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Some remarks on Gill's theorems on Young modules

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ABSTRACT

In a recent paper, [10], Gill, using methods of the modular representation theory of finite groups, describes some results on the tensor product of Young modules for symmetric groups. We here give an alternative approach using the polynomial representation theory of general linear groups and the Schur functor. The main result is a formula for the multiplicities of Young modules in a tensor product in terms of the characters of the simple polynomial modules for general linear groups. Our approach is also valid for Young modules for Hecke algebras of type A at roots of unity and here the formula involves the characters of the simple polynomial modules for quantised general linear groups.

1. Introduction

In [10], Gill considers the coefficients obtained by expressing the tensor product of Young modules, for a symmetric group, as a direct sum of Young modules: $Y(\lambda)\otimes Y(\mu) = \bigoplus_\nu y^\nu_{\lambda,\mu} Y(\nu)$. Working over a field of positive characteristic p, and using methods of the modular representation theory of finite groups, he shows in particular that $y^{p\nu}_{p\lambda,p\mu} = y^\nu_{\lambda,\mu}$, [10], Theorem 3.6. He also gives a lower bound for the Cartan invariant $c_{\lambda,\mu}$ of the Schur algebra in terms of the base p expansions of λ and μ, [10] Theorem 4.1. We here adopt a different approach, by first considering the corresponding problems in the category of

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polynomial representations and then applying the Schur functor, as in [11], Chapter 6, or [6], Section 2.1. Our main result, see Theorem 3.2, is a formula for the coefficients $y^\nu_{\lambda,\mu}$ in terms of what we call modular Kostka numbers, which describe the weight space multiplicities of the irreducible polynomial modules. From this we obtain the first of Gill's results mentioned above. We also obtain the lower bound for the Cartan numbers.

It is no more difficult, from our point of view, to work with quantised general linear groups and Hecke algebras, so we adopt this point of view from the outset. However, there is one interesting difference in the quantum case. Whereas the tensor product $Y(\lambda) \otimes Y(\mu)$ has a natural module structure in the classical case, we only assign the tensor product of Young modules for the Hecke algebra a meaning as a virtual module, and we give an example (in Section 6) to show that this is not represented by a direct sum of Young modules in general.

2. Preliminaries

2.1. We begin by introducing the usual notation for the combinatorics associated with polynomial representations, for the most part following Green, [11]. By a partition we mean an infinite sequence of weakly decreasing non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $\lambda_i = 0$ for $i \gg 0$. The length of the zero partition is 0 and a non-zero partition λ has length l if $\lambda_l = 0$ and $\lambda_i = 0$ for $i > l$. We identify the partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ with the finite sequence $(\lambda_1, \ldots, \lambda_n)$ if $\lambda_{n+1} = 0$. By the degree of a finite sequence of non-negative integers we mean the sum of its terms.

Fix a positive integer n and a non-negative integer r. We denote by $\Lambda(n)$ the set of n-tuples of non-negative integers. We denote by $\Lambda(n,r)$ set of elements of $\Lambda(n)$ of degree r. We denote by $\Lambda^+(n)$ the set of partitions of length at most n and denote by $\Lambda^+(n,r)$ the set of partitions of length at most n and degree r. We write $P(r)$ for the set of all partitions of r (so $P(r) = \Lambda^+(n,r)$, for $n \geq r$). Elements of $\Lambda(n)$ will be called (polynomial) weights and elements of $\Lambda^+(n)$ called dominant (polynomial) weights. The set $\Lambda(n,r)$ has a natural partial order, namely the dominance order: for $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ we have $\alpha \leq \beta$ if α and β have the same degree and $\sum_{j=1}^i \alpha_j \leq \sum_{j=1}^i \beta_j$, for all $1 \leq i \leq n$.

We write $\mathrm{Sym}(r)$ for the group of symmetries of $\{1, \ldots, r\}$. Then $W = \mathrm{Sym}(n)$ acts naturally on $\Lambda(n)$ by place permutation. The integral monoid ring of $\Lambda(n)$ will be identified with the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$. The natural action of W on $x_1, \ldots, x_n$ extends to an action on $\mathbb{Z}[x_1, \ldots, x_n]$ by ring automorphisms. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Lambda(n)$ we set $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ and, as in [13], define the monomial symmetric function

$$m_\lambda = \sum_{\alpha \in W\lambda} x^\alpha$$

for $\lambda \in \Lambda^+(n)$.

2.2. We now recall some facts about the representation theory of Hecke algebras of type A. Let r be a positive integer. We fix a field K and $0 \neq q \in K$. We write $\mathrm{Hec}(r)$ for the Hecke algebra of degree r over K with standard generators $T_1, \ldots, T_{r-1}$ and parameter q, as in [6]. For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, of degree r, we write $\mathrm{Hec}(\lambda)$ for the "Young Hecke subalgebra", i.e., the algebra $\mathrm{Hec}(\lambda_1) \otimes \cdots \otimes \mathrm{Hec}(\lambda_n)$ regarded as a subalgebra of $\mathrm{Hec}(r)$ in the natural way. We write $\mathrm{Sym}(\lambda)$ for the group $\mathrm{Sym}(\lambda_1) \times \cdots \times \mathrm{Sym}(\lambda_n)$ regarded as a subgroup of $\mathrm{Sym}(r)$ in the natural way.

We write K also for the one dimensional $\mathrm{Hec}(\lambda)$-module on which each standard generator $T_i \in \mathrm{Hec}(\lambda)$ act as multiplication by q. We write $M(\lambda)$ for the "permutation module" $\mathrm{Hec}(r) \otimes_{\mathrm{Hec}(\lambda)} K$. The Young modules $Y(\mu)$, μ a partition of r, are the indecomposable $\mathrm{Hec}(r)$-module summands, of the permutation modules. More precisely, we have

$$M(\lambda) = Y(\lambda) \oplus Z$$

where Z is a direct sum of modules $Y(\mu)$ with μ bigger than λ in the dominance order, see [6], 2.1, (9) and 4.4, (1).

We write $\mathrm{Rep}(\mathrm{Hec}(r))$ for the Grothedieck group of the category $\mathrm{mod}(\mathrm{Hec}(r))$, of finite dimensional left $\mathrm{Hec}(r)$-modules and write $\mathrm{Rep}_{\mathrm{Young}}(\mathrm{Hec}(r))$ for the subgroup of $\mathrm{Rep}(\mathrm{Hec}(r))$ generated by the classes of the Young modules. Thus we have

$$\mathrm{Rep}_{\mathrm{Young}}(\mathrm{Hec}(r)) = \bigoplus_{\lambda \in P(r)} \mathbb{Z}[Y(\lambda)] = \bigoplus_{\lambda \in P(r)} \mathbb{Z}[M(\lambda)] \tag{1}$$

where $[X]$ denotes the class in $\mathrm{Rep}(\mathrm{Hec}(r))$ of a finite dimensional $\mathrm{Hec}(r)$-module X.

We write $\mathrm{Ch}(\mathrm{Sym}(r))$ for the group of generalised characters of the symmetric group $\mathrm{Sym}(r)$. Then $\mathrm{Ch}(\mathrm{Sym}(r))$ is free on the characters $\eta_\lambda = 1 \uparrow_{\mathrm{Sym}(\lambda)}^{\mathrm{Sym}(r)}$, as λ runs over partition of r. Thus we have an isomorphism

$$\theta : \mathrm{Rep}_{\mathrm{Young}}(\mathrm{Hec}(r)) \to \mathrm{Ch}(\mathrm{Sym}(r)) \tag{2}$$

taking $[M(\lambda)]$ to η_λ.

Note that in case $K = \mathbb{C}$ and $q = 1$ the map θ assigns to $[M(\lambda)]$, the ordinary character of $M(\lambda)$ and it follows by $\mathbb{Z}$-linearity that $\theta([X])$ is the ordinary character of X for any finite dimensional $K\mathrm{Sym}(r)$-module in this case.

For more on the assignment of ordinary characters of symmetric groups to modules for Hecke algebra, see [7].

2.3. In this section we take $q = 1$ so that $\mathrm{Hec}(r) = K\mathrm{Sym}(r)$. From Mackey's tensor product theorem, for partitions λ, μ of r we have

$$M(\lambda) \otimes M(\mu) = \bigoplus_{\delta \in \Delta} K \uparrow_{\delta \mathrm{Sym}(\lambda) \delta^{-1} \cap S_\mu}^{\mathrm{Sym}(r)}$$

where Δ is a set of $(\mathrm{Sym}(\lambda), \mathrm{Sym}(\mu))$ double coset representatives in $\mathrm{Sym}(r)$. Thus we have

$$M(\lambda) \otimes M(\mu) = \bigoplus_{\nu \in P(r)} m^{\nu}_{\lambda,\mu} M(\nu)$$

where $m^{\nu}_{\lambda,\mu}$ is the number of $\delta \in \Delta$ such that $\delta S_\lambda \delta^{-1} \cap S_\mu$ is conjugate to S_ν. This applies in particular when K has characteristic 0 so that the product of permutation characters is described in the same way, i.e.,

$$\eta_\lambda \cdot \eta_\mu = \sum_{\nu} m^{\nu}_{\lambda,\mu} \eta_\nu. \tag{3}$$

If K has arbitrary characteristic (and $q = 1$) the group $\mathrm{Rep}_{\mathrm{Young}}(K\mathrm{Sym}(r))$ has a natural ring structure (given by the tensor product) and it follows that the map (2) is then a ring isomorphism.

The integers $m^{\nu}_{\lambda,\mu}$ have a simple combinatorial interpretation. By [12], 1.3.8, 1.3.9, we have that $m^{\nu}_{\lambda,\mu}$, is the cardinality of the set $Z(\nu, \lambda, \mu)$ of sufficiently large matrices with non-negative entries that have row sums $\lambda_1, \lambda_2, \ldots$ and column sums $\mu_1, \mu_2, \ldots$ and such that listing all the entries in weakly descending order gives ν.

Let t be a positive integer. Let γ be a partition of r and let λ, μ be partitions of tr. If $Z(t\gamma, t\lambda, t\mu)$ is not empty then there is some matrix Q of non-negative integers with row sums $\lambda_1, \lambda_2, \ldots$ and column sums $\mu_1, \mu_2, \ldots$ such that the entries of Q, listed in descending order, gives $t\gamma$. Thus all entries in Q are divisible by t and the row sums and column sums are divisible by t. Thus we may write $\lambda = t\alpha$, $\mu = t\beta$, for partitions λ, μ of r. Moreover, if $P \in Z(\nu, \lambda, \mu)$ then $tP \in Z(t\nu, t\lambda, t\mu)$ and if $Q \in Z(t\lambda, t\mu, t\nu)$ then $\frac{1}{t}Q \in Z(\nu, \lambda, \mu)$. In particular the sets $Z(\nu, \alpha, \beta)$ and $Z(t\nu, t\alpha, t\beta)$ are in bijective correspondence. So, for a partition γ of r and partitions λ, μ of tr we have:

$$m^{t\gamma}_{\lambda,\mu} = \begin{cases} m^{\gamma}_{\alpha,\beta}, & \text{if } \lambda, \mu \text{ are divisible by } t \text{ with } \lambda = t\alpha, \mu = t\beta; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Theorem 3.5 of [10] may be regarded as an analogue of this for Young modules.

3. Modular Kostka numbers and tensor products of Young modules

Let K be a field. We regard the category of quantum groups as the opposite of the category of Hopf algebras over K. Less formally, we use the expression "G is a quantum group over K" to indicate that we in mind a Hopf algebra over K, which we call the coordinate algebra of G and denote $K[G]$, cf. the approach of Parshall–Wang, [14]. Fix a positive integer n and $0 \neq q \in K$. We omit q from notation as much as possible. We write simply $G(n)$ for the quantum general linear group as in [3]. We recall briefly the construction of $G(n)$. The algebra $A(n)$ is given by generators c_{ij}, $1 \leq i, j \leq n$, subject to certain relations, see [6], 0.20. Then $A(n)$ has a bialgebra structure with comultiplication $\delta : A(n) \to A(n) \otimes A(n)$ and evaluation $\epsilon : A(n) \to K$ satisfying

$$\delta(c_{ij}) = \sum_{h=1}^{n} c_{ih} \otimes c_{hj} \quad \text{and} \quad \epsilon(c_{ij}) = \delta_{ij}$$

for $1 \leq i,\ j \leq n$, where δ_{ij} is the Kronecker delta. We call the c_{ij} the coefficient elements. The K-algebra $A(n)$ has a natural grading $A(n) = \bigoplus_{r \geq 0} A(n,r)$, in which each coefficient element has degree 1. Furthermore, each $A(n,r)$ is a subcoalgebra.

The determinant element

$$d = \sum_{\pi \in \mathrm{Sym}(n)} \mathrm{sgn}(\pi) c_{1,\pi(1)} \dots c_{n,\pi(n)}$$

of $A(n)$ is group-like and the Ore localisation $A(n)_d$ is a Hopf algebra. The quantum general linear group $G(n)$ is the quantum group over K whose coordinate algebra $K[G(n)]$ is $A(n)_d$.

Recall that, for a quantum group G over K, a (left) G-module is, by definition, a right $K[G]$-comodule. A $G(n)$-module V, with structure map $\tau : V \otimes A(n)$, is called polynomial (resp. polynomial of degree r) if $\tau(V) \leq V \otimes A(n)$ (resp. $\tau(V) \leq A(n,r)$). We regard a $G(n)$-module that is polynomial also as an $A(n)$-comodule and a $G(n)$-module that is polynomial of degree r also as an $A(n,r)$-comodule. The dual algebra $S(n,r) = A(n,r)^*$ is called the Schur algebra. The category of $G(n)$-modules that are polynomial of degree r is naturally equivalent to the category of $A(n,r)$-comodules and hence to the category of $S(n,r)$-modules. When we need background results on polynomial modules we shall ofter refer to [6], where the results are most often expressed in terms of $S(n,r)$-modules.

Recall that for a polynomial $G(n)$-module V and $\alpha \in \Lambda(n)$ we have the weight space V^α and that $V = \oplus_{\alpha \in \Lambda(n)} V^\alpha$. To a finite dimensional polynomial $G(n)$-module V we assign its character

$$\mathrm{ch}\, V = \sum_{\alpha \in \Lambda(n)} (\dim V^\alpha) x^\alpha \in \mathbb{Z}[x_1, \dots, x_n].$$

For each $\lambda \in \Lambda(n)$ there is an irreducible $G(n)$-module $L(\lambda)$ such that $\dim L(\lambda)^\lambda = 1$ and, for $\alpha \in \Lambda(n)$, we have $\dim L(\lambda)^\alpha = 0$ unless $\alpha \leq \lambda$. The modules $L(\lambda)$, $\lambda \in \Lambda^+(n)$ (resp. $\lambda \in \Lambda^+(n,r)$), form a complete set of pairwise non-isomorphic irreducible $G(n)$-modules that are polynomial (resp. polynomial of degree r, for $r \geq 0$). When we wish to emphasise the role of n we write $L_n(\lambda)$, for $L(\lambda)$, $\lambda \in \Lambda^+(n)$.

The classical case $q = 1$ is of course very important. In the classical case we write $\bar{c}_{ij}$ for c_{ij}, $1 \leq i,\ j \leq n$, write $\bar{G}(n)$ for $G(n)$ and write $\bar{L}(\lambda)$ for $L(\lambda)$, $\lambda \in \Lambda^+(n)$. If K is infinite then the polynomial representation theory as described above corresponds to the polynomial representation theory of $\mathrm{GL}_n(K)$ as in [11], in the following way. We identify $\bar{c}_{ij}$ with the K-valued function taking $g \in \mathrm{GL}_n(K)$ to the (i,j)-coefficient of g

and in this way identify $A(n)$ with an algebra of K-valued functions on $\mathrm{GL}_n(K)$. Given a finite dimensional polynomial $\mathrm{GL}_n(K)$-module (in the sense of Green, [11]) V with basis $v_1, \ldots, v_m$ we obtain coefficient functions $f_{ij} \in A(n)$ defined by the equations

$$gv_i = \sum_{j=1}^{m} f_{ji}(g) v_j$$

for $1 \leq i \leq m$ and thereby identify V with the $A(n)$-comodule with structure map $\tau : V \to V \otimes A(n)$ given by $\tau(v_i) = \sum_{j=1}^{m} v_j \otimes f_{ji}$ for $1 \leq i \leq m$.

For $\lambda, \mu \in \Lambda^+(n)$ we define $a^{(n)}_{\lambda,\mu} = \dim L_n(\lambda)^\mu$. We call the integers $a^{(n)}_{\lambda,\mu}$ the modular Kostka numbers. Note that it follows from [11], (6.5f), that, regarding λ, μ as elements of $\Lambda^+(N, r)$, for $N \geq n$, we have $a^{(N)}_{\lambda,\mu} = a^{(n)}_{\lambda,\mu}$. Therefore, for partitions λ, μ of the same degree write simply $a_{\lambda,\mu}$ for $a^{(n)}_{\lambda,\mu}$, if λ and μ have at most n parts.

Since we will be comparing the case of arbitrary q with the classical situation it is useful to have a different notation in that case, so we write $\bar{a}_{\lambda,\mu}$ for $a_{\lambda,\mu}$ when $q = 1$. For a finite dimensional polynomial $G(n)$-module V and $\alpha \in \Lambda(n)$, $w \in W$, we have $\dim V^{w\alpha} = \dim V^\alpha$. It follows that, for $\lambda \in \Lambda^+(n, r)$, we have

$$\mathrm{ch}\, L(\lambda) = \sum_{\mu \in \Lambda^+(n,r)} a_{\lambda,\mu} m_\mu \tag{5}$$

(with m_μ is as in Section 2.1).

If K has characteristic 0 and $q = 1$ then the character of $L(\lambda)$ is Schur's symmetric function s_λ, see [11], Section 3.5, and so $a_{\lambda,\mu}$ is the Kostka number $K_{\lambda,\mu}$, as in [13]. In general the matrix $(a_{\lambda,\mu})$, with rows ordered consistently with the dominance order, is unitriangular so there exist uniquely determined integers $b_{\lambda,\mu}$, for $\lambda, \mu \in \Lambda^+(n, r)$, such that

$$m_\lambda = \sum_{\mu \in \Lambda^+(n,r)} b_{\lambda,\mu}\, \mathrm{ch}\, L(\mu) \tag{6}$$

with $b_{\lambda,\lambda} = 1$ and $b_{\lambda,\mu} = 0$ for $\lambda \not\leq \mu$. The matrices $(a_{\lambda,\mu})_{\lambda,\mu \in \Lambda^+(n,r)}$ and $(b_{\lambda,\mu})_{\lambda,\mu \in \Lambda^+(n,r)}$ are inverse to each other. We call the $b_{\lambda,\mu}$ the inverse modular Kostka numbers. We write $\bar{b}_{\lambda,\mu}$ for $b_{\lambda,\mu}$ when $q = 1$.

Note that the expression "modular Kostka number" is used for something different in [9] and related papers.

We shall need another interpretation of the modular Kostka numbers. We denote by E the natural $G(n)$-module, i.e., the K-vector space on basis $e_1, \ldots, e_n$ regarded as a $G(n)$-module via the comodule structure map $\tau : E \to E \otimes A(n)$ given by $\tau(e_i) = \sum_{j=1}^{n} e_j \otimes c_{ji}$ for $1 \leq i \leq n$. For $r \geq 0$, the rth symmetric power $S^r E$ is naturally a $G(n)$-module and for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Lambda(n, r)$ we write $S^\alpha E$ for the $G(n)$-module $S^{\alpha_1} E \otimes \cdots \otimes S^{\alpha_n} E$. The module $S^\alpha E$ is polynomial of degree r. For $\lambda \in \Lambda^+(n)$ we write

$I(\lambda)$ for the injective hull of $L(\lambda)$ in the category of polynomial $G(n)$-module (i.e., $I(\lambda)$ is the injective hull of $L(\lambda)$ as an $A(n)$-comodule).

By [6], 2.1, (9), for any $G(n)$-module X which is polynomial of degree r we have that $\mathrm{Hom}_{G(n)}(X, S^\alpha E)$ is isomorphic to X^α. In particular the functor $\mathrm{Hom}_{G(n)}(-, S^\alpha E)$ is exact on finite dimensional polynomial modules of degree r and $S^\alpha E$ is injective in the polynomial category. Thus, for $\lambda \in \Lambda^+(n,r)$, we write

$$S^\lambda E = \bigoplus_{\mu\in\Lambda^+(n,r)} h_\mu I(\mu)$$

for certain non-negative integers h_μ. Applying $\mathrm{Hom}_{G(n)}(L(\mu), -)$ we find that

$$h_\mu = \dim \mathrm{Hom}_{G(n)}(L(\mu), S^\lambda E) = \dim L(\mu)^\lambda = a_{\mu,\lambda}$$

for $\mu \in \Lambda^+(n,r)$. Hence we have

$$S^\lambda E = \bigoplus_{\mu\in\Lambda^+(n,r)} a_{\mu,\lambda} I(\mu). \tag{7}$$

Now assume that $n \geq r$. Then we have the Schur functor $f : M(n,r) \to \mathrm{mod}(\mathrm{Hec}(r))$, where $M(n,r)$ is the category of finite dimensional $G(n)$-modules which are polynomial of degree r. The functor f is exact and takes $S^\lambda E$ to $M(\lambda)$ and takes $I(\lambda)$ to $Y(\lambda)$, for $\lambda \in \Lambda^+(n,r)$, by [6], Section 2.1, (20)(i) and Section 4.4, (1). Applying f to (7) gives

$$M(\lambda) = \bigoplus_{\mu\in P(r)} a_{\mu,\lambda} Y(\mu) \tag{8}$$

for $\lambda \in \Lambda^+(n,r)$. Interpreting this as an equation in $\mathrm{Rep}(\mathrm{Hec}(r))$ and inverting we thus get

$$[Y(\lambda)] = \sum_{\mu\in P(r)} b_{\mu,\lambda}[M(\mu)] \tag{9}$$

for $\lambda \in \Lambda^+(n,r)$.

In general we give $\mathrm{Rep}_{\mathrm{Young}}(\mathrm{Hec}(r))$ a ring structure using the group isomorphism $\theta : \mathrm{Rep}_{\mathrm{Young}}(\mathrm{Hec}(r)) \to \mathrm{Ch}(\mathrm{Sym}(r))$ of (2) above. As remarked in Section 2.3 this agrees with the natural structure given by the tensor product in the classical case. We have

$$[M(\lambda)].[M(\mu)] = \sum_{\nu\in P(r)} m^\nu_{\lambda,\mu}[M(\nu)]$$

by (3). Thus, from (9), for $\lambda, \mu \in P(r)$, we have

$$[Y(\lambda)].[Y(\mu)] = \sum_{\alpha,\beta\in P(r)} b_{\alpha,\lambda} b_{\beta,\mu}[M(\alpha)].[M(\beta)].$$

Now from (8) we obtain one of our main results.

Theorem 3.1. *For $\lambda, \mu \in P(r)$ we have*

$$[Y(\lambda)].[Y(\mu)] = \sum_{\nu \in P(r)} y^{\nu}_{\lambda,\mu}[Y(\nu)]$$

where $y^{\nu}_{\lambda,\mu} = \sum_{\alpha,\beta,\gamma \in P(r)} b_{\alpha,\lambda} b_{\beta,\mu} m^{\gamma}_{\alpha,\beta} a_{\nu,\gamma}$.

In the case $q = 1$, we may replace the product of classes in $\mathrm{Rep}_{\mathrm{Young}}(\mathrm{Hec}(r))$ by the tensor product so we have the following.

Theorem 3.2. *For $\lambda, \mu \in P(r)$ the tensor product of the Young modules $Y(\lambda)$, $Y(\mu)$ for the symmetric group* $\mathrm{Sym}(r)$ *is given by*

$$Y(\lambda) \otimes Y(\mu) = \bigoplus_{\nu \in P(r)} y^{\nu}_{\lambda,\mu} Y(\nu)$$

where $y^{\nu}_{\lambda,\mu} = \sum_{\alpha,\beta,\gamma \in P(r)} b_{\alpha,\lambda} b_{\beta,\mu} m^{\gamma}_{\alpha,\beta} a_{\nu,\gamma}$.

4. The influence of Steinberg's tensor product theorem

If q is not a root of unity then $\mathrm{Hec}(\lambda)$ is semisimple, see for example [6], Section 2.1, Remark (iii). In that case the Young modules are the irreducible modules and the map $\theta : \mathrm{Rep}_{\mathrm{Young}}(\mathrm{Hec}(r)) \to \mathrm{Ch}(\mathrm{Sym}(r))$ takes $[Y(\lambda)]$ to the irreducible character χ^{λ} labelled by λ, for $\lambda \in \Lambda^+(n, r)$. The problem of describing the $y^{\nu}_{\lambda,\mu}$ is then equivalent to that of describing the well known problem of decomposing the product of characters $\chi^{\lambda}\chi^{\mu}$. For a recent contribution to this difficult problem see [1].

We therefore assume from now on that q is a root of unity. Let l be the smallest positive integer such that $1 + q + \cdots + q^{l-1} = 0$.

We have, as in [6], Section 3.2, the Frobenius morphism $F : G(n) \to \bar{G}(n)$, whose comorphism $F^{\sharp} : K[\bar{G}(n)] \to K(G(n)]$ satisfies $F^{\sharp}(\bar{c}_{ij}) = c^{l}_{ij}$, for $1 \leq i,\ j \leq n$. For a $\bar{G}(n)$-module V we write V^F for the corresponding $G(n)$-module. Thus, if V has structure map $\tau : V \to V \otimes K[\bar{G}(n)]$ then V^F is the K-space V regarded as a $G(n)$-module via the structure map $(\mathrm{id}_V \otimes F^{\sharp}) \circ \tau : V \to V \otimes K[G(n)]$, where $\mathrm{id}_V : V \to V$ is the identity map.

For $\lambda \in \Lambda^+(n)$ we write $\bar{L}(\lambda)$ for the irreducible $\bar{G}(n)$-module of high weight λ. We write $X^0(n)$ for the set of all $\lambda \in \Lambda^+(n)$ such that $0 \leq \lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n, \lambda_n < l$. Then an element $\lambda \in \Lambda^+(n)$ may be uniquely expressed $\lambda = \lambda^0 + l\bar{\lambda}$, with $\lambda^0 \in X^0(n)$, $\bar{\lambda} \in \Lambda^+(n)$.

Theorem 4.1 *(Steinberg's tensor product theorem). Let $\lambda^0 \in X^0(n)$, $\bar{\lambda} \in \Lambda^+(n)$ and $\lambda = \lambda^0 + l\bar{\lambda}$. Then we have*

$$L(\lambda) = L(\lambda^0) \otimes \bar{L}(\bar{\lambda})^F.$$

See e.g., [6], Section 3.2(5).

We also define the Frobenius twist of an element of $\mathbb{Z}[x_1,\dots,x_n]$. For $\theta = \sum_{\alpha\in\Lambda(n)} c_\alpha x^\alpha$ we set $\theta^F = \sum_{\alpha\in\Lambda(n)} c_\alpha x^{l\alpha}$. Then it is easy to check that, for a finite dimensional polynomial $G(n)$-module V, we have $\operatorname{ch} V^F = (\operatorname{ch} V)^F$. Hence we have

$$\operatorname{ch} L(\lambda^0 + l\bar\lambda) = \operatorname{ch} L(\lambda^0).(\operatorname{ch} \bar L(\bar\lambda))^F \tag{10}$$

for $\lambda^0 \in X^0(n)$, $\bar\lambda \in \Lambda^+(n)$ and in particular

$$\operatorname{ch} L(l\bar\lambda) = (\operatorname{ch} \bar L(\bar\lambda))^F. \tag{11}$$

In view of (11), for $\lambda \in \Lambda^+(n)$ divisible by l with $\lambda = l\alpha$, $\alpha \in \Lambda^+(n)$, we have

$$a_{\lambda,\mu} = \begin{cases} \bar a_{\alpha,\beta}, & \text{if } \mu = l\beta \text{ for some } \beta \in \Lambda^+(n); \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Equally, we observe that, for $\lambda \in \Lambda^+(n)$, we have $m_\lambda^F = m_{l\lambda}$ so we that for $\lambda \in \Lambda^+(n)$ divisible by l with $\lambda = l\alpha$, $\alpha \in \Lambda^+(n)$, we have

$$b_{\lambda,\mu} = \begin{cases} \bar b_{\alpha,\beta}, & \text{if } \mu = l\beta \text{ for some } \beta \in \Lambda^+(n); \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

We now obtain our version of Gill's Theorem, [10], Theorem 3.6, on the Young module multiplicities.

Theorem 4.2. *Let λ, μ, ν be partitions of r. Then we have*

$$y^{l\nu}_{l\lambda,l\mu} = \bar y^{\nu}_{\lambda,\mu}.$$

Proof. We fix $n \geq lr$. By Theorem 3.1, we have

$$y^{l\nu}_{l\lambda,l\mu} = \sum_{\alpha,\beta,\gamma\in\Lambda^+(n,lr)} b_{\alpha,l\lambda} b_{\beta,l\mu} m^{\gamma}_{\alpha,\beta} a_{l\nu,\gamma}.$$

Now from (12) we get

$$y^{l\nu}_{l\lambda,l\mu} = \sum_{\alpha,\beta\in\Lambda^+(n,lr),\gamma\in\Lambda^+(n,r)} b_{\alpha,l\lambda} b_{\beta,l\mu} m^{l\gamma}_{\alpha,\beta} \bar a_{\nu,\gamma},$$

and so by (4) and (13) we have

$$y^{l\nu}_{l\lambda,l\mu} = \sum_{\alpha,\beta,\gamma\in\Lambda^+(n,r)} \bar b_{\alpha,\lambda} \bar b_{\beta,\mu} m^{\gamma}_{\alpha,\beta} \bar a_{\nu,\gamma}$$

and from Theorem 3.1 we get

$$y^{l\nu}_{l\lambda,l\mu} = \bar{y}^{\nu}_{\lambda,\mu}$$

as required. $\square$

Remark 1. In the classical case $q = 1$ with K a field of characteristic $p > 0$ this says $(Y(p\lambda) \otimes Y(p\mu) \mid Y(p\nu)) = (Y(\lambda) \otimes Y(\nu) \mid Y(\nu))$. (Here, for finite dimensional modules X, Y over $K\mathrm{Sym}(r)$ with Y indecomposable we are writing $(X \mid Y)$ for the number of components of X isomorphic to Y in a decomposition of X as a direct sum of indecomposable modules.)

Remark 2. Note that actually the only thing we have used from Steinberg's tensor product is the trivial special case: $L(l\bar{\lambda}) = \bar{L}(\bar{\lambda})^F$, for $\bar{\lambda} \in \Lambda^+(n)$. For general $\lambda, \mu \in \Lambda^+(n)$ one could write $\lambda = \lambda^0 + l\bar{\lambda}$, $\mu = \mu^0 + l\bar{\mu}$ (with λ^0, μ^0 restricted, $\bar{\lambda}, \bar{\mu} \in \Lambda^+(n)$) and carry through the above analysis using the full force of Steinberg's tensor product theorem and Theorem 3.1 to give a reduction formula for $y^{\nu}_{\lambda,\mu}$ generalising Theorem 4.2. The result would be rather complicated and, in the absence of obvious applications, we do not pursue this line of argument here.

5. Cartan invariants

We give a formula for the Cartan invariants, similar to our formula for Young module multiplicities, Theorem 3.1. Let n be a positive integer. Let $\lambda \in \Lambda^+(n)$. Then we have the induced module $\nabla(\lambda)$ and the character of $\nabla(\lambda)$ is given by Weyl's character formula, see [6], 0.16. A filtration $0 = V_0 \leq V_1 \leq \cdots \leq V_m = V$ of a finite dimensional polynomial $G(n)$-module V is called good if for each $0 < i \leq m$, the section V_i/V_{i-1} is isomorphic to $\nabla(\lambda)$ for some $\lambda \in \Lambda^+(n)$ (which may depend on i). For a finite dimensional polynomial module V admitting a good filtration and $\lambda \in \Lambda^+(n)$ the number of sections in a good filtration is independent of the good filtration (it is determined by the character of V) and will be denoted $(V : \nabla(\lambda))$.

Let $\lambda, \mu \in \Lambda^+(n, r)$, for some $r \geq 0$. The Cartan number $c_{\lambda,\mu}$ is the multiplicity of $L(\mu)$ as a composition factor of $I(\lambda)$ and from Schur's Lemma one has also $c_{\lambda,\mu} = \dim \mathrm{Hom}_{G(n)}(I(\lambda), I(\mu))$. We shall write $c^{(n)}_{\lambda,\mu}$ for the Cartan integer, $\nabla_n(\lambda)$, for the induced module and $I_n(\lambda)$ the polynomial injective module and $L_n(\lambda)$ for the simple module corresponding to λ, for the moment, to emphasise dependence on n, for $\lambda \in \Lambda^+(n, r)$. Let $N \geq n$. The injective module $I_N(\lambda)$ has a filtration with sections $\nabla_N(\xi)$, $\xi \in \Lambda^+(N, r)$, and the multiplicity of $\nabla_N(\xi)$ in such a filtration is equal to the multiplicity $[\nabla_N(\xi) : L_N(\lambda)]$, of $L_N(\lambda)$ as a composition factor of $\nabla_N(\lambda)$, [5], Section 4, (6). Hence we have

$$c^{(N)}_{\lambda.\mu} = [I_N(\lambda) : L_N(\mu)] = \sum_{\mu \in \Lambda^+(N,r)} (I_N((\lambda) : \nabla_N(\xi)).[\nabla_N(\xi) : L_N(\mu)].$$

By reciprocity, [5], Section 4, (6), we have $(I_N((\lambda) : \nabla_N(\xi) = [\nabla_N(\xi) : L_N(\lambda)]$ and moreover, if $[\nabla_N(\xi) : L_N(\mu)] \neq 0$ then $\xi \geq \mu$, which implies that ξ has at most n parts. Hence we have

$$c^{(N)}_{\lambda.\mu} = \sum_{\mu \in \Lambda^+(n,r)} [\nabla_N(\xi) : L_N(\lambda)]).[\nabla_N(\xi) : L_N(\mu)].$$

Furthermore, we have $[\nabla_N(\xi) : L_N(\nu)] = [\nabla_n(\xi) : L_n(\nu)]$, for $\nu \in \Lambda^+(n,r)$, by [6], 4.2, (6) (and [11], (6.6e) Theorem, in the classical case). Hence we have

$$c^{(N)}_{\lambda.\mu} = \sum_{\mu \in \Lambda^+(n,r)} [\nabla_n(\xi) : L_n(\lambda)]).[\nabla_n(\xi) : L_n(\mu)]$$

which is $c^{(n)}_{\lambda,\mu}$. Thus, for $\lambda, \mu \in \Lambda^+(n,r)$ and $N \geq n$ we have

$$c^{(N)}_{\lambda.\mu} = c^{(n)}_{\lambda,\mu}. \tag{14}$$

From now on for partitions λ, μ of the same degree we write $c_{\lambda,\mu}$ for the stable value $c^{(n)}_{\lambda,\mu}$, where n is such that λ and μ have at most n parts.

Remark 5.1. Suppose $\lambda, \mu \in \Lambda^+(n,r)$ and suppose that $n \geq r$. It follows from [6], 2.1, (8) that $S^\lambda E$ is a polynomially injective module with has $I(\lambda)$ as a component and so by [6], (2.1), (16)(ii) and (i), (ii) we have that the Schur functor f induces an isomorphism $\mathrm{Hom}_{G(n)}(I(\lambda), I(\mu)) \to \mathrm{Hom}_{\mathrm{Hec}(r)}(Y(\lambda), Y(\mu))$ so that $c_{\lambda,\mu} = \dim \mathrm{Hom}_{\mathrm{Hec}(r)}(Y(\lambda), Y(\mu))$.

We now describe the Cartan numbers in terms of the inverse modular Kostka numbers.

Proposition 5.2. *For partitions λ and μ of the same degree r, say, we have*

$$c_{\lambda,\mu} = \sum_{\nu,\alpha,\beta} m^\nu_{\alpha,\beta} b_{\alpha,\lambda} b_{\beta,\mu},$$

where ν, α, β range over all partitions of degree r.

Proof. We fix $n \geq r$. Let λ, μ be partitions of degree r. We claim that

$$\dim \mathrm{Hom}_{G(n)}(S^\lambda E, S^\mu E) = \sum_\nu m^\nu_{\lambda,\mu}. \tag{15}$$

The dimension of $\mathrm{Hom}_{G(n)}(S^\lambda E, S^\mu E)$ is the dimension of the μ-weight space of $S^\lambda E$, by [6], 2.1, (8) and this is easily seen to be independent of K and q. Thus, in checking (15), we may (and do) take K to be the field of complex numbers and $q = 1$. But then, applying the Schur functor to $S^\lambda E$, $S^\mu E$ and using [6], 2.1, (16)(ii), we get

$$\dim \operatorname{Hom}_{G(n)}(S^\lambda E, S^\mu E) = \dim \operatorname{Hom}_{\mathrm{Sym}(r)}(M(\lambda), M(\mu) = (\eta_\lambda, \eta_\mu)$$

and

$$(\eta_\lambda, \eta_\mu) = (\eta_\lambda \cdot \eta_\mu, 1) = \sum_\nu (m^\nu_{\lambda,\mu}\eta_\nu, 1) = \sum_\nu m^\nu_{\lambda,\mu}$$

(with the sum over partitions of r) as required.

We suppose once more that K is an arbitrary field and q is an arbitrary non-zero element of K. By (7) we have

$$\dim \operatorname{Hom}_{G(n)}(S^\lambda E, S^\mu E) = \sum_{\alpha,\beta} a_{\alpha,\lambda} a_{\beta,\mu} \dim \operatorname{Hom}_{G(n)}(I(\alpha), I(\beta))$$

i.e.,

$$\sum_{\alpha,\beta\in\Lambda^+(n,r)} a_{\alpha,\lambda} a_{\beta,\mu} c_{\alpha,\beta} = \sum_{\nu\in\Lambda^+(n,r)} m^\nu_{\lambda,\mu}.$$

Let $\theta, \eta \in \Lambda^+(n,r)$. Multiplying by $b_{\lambda,\theta} b_{\mu,\eta}$ and summing over λ, μ we get $c_{\theta,\eta} = \sum m^\nu_{\lambda,\beta} b_{\alpha,\lambda} b_{\beta,\mu}$ and hence

$$c_{\lambda,\mu} = \sum_{\nu,\alpha,\beta\in\Lambda^+(n,r)} m^\nu_{\alpha,\beta} b_{\alpha,\lambda} b_{\beta,\mu}$$

as required. □

Remark. We discuss the observation of Gill, [10], Theorem 4.1, giving a lower bound for the Cartan integer $c_{\lambda,\mu}$ in terms of the base p-expansion of $\lambda, \mu \in \Lambda^+(n)$. In our context this is explained as follows. We write $\lambda = \lambda^0 + l\bar{\lambda}$, $\mu = \mu^0 + l\bar{\mu}$, with $\lambda^0, \mu^0, \bar{\lambda}, \bar{\mu} \in \Lambda^+(n)$ and λ^0, μ^0 being l-restricted. The according to [8], Lemma 3.2(i), (and [8], Section 5), the module $I(\lambda^0) \otimes \bar{I}(\bar{\lambda})^F$ embeds in $I(\lambda)$. Here $\bar{I}(\bar{\lambda})$ denotes the injective envelope of $\bar{L}(\bar{\lambda})$ (as a polynomial $\bar{G}(n)$-module).

From this it is clear that the multiplicity of $L(\mu) = L(\mu^0) \otimes \bar{L}(\bar{\mu})^F$ as a composition factor is at least the product of the multiplicity of $L(\mu^0)$ as a composition factor of $I(\lambda^0)$ and the multiplicity of $\bar{L}(\bar{\mu})$ as a composition factor of $\bar{I}(\bar{\lambda})$ i.e., that

$$c_{\lambda,\mu} \geq c_{\lambda^0,\mu^0} . c_{\bar{\lambda},\bar{\mu}}. \tag{16}$$

Iterating we get (the q-analogue of) [10], Theorem 4.1 and the special case $\lambda^0 = \mu^0 = 0$ gives (the q-analogue of) [10], Theorem 4.3.

6. An example

In the classical situation we have $y^\nu_{\lambda,\mu} = (Y(\lambda) \otimes Y(\mu) \,|\, Y(\nu))$, in particular each coefficient $y^\nu_{\lambda,\mu}$ is non-negative (for partitions λ, μ, ν of the same degree). We give an example to show that this is not always so in the quantised case.

Assume $n \geq r$. We use the general fact that if I is a finite dimensional $G(n)$-module that is polynomial of degree r and $\mathrm{ch}\, I = \sum_{\mu \in P(r)} c_\mu s_\mu$ then $\theta([fI]) = \sum_{\mu \in P(r)} c_\mu \chi^\mu$. By additivity it is enough to prove for $I = I(\lambda)$, $\lambda \in P(r)$. Since the modules $S^\lambda E$, $\lambda \in P(r)$, and $I(\lambda)$, $\lambda \in P(r)$ are related by a unitriangular matrix (see (7)) it is enough to prove for $I = S^\lambda E$, $\lambda \in P(r)$. However the character of $S^\lambda E$ is independent of characteristic, and $\theta(f[S^\lambda E]) = \theta(M(\lambda))$ so it is enough to prove the result in the case $K = \mathbb{C}$ and $q = 1$. In that case $I(\lambda)$ is the irreducible polynomial module labelled by λ and $fI(\lambda) = Y(\lambda)$ is the irreducible $K\mathrm{Sym}(r)$-module labelled by λ and the result is clear.

We now take $r = 4$ and $l = 4$ and note that $[Y(2,2)].[Y(2,2)]$ is not a sum of terms $[Y(\nu)]$, with ν a partition of 4. We choose $n \geq 4$. Then $(2,2)$ is the unique element of $\Lambda^+(n,4)$ in the block of $(2,2)$, see e.g., [2], Theorem 5.3. We therefore have $I(2,2) = L(2,2) = \nabla(2,2)$. For $\lambda \in \Lambda^+(n)$ we have $(I(\lambda) : \nabla(\lambda)) = 1$ and $(I(\lambda) : \nabla(\mu)) = 0$ for $\not\geq \lambda$. In particular we have $I(4) = S^4 E$. Also, it follows, for example from Cox's description of the blocks, [2], Theorem 5.3, that $I(\lambda) = \nabla(\lambda)$ for λ a partition of degree at most 3. Also, from [6], 2.1, (i), it follows that $I(\lambda) \otimes I(\mu) = \nabla(\lambda) \otimes \nabla(\mu)$ is injective for non-zero partitions λ and μ with $\deg(\lambda) + \deg(\mu) = 4$. Now we have $L(4) = L(1)^F = E^F$, by Steinberg's tensor product theorem. Thus $\nabla(4)/L(4)$ has highest weight $(3,1)$, and this weight occurs with multiplicity one, so that $[\nabla(4) : L(3,1)] = 1$ and so, by reciprocity, [5], Section 4, (6), we have $(I(3,1) : \nabla(4)) = 1$. It follows that $I(3,1) = E \otimes S^3 E$ and so has character $s_{3,1} + s_4$. Now $\bigwedge^2 E \otimes S^2 E$ is injective and has character $s_{3,1} + s_{2,1,1}$. By the character calculation for $I(3,1)$ just done we see that $I(3,1)$ does not occurs as a component of $\bigwedge^2 E \otimes S^2 E$ and hence we have $I(2,1,1) = \bigwedge^2 E \otimes S^2 E$. Finally, by a similar observation, we have $I(1^4) = E \otimes \bigwedge^3 E$. Applying the Schur functor we get $\mathrm{ch}\, Y(4) = \chi^4$, $\mathrm{ch}\, Y(3,1) = \chi^{3,1} + \chi^4$, $\mathrm{ch}\, Y(2,2) = \chi^{2,2}$, $\mathrm{ch}\, Y(2,1,1) = \chi^{2,1,1} + \chi^{3,1}$ and $\mathrm{ch}\, Y(1^4) = \chi^{2,1,1} + \chi^{1^4}$.

Now we have $\mathrm{ch}[Y(2,2)]^2 = (\chi^{2,2})^2 = \chi^4 + \chi^{2,2} + \chi^{1^4}$ and so

$$[Y(2,2)]^2 = [Y(1^4)] - [Y(2,1,1)] + [Y(3,1)] + [Y(2,2)].$$

Our approach in this example is, in keeping with the spirit of the paper, to work from the quantum general linear group to the Hecke algebra. Some readers may prefer a more direct approach via the work of Dipper and Du, [4]. One has in particular, that the Young modules labelled by l-restricted partitions are projective, see [3], 5.8 Theorem. Using this and the fact and that there is only one non-trivial l-block of degree 4, and its composition multiplicities are well known, one quickly obtains the above.

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