA = (kx_1 + lx_2)A = kx_1A + lx_2A = kx_1 + lx_2 = x.$$

Namely, $\phi = id$. Therefore, the claim holds.

The example below shows that the fundamental group of a Klein bottle has a nonidentity automorphism with fixed subgroup of rank 2, hence it does not satisfy the conclusion of Theorem 1.3.

Example 6.2. Let $G = \langle a, b | bab^{-1}a\rangle$ be the fundamental group of a Klein bottle, and ϕ an endomorphism of G. Since $a^{-1}b^{\epsilon} = b^{\epsilon}a, ab^{\epsilon} = b^{\epsilon}a^{-1}$ in G for $\epsilon = \pm 1$, any element g of G can be write uniquely as $b^m a^n$. Suppose

$$\phi(a) = b^s a^t, \qquad \phi(b) = b^p a^q.$$

We have

$$\phi\bigl(bab^{-1}a\bigr) = b^p a^q b^s a^t a^{-q} b^{-p} b^s a^t = b^{2s} a^{(-1)^{s-p}[(-1)^s q + t - q] + t} = 1.$$

Thus $s = 0$ and $(-1)^p t + t = 0$. There are two cases.

Case (1). $t = 0$. Then $\phi(a) = 1, \phi(b) = b^p a^q$.

If there exists $1 \neq g = b^m a^n \in G$ fixed by ϕ, then

$$b^m a^n = g = \phi(g) = \phi\bigl(b^m a^n\bigr) = \phi\bigl(b^m\bigr) = \bigl(b^p a^q\bigr)^m = b^{mp} a^k.$$

We have $m = 0$ or $p = 1$. If $m = 0$, then $g = a^n$ and $g = \phi(g) = \phi(a^n) = 1$ contradicts to that g is nontrivial.

So $p = 1$, namely $\phi(a) = 1, \phi(b) = ba^q$. $\operatorname{Fix}\phi$ is a cyclic subgroup generated by ba^q.

Case (2). $t \neq 0$ and p is odd. Then $\phi(a) = a^t, \phi(b) = b^p a^q$.

If there exists $1 \neq g = b^m a^n \in G$ fixed by ϕ, then a same argument as in Case (1) implies $m = 0$ or $p = 1$. There are two subcases.

Subcase (2.1). If $p \neq 1$ which means $m = 0$, then $g = a^n \neq 1$ and $g = \phi(g) = \phi(a^n) = a^{tn}$. We have $t = 1$ and $\phi(a) = a, \phi(b) = b^p a^q$. $\operatorname{Fix}\phi$ is a cyclic subgroup generated by a.

Subcase (2.2). If $p = 1$, then $\phi(a) = a^t, \phi(b) = ba^q$.

If $t = 1$ and $q = 0$, then $\phi = id$ and $\operatorname{Fix}\phi = G$.

If $t = 1$ and $q \neq 0$, then $\operatorname{Fix}\phi$ is generated by a and b^2 which is isomorphic to a rank two free abelian group $\mathbb{Z} \oplus \mathbb{Z}$.

If $t \neq 1$, then we have

$$b^m a^n = g = \phi(g) = \phi\bigl(b^m a^n\bigr) = \bigl(ba^q\bigr)^m a^{tn} = \begin{cases} b^m a^{q+tn}, & m \text{ is odd} \\ b^m a^{tn}, & m \text{ is even.} \end{cases}$$

Note that for any $k \in \mathbb{Z}$, $b^2 = (ba^k)(ba^k)$. So if $\frac{q}{1-t} \in \mathbb{Z}$, then $\operatorname{Fix}\phi$ is a cyclic subgroup generated by $ba^{\frac{q}{1-t}}$; if $\frac{q}{1-t}$ is not an integer then $\operatorname{Fix}\phi$ is a cyclic subgroup generated by b^2.

In conclusion, we have prove that $\operatorname{Fix}\phi$ is either G, $\langle a, b^2\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, $\mathbb{Z}$ or trivial for any endomorphism ϕ of G. So $\operatorname{Fix}\mathcal{B}$ is also one of such subgroups for any family of endomorphisms $\mathcal{B}$.

The following example shows that Theorem 1.3 is sharp.

Example 6.3. Let the surface group $G = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i]\rangle$. Consider the automorphism $\phi_n : G \to G$ induced by a Dehn twist:

$$a_i \mapsto a_i, \quad i = 1, \ldots, g;$$

$$b_j \mapsto b_j, \quad j = 1, \ldots, g-1; \qquad b_g \mapsto a_g^n b_g.$$

Then

$$\bigcap_{n=1}^{\infty} \operatorname{Fix} \phi_n = \langle a_1, b_1, \dots, a_{g-1}, b_{g-1}, a_g \rangle \cong F_{2g-1},$$

a free group with rank $2g - 1$.

The example below shows that the intersection of two retracts of a surface group is not a retract, which is not similar to the case of free groups, see Proposition 4.2.

Example 6.4. Let $G = \langle a, b, c, d | a^{-1}b^{-1}abc^{-1}d^{-1}cd \rangle$ be a surface group, and

$$A = \langle a, b \rangle < G, \qquad B = \langle c, d \rangle < G.$$

Note that

$$\phi : G \to A, \quad a, d \mapsto a, \quad b, c \mapsto b$$

and

$$\psi : G \to B, \quad a, d \mapsto d, \quad b, c \mapsto c$$

are two retractions. Then A and B are two retracts of G. But the intersection

$$A \cap B = \langle a^{-1}b^{-1}ab \rangle < G$$

is not a retract. Indeed, if there is a retraction $\pi : G \to \langle a^{-1}b^{-1}ab \rangle$, then

$$\pi(a^{-1}b^{-1}ab) = \pi(a^{-1})\pi(b^{-1})\pi(a)\pi(b) = \pi(a^{-1})\pi(a)\pi(b^{-1})\pi(b) = 1 \in \langle a^{-1}b^{-1}ab \rangle,$$

the second equality holds since $\langle a^{-1}b^{-1}ab \rangle$ is cyclic. It contradicts to $\pi|_{\langle a^{-1}b^{-1}ab \rangle} = id$.

On retracts of surface groups, we have a question below generalized from [1, Question 20]: Is every retract R of a finitely generated free group F inert in F?

Question 6.5. *Is every retract H of a surface group G inert in G? Namely, is*

$$\operatorname{rank}(H \cap K) \leq \operatorname{rank} K$$

for any subgroup $K \leq G$?

If K is also a surface group, then the answer is affirmative.

Indeed, we have a finite covering $p : \tilde{S} \to S$ of closed surfaces such that $G = \pi_1(S)$ and $K = p_*(\pi_1(\tilde{S})) \leq G$. Since H is a proper retract of G, H is a free subgroup with

$\operatorname{rank} H \leq \frac{1}{2} \operatorname{rank} G$ according to Lemma 4.5. Thus there is a noncompact surface F' and a covering $f' : F' \to S$ such that $H = f'_*(\pi_1(F'))$. Pick a compact incompressible subsurface $F \subset F'$ such that $\pi_1(F) = \pi_1(F')$, then the map $f = f'|_F : F \to S$ is π_1-injective and $H = f_*(\pi_1(F))$. As in the proof of Theorem 3.1, consider the pull back map $p' : M \to F$ of p and f, where

$$M = \{(x, y) \in F \times \tilde{S} \mid f(x) = p(y)\}$$

such that $p'((x, y)) = x$. Since $p : \tilde{S} \to S$ is a finite covering, p' is also a finite covering. Let $M_0 \subseteq M$ be the component containing the base point. Then $p'|_{M_0} : M_0 \to F$ is a covering of compact surfaces with nonempty boundary of sheets $\chi(M_0)/\chi(F) \leq \chi(\tilde{S})/\chi(S)$. It implies that

$$\frac{1 - \operatorname{rank} \pi_1(M_0)}{1 - \operatorname{rank} \pi_1(F)} \leq \frac{2 - \operatorname{rank} \pi_1(\tilde{S})}{2 - \operatorname{rank} \pi_1(S)}.$$

Note that $H \cap K = f_* p'_*(\pi_1(M_0)) \cong \pi_1(M_0)$, thus

$$\operatorname{rank}(H \cap K) - 1 \leq \frac{(\operatorname{rank} H - 1)(\operatorname{rank} K - 2)}{\operatorname{rank} G - 2}.$$

Hence

$$\operatorname{rank}(H \cap K) \leq \frac{1}{2} \operatorname{rank} K \leq \operatorname{rank} K.$$

If K is free, and the subgroup $\langle H, K\rangle \leq G$ generated by H and K is also free, then Question 6.5 becomes [1, Question 20].

If K is free, and $\langle H, K\rangle \leq G$ is a surface group, what will happen?

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References

[1] G. Bergman, Supports of derivations, free factorizations and ranks of fixed subgroups in free groups, Trans. Amer. Math. Soc. 351 (1999) 1531–1550.

[2] M. Bestvina, M. Handel, Train tracks and automorphisms of free groups, Ann. of Math. 135 (1992) 1–51.

[3] S. Brodskii, Anomalous products of locally indicable groups in: Algebraic Systems, Ivanov. Gos. Univ., Ivanovo, 1981, pp. 51–77 (in Russian).
[4] D. Collins, Intersections of Magnus subgroups of one-relator groups, in: T.W. Müller (Ed.), Groups: Topological, Combinatorial and Arithmetic Aspects, in: London Math. Soc. Lecture Note Ser., vol. 311, Cambridge Univ. Press, Cambridge, 2004, pp. 255–296.
[5] D. Collins, Intersections of conjugates of Magnus subgroups of one-relator groups, Geom. Topol. Monogr. 14 (2008) 135–171.
[6] W. Dicks, E. Ventura, The Group Fixed by a Family of Injective Endomorphisms of a Free Group, Contemp. Math., vol. 195, American Mathematical Society, Providence, 1996.
[7] J. Friedman, Sheaves on graphs and a proof of the Hanna Neumann conjecture, arXiv:1105.0129v1.
[8] A. Hatcher, Algebraic Topology, Cambridge University Press, 2002.
[9] J. Howie, Magnus intersections in one-relator products, Michigan Math. J. 53 (2005) 597–623.
[10] B. Jiang, Lectures on Nielsen Fixed Point Theory, Contemp. Math., vol. 14, American Mathematical Society, Providence, 1983.
[11] B. Jiang, J. Guo, Fixed points of surface diffeomorphisms, Pacific J. Math. 160 (1) (1993) 67–89.
[12] B. Jiang, S. Wang, Q. Zhang, Bounds for fixed points and fixed subgroups on surfaces and graphs, Algebr. Geom. Topol. 11 (2011) 2297–2318.
[13] L. Louder, Nielsen equivalence of generating sets for closed surface groups, http://arxiv.org/abs/1009.0454v1.
[14] R. Lyndon, P. Schupp, Combinatorial Group Theory, Ergeb. Math. Grenzgeb., vol. 89, Springer, Berlin, 1977.
[15] W. Magnus, A. Karrass, D. Solitar, Combinatorial Group Theory, Wiley, New York, 1966.
[16] I. Mineyev, The topology and analysis of the Hanna Neumann conjecture, J. Topol. Anal. 3 (2011) 307–376, http://dx.doi.org/10.1142/S1793525311000611.
[17] I. Mineyev, Submultiplicativity and the Hanna Neumann conjecture, Ann. of Math. 175 (2012) 393–414, http://dx.doi.org/10.4007/annals.2012.175.1.11.
[18] H. Neumann, On the intersection of finitely generated free groups, Publ. Math. (Debr.) 4 (1956) 186–189.
[19] H. Neumann, On the intersection of finitely generated free groups, in: Addendum, in: Publ. Math. (Debr.), vol. 5, 1957, p. 128.
[20] P. Scott, Subgroups of surface groups are almost geometric, J. Lond. Math. Soc. (2) 17 (1978) 555–565.
[21] T. Soma, Intersection of finitely generated surface groups, J. Pure Appl. Algebra 66 (1990) 81–95.
[22] T. Soma, Intersection of finitely generated surface groups II, Bull. Kyushu Inst. Technol. (Math. Natur. Sci.) 38 (1991) 13–22.
[23] W.P. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces, Bull. Amer. Math. Soc. 19 (1988) 417–431.
[24] H. Zieschang, Über einfache Kurven auf Vollbrezeln, Abh. Math. Semin. Univ. Hambg. 25 (1961/1962) 231–250.
[25] H. Zieschang, Über die Nielsensche Kürzungsmethode in freien Produkten mit Amalgam, Invent. Math. 10 (1970) 4–37.