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Additive group actions on affine \mathbb{T} -varieties of complexity one in arbitrary characteristic

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ABSTRACT

Let X be a normal affine \mathbb{T} -variety of complexity at most one over a perfect field \mathbf{k} , where $\mathbb{T} = \mathbb{G}_m^n$ stands for the split algebraic torus. Our main result is a classification of additive group actions on X that are normalized by the \mathbb{T} -action. This generalizes the classification given by the second author in the particular case where \mathbf{k} is algebraically closed and of characteristic zero.

With the assumption that the characteristic of \mathbf{k} is positive, we introduce the notion of rationally homogeneous locally finite iterative higher derivations which corresponds geometrically to additive group actions on affine \mathbb{T} -varieties normalized up to a Frobenius map. As a preliminary result, we provide a complete description of these \mathbb{G}_a -actions in the toric situation.

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Introduction

Let \mathbf{k} be an arbitrary field. In this paper a variety X is an integral separated scheme of finite type over the field \mathbf{k} . We assume further that \mathbf{k} is algebraically closed in the field of rational functions $\mathbf{k}(X)$. A point in X is a not necessarily rational closed point. A variety is called normal if all its local rings are integrally closed domains. All algebraic group actions are, in particular, regular morphisms.

Let $\mathbb{T} = \mathbb{G}_m^n$ be the n -dimensional split algebraic torus, where \mathbb{G}_m stands for the multiplicative group of \mathbf{k} . A \mathbb{T} -variety is a normal variety endowed with an effective action of \mathbb{T} . The complexity of a \mathbb{T} -variety X is the non-negative integer $\dim X - \dim \mathbb{T}$. If the base field \mathbf{k} is algebraically closed, then the complexity of X can be read off geometrically as the codimension of the generic orbit. The best known examples of \mathbb{T} -varieties are those of complexity zero, called toric varieties.

Let \mathbb{G}_a be the additive group of the field \mathbf{k} . The main result of this paper is a classification of the \mathbb{G}_a -actions on an affine \mathbb{T} -variety X that are normalized by \mathbb{T} in the cases where X is of complexity zero or one. This generalizes a paper by the second author [23], where the same result is obtained in the particular case where \mathbf{k} is algebraically closed and of characteristic zero. The case of normalized \mathbb{G}_a -actions on an affine \mathbb{G}_m -surface over the field of complex numbers was first studied in [16].

Let M be the character lattice of \mathbb{T} and let N be the lattice of one-parameter subgroups. We have a natural duality $M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}$ given by $(m, v) \mapsto \langle m, v \rangle$ between the vector spaces $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Recall that \mathbb{T} -actions on an affine variety corresponds to M -gradings on its coordinate ring.

Affine \mathbb{T} -varieties can be described in combinatorial terms. In the case of toric varieties, there is the well-known description of affine toric varieties via strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ [12,30]. In 2006, Altmann and Hausen gave a combinatorial description of affine \mathbb{T} -varieties of arbitrary complexity over an algebraically closed field of characteristic zero [1]. This intersects with previous works by several authors [18,13,34,15,35] (see also [2,3] for the theory of non-necessarily affine \mathbb{T} -varieties). Furthermore, in a recent paper, the first author generalized the combinatorial description due to Altmann and Hausen to the case of affine \mathbb{T} -varieties of complexity one over an arbitrary field [21].

The combinatorial description of affine \mathbb{T} -varieties of complexity one that we will use in this paper encodes an affine \mathbb{T} -variety X with a triple $(C, \sigma, \mathfrak{D})$, where C is a regular curve, σ is a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$ and \mathfrak{D} is a σ -polyhedral divisor on C , i.e., a divisor in C whose coefficients instead of integers are polyhedra in $N_{\mathbb{R}}$ that can be decomposed as a Minkowski sum $Q + \sigma$ with Q a compact polyhedron (see Section 1 for details).

It is well known that the additive group actions on an affine variety $X = \operatorname{Spec} A$ are in one to one correspondence with certain sequences $\partial = \{\partial^{(i)} : A \rightarrow A\}_{i \in \mathbb{Z}_{\geq 0}}$ of \mathbf{k} -linear operators on A called locally finite iterative higher derivations [27,8,9], or LFIHDs for short (see Definition 2.1 for details). Now, assume that $X = \operatorname{Spec} A$ is an affine \mathbb{T} -variety and let ∂ be an LFIHD on A . The LFIHD ∂ is called homogeneous of degree $e \in M$ if every $\partial^{(i)}$ is homogeneous of degree ie . Furthermore, in positive characteristic, we introduce the technical notion of rationally homogeneous LFIHDs as follows: let $p > 0$ be the characteristic of \mathbf{k} and let $r \in \mathbb{Z}_{\geq 0}$, then ∂ is called rationally homogeneous of degree e/p^r if $\partial^{(ip^r)}$ is homogeneous of degree ie and $\partial^{(j)} = 0$ whenever p^r does not divide j .

In the case where \mathbf{k} is algebraically closed, the notion of (rationally) homogeneous LFIHD translates into geometric terms in the following way. An LFIHD on A is homogeneous if and only if the corresponding \mathbb{G}_a -action on X is normalized by the \mathbb{T} -action. Moreover, let $F_{p^r} : \mathbb{G}_a \rightarrow \mathbb{G}_a$ be the Frobenius map sending $t \mapsto t^{p^r}$. If ∂ is an LFIHD and $\phi : \mathbb{G}_a \rightarrow \operatorname{Aut}(X)$ is the corresponding \mathbb{G}_a -action, then ∂ is rationally homogeneous if and only if $\phi \circ F_{p^r}^{-1}$ is normalized by the \mathbb{T} -action for some $r \in \mathbb{Z}_{\geq 0}$ (see Proposition 2.8). In this case we say that ϕ is normalized by the \mathbb{T} -action up to a Frobenius map.

The kernel $\ker \partial$ of an LFIHD ∂ is defined as the intersection of $\ker \partial^{(i)}$ for all $i \in \mathbb{Z}_{>0}$; it is equal to the ring $\mathbf{k}[X]^{\mathbb{G}_a}$ of \mathbb{G}_a -invariant regular functions on X and $\operatorname{Frac}(\ker \partial)$ corresponds to the field $\mathbf{k}(X)^{\mathbb{G}_a}$ of \mathbb{G}_a -invariant rational functions on X . Denote by $\mathbf{k}(X)^{\mathbb{T}}$ the field of \mathbb{T} -invariant rational functions on X . A (rationally) homogeneous LFIHD is called vertical if $\mathbf{k}(X)^{\mathbb{T}} \subseteq \mathbf{k}(X)^{\mathbb{G}_a}$ and horizontal otherwise. When \mathbf{k} is algebraically closed, the horizontal condition means geometrically that the general \mathbb{G}_a -orbits are transverse to the rational fibration defined by the \mathbb{T} -action.

Let $X = \operatorname{Spec} A$ be the affine toric variety given by the strongly convex rational cone $\sigma \subseteq N_{\mathbb{R}}$. We denote by $\sigma(1)$ the set of extremal rays of the cone σ . In Theorem 3.5 we classify normalized \mathbb{G}_a -actions on affine toric varieties. They are described by Demazure roots of the cone σ , i.e., vectors $e \in M$ such that there exists $\rho \in \sigma(1)$ with $\langle e, \rho \rangle = -1$ and $\langle e, \rho' \rangle \geq 0$, for all $\rho' \in \sigma(1)$ different from ρ . We also classify \mathbb{G}_a -actions on affine toric varieties that are normalized up to a Frobenius map (see Corollary 3.7). Let us mention some developments from the theory of Demazure roots. The reader may consult [12,10,29,7,11,5] for the study of automorphisms of complete \mathbb{T} -varieties via Demazure's roots and [25,19] for the roots of the affine Cremona groups. See also [22] for a geometric description in the setting of affine spherical varieties.

Let now $X = \operatorname{Spec} A$ be an affine \mathbb{T} -variety of complexity one given by the triple $(C, \sigma, \mathfrak{D})$. The classification of normalized \mathbb{G}_a -actions on such an X is divided into two theorems corresponding to vertical and horizontal LFIHDs. The classification of vertical LFIHDs on A is given in Theorem 4.4. They are described by pairs (e, φ) , where e is a Demazure root of σ and φ is a global section of the invertible sheaf $\mathcal{O}_C(\mathfrak{D}(e))$. The \mathbb{Q} -divisor $\mathfrak{D}(e)$ is uniquely determined by \mathfrak{D} and e in a combinatorial way. The classification of horizontal LFIHDs on A is only available when \mathbf{k} is perfect, see Theorem 5.11.

Its combinatorial counterpart is different from the characteristic zero case (compare with [23, Theorem 3.28]) and is related to the description of rationally homogeneous LFIHDs on affine toric varieties.

The content of the paper is the following. In Section 1 we present the combinatorial description of affine \mathbb{T} -varieties of complexity one that will be used in this paper. In Section 2 we introduced the background results on \mathbb{G}_a -actions. In Section 3 we obtain our classification result for toric varieties. Finally, the classification of normalized \mathbb{G}_a -actions on affine \mathbb{T} -varieties of complexity one is divided in Sections 4 and 5 corresponding to the vertical and horizontal cases, respectively.

1. Generalities on affine \mathbb{T} -varieties of complexity one

In this section, we recall a combinatorial description of affine \mathbb{T} -varieties of complexity one over an arbitrary field [21, Section 3]. Let \mathbf{k} be field and let $X = \operatorname{Spec} A$ be an affine variety over \mathbf{k} . We start by introducing some notation from convex geometry (see e.g. [30] or [1, Section 1]).

1.1. Let $\mathbb{T} \simeq \mathbb{G}_m^n$ be a split algebraic torus over \mathbf{k} . Denote by $M = \operatorname{Hom}(\mathbb{T}, \mathbb{G}_m)$ the character lattice of \mathbb{T} and let $N = \operatorname{Hom}(\mathbb{G}_m, \mathbb{T})$ be the lattice of one-parameter subgroups. We have a natural duality $M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}$ given by $(m, v) \mapsto \langle m, v \rangle$, where $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ are the associated real vector spaces. We also let $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$ and $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ be the corresponding rational vector spaces.

A *rational cone* in $N_{\mathbb{R}}$ is a cone generated by a finite subset of N . If $\sigma \subseteq N_{\mathbb{R}}$ is a rational cone, then we let $\sigma^{\vee} \subseteq M_{\mathbb{R}}$ be its dual cone, i.e., the cone of real linear forms on $M_{\mathbb{R}}$ that are non-negative on σ . Recall that the dual cone σ^{\vee} of a rational cone is again rational. The relative interior of a rational cone $\sigma \subseteq N_{\mathbb{R}}$, denoted by $\operatorname{rel.int}(\sigma)$, is the topological interior of σ in the span of σ inside $N_{\mathbb{R}}$.

For any face $F \subseteq \sigma$ the set F^{\star} stands for the dual face of F in σ^{\vee} , i.e., $F^{\star} = F^{\perp} \cap \sigma^{\vee}$. A rational cone σ is *strongly convex* if 0 is a face of σ . This is equivalent to say that the dual $\sigma^{\vee} \subseteq M_{\mathbb{R}}$ is full dimensional. For any rational cone $\omega \subseteq M_{\mathbb{R}}$ we let $\omega_M = \omega \cap M$.

Furthermore, given a subsemigroup $S \subseteq M$ we let

$$\mathbf{k}[S] = \bigoplus_{m \in S} \mathbf{k}\chi^m$$

be the *semigroup algebra* of S defined by the relations $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$ for all $m, m' \in S$ and $\chi^0 = 1$.

For any integer $d \geq 0$ and any polyhedron $\Delta \subseteq N_{\mathbb{R}}$ we let $\Delta(d)$ be the set of faces of dimension d . In particular, $\Delta(0)$ is the set of vertices of Δ .

Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational cone. We define $\operatorname{Pol}_{\sigma}(N_{\mathbb{R}})$ as the set of polyhedra in $N_{\mathbb{R}}$ that can be written as a Minkowski sum $Q + \sigma$, where $Q \subseteq N_{\mathbb{R}}$ is a rational polytope, i.e., a bounded polyhedron having its vertices in the rational vector space $N_{\mathbb{Q}}$.

1.2. A \mathbb{T} -variety is a normal variety endowed with an effective action of the algebraic torus \mathbb{T} . Recall that a \mathbb{T} -action $X = \operatorname{Spec} A$ is equivalent to an M -grading of the algebra A . In algebraic terms, a \mathbb{T} -action on X is effective if and only if the semigroup of weights of A generates M . In this case the weight cone σ^\vee of A is the dual of a strongly convex rational cone $\sigma \subseteq N_{\mathbb{R}}$.

1.3. Let $X = \operatorname{Spec} A$ be an affine \mathbb{T} -variety. Letting $K_0 = \mathbf{k}(X)^{\mathbb{T}}$ be the field of \mathbb{T} -invariant rational functions on X we can write

$$A = \bigoplus_{m \in \sigma_M^\vee} A_m \chi^m$$

as an M -graded subalgebra of $K_0[M]$. Here, $\sigma^\vee \subseteq M_{\mathbb{R}}$ is the weight cone of A , χ^m is a weight vector in $\mathbf{k}(X)$, $A_0 = K_0 \cap A$, and A_m is an A_0 -module contained in K_0 . Furthermore, the weight vectors satisfy $\chi^0 = 1$, and $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$ for all $m, m' \in M$.

The *complexity* of the \mathbb{T} -variety X is the transcendence degree of the field extension K_0/\mathbf{k} . Since the action is effective, it is also equal to $\operatorname{rank} M - \dim X$. In geometrical terms, when $\mathbf{k} = \bar{\mathbf{k}}$ is algebraically closed the complexity is the codimension of the generic \mathbb{T} -orbit.

A *toric variety* is a \mathbb{T} -variety of complexity zero. An affine toric variety $X = \operatorname{Spec} A$ is completely determined by the weight cone σ^\vee of A . Conversely, given a strongly convex rational cone $\sigma \subseteq N_{\mathbb{R}}$, we can define an affine toric variety by letting $X_\sigma := \operatorname{Spec} \mathbf{k}[\sigma_M^\vee]$.

Another important class of affine \mathbb{T} -varieties is provided by the surface case. If X is an affine \mathbb{G}_m -surface, then the coordinate ring $A = \mathbf{k}[X]$ is endowed with a \mathbb{Z} -grading. Up to reversing the grading, we can assume that the subspace $A_+ = \bigoplus_{m \in \mathbb{Z}_{>0}} A_m \chi^m$ is nonzero. We distinguish three cases (see [14]).

- (i) The elliptic case: $A_- = \bigoplus_{m \in \mathbb{Z}_{<0}} A_m \chi^m = 0$ and $A_0 = \mathbf{k}$.
- (ii) The parabolic case: $A_- = 0$ and $A_0 \neq \mathbf{k}$.
- (iii) The hyperbolic case: $A_- \neq 0$.

More generally, an affine \mathbb{T} -variety $X = \operatorname{Spec} A$ of complexity one is called *elliptic* if $A_0 = \mathbf{k}$ (see [23, Section 1.1]).

To provide a description of affine \mathbb{T} -varieties of complexity one, we need to consider the Weil divisors theory on regular algebraic curves. In the next paragraph, we recall the definitions we need.

1.4. Let C be a regular curve over \mathbf{k} . By a point belonging to C we mean a closed point. Letting $z \in C$ we let $[\kappa_z : \mathbf{k}]$ be the *degree* of the point z defined as the dimension of residue field κ_z of z over \mathbf{k} (see [33, Proposition 1.1.15]). A point $z \in C$ of degree one is called a *rational point*. For a nonzero rational function $f \in \mathbf{k}(C)^*$ the associated principal divisor is

$$\operatorname{div} f = \sum_{z \in C} \operatorname{ord}_z f \cdot z,$$

where $\operatorname{ord}_z f$ is the order of f at the point z . The *degree* of a Weil \mathbb{Q} -divisor $D = \sum_{z \in C} a_z \cdot z$ is the rational number

$$\deg D = \sum_{z \in C} [\kappa_z : \mathbf{k}] \cdot a_z.$$

If C is projective, then we have $\deg \operatorname{div} f = 0$ (see [33, Theorem 1.4.11]). In addition, we let $\lfloor D \rfloor = \sum_{z \in C} \lfloor a_z \rfloor \cdot z$ be the integral Weil divisor obtained by taking the integral part of each coefficient of D . Similarly, the \mathbb{Q} -divisor $\{D\} = D - \lfloor D \rfloor$ stands for the fractional part of D . The space of global sections of the \mathbb{Q} -divisor D is defined by

$$H^0(C, \mathcal{O}_C(D)) := H^0(C, \mathcal{O}_C(\lfloor D \rfloor)) = \{f \in \mathbf{k}(C)^* \mid \operatorname{div} f + D \geq 0\} \cup \{0\}.$$

When C is projective, $H^0(C, \mathcal{O}_C(D))$ is usually called the *Riemann–Roch space* of D .

The following has been introduced in [1] for any complexity in the case where \mathbf{k} is algebraically closed of characteristic zero. In our context, we give a similar definition.

Definition 1.5. Let C be a regular curve over \mathbf{k} . Consider $\sigma \subseteq N_{\mathbb{R}}$ a strongly convex rational cone. A σ -polyhedral divisor over C is a formal sum $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$, where each $\Delta_z \in \operatorname{Pol}_{\sigma}(N_{\mathbb{R}})$ and $\Delta_z = \sigma$ for all but finitely number of z . For every coefficient Δ_z of the σ -polyhedral divisor \mathfrak{D} we define h_z as the piecewise linear map $h_z : M_{\mathbb{R}} \rightarrow \mathbb{R}$ given by $m \mapsto \min_{v \in \Delta_z(0)} \langle m, v \rangle$. We remark that h_z restricted to $\sigma^{\vee} \subseteq M_{\mathbb{R}}$ corresponds to the support function of Δ_z .

For any $m \in M_{\mathbb{Q}}$ we define the *evaluation* of \mathfrak{D} as the \mathbb{Q} -divisor

$$\mathfrak{D}(m) = \sum_{z \in C} h_z(m) \cdot z.$$

We denote by $\Lambda(\mathfrak{D})$ the coarsest refinement of the quasifan of σ^{\vee} such that the map $m \mapsto \mathfrak{D}(m)$ is linear in each cone. We also define the *degree* of \mathfrak{D} as

$$\deg \mathfrak{D} = \sum_{z \in C} [\kappa_z : \mathbf{k}] \cdot \Delta_z \in \operatorname{Pol}_{\sigma}(N_{\mathbb{R}}).$$

A σ -polyhedral divisor $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ is called *proper* if it satisfies one of the following conditions.

- (i) the curve C is affine, or
- (ii) the curve C is projective, the polyhedron $\deg \mathfrak{D}$ is a proper subset of σ , and for every $m \in \sigma_M^{\vee}$ such that $\deg \mathfrak{D}(m) = 0$, a nonzero integral multiple of $\mathfrak{D}(m)$ is principal.

Actually, polyhedral divisors are combinatorial objects that allow us to construct multigraded algebras, as explained in the following.

Notation 1.6. To a σ -polyhedral divisor $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ over C we associate the rational \mathbb{T} -submodule

$$A[C, \mathfrak{D}] = \bigoplus_{m \in \sigma_M^\vee} A_m \cdot \chi^m \subseteq K_0[M],$$

$$\text{where } A_m = H^0(C, \mathcal{O}_C(\mathfrak{D}(m))) \text{ and } K_0 = \mathbf{k}(C).$$

Given $m, m' \in \sigma_M^\vee$, the evaluations satisfy $\mathfrak{D}(m) + \mathfrak{D}(m') \leq \mathfrak{D}(m + m')$. Hence, for every $f \in A_m$ and every $g \in A_{m'}$, the product fg lies on $A_{m+m'}$. This multiplication rule turns the vector space $A[C, \mathfrak{D}]$ into an M -graded subalgebra.

For a non-empty open subset $C_0 \subseteq C$ we let

$$\mathfrak{D}|_{C_0} = \sum_{z \in C_0} \Delta_z \cdot z$$

be the *restriction* of \mathfrak{D} to C_0 .

The following yields a description of the coordinate ring of an affine \mathbb{T} -variety of complexity one (for a proof see [21, Theorem 4.3]). This description intersects with some classical cases; see [35,34] for complexity one case, [1] for higher complexity, and [15] for the Dolgachev–Pinkham–Demazure presentation of affine complex \mathbb{C}^* -surfaces. For the functorial properties of this description see [21, Proposition 4.5].

Theorem 1.7.

- (i) *If \mathfrak{D} is a proper σ -polyhedral divisor on a regular curve C over \mathbf{k} , then the M -graded algebra $A[C, \mathfrak{D}] = \bigoplus_{m \in \sigma^\vee \cap M} A_m$, where*

$$A_m = H^0(C, \mathcal{O}_C(\mathfrak{D}(m))),$$

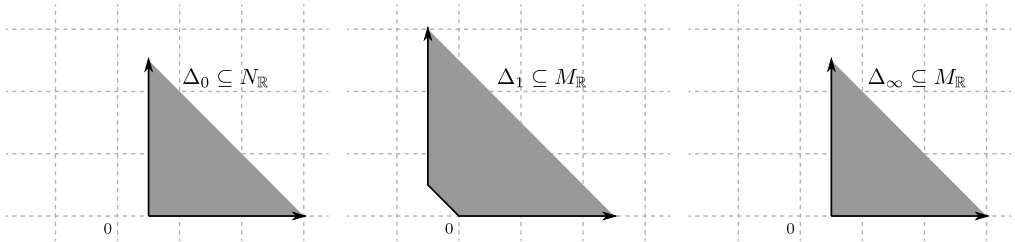
is the coordinate ring of an affine \mathbb{T} -variety of complexity one over \mathbf{k} .

- (ii) *Conversely, to any affine \mathbb{T} -variety $X = \operatorname{Spec} A$ of complexity one over \mathbf{k} , one can associate a pair $(C_X, \mathfrak{D}_{X,\gamma})$ as follows.*

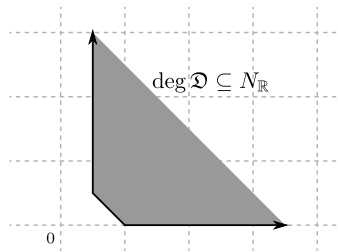
- (a) *C_X is the abstract regular curve over \mathbf{k} defined by the conditions $\mathbf{k}[C_X] = \mathbf{k}[X]^\mathbb{T}$ and $k(C_X) = k(X)^\mathbb{T}$.*
 (b) *$\mathfrak{D}_{X,\gamma}$ is a proper σ_X -polyhedral divisor over C_X , which is uniquely determined by X and by a sequence $\gamma = (\chi^m)_{m \in M}$ of $k(X)$ as in 1.3.*

We have a natural identification $A = A[C_X, \mathfrak{D}_{X,\gamma}]$ of M -graded algebras with the property that every homogeneous element $f \in A$ of degree m is equal to $f_m \chi^m$, for a unique global section f_m of the sheaf $\mathcal{O}_{C_X}(\mathfrak{D}_{X,\gamma}(m))$.

Example 1.8. Let $M = \mathbb{Z}^2$ and let σ be the first quadrant in the vector space $N_{\mathbb{R}} = \mathbb{R}^2$. We also let $\Delta_0 = (1/2, 0) + \sigma$, $\Delta_1 = L + \sigma$ and $\Delta_{\infty} = (1/2, 0) + \sigma$, where L is the line segment joining the points $(0, 0)$ and $(-1/2, 1/2)$.



Letting \mathbf{k} be an arbitrary field and $C = \mathbb{P}_{\mathbf{k}}^1$ we let \mathfrak{D} be the σ -polyhedral divisor $\mathfrak{D} = \Delta_0 \cdot [0] + \Delta_1 \cdot [1] + \Delta_{\infty} \cdot [\infty]$ over C . The degree of \mathfrak{D} is $\deg \mathfrak{D} = L' + \sigma$, where L' is the line segment joining the points $(1, 0)$ and $(1/2, 1/2)$.



Hence $\deg \mathfrak{D} \subsetneq \sigma$ and \mathfrak{D} is proper. Let $A = A[C, \mathfrak{D}]$ and $X = \text{Spec } A$. A direct computation shows that the elements

$$u_1 = \frac{t-1}{t} \cdot \chi^{(2,0)}, \quad u_2 = \chi^{(0,1)}, \quad u_3 = \chi^{(1,1)}, \quad u_4 = \frac{(t-1)^2}{t} \cdot \chi^{(2,0)}, \quad \text{and} \\ u_5 = \frac{(t-1)^2}{t} \cdot \chi^{(3,0)}$$

generate the algebra A . Furthermore, a minimal set of relations satisfied by these generators is given by $u_2 u_5 - u_3 u_4 = 0$, $u_3 u_5 - u_1^2 u_2 - u_1 u_2 u_4 = 0$ and $u_5^2 - u_1^2 u_4 - u_1 u_4^2 = 0$. Hence

$$A \simeq k[x_1, x_2, x_3, x_4, x_5] / (x_2 x_5 - x_3 x_4, x_3 x_5 - x_1^2 x_2 - x_1 x_2 x_4, x_5^2 - x_1^2 x_4 - x_1 x_4^2).$$

The following result provides a calculation of the Altmann–Hausen presentation in terms of polyhedral divisors when we extend the scalars to an algebraic closure of \mathbf{k} , see [21, Proposition 3.9].

Lemma 1.9. Assume that \mathbf{k} is a perfect field, and let $\bar{\mathbf{k}}$ be an algebraic closure of \mathbf{k} . The absolute Galois group of $\mathfrak{S}_{\bar{\mathbf{k}}/\mathbf{k}}$ acts on the closed points of the curve

$$C_{\bar{\mathbf{k}}} = C \times_{\text{Spec } \mathbf{k}} \text{Spec } \bar{\mathbf{k}}$$

which can be identified with the set of the $\bar{\mathbf{k}}$ -rational points of $C(\bar{\mathbf{k}})$. The orbit space $C(\bar{\mathbf{k}})/\mathfrak{S}_{\bar{\mathbf{k}}/\mathbf{k}}$ can be identified with C . We denote by $S : C(\bar{\mathbf{k}}) \rightarrow C$ the quotient map. If $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ is a proper σ -polyhedral divisor over C , then

$$A[C, \mathfrak{D}] \otimes_{\mathbf{k}} \bar{\mathbf{k}} = A[C(\bar{\mathbf{k}}), \mathfrak{D}_{\bar{\mathbf{k}}}],$$

where $\mathfrak{D}_{\bar{\mathbf{k}}}$ is the proper σ -polyhedral divisor over $C(\bar{\mathbf{k}})$ defined by

$$\mathfrak{D}_{\bar{\mathbf{k}}} = \sum_{z \in C} \Delta_z \cdot S^*(z) \quad \text{with} \quad S^*(z) = \sum_{z' \in S^{-1}(z)} z'.$$

The proof of the following result is exactly the same as in [23, Lemma 1.6].

Lemma 1.10. Let $A = A[C, \mathfrak{D}]$, where C is a regular curve over \mathbf{k} with field of rational functions K_0 and $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ is a proper σ -polyhedral divisor. Consider the normalization A' of the cyclic extension $A[s\chi^e]$, where $e \in M$, $s^d \in A$ homogeneous of degree de , and $d \in \mathbb{Z}_{>0}$. If \mathbf{k} is algebraically closed in A' , then $A' = A[C', \mathfrak{D}']$ where C' and \mathfrak{D}' are defined by the following.

- (i) If A is elliptic, then A' is also and C' is the regular projective curve associated with the algebraic function field $K_0[s]$.
- (ii) If A is non-elliptic, then A' is also and $C' = \text{Spec } A'_0$, where A'_0 is the normalization of A_0 in $K_0[s]$.
- (iii) In both cases $\mathfrak{D}' = \sum_{z \in C} \Delta_z \cdot \pi^*(z)$, where $\pi : C' \rightarrow C$ is the natural projection.

2. Generalities on \mathbb{G}_a -actions

Let $X = \text{Spec } A$ be an affine \mathbb{T} -variety over an arbitrary field \mathbf{k} . In this section, we study the relation between \mathbb{G}_a -actions on X that are normalized by the torus action and homogeneous locally finite iterative higher derivations.

Definition 2.1. Let $\partial = \{\partial^{(i)}\}_{i \in \mathbb{Z}_{\geq 0}}$ be a sequence of \mathbf{k} -linear operators on A . We say that ∂ is a *locally finite iterative higher derivation* (LFIHD for short) if it satisfies the following conditions:

- (i) The operator $\partial^{(0)}$ is the identity map.

(ii) For any $i \in \mathbb{Z}_{\geq 0}$ and for all $f_1, f_2 \in A$ we have the *Leibniz rule*

$$\partial^{(i)}(f_1 \cdot f_2) = \sum_{j=0}^i \partial^{(j)}(f_1) \cdot \partial^{(i-j)}(f_2).$$

(iii) The sequence ∂ is locally finite, i.e. for any $f \in A$ there exists a positive integer r such that for any $i \geq r$, $\partial^{(i)}(f) = 0$.

(iv) For all $i, j \in \mathbb{Z}_{\geq 0}$ and for any regular function $f \in A$ we have

$$\left(\partial^{(i)} \circ \partial^{(j)}\right)(f) = \binom{i+j}{i} \partial^{(i+j)}(f).$$

Furthermore, if ∂ verifies only (i), (ii), (iv), we say that ∂ is a *iterative higher derivation*. If ∂ verifies only (i), (ii), we say ∂ is a *Hasse–Schmidt derivation* (see [36]).

Consider an action

$$\phi : \mathbb{G}_a \times X \rightarrow X$$

of the additive group \mathbb{G}_a over \mathbf{k} . Then the comorphism ϕ^* gives a sequence $\partial = \{\partial^{(i)}\}_{i \in \mathbb{Z}_{\geq 0}}$ of \mathbf{k} -linear operators on A defined by the following way. For any $f \in A$ we write

$$\phi^*(f) = \sum_{i=0}^{\infty} \partial^{(i)}(f) \cdot x^i \in A \otimes_{\mathbf{k}} \mathbf{k}[x], \quad \text{where } \mathbf{k}[x] = \mathbf{k}[\mathbb{G}_a]$$

is the polynomial algebra in one variable. An easy computation shows that ∂ is an LFIHD [27]. Conversely, given an LFIHD ∂ on A , its *exponential map*

$$e^{x\partial} := \sum_{i=0}^{\infty} \partial^{(i)} x^i$$

is the comorphism of a \mathbb{G}_a -action on $X = \text{Spec } A$.

Remark 2.2. Consider an LFIHD ∂ on A . For a positive integer i we let

$$\left(\partial^{(1)}\right)^{\circ i} = \partial^{(1)} \circ \dots \circ \partial^{(1)}$$

be the composition of i copies of $\partial^{(1)}$. Denoting by p the characteristic of the field \mathbf{k} , we have the equality

$$\partial^{(i)} = \frac{\left(\partial^{(1)}\right)^{\circ i_0} \circ \left(\partial^{(p)}\right)^{\circ i_1} \circ \dots \circ \left(\partial^{(p^r)}\right)^{\circ i_r}}{(i_0)!(i_1)!\dots(i_r)!},$$

where $i = \sum_{j=0}^r i_j \cdot p^j$ is the p -adic expansion¹ of i . If further $p = 0$, then the \mathbb{G}_a -action is therefore uniquely determined by the locally nilpotent derivation $\partial^{(1)}$.

In characteristic zero, the algebra of invariants of a \mathbb{G}_a -action on the variety $X = \text{Spec } A$ is the kernel of the associated locally nilpotent derivation on A . The following definition describes the arbitrary characteristic case.

Definition 2.3. For an LFIHD ∂ on the algebra A its *kernel* is the subset

$$\ker \partial := \left\{ f \in A \mid \partial^{(i)}(f) = 0, \text{ for all } i \in \mathbb{Z}_{>0} \right\}.$$

This is the subalgebra of invariants $A^{\mathbb{G}_a} \subseteq A$ for the \mathbb{G}_a -action corresponding to ∂ . The LFIHD ∂ is *non-trivial* if $\ker \partial \neq A$. A subspace $V \subseteq A$ is called ∂ -invariant if for any $i \in \mathbb{Z}_{\geq 0}$, we have the inclusion $\partial^{(i)}(V) \subseteq V$. In particular, the subspace $\ker \partial$ is ∂ -invariant. For any $f \in A$ we define the multiplication $f\partial$ as the sequence of \mathbf{k} -linear operators $f\partial = \{f^i \partial^{(i)}\}_{i \in \mathbb{Z}_{\geq 0}}$. It is easy to check that $f\partial$ is an LFIHD if and only if $f \in \ker \partial$.

The next result provides some useful properties of \mathbb{G}_a -actions, see [9, 2.1, 2.2] and [8, Example 3.5].

Proposition 2.4. For every non-trivial LFIHD ∂ on the algebra A the following hold.

- (a) The subring $\ker \partial \subseteq A$ is factorially closed, i.e., for all $f_1, f_2 \in A$ we have $f_1 f_2 \in \ker \partial \setminus \{0\}$ implies $f_1, f_2 \in \ker \partial$.
- (b) The subring $\ker \partial$ is algebraically closed in A .
- (c) The subring $\ker \partial$ is a subring of codimension one in A .
- (d) If $\text{char}(\mathbf{k}) = p > 0$ and $A = \mathbf{k}[y]$ is the polynomial ring in one variable, then there are some $c_1, \dots, c_r \in \mathbf{k}^*$ and some integers $0 \leq s_1 < \dots < s_r$ such that

$$e^{x\partial}(y) = y + \sum_{i=1}^r c_i \cdot x^{p^{s_i}}.$$

- (e) If A^* is the set of units of A , then $A^* \subseteq \ker \partial$ so that $A^* = (\ker \partial)^*$.
- (f) A principal ideal $(f) = fA$ is ∂ -invariant if and only if $f \in \ker \partial$.

Proof. Assertions (a), (b) and (c) are obtained by using the degree function

$$A \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}, \quad f \mapsto \deg_x e^{x\partial}(f).$$

¹ When $p = 0$ we make the convention that the p -adic expansion is $i = i_0$.

In particular, we remark that (b) implies that the ring $\ker \partial$ is normal whenever A is normal. Assertion (d) is proven in [8, Example 3.5]. Assertion (e) is an easy consequence of (a).

Using arguments from [15, 2, 1.2 (b)] we give a short proof of (f). Assume that f is nonzero. By Definition 2.1 (iii) we can consider $d \in \mathbb{Z}_{\geq 0}$ such that $f' := \partial^{(d)}(f) \neq 0$ and belongs to $\ker \partial$. If the ideal (f) is ∂ -invariant, then $f' \in \ker \partial \cap (f)$ so that $f' = af$ for some $a \in A$. By Proposition 2.4 (a) we obtain $f \in \ker \partial$. Conversely, let $a' \in A$. By Definition 2.1 (ii), for any $i \in \mathbb{Z}_{\geq 0}$ we have $\partial^{(i)}(a'f) = \partial^{(i)}(a')f$ and so the ideal (f) is ∂ -invariant. \square

In the next lemma, we study the extensions of LFIHDs on the algebra A to the localization ring $T^{-1}A$ given by a multiplicative system $T \subseteq A$. We were inspired by well-known computations with the Hasse–Teichmüller derivatives (cf. [17, Section 2]). For this lemma, we let

$$E(i, j) = \left\{ (s_1, \dots, s_j) \in \mathbb{Z}_{>0}^j \mid \sum_{\ell=1}^j s_\ell = i \right\} \quad \text{for all integers } i, j \in \mathbb{Z}_{>0}, \text{ such that } j \leq i.$$

Lemma 2.5. *Let T be a subset of A stable under multiplication such that $0 \notin T$ and $1 \in T$.*

- (i) *If ∂ be an iterative higher derivation on the algebra A , then ∂ extends to a unique iterative higher derivation $\bar{\partial} = \{\bar{\partial}^{(i)}\}_{i \in \mathbb{Z}_{\geq 0}}$ on the algebra $T^{-1}A$ given by*

$$\bar{\partial}^{(i)} \left(\frac{1}{f} \right) = \sum_{j=1}^i \frac{(-1)^j}{f^{j+1}} \sum_{(s_1, \dots, s_j) \in E(i, j)} \partial^{(s_1)}(f) \dots \partial^{(s_j)}(f)$$

for all $f \in T$ and all $i \in \mathbb{Z}_{>0}$.

- (ii) *Furthermore, if ∂ is an LFIHD on A and if $T \subseteq \ker \partial$, then the extension $\bar{\partial}$ on $T^{-1}A$ is an LFIHD.*

Proof. The existence and the uniqueness of $\bar{\partial}$ is given in [26, 3.7, 5.8], [36, Section 3]. Proceeding by induction the computation of $\bar{\partial}^{(i)}(\frac{1}{f})$ is an easy consequence of Definition 2.1 (ii). The rest of the proof is straightforward. \square

As a consequence of the previous lemma, we obtain a result on equivariant cyclic coverings of an affine variety with a \mathbb{G}_a -action (see also [16, Lemma 1.8]).

Corollary 2.6. *Let $K = \text{Frac } A$. Consider an LFIHD ∂ on A and let $f \in \ker \partial$ be a nonzero element. Let $d \in \mathbb{Z}_{>0}$ be an integer and let u be an algebraic element over K satisfying $u^d - f = 0$. If B is the integral closure of $A[u]$ in its field of fractions, then ∂ extends to a unique LFIHD ∂' on the algebra B such that $u \in \ker \partial'$.*

Proof. By Lemma 2.5 we can extend the LFIHD ∂ on A to an iterative higher derivation on the field K , and on the polynomial ring $K[t]$ by letting $\bar{\partial}^{(i)}(t) = 0$ for any $i \geq 1$. Consider the morphism of K -algebras $\phi : K[t] \rightarrow K[u]$, $t \mapsto u$. Let $P \in K[t]$ be the monic polynomial generating the ideal $\ker \phi$.

We can write $t^d - f = FP$, for some $F \in K[t]$. Remark that F is monic since P and $t^d - f$ are monic. Since A is integrally closed, we obtain $F, P \in A[t]$. Furthermore, for any $i \in \mathbb{Z}_{>0}$ we have $\bar{\partial}^{(i)}(FP) = \bar{\partial}^{(i)}(t^d - f) = 0$. Note that $A[t]$ is $\bar{\partial}$ -invariant and the restriction of $\bar{\partial}$ to $A[t]$ is an LFIHD. Therefore, by Proposition 2.4 (a), we have $P \in A[t] \cap \ker \bar{\partial}$ defining an iterative higher derivation ∂' on $K[u]$. Clearly, the normalization B of the ring $A[u]$ is again ∂' -invariant. The rest of the proof is straightforward and we omitted it. \square

In the sequel, we let

$$A = \bigoplus_{m \in \sigma_M^\vee} A_m \chi^m \subseteq K_0[M]$$

as in Section 1, where χ^m is also seen as the character of the split torus \mathbb{T} corresponding to the lattice vector $m \in M$. Let us introduce the notion of homogeneous iterative higher derivations.

Definition 2.7. Let ∂ be an iterative higher derivation. The sequence ∂ is *homogeneous* if there exists $e \in M$ such that

$$\partial^{(i)}(A_m \chi^m) \subseteq A_{m+ie} \chi^{m+ie} \quad \text{for all } i \in \mathbb{Z}_{\geq 0} \text{ and } m \in M.$$

If ∂ is non-trivial, then the vector e is called the *degree* of ∂ and is denoted by $\deg \partial$. For the case where \mathbf{k} is of characteristic $p > 0$ we have the more general definition. Given $r \in \mathbb{Z}_{\geq 0}$ we say that ∂ is *rationally homogeneous* of degree e/p^r (or of bidegree (e, p^r) if we need to emphasize the vector e) if it satisfies the following.

- (i) $\partial^{(ip^r)}(A_m \chi^m) \subseteq A_{m+ie} \chi^{m+ie}$, for all $i \in \mathbb{Z}_{\geq 0}$, and $m \in M$.
- (ii) $\partial^{(j)} = 0$ whenever p^r does not divide j .

In [23, Section 1.2] it is shown that a usual derivation on a multigraded algebra which sends graded pieces into graded pieces is homogeneous. However this does not hold for higher derivations. Note also that the kernel of a homogeneous LFIHD ∂ on A is an M -graded subalgebra of A . In the sequel, we introduce some notation in order to have a geometrical interpretation of homogeneous and rationally homogeneous LFIHDs in the case where \mathbf{k} is an algebraically closed field.²

² Note that the Notation 2.8 and Proposition 2.9 can be generalized in the setting of group schemes and of Hopf algebras when \mathbf{k} is arbitrary.

Notation 2.8. Assume that \mathbf{k} is algebraically closed. Letting $e \in M$ be a vector we denote by G_e the group whose underlying set is $\mathbb{T} \times \mathbb{G}_a$ and multiplication law is defined by

$$(t_1, \alpha_1) \cdot (t_2, \alpha_2) = (t_1 \cdot t_2, \chi^{-e}(t_2) \cdot \alpha_1 + \alpha_2),$$

where $t_i \in \mathbb{T}$ and $\alpha_i \in \mathbb{G}_a$. Actually, every semidirect product of $\mathbb{T} \ltimes \mathbb{G}_a$ given by a character $\mathbb{T} \rightarrow \text{Aut } \mathbb{G}_a \simeq \mathbb{G}_m$ is isomorphic to some G_e .

The following proposition is similar to [16, Lemma 2.2]. For the convenience of the reader we give a short proof.

Proposition 2.9. *Assume that the field \mathbf{k} is algebraically closed.*

- (i) *If A is M -graded and ∂ is a homogeneous LFIHD on A of degree e , then the corresponding \mathbb{G}_a -action is normalized by the \mathbb{T} -action. This means that the actions of the torus and the additive group induce a G_e -action with comorphism given by*

$$\psi^*(t, \alpha) = t \cdot e^{\alpha\partial}(f),$$

where $(t, \alpha) \in G_e$ and $f \in A$.

- (ii) *Conversely, if G_e acts on $X = \text{Spec } A$, then the actions of the subgroups \mathbb{T} and \mathbb{G}_a give an M -grading on A and a homogeneous LFIHD of degree e .*
- (iii) *Assume further that $\text{char}(\mathbf{k}) = p > 0$. Let $F_{p^r} : \mathbb{G}_a \rightarrow \mathbb{G}_a$, $t \mapsto t^{p^r}$ be the Frobenius map. Giving a rationally homogeneous LFIHD ∂ on A of degree e/p^r is equivalent to having a \mathbb{G}_a -action on X equal to $\phi \circ (F_{p^r}, \text{id}_X)$, where ϕ is a \mathbb{G}_a -action normalized by \mathbb{T} .*

Proof. (i) Given $(t, \alpha) \in G_e$ and $f \in A$, by homogeneity of ∂ we have

$$t \cdot \partial^{(i)}(f) = \chi^{ie}(t) \partial^{(i)}(t \cdot f), \quad \forall i \in \mathbb{Z}_{\geq 0}. \quad (1)$$

This gives

$$t \cdot e^{\alpha\partial}(f) = \sum_{i=0}^{\infty} \chi^{ie}(t) \alpha^i \partial^{(i)}(t \cdot f) = e^{\chi^e(t)\alpha\partial}(t \cdot f).$$

Hence for all $(t_1, \alpha_1), (t_2, \alpha_2) \in G_e$ we obtain

$$\psi^*((t_1, \alpha_1) \cdot (t_2, \alpha_2))(f) = e^{\chi^e(t_1)\alpha_1\partial} \circ e^{\chi^e(t_2)\alpha_2\partial}(t_1 t_2 \cdot f) = \psi^*(t_1, \alpha_1)(\psi^*(t_2, \alpha_2)(f)).$$

We conclude that ψ^* defines a G_e -action on the variety $X = \text{Spec } A$.

(ii) The action of the subgroup $\mathbb{G}_a \subseteq G_e$ yields an LFIHD ∂ on the algebra A . For $\alpha \in \mathbb{G}_a$ and $f \in A$ we have $\psi^*(1, \alpha)(f) = e^{\alpha\partial}(f)$. So for any $t \in \mathbb{T}$ we have

$$t \cdot e^{\alpha\partial}(f) = \psi^*((1, \chi^e(t)\alpha) \cdot (t, 0))(f) = e^{\chi^e(t)\alpha\partial}(t \cdot f).$$

Identifying the coefficients we obtain (1). Thus the LFIHD ∂ is homogeneous for the M -grading given by the action of the subgroup $\mathbb{T} \subseteq G_e$.

Assertion (iii) follows immediately from (i) and (ii). \square

For an arbitrary field \mathbf{k} we consider the following natural definition.

Definition 2.10. Assume that the torus \mathbb{T} acts on $X = \operatorname{Spec} A$. A \mathbb{G}_a -action on X is *normalized* (resp. *normalized up to a Frobenius map*) by the \mathbb{T} -action if the corresponding LFIHD ∂ is homogeneous (resp. rationally homogeneous).

To classify normalized \mathbb{G}_a -action it is convenient to separate them into two types (see [16, 3.11] and [23, Lemma 1.11] for special cases).

Definition 2.11. A homogeneous LFIHD ∂ is of *vertical type* (or of *fiber type*) if $\bar{\partial}^{(i)}(K_0) = \{0\}$ for any $i \in \mathbb{Z}_{>0}$. Otherwise ∂ is of *horizontal type*. We use similar terminology for normalized \mathbb{G}_a -actions. An affine \mathbb{T} -variety endowed with a non-trivial vertical (resp. horizontal) \mathbb{G}_a -action is called *vertical* (resp. *horizontal*).

A homogeneous LFIHD of horizontal type is automatically non-trivial. In the vertical case, one can extend a homogeneous LFIHD on A to an LFIHD on the semigroup algebra $K_0[\sigma_M^\vee]$.

Lemma 2.12. Let ∂ be a homogeneous LFIHD of vertical type on the M -graded algebra A . Then ∂ extends to a unique homogeneous locally finite iterative higher K_0 -derivation on the semigroup algebra $K_0[\sigma_M^\vee]$.

Proof. By Lemma 2.5, the LFIHD ∂ extends to an iterative higher derivation ∂' on $K_0[M]$. Since ∂ is of vertical type, Definition 2.1 (ii) implies that each $\partial'^{(i)}$ is K_0 -linear. Consequently, if $S \subseteq M$ is the subsemigroup of weights of the M -graded algebra A , then $B := K_0[S] = A \otimes_{\mathbf{k}} K_0$ is ∂' -invariant.

Let us show that $\partial'|_B$ is an LFIHD on B . Let $f\chi^m \in B$ be a homogeneous element with $f \in K_0^*$. Write $f\chi^m = f'h\chi^m$ for some $f' \in K_0$ and for some $h \in A_m$. There exists $r \in \mathbb{Z}_{>0}$ such that for any $i \geq r$,

$$\partial'^{(i)}(f\chi^m) = f'\partial^{(i)}(h\chi^m) = 0.$$

Since every element of B is a sum of homogeneous elements we conclude that $\partial'|_B$ is a locally finite iterative higher K_0 -derivation on B . Thus, $\partial'|_B$ extends to an LFIHD on the integral closure $\bar{B} = K_0[\sigma_M^\vee]$. \square

In the next lemma, we prove an elementary result concerning the LFIHDs of the polynomial algebra in one variable. It will be useful in order to study horizontal \mathbb{G}_a -actions in Section 5. We let ord_0 be the natural valuation

$$\text{ord}_0 : \mathbf{k}[t] \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}, \quad \sum_i a_i t^i \mapsto \min\{i \mid a_i \neq 0\}.$$

Lemma 2.13. *Assume that $\text{char}(\mathbf{k}) = p > 0$. Let ∂ be an LFIHD on the polynomial algebra $\mathbf{k}[t]$ in one variable such that*

$$e^{x\partial}(t) = t + \sum_{i=1}^r \lambda_i x^{p^{s_i}},$$

where $\lambda_i \in \mathbf{k}^*$ and $0 \leq s_1 < \dots < s_r$ are integers. We also fix a non-negative integer $i \in \mathbb{Z}_{\geq 0}$.

If $\ell \in \mathbb{Z}_{\geq 0}$ verifies $\ell \geq ip^{s_1}$, then

$$\partial^{(ip^{s_1})}(t^\ell) = \lambda_1^i \binom{\ell}{i} t^{\ell-i}$$

and therefore $\text{ord}_0 \partial^{(ip^{s_1})}(t^\ell) = \ell - i$ whenever $\binom{\ell}{i} \neq 0$.

Proof. First of all, we have

$$\begin{aligned} e^{x\partial}(t^\ell) &= e^{x\partial}(t)^\ell = \left(t + \sum_{i=1}^r \lambda_i x^{p^{s_i}}\right)^\ell \\ &= \sum_{i_0 + \dots + i_r = \ell, i_0, \dots, i_r \geq 0} \binom{\ell}{i_0 \dots i_r} t^{i_0} \prod_{\alpha=1}^r (\lambda_\alpha x^{p^{s_\alpha}})^{i_\alpha}. \end{aligned}$$

Considering the term of degree ip^{s_1} in x of the previous sum, we get the following conditions:

$$ip^{s_1} = i_1 p^{s_1} + \dots + i_r p^{s_r} \quad \text{and} \quad i_0 + i_1 + \dots + i_r = \ell, \quad (2)$$

where $(i_0, i_1, \dots, i_r) \in \mathbb{Z}_{\geq 0}^{r+1}$. Note that such an $(r+1)$ -tuple (i_0, i_1, \dots, i_r) exists since $\ell \geq ip^{s_1}$ and so we can take

$$(i_0, i_1, \dots, i_r) = (\ell - i, i, 0, \dots, 0).$$

Let us show that this is the minimal choice for $i_0 \in \mathbb{Z}_{\geq 0}$. Indeed, let $(\gamma_0, \gamma_1, \dots, \gamma_r) \in \mathbb{Z}_{\geq 0}^r$ be an $(r+1)$ -uplet satisfying (2) with γ_0 minimal. Then we have

$$\ell - i = \ell - \sum_{\alpha=1}^r \gamma_\alpha p^{s_\alpha - s_1} \leq \ell - \sum_{\alpha=1}^r \gamma_\alpha = \gamma_0.$$

Hence by minimality, $\gamma_0 = \ell - i$, so that $i = \sum_{\alpha=1}^r \gamma_\alpha$. Thus,

$$\left(\sum_{\gamma_\alpha}^r \gamma_\alpha \right) p^{s_1} = \sum_{\alpha=1}^r \gamma_\alpha p^{s_\alpha}.$$

We obtain $(\gamma_0, \gamma_1, \dots, \gamma_r) = (\ell - i, i, 0, \dots, 0)$. This implies in particular that $\partial^{(ip^{s_1})}(t^\ell) = \lambda_1^i \binom{\ell}{i} t^{\ell-i}$ as required. \square

3. \mathbb{G}_a -actions on affine toric varieties

Let \mathbf{k} be a field. In this section, we present a combinatorial description of normalized \mathbb{G}_a -actions up to a Frobenius map on affine toric varieties over \mathbf{k} .

For a rational cone $\sigma \subseteq N_{\mathbb{R}}$ we recall that $\sigma(1)$ denotes its set of extremal rays. As usual we write by the same letter a ray of σ and its primitive vector. The following is a classical definition, see for instance [12,23,4].

Definition 3.1. Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational cone. A vector $e \in M$ is called a *Demazure's root* (or for simplicity called *root*) if the following hold.

- (i) There exists $\rho \in \sigma(1)$ such that $\langle e, \rho \rangle = -1$.
- (ii) For any $\rho' \in \sigma(1) \setminus \{\rho\}$ we have $\langle e, \rho' \rangle \geq 0$.

The extremal ray ρ satisfying $\langle e, \rho \rangle = -1$ is called the *distinguished ray* of the root $e \in M$. We denote by $\text{Rt } \sigma$ the set of Demazure's roots of the cone σ . By [23, Remark 2.5] every element of $\sigma(1)$ is the distinguished ray of a root of $\text{Rt } \sigma$.

Since the subset $\mathbf{k}[\mathbb{T}]^*$ generates the algebra $\mathbf{k}[\mathbb{T}]$, Proposition 2.4 (e) implies that $\mathbf{k}[\mathbb{T}]$ has no non-trivial LFIHDs. So without loss of generality, in the sequel, we may only consider toric varieties $X_\sigma = \text{Spec } \mathbf{k}[\sigma_M^\vee]$ given by a nonzero strongly convex rational cone $\sigma \subseteq N_{\mathbb{R}}$.

Example 3.2. Let $e \in \text{Rt } \sigma$ be a root. Consider the homogeneous derivation $\partial_e^{(1)}$ on the semigroup algebra $\mathbf{k}[\sigma_M^\vee]$ given by

$$\partial_e^{(1)}(\chi^m) = \langle m, \rho \rangle \chi^{m+e} \quad \text{for all } m \in \sigma_M^\vee,$$

where ρ is the distinguished ray of e . Then $\partial_e^{(1)}$ is locally nilpotent and yields a \mathbb{G}_a -action on X_σ in the following natural way: the homogeneous LFIHD ∂_e is given by the formula³

$$\partial_e^{(i)}(\chi^m) = \binom{\langle m, \rho \rangle}{i} \cdot \chi^{m+ie} \quad \text{for all } i \in \mathbb{Z}_{\geq 0} \quad \text{and } m \in \sigma_M^\vee.$$

The kernel of ∂_e is $\mathbf{k}[\rho_M^\star]$, where $\rho^\star \subseteq \sigma^\vee$ is the dual face of ρ .

³ We set the convention that $\binom{r_1}{r_2} = 0$, for all $r_1, r_2 \in \mathbb{Z}_{\geq 0}$ with $r_1 < r_2$.

Assume now that $\text{char}(\mathbf{k}) = p > 0$. Starting from ∂_e and an integer $r \in \mathbb{Z}_{\geq 0}$ we can also define a rationally homogeneous LFIHD $\partial_{e,r}$ of degree $e/p^r \in M_{\mathbb{Q}}$. Its exponential map is

$$e^{x\partial_{e,r}} = \sum_{i=0}^{\infty} \partial_e^{(i)} x^{ip^r}.$$

We check easily that $\ker \partial_{e,r} = \mathbf{k}[\rho_M^*]$. In addition, for any $m \in \sigma_M^{\vee}$ we have

$$\deg_x e^{x\partial_{e,r}}(\chi^m) = p^r \langle m, \rho \rangle.$$

We start by describing the kernel and the possible degree vectors of a homogeneous LFIHD on $\mathbf{k}[\sigma_M^{\vee}]$, where σ is a nonzero strongly convex rational cone.

Lemma 3.3. *Consider a non-trivial homogeneous LFIHD ∂ on $\mathbf{k}[\sigma_M^{\vee}]$. Then the following statements hold.*

- (i) *There exists $\rho \in \sigma(1)$ such that $\ker \partial = \mathbf{k}[\rho^* \cap M]$.*
- (ii) *The degree $e \in M$ of the sequence ∂ is a Demazure's root of σ and ρ is the distinguished ray of e .*

Proof. (i) By Proposition 2.4 (a) we have $\ker \partial = \mathbf{k}[W \cap \sigma_M^{\vee}]$ for some linear subspace $W \subseteq M_{\mathbb{R}}$. Assume that $W \cap \sigma^{\vee}$ is not a face of σ^{\vee} . Then W divides σ^{\vee} into two parts. We can find $m \in \sigma_M^{\vee}$ such that for any $r \in \mathbb{Z}_{\geq 0}$, $m + re \notin W$. Since $\chi^m \notin \ker \partial$, there is some $r_0 \in \mathbb{Z}_{>0}$ satisfying $\partial^{(r_0)}(\chi^m) \neq 0$. Hence $\partial^{(r_0)}(\chi^m)$ is homogeneous of degree $m + r_0 e$. By the previous argument

$$\partial^{(r'_1)} \circ \partial^{(r_0)}(\chi^m) \neq 0 \quad \text{for some } r'_1 \in \mathbb{Z}_{>0}.$$

By Definition 2.1 (iv) we have $\partial^{(r_0+r'_1)}(\chi^m) \neq 0$ and so we let $r_1 = r_0 + r'_1$. Proceeding by induction we can build a strictly increasing sequence of positive integers $\{r_j\}_{j \in \mathbb{Z}_{\geq 0}}$ verifying $\partial^{(r_j)}(\chi^m) \neq 0$ for any $j \in \mathbb{Z}_{\geq 0}$. This contradicts the fact that ∂ is an LFIHD. Thus $W \cap \sigma^{\vee}$ is a face of σ^{\vee} . Since $\ker \partial$ is a subring of codimension one, we have $W \cap \sigma_M^{\vee} = \rho^* \cap M$ for some extremal ray $\rho \in \sigma(1)$.

(ii) If $e \in \sigma_M^{\vee}$, then the same argument as before gives a contradiction. The rest of the proof follows as in [23, Lemma 2.4]. \square

In the following lemma, we state some properties of a homogeneous LFIHD on $\mathbf{k}[\sigma_M^{\vee}]$.

Lemma 3.4. *Let ∂ be a non-trivial homogeneous LFIHD on $\mathbf{k}[\sigma_M^{\vee}]$ of degree e and with distinguished ray ρ . For every $i \in \mathbb{Z}_{\geq 0}$ we let $c_i : \sigma_M^{\vee} \rightarrow \mathbf{k}$ be such that $\partial^{(i)}(\chi^m) = c_i(m)\chi^{m+ie}$. Then the sequence $\{c_i\}_{i \in \mathbb{Z}_{\geq 0}}$ of functions on σ_M^{\vee} satisfies the following conditions.*

- (i) The map c_0 is the constant map $m \mapsto 1$.
(ii) For all $m, m' \in \sigma_M^\vee$ we have

$$c_i(m + m') = \sum_{j=0}^i c_{i-j}(m) \cdot c_j(m'). \quad (3)$$

- (iii) For every $m \in \sigma_M^\vee$ there exists $r \in \mathbb{Z}_{\geq 0}$ such that $c_i(m) = 0$ for all $i \geq r$.
(iv) For every $i, j \in \mathbb{Z}_{\geq 0}$ we have

$$\binom{i+j}{i} c_{i+j}(m) = c_i(m + je) \cdot c_j(m) \quad \text{for all } m \in \sigma_M^\vee.$$

- (v) For every $i \in \mathbb{Z}_{\geq 0}$ we have $c_i(m + m') = c_i(m)$ for all $m \in \sigma_M^\vee$ and all $m' \in \rho^* \cap M$.

Proof. Assertions (i), (ii), (iii) and (iv) follow from the definition of LFIHD. Let us show (v). Since $\chi^{m'} \in \ker \partial$, for any $j \in \mathbb{Z}_{>0}$ we have $c_j(m') = 0$. Applying (3) we obtain $c_i(m + m') = c_i(m)$. \square

The next result provides a classification of normalized \mathbb{G}_a -actions on X_σ . See [23, Theorem 2.7] for the case where $\text{char}(\mathbf{k}) = 0$.

Theorem 3.5. *Let $\sigma \subseteq N_{\mathbb{R}}$ be a nonzero strongly convex rational cone. Every non-trivial \mathbb{G}_a -action on X_σ normalized by the \mathbb{T} -action is given by a homogeneous LFIHD of the form $\lambda \partial_e$, where ∂_e is as in Example 3.2, $e \in \text{Rt } \sigma$ and $\lambda \in \mathbf{k}^*$.*

Proof. Let ∂ be a non-trivial homogeneous LFIHD of degree e on $\mathbf{k}[\sigma_M^\vee]$. By Lemma 3.3, e is a root of σ and $\ker \partial = \mathbf{k}[\rho^* \cap M]$, where $\rho \in \sigma(1)$ is the distinguished ray of the root e .

Let us first show that there exists a lattice vector $m \in \sigma_M^\vee$ such that $\langle m, \rho \rangle = 1$. Let $m' \in \sigma_M^\vee$ not contained in the face ρ^* so that $\langle m', \rho \rangle > 1$. By [23, Lemma 2.4], we have that $m := m' + (\langle m', \rho \rangle - 1) \cdot e \in \sigma_M^\vee$ satisfies $\langle m, \rho \rangle = 1$.

We let $c_i : \sigma_M^\vee \rightarrow \mathbf{k}$ be the maps defined in Lemma 3.4. Let $B = \mathbf{k}[x]$ be the polynomial algebra of one variable. Using the basis $(1, x, x^2, \dots)$ we define a sequence of linear operators $\bar{\partial} = \{\bar{\partial}^{(i)}\}_{i \in \mathbb{Z}_{\geq 0}}$ on the \mathbf{k} -linear space B as follows: fixing a vector $m \in \sigma_M^\vee$ verifying $\langle m, \rho \rangle = 1$ we define

$$\bar{\partial}^{(i)}(x^r) = c_i(rm)x^{r-i} \quad \text{for all } i, r \in \mathbb{Z}_{\geq 0}.$$

We claim that $\bar{\partial}$ is well defined. Indeed, let $i, r \in \mathbb{Z}_{\geq 0}$ be such that $i > r$, then

$$\begin{aligned} \partial^{(i)}(\chi^{rm}) &= c_i(rm)\chi^{rm+ie} \in \mathbf{k}[\sigma_M^\vee] \quad \text{and} \quad \langle rm + ie, \rho \rangle = r - i < 0 \\ \text{so that} \quad c_i(rm) &= 0. \end{aligned}$$

Hence, $\bar{\partial}^{(i)}(x^r) = c_i(rm)x^{r-i} = 0$ for all $i > r$.

By Lemma 3.4, the sequence of operators $\bar{\partial}$ is an LFIHD on B . For instance, let us show that $\bar{\partial}$ satisfies Definition 2.1 (iv). Letting $i, j \in \mathbb{Z}_{\geq 0}$ we have

$$\bar{\partial}^{(i)} \circ \bar{\partial}^{(j)}(x^r) = \bar{\partial}^{(i)}(c_j(rm)x^{r-j}) = c_i((r-j)m) \cdot c_j(rm)x^{r-i-j}.$$

Since $e \in \text{Rt } \sigma$ is a root having ρ as distinguished ray, it follows that

$$v := rm + je - (r-j)m = j(m+e) \in \rho^* \cap M.$$

By Lemma 3.4 (v), we have

$$c_i((r-j)m) = c_i((r-j)m + v) = c_i(rm + je).$$

Therefore by Lemma 3.4 (iv), we conclude that

$$\bar{\partial}^{(i)} \circ \bar{\partial}^{(j)}(x^r) = \binom{i+j}{i} c_{i+j}(rm)x^{r-i-j} = \binom{i+j}{i} \bar{\partial}^{(i+j)}(x^r),$$

as required. Conditions (i), (ii), (iii) of Definition 2.1 follow from similar straightforward computations.

Since $\bar{\partial}$ is homogeneous for the natural graduation of B , by Proposition 2.4 (d) there exists $\lambda \in \mathbf{k}$ such that every c_i verifies

$$c_i(rm) = \binom{r}{i} \lambda^i$$

for any $r \in \mathbb{Z}_{\geq 0}$. We use the convention $\lambda^0 = 1$ whenever $\lambda = 0$. Let $w \in \sigma_M^\vee$ be a lattice vector. The elements

$$w + \langle w, \rho \rangle e, \langle w, \rho \rangle e + \langle w, \rho \rangle m$$

belong to $\rho^* \cap M$. By Lemma 3.4 (v) this implies

$$c_i(w) = c_i(w + \langle w, \rho \rangle e + \langle w, \rho \rangle m) = c_i(\langle w, \rho \rangle m) = \binom{\langle w, \rho \rangle}{i} \lambda^i. \quad (4)$$

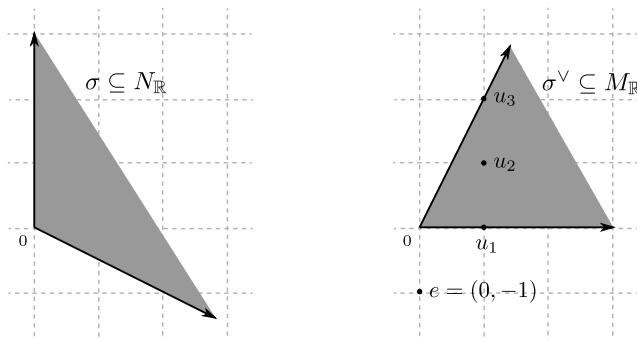
Since ∂ is non-trivial, we have $\lambda \in \mathbf{k}^*$. By virtue of (4) the sequence ∂ is given by the LFIHD $\lambda \partial_e$ (see Example 3.2). \square

Example 3.6. Let $M = \mathbb{Z}^2$ and let σ be the strongly convex rational cone generated in the vector space $N_{\mathbb{R}} = \mathbb{R}^2$ by the vectors and $\rho = (0, 1)$ and $\rho' = (2, -1)$. The dual cone σ^\vee is the cone in $M_{\mathbb{R}}$ generated by the vectors $(1, 0)$ and $(1, 2)$. Let $A = \mathbf{k}[\sigma_M^\vee]$ and

let $X = \operatorname{Spec} A$ be the corresponding toric variety. The algebra A is generated by the elements

$$u_1 = \chi^{(1,0)}, \quad u_2 = \chi^{(1,1)} \quad \text{and} \quad u_3 = \chi^{(1,2)}.$$

The generators satisfy the relation $u_1 u_3 = u_2^2$ and so $A = \mathbf{k}[x, y, z]/(xz - y^2)$. The lattice vector $e = (0, -1) \in M$ is a root of σ since $\langle e, \rho \rangle = -1$ and $\langle e, \rho' \rangle = 1$.



The corresponding LFIHD ∂_e of Example 3.2 is given by

$$\begin{aligned} \partial_e^{(0)}(x) &= x, & \partial_e^{(i)}(x) &= 0, & \text{for all } i > 0; \\ \partial_e^{(0)}(y) &= y, & \partial_e^{(1)}(y) &= x, & \partial_e^{(i)}(y) &= 0, & \text{for all } i > 1; \\ \partial_e^{(0)}(z) &= z, & \partial_e^{(1)}(z) &= 2y, & \partial_e^{(2)}(z) &= x, & \partial_e^{(i)}(z) &= 0, & \text{for all } i > 2. \end{aligned}$$

Hence, the corresponding normalized \mathbb{G}_a -action ϕ is defined by

$$\phi : \mathbb{G}_a \times X \rightarrow X, \quad \text{where} \quad (\lambda, (x, y, z)) \mapsto (x, y + \lambda x, z + 2\lambda y + \lambda^2 z).$$

As an immediate consequence of Theorem 3.5, we obtain a description of all normalized \mathbb{G}_a -actions up to a Frobenius map.

Corollary 3.7. *Let $\sigma \subseteq N_{\mathbb{R}}$ be a nonzero strongly convex rational cone. Then for every root $e \in \operatorname{Rt} \sigma$ with distinguished ray ρ , every integer $r \in \mathbb{Z}_{\geq 0}$, and every scalar $\lambda \in \mathbf{k}^*$, there is a non-trivial rationally homogeneous LFIHD ∂ on the algebra $\mathbf{k}[\sigma_M^{\vee}]$ whose exponential is given by*

$$e^{x\partial}(\chi^m) = \sum_{i=0}^{\infty} \binom{\langle m, \rho \rangle}{i} \lambda^i \chi^{m+ie} x^{ip^r} \quad \text{for all } m \in \sigma_M^{\vee}.$$

Conversely, every rationally homogeneous LFIHD on $\mathbf{k}[\sigma_M^{\vee}]$ arises in this way.

In the next corollary, we generalize to the case of positive characteristic some results in [23, Section 2]. See also [20, Corollary 3.5] for a more general statement in the characteristic zero case. The proofs are similar to those in [23] so we omit them.

Corollary 3.8. *Let $\sigma \subseteq N_{\mathbb{R}}$ be a strongly convex rational, then the following hold.*

- (i) *For any normalized up to a Frobenius map \mathbb{G}_a -actions in $\mathrm{Spec} \mathbf{k}[\sigma_M^{\vee}]$ the algebra of invariants is finitely generated.*
- (ii) *There is a finite number of rationally homogeneous LFIHDs on $\mathbf{k}[\sigma_M^{\vee}]$ with pairwise distinct kernels.*

4. \mathbb{G}_a -actions of vertical type

In this section, we classify normalized \mathbb{G}_a -actions of vertical type on an affine \mathbb{T} -variety $X = \mathrm{Spec} A$ of complexity one over a field \mathbf{k} . See [24] for higher complexity when the base field is algebraically closed of characteristic zero.

To achieve our classification, we place ourselves in the context of Section 1 by letting $A = A[C, \mathfrak{D}]$, where C is a regular curve over \mathbf{k} and $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ is a proper σ -polyhedral divisor. Hence,

$$A[C, \mathfrak{D}] = \bigoplus_{m \in \sigma_M^{\vee}} A_m \cdot \chi^m \subseteq K_0[M],$$

$$\text{where } A_m = H^0(C, \mathcal{O}_C(\mathfrak{D}(m))) \text{ and } K_0 = \mathbf{k}(C).$$

The following result gives some general properties of homogeneous LFIHDs on the M -graded algebra A . Recall that the affine \mathbb{T} -variety $X = \mathrm{Spec} A$ is called elliptic if $A_0 = \mathbf{k}$.

Lemma 4.1. *Let ∂ be a homogeneous LFIHD on A of degree e . Then the following statements hold.*

- (i) *If ∂ is vertical, then $e \notin \sigma^{\vee}$ and $\ker \partial = \bigoplus_{m \in \tau_M} A_m \chi^m$ for some codimension one face τ of the cone σ^{\vee} . In particular, the algebra $\ker \partial$ is finitely generated.*
- (ii) *If A is non-elliptic, then ∂ is vertical if and only if $e \notin \sigma^{\vee}$.*

Proof. (i) By Lemma 2.12 we may extend ∂ to a homogeneous LFIHD $\bar{\partial}$ on the semigroup K_0 -algebra $K_0[\sigma_M^{\vee}]$. By Lemma 3.3 we have $e \in \mathrm{Rt} \sigma$ and so $e \notin \sigma^{\vee}$. Moreover, we obtain $\ker \bar{\partial} = K_0[\tau_M]$ for some codimension one face τ of σ^{\vee} . Thus,

$$\ker \partial = A \cap \ker \bar{\partial} = \bigoplus_{m \in \tau_M} A_m \chi^m.$$

As a consequence of [1, Lemma 4.1], the algebra $\ker \partial$ is finitely generated.

(ii) Assume that A is non-elliptic and let $\bar{\partial}$ be the extension of ∂ on the K_0 -algebra $K_0[M]$. If $e \notin \sigma^\vee$, then for any $i \in \mathbb{Z}_{>0}$ we have $\partial^{(i)}(A_0) = A_{ie} = \{0\}$. Since $K_0 = \text{Frac } A_0$, we conclude that ∂ is vertical. \square

As remarked in [23, Remark 3.2], in the elliptic case, the M -graded algebra admits in general LFIHDs ∂ of horizontal type satisfying $\deg \partial \notin \sigma^\vee$.

In the following, we introduce some combinatorial data on $A = A[C, \mathfrak{D}]$ in order to describe its vertical normalized \mathbb{G}_a -actions.

Notation 4.2. Let $e \in \text{Rt } \sigma$ be a root of σ with distinguished ray ρ and recall that $\mathfrak{D}(e) = \sum_{z \in C} \min_{v \in \Delta_z(0)} \langle e, v \rangle \cdot z$. We denote by Φ_e the A_0 -module $H^0(C, \mathcal{O}_C(\mathfrak{D}(e)))$. Furthermore, if $\varphi \in \Phi_e$ is a nonzero section, then for any vector $m \in \sigma^\vee$ belonging to $M_{\mathbb{Q}}$ we have

$$\text{div } \varphi \geq -\mathfrak{D}(e) \geq \mathfrak{D}(m) - \mathfrak{D}(m + e). \quad (5)$$

Starting with the previous combinatorial data, we may construct a homogeneous LFIHD of vertical type, as follows:

Lemma 4.3. Let $e \in \text{Rt } \sigma$ be a root of σ with distinguished ray ρ and let $\varphi \in \Phi_e$ be a section. Denote $\bar{\partial} = \varphi \partial_e$, where ∂_e is the LFIHD on the K_0 -algebra $K_0[\sigma_M^\vee]$ corresponding to the root e as in Example 3.2. Then for any $i \in \mathbb{Z}_{\geq 0}$ we have $\bar{\partial}^{(i)}(A) \subseteq A$. Consequently, the sequence

$$\partial_{e,\varphi} := \left\{ \bar{\partial}^{(i)}|_A : A \rightarrow A \right\}_{i \in \mathbb{Z}_{\geq 0}}$$

defines a homogeneous LFIHD of vertical type on A .

Proof. Fix $i \in \mathbb{Z}_{>0}$ and let $f \in A_m$ be nonzero such that $\text{div } f + \lfloor \mathfrak{D}(m) \rfloor \geq 0$. If $\partial^{(i)}(f\chi^m) \neq 0$ and $\varphi \neq 0$, then by (5) we have

$$\begin{aligned} & \text{div} \left(\partial^{(i)}(f\chi^m) / \chi^{m+ie} \right) + \lfloor \mathfrak{D}(m + ie) \rfloor \\ &= i \text{div } \varphi + \text{div } f + \lfloor \mathfrak{D}(m + ie) \rfloor \geq i(\mathfrak{D}(m/i) - \mathfrak{D}(m/i + e)) - \lfloor \mathfrak{D}(m) \rfloor + \lfloor \mathfrak{D}(m + ie) \rfloor \\ &\geq \{\mathfrak{D}(m)\} - \{\mathfrak{D}(m + ie)\}. \end{aligned}$$

Since the coefficients of the \mathbb{Q} -divisor $\{\mathfrak{D}(m)\} - \{\mathfrak{D}(m + ie)\}$ belong to $] -1, 1[$ we have

$$\text{div} \left(\partial^{(i)}(f\chi^m) / \chi^{m+ie} \right) + \lfloor \mathfrak{D}(m + ie) \rfloor \geq 0,$$

proving that A is ∂ -invariant. The rest of the proof is straightforward and left to the reader. \square

Our next theorem gives a classification of normalized vertical \mathbb{G}_a -actions on an affine \mathbb{T} -variety $X = \operatorname{Spec} A[C, \mathfrak{D}]$ of complexity one.

Theorem 4.4. *Let $A = A[C, \mathfrak{D}]$. If $e \in \operatorname{Rt} \sigma$ is a root of σ with distinguished ray ρ and $\varphi \in \Phi_e$ is a section, then $\partial_{e, \varphi}$ is a homogeneous vertical LFIHD on A . Conversely, every homogeneous vertical LFIHD on A is of the form $\partial_{e, \varphi}$, where $e \in \operatorname{Rt} \sigma$ and $\varphi \in \Phi_e$.*

Proof. The direct implication corresponds to Lemma 4.3. To prove the converse statement, let ∂ be a non-trivial homogeneous vertical LFIHD on A . By Lemma 2.12, ∂ extends to a locally finite iterative higher K_0 -derivation $\bar{\partial}$ on the semigroup algebra $K_0[\sigma_M^\vee]$. By Theorem 3.5, $\bar{\partial}$ is given by $\varphi \partial_e$ as in Example 3.2, for some $\varphi \in K_0^*$ and some root $e \in \operatorname{Rt} \sigma$.

To conclude the proof, let us show that $\varphi \in \Phi_e$. Let ρ be the distinguished ray of e . For every point $z \in C$ we let v_z be a vertex of Δ_z where the minimum $\min_{v \in \Delta_z(0)} \langle e, v \rangle$ is achieved so that

$$\mathfrak{D}(e) = \sum_{z \in C} \langle e, v_z \rangle \cdot z.$$

For every $z \in C$ we let $\omega_z = \{m \in \sigma^\vee \mid h_{\Delta_z}(m) = \langle m, v_z \rangle\}$. The set $\omega_z \subseteq M_{\mathbb{R}}$ is a full dimensional cone in $M_{\mathbb{R}}$ (see [1, Section 1]).

Let also $m_z \in \sigma_M^\vee \setminus \rho_M^\star$ be a lattice vector such that m_z and $m_z + e$ belong to ω_z , $\deg \mathfrak{D}(m_z) \geq g$ and $\langle m_z, \rho \rangle \notin p\mathbb{Z}$, where p is characteristic of the field \mathbf{k} and g the genus of the curve C . It is always possible to choose such m_z since ω_z is full dimensional, the polyhedral divisor \mathfrak{D} is proper, and the lattice vector ρ is primitive. According to the Riemann–Roch Theorem we have $A_{m_z} \neq \{0\}$.

Furthermore, the inclusion $\partial^{(1)}(A_{m_z} \chi^{m_z}) \subseteq A_{m_z+e} \chi^{m_z+e}$ implies $\varphi A_{m_z} \subseteq A_{m_z+e}$. Consequently, for any $z \in C$ we have

$$\operatorname{div} \varphi \geq \mathfrak{D}(m_z) - \mathfrak{D}(m_z + e).$$

The coefficient of the divisor $\mathfrak{D}(m_z) - \mathfrak{D}(m_z + e)$ at the point $z \in C$ is $-\langle v_z, e \rangle$. Thus, $\operatorname{div} \varphi \geq -\mathfrak{D}(e)$ and we have $\varphi \in \Phi_e$, as required. \square

In analogy with the toric case, the family of vertical normalized \mathbb{G}_a -actions on $X = \operatorname{Spec} A$ having pairwise distinct kernels is a finite set. The next result provides a combinatorial criterion for A to admit a homogeneous non-trivial LFIHD of vertical type.

Corollary 4.5. *Let $A = A[C, \mathfrak{D}]$ and let $\rho \subseteq \sigma$ be an extremal ray. Then, the M -graded algebra A admits a non-trivial vertical homogeneous LFIHD such that the distinguished ray of $e = \deg \partial \in \operatorname{Rt} \sigma$ is ρ if and only if one of the following conditions holds.*

- (i) C is affine, or
- (ii) C is projective and $\rho \cap \deg \mathfrak{D} = \emptyset$.

Proof. If C is an affine curve, then every divisor on C has a global nonzero section and so for any $e \in \text{Rt } \sigma$ we have $\dim \Phi_e > 0$. In this case, the corollary follows from [Theorem 4.4](#).

Assume that C is projective and fix a root $e \in \text{Rt } \sigma$ with distinguished ray ρ . Let us notice that for any $m \in \rho_M^*$ we have $e + m \in \text{Rt } \sigma$. Furthermore

$$\mathfrak{D}(e + m) \geq \mathfrak{D}(m) + \mathfrak{D}(e) \quad \text{and so} \quad \deg \mathfrak{D}(m + e) \geq \deg \mathfrak{D}(m) + \deg \mathfrak{D}(e).$$

Hence, if $\rho \cap \deg \mathfrak{D} = \emptyset$, then we have $\dim \Phi_{e+m} > 0$ for some $m \in \rho_M^*$, by the Riemann–Roch Theorem and by the properness of \mathfrak{D} .

Conversely, assume that $\rho \cap \deg \mathfrak{D} \neq \emptyset$. Since we have $\langle e, \rho \rangle = -1$, there exists a vertex v of $\deg \mathfrak{D}$ such that $\langle e, v \rangle < 0$ and therefore $\deg \mathfrak{D}(e) < 0$. Under these latter conditions we have $\dim \Phi_e = 0$. Again, we conclude by [Theorem 4.4](#) in the case where C is projective. \square

Example 4.6. Let the notation be as in [Example 1.8](#). Let ρ be the ray of σ spanned by $(1, 0)$ and let ρ' be the ray of σ spanned by $(0, 1)$. We have $\deg \mathfrak{D} \cap \rho \neq \emptyset$ and $\deg \mathfrak{D} \cap \rho' = \emptyset$. Hence, [Corollary 4.5](#) shows that only ρ' can be the distinguished ray of the degree e of an LFIHD ∂ of vertical type.

5. \mathbb{G}_a -actions of horizontal type

The purpose of this section is to classify all horizontal \mathbb{G}_a -actions on affine \mathbb{T} -varieties of complexity one over a perfect field in terms of polyhedral divisors. The reader may consult [\[23, Section 3.2\]](#) for the case where \mathbf{k} is algebraically closed and of characteristic zero. Let as before $A = A[C, \mathfrak{D}]$, where C is a regular curve over \mathbf{k} and $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ is a proper σ -polyhedral divisor. Hence,

$$A[C, \mathfrak{D}] = \bigoplus_{m \in \sigma_M^\vee} A_m \cdot \chi^m \subseteq K_0[M], \quad \text{where} \quad A_m = H^0(C, \mathcal{O}_C(\mathfrak{D}(m))) \quad \text{and} \quad K_0 = \mathbf{k}(C).$$

In this section, several results will require the assumption that \mathbf{k} is perfect so the classification will only hold in this case. Nevertheless, the statements that we can prove without asking for \mathbf{k} to be perfect are stated in general.

According to the Rosenlicht Theorem [\[31\]](#), in the case where \mathbf{k} is algebraically closed, the following lemma implies in particular that an affine horizontal \mathbb{T} -variety of complexity one has an open orbit for its corresponding $\mathbb{T} \ltimes \mathbb{G}_a$ -action.

Lemma 5.1. *Let $X = \text{Spec } A$, where $A = A[C, \mathfrak{D}]$ and let ∂ be a homogeneous LFIHD on A . Then ∂ is horizontal if and only if $\mathbf{k}(X)^{\mathbb{G}_a} \cap \mathbf{k}(X)^{\mathbb{T}} = \mathbf{k}$.*

Proof. Let $L = \mathbf{k}(X)^{\mathbb{G}_a} \cap \mathbf{k}(X)^{\mathbb{T}}$. Assume that ∂ is horizontal and that $\mathbf{k}(X)^{\mathbb{T}}/L$ is an algebraic field extension. Consider $F \in \mathbf{k}(X)^{\mathbb{T}}$ a nonzero invariant rational function.

Remarking that $\mathbf{k}(X)^{\mathbb{G}_a}$ is the field of fractions of the ring $\ker \partial$, we can find $a \in \ker \partial$ such that aF is integral over $\ker \partial$. Since A is normal, $aF \in A$, and by Proposition 2.4 (b) we have $aF \in \ker \partial$. Hence $F \in \mathbf{k}(X)^{\mathbb{G}_a}$, contradicting the fact that ∂ is of horizontal type. Since $\mathbf{k}(X)^{\mathbb{T}}/\mathbf{k}$ is of transcendence degree one, we have that L/\mathbf{k} is algebraic. By our convention \mathbf{k} is algebraically closed in $\mathbf{k}(X)$ which yields $L = \mathbf{k}$. The converse implication follows directly from the definition of horizontal and vertical LFIHDs. \square

Our next lemma shows that the existence of a homogeneous LFIHD on the algebra $A = A[C, \mathfrak{D}]$ imposes some restrictions on the curve C . We refer the reader to [16, 3.5], [23, 3.16] for the case where the base field is algebraically closed of characteristic zero.

Lemma 5.2. *Assume that $A = A[C, \mathfrak{D}]$ admits a homogeneous LFIHD ∂ of horizontal type. Consider ω (resp. L) the cone (resp. sublattice) generated by the weights of $\ker \partial$ and let $\omega_L = \omega \cap L$. Then the following statements hold.*

(i) *The kernel of ∂ is a semigroup algebra, i.e.,*

$$\ker \partial = \bigoplus_{m \in \omega_L} \mathbf{k} \cdot \varphi_m \chi^m, \quad \text{where } \varphi_m \in \mathbf{k}(C)^*.$$

(ii) *We have $C \simeq \mathbb{P}_{\mathbf{k}}^1$, in the case where A is elliptic.*

(iii) *If \mathbf{k} is perfect, then $C \simeq \mathbb{A}_{\mathbf{k}}^1$ in the case where A is non-elliptic.*

Proof. (i) Let $a, a' \in \ker \partial \setminus \{0\}$ be homogeneous elements of the same degree. By Lemma 5.1, we have $a/a' \in \mathbf{k}(X)^{\mathbb{G}_a} \cap \mathbf{k}(X)^{\mathbb{T}} = \mathbf{k}$. Thus $\ker \partial$ is a semigroup algebra. By Proposition 2.4 (b) we have that $\ker \partial$ is integrally closed, hence normal. This yields (i).

(ii) Let $K = \text{Frac } A$ and consider $E = K^{\mathbb{G}_a}$. By [9, Lemma 2.2] there exists a variable x over the field E such that $E(x) = K$. By (i), the extension E/\mathbf{k} is purely transcendental and so is K/\mathbf{k} . Since $\mathbf{k}(C) \subseteq K$, the regular projective curve C is unirational. According to the Luröth Theorem, it follows that $C \simeq \mathbb{P}_{\mathbf{k}}^1$.

(iii) Assume that A is non-elliptic. Let $\bar{\mathbf{k}}$ be an algebraic closure of \mathbf{k} , so that the field extension $\bar{\mathbf{k}}/\mathbf{k}$ is separable. Let B be the algebra $A \otimes_{\mathbf{k}} \bar{\mathbf{k}}$. Then B is a normal finitely generated M -graded domain (see Lemma 1.9). Note that the graded piece B_0 is $A_0 \otimes_{\mathbf{k}} \bar{\mathbf{k}}$. Consequently, ∂ extends to a homogeneous LFIHD of horizontal type on the $\bar{\mathbf{k}}$ -algebra B . Now, we can apply the geometrical argument in [23, Lemma 3.16] to conclude that we have $B_0 \simeq \bar{\mathbf{k}}[t]$, for some variable t over $\bar{\mathbf{k}}$. By separability of $\bar{\mathbf{k}}/\mathbf{k}$, this yields $A_0 \simeq \mathbf{k}[t]$ (see e.g. [32, 6]). \square

The preceding lemma implies that the kernel of a horizontal homogeneous LFIHD on A is finitely generated. This result can be obtained independently from [20, Theorem 1.3] in the characteristic zero case. Note also that the kernel of a non-trivial LFIHD on a normal unirational surface V over a perfect field \mathbf{k} such that $\mathbf{k}[V]^* = \mathbf{k}^*$ is a polynomial algebra (see [28, Theorem 2]).

5.3. In view of the above results, in the following we let $C = \mathbb{A}_{\mathbf{k}}^1$ or $C = \mathbb{P}_{\mathbf{k}}^1$. Assume that A has a homogeneous LFIHD ∂ of horizontal type and let

$$\ker \partial = \bigoplus_{m \in \omega_L} \mathbf{k} \cdot \varphi_m \chi^m$$

be the kernel of ∂ . We also assume that $\mathbf{k}(C) = \mathbf{k}(t)$ for some local parameter t and, when C is affine, we let $\mathbf{k}[C] = \mathbf{k}[t]$ be its coordinate ring.

Lemma 5.4. *Keeping the notation as above, the following statements hold.*

- (i) *If $C = \mathbb{A}_{\mathbf{k}}^1$, then for any $m \in \omega_L$ we have $\operatorname{div} \varphi_m + \mathfrak{D}(m) = 0$.*
- (ii) *Assume that $C = \mathbb{P}_{\mathbf{k}}^1$. Then there exists a point $z_\infty \in C$ such that for any $m \in \omega_L$ the effective \mathbb{Q} -divisor $\operatorname{div} \varphi_m + \mathfrak{D}(m)$ has at most z_∞ in its support.*
- (iii) *The cone ω is a maximal cone of the quasifan $\Lambda(\mathfrak{D})$ (see [Definition 1.5](#)) in the non-elliptic case, and of $\Lambda(\mathfrak{D}|_{\mathbb{P}_{\mathbf{k}}^1 \setminus \{z_\infty\}})$ for the elliptic case.*
- (iv) *The rank of the lattice L is equal to $n = \operatorname{rank} M$. The lattice M is spanned by $e := \deg \partial$ and L . Furthermore, if d is the smallest positive integer such that $de \in L$, then we can write every vector $m \in M$ in an unique way as $m = l + re$ for some $l \in L$ and some $r \in \mathbb{Z}$ such that $0 \leq r < d$.*
- (v) *If \mathbf{k} is perfect, then the point z_∞ in (ii) is rational, i.e., the residue field of z_∞ is \mathbf{k} .*

Proof. (i) Given a lattice vector $m \in \sigma_M^\vee$ we let

$$A_m = f_m \cdot \mathbf{k}[t],$$

where $f_m \in \mathbf{k}(t)$. Assume that $m \in \omega_L$. Then we have $\varphi_m = Ff_m$, for some nonzero $F \in \mathbf{k}[t]$. By [Proposition 2.4\(a\)](#) the polynomial F is constant. Hence,

$$\operatorname{div} \varphi_m + [\mathfrak{D}(m)] = 0.$$

Consequently, for any $r \in \mathbb{Z}_{\geq 0}$ we obtain

$$r[\mathfrak{D}(m)] = -r \operatorname{div} \varphi_m = -\operatorname{div} \varphi_{rm} = [\mathfrak{D}(rm)].$$

This shows that $\mathfrak{D}(m)$ is integral when $m \in \omega_L$.

(ii) Assume that there exists $m \in \omega_L$ such that

$$\operatorname{div} \varphi_m + \mathfrak{D}(m) \geq [z_\infty] + [z_0],$$

where z_0, z_∞ are distinct points of C . Denote by ∞ the point at the infinity in $C = \mathbb{P}_{\mathbf{k}}^1$ for the local parameter t . Let $p_0(t), p_\infty(t) \in \mathbf{k}(t)$ be two rational functions verifying the following: if the point z_0 (resp. z_∞) belongs to $\mathbb{A}_{\mathbf{k}}^1 = \operatorname{Spec} \mathbf{k}[t]$, then $p_0(t)$ (resp. $p_\infty(t)$) is

the monic polynomial generator of the ideal of z_0 (resp. z_∞) in $\mathbf{k}[t]$. Otherwise, $z_0 = \infty$ (resp. $z_\infty = \infty$) and we let $p_0(t) = 1/t$ (resp. $p_\infty(t) = 1/t$).

Let $f := p_0(t)/p_\infty(t)$. The rational functions $f\varphi_m$ and $f^{-1}\varphi_m$ belong to A_m . By [Proposition 2.4](#) (a) we have

$$f\varphi_m\chi^m \cdot f^{-1}\varphi_m\chi^m = \varphi_{2m}\chi^{2m} \in \ker \partial, \quad \text{and so} \quad f\varphi_m\chi^m, f^{-1}\varphi_m\chi^m \in \ker \partial,$$

yielding a contradiction with [Lemma 5.2](#) (i). Hence, $\operatorname{div} \varphi_m + \mathfrak{D}(m)$ is supported in at most one point.

(iii) By (i) and (ii), the map $m \mapsto \mathfrak{D}(m)$ in the non-elliptic case, and the map $m \mapsto \mathfrak{D}|_{\mathbb{P}_{\mathbf{k}}^1 \setminus \{z_\infty\}}(m)$ in the elliptic case, are linear in the cone ω . This implies that there exists a maximal cone ω_0 belonging to $\Lambda(\mathfrak{D})$ in the non-elliptic case, and belonging to $\Lambda(\mathfrak{D}|_{\mathbb{P}_{\mathbf{k}}^1 \setminus \{z_\infty\}})$ in the elliptic case, such that $\omega \subseteq \omega_0$.

Let us show the reverse inclusion $\omega_0 \subseteq \omega$. Let $m \in \omega_0$. Changing m by an integral multiple, we may assume $m \in L$ and $\mathfrak{D}(m)$ integral. By [Lemma 5.2](#) (i) and [Proposition 2.4](#) (c), the cone ω is full dimensional in $M_{\mathbb{R}}$. Hence, there exists $m' \in \omega_L$ such that $m + m' \in \omega_L$. Consider a nonzero section $f_m \in A_m$ such that

$$\operatorname{div} f_m + \mathfrak{D}(m) = 0$$

in the non-elliptic case, and such that

$$(\operatorname{div} f_m + \mathfrak{D}(m))|_{\mathbb{P}_{\mathbf{k}}^1 \setminus \{z_\infty\}} = 0$$

in the elliptic case. It follows that

$$f_m\chi^m \cdot \varphi_{m'}\chi^{m'} = \lambda\varphi_{m+m'}\chi^{m+m'}$$

for some $\lambda \in \mathbf{k}^*$. Therefore, $f_m\chi^m \in \ker \partial$ and again by [Proposition 2.4](#) (a) we have $m \in \omega$.

(iv) According to the fact that σ_M^\vee spans M and that ∂ is a homogeneous LFIHD on A , for any $m \in M$ we have $m + se \in L$ for some $s \in \mathbb{Z}$. Changing $r := -s$ by the remainder of the Euclidean division of r by d , if necessary, we obtain $m = l + re$, where $l \in L$ and $0 \leq r < d$. The minimality of d implies that this latter decomposition is unique.

(v) Assume that \mathbf{k} is perfect and fix $\bar{\mathbf{k}}$ an algebraic closure of \mathbf{k} . Consider the algebra $B = A \otimes_{\mathbf{k}} \bar{\mathbf{k}}$. If we let $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$, then by [Lemma 1.9](#) the polyhedral divisor

$$\mathfrak{D}_{\bar{\mathbf{k}}} = \sum_{z \in C} \Delta_z \cdot S^*(z)$$

over $\mathbb{P}_{\bar{\mathbf{k}}}^1$ satisfies

$$B = \bigoplus_{m \in \sigma_M^\vee} B_m \chi^m, \quad \text{where } B_m = H^0(\mathbb{P}_{\bar{\mathbf{k}}}^1, \mathcal{O}_{\mathbb{P}_{\bar{\mathbf{k}}}^1}(\mathfrak{D}_{\bar{\mathbf{k}}}(m))).$$

We can also extend ∂ to a homogeneous LFIHD $\partial_{\bar{\mathbf{k}}}$ of horizontal type on B . For any $m \in \omega_L$ we have $\varphi_m \chi^m \in \ker \partial_{\bar{\mathbf{k}}}$ and there exists a rational non-negative number λ_m such that

$$\operatorname{div} \varphi_m + \mathfrak{D}(m) = \lambda_m \cdot z_\infty.$$

Applying S^* to the previous equality we obtain

$$\operatorname{div}_{\bar{\mathbf{k}}} \varphi_m + \mathfrak{D}_{\bar{\mathbf{k}}}(m) = \lambda_m \cdot S^*(z_\infty).$$

Assume that z_∞ is not a rational point and that $\lambda_m > 0$ for some lattice vector $m \in \omega_L$. Changing m by a multiple we may suppose that λ_m is greater than 1. Since the field extension $\bar{\mathbf{k}}/\mathbf{k}$ is separable, the polynomial $p_{z_\infty}(t)$ in the proof of (ii) has at least two distinct roots, say $z_1, z_2 \in \bar{\mathbf{k}}$. Note that the points z_1, z_2 belong to the support of $S^*(z_\infty)$. Considering the non-constant rational function

$$f = (t - z_1)/(t - z_2),$$

we fall again into a contradiction with [Lemma 5.2](#) (i) since

$$f \varphi_m \chi^m \cdot f^{-1} \varphi_m \chi^m = \varphi_{2m} \chi^{2m} \in \ker \partial_{\bar{\mathbf{k}}}, \quad \text{and so } f \varphi_m \chi^m, f^{-1} \varphi_m \chi^m \in \ker \partial_{\bar{\mathbf{k}}}. \quad \square$$

In the sequel, we let the notation be as in [5.3](#). Without loss of generality, whenever \mathbf{k} is perfect, in the elliptic case we can assume that z_∞ is the rational point ∞ for the local parameter t .

Lemma 5.5. *Let \mathbf{k} be a perfect field. The following statements hold.*

- (i) *If $C = \mathbb{P}_{\bar{\mathbf{k}}}^1$, then the normalization of the subalgebra $A[t] \subseteq \mathbf{k}(t)[M]$ is $A' = A[\mathbb{A}_{\bar{\mathbf{k}}}^1, \mathfrak{D}|_{\mathbb{A}_{\bar{\mathbf{k}}}^1}]$, where $\mathbb{A}_{\bar{\mathbf{k}}}^1 = \operatorname{Spec} \mathbf{k}[t]$.*
- (ii) *If the degree of ∂ belongs to ω and the evaluation of the polyhedral divisor $\mathfrak{D}|_{\mathbb{A}_{\bar{\mathbf{k}}}^1}$ is linear, then ∂ extends to a homogeneous LFIHD ∂' on A' of horizontal type. Furthermore, we have $\ker \partial = \ker \partial'$.*
- (iii) *Let d be the smallest positive integer such that for any $m \in \omega_M$ the divisor $\mathfrak{D}(d \cdot m)$ is integral. Then we have $d \cdot M \subseteq L$.*

Proof. (i) This follows from [\[21, Theorem 2.5\]](#).

(ii) Letting

$$A' = \bigoplus_{m \in \sigma_M^\vee} A'_m \chi^m, \quad \text{where } A'_m = H^0(\mathbb{A}_{\bar{\mathbf{k}}}^1, \mathcal{O}_{\mathbb{A}_{\bar{\mathbf{k}}}^1}(\mathfrak{D}|_{\mathbb{A}_{\bar{\mathbf{k}}}^1}(m))),$$

for any $m \in \sigma_M^\vee$ we can write $A'_m = \varphi_m \cdot \mathbf{k}[t]$ with φ_m is a nonzero rational function satisfying

$$\operatorname{div} \varphi_m + [\mathfrak{D}|_{\mathbb{A}_{\mathbf{k}}^1}(m)] = 0.$$

If $m \in \omega_L$, we can assume that φ_m is as in Lemma 5.4 (ii).

By Lemma 2.5, we may extend ∂ to a homogeneous iterative higher derivation ∂' on the semigroup algebra $\mathbf{k}(t)[M]$. Denote by $\partial'^{(i)}$ the i -th term of ∂' . Consider $f \in A'_m$ for a lattice vector $m \in \sigma_M^\vee$ and fix an integer $i \in \mathbb{Z}_{>0}$. We will show that $\partial'^{(i)}(f\chi^m) \in A'$.

By the properness of \mathfrak{D} and Lemma 5.4 (ii) with $z_\infty = \infty$, we can find a lattice vector $m' \in \omega_L$ verifying the following. The vectors m, m' belong to a same maximal cone of $\Lambda(\mathfrak{D})$ and the coefficient in ∞ of the divisor $\operatorname{div} \varphi_{m'} + \mathfrak{D}(m')$ is integral, positive, and greater than that of $-\operatorname{div} f - [\mathfrak{D}(m)]$. Therefore

$$\operatorname{div} f\varphi_{m'} + [\mathfrak{D}(m' + m)] = \operatorname{div} f + [\mathfrak{D}(m)] + \operatorname{div} \varphi_{m'} + \mathfrak{D}(m') \geq 0.$$

In particular, $\varphi_{m'}f$ belongs to $A_{m+m'}$. Hence it follows that

$$\varphi_{m'}\chi^{m'}\partial'^{(i)}(f\chi^m) = \partial^{(i)}(\varphi_{m'}f\chi^{m'+m}) \in A_{m'+m+ie}\chi^{m'+m+ie}.$$

By our assumption we have $e \in \omega = \sigma^\vee$ so that $m + ie \in \sigma_M^\vee$. Since $\mathfrak{D}|_{\mathbb{A}_{\mathbf{k}}^1}$ is linear and $\mathfrak{D}(m')$ is integral, we obtain the following identities of \mathbb{Q} -divisors over $\mathbb{A}_{\mathbf{k}}^1$:

$$-\operatorname{div} \varphi_{m'+m+ie} = [\mathfrak{D}|_{\mathbb{A}_{\mathbf{k}}^1}(m' + m + ie)] = [\mathfrak{D}|_{\mathbb{A}_{\mathbf{k}}^1}(m')] + [\mathfrak{D}|_{\mathbb{A}_{\mathbf{k}}^1}(m + ie)].$$

Hence,

$$\varphi_{m'+m+ie} = \lambda \varphi_{m'} \cdot \varphi_{m+ie} \quad \text{for some } \lambda \in \mathbf{k}^*.$$

Consequently, this implies

$$\varphi_{m'}\chi^{m'}\partial'^{(i)}(f\chi^m) \in A_{m'+m+ie}\chi^{m'+m+ie} \subseteq \varphi_{m'} \cdot \varphi_{m+ie} \cdot \mathbf{k}[t]\chi^{m'+m+ie}.$$

This yields

$$\partial'^{(i)}(f\chi^m) \in \varphi_{m+ie} \cdot \mathbf{k}[t]\chi^{m+ie} = A'_{m+ie}\chi^{m+ie} \subseteq A',$$

as required. We conclude that the subalgebra A' is ∂' -invariant.

Next, we show that ∂' is a homogeneous LFIHD on A' . Let m' be as above. We have $t\varphi_{m'}\chi^{m'} \in A$. Thus, there exists $s \in \mathbb{Z}_{>0}$ such that

$$\varphi_{m'}\chi^{m'}\partial'^{(i)}(t) = \partial^{(i)}(t\varphi_{m'}\chi^{m'}) = 0 \quad \text{for any } i \geq s.$$

Hence ∂' acts locally finitely on t and so the same holds for $A[t]$. Let $f \in A'_m$ and choose $s' \in \mathbb{Z}_{>0}$ such that the sheaf $\mathcal{O}_{\mathbb{P}_{\mathbf{k}}^1}([\mathfrak{D}(m + s'm')])$ is globally generated. Thus,

$$\varphi_{s'm'} f \chi^{m+s'm'} \in A'_{m+s'm'} = \mathbf{k}[t] \otimes_{\mathbf{k}} A_{m+s'm'} \subseteq A[t].$$

Since $\varphi_{s'm'} \chi^{s'm'}$ is in the kernel of ∂ we conclude that ∂' acts locally finitely on $f \chi^m$. This proves that ∂' is an LFIHD. The fact that ∂' is of horizontal type is straightforward and the proof is left to the reader.

It remains to show that $\ker \partial = \ker \partial'$. By Lemma 5.2 (i) the kernel $\ker \partial'$ is the semigroup algebra given by $\omega_{L'}$, where L' is a sublattice of maximal rank. Since $\ker \partial \subseteq \ker \partial'$ we have $L \subseteq L'$ and L'/L is a finite abelian group. Let

$$\ker \partial = \bigoplus_{m \in \omega_L} \mathbf{k} \cdot \varphi_m \chi^m \quad \text{and} \quad \ker \partial' = \bigoplus_{m \in \omega_{L'}} \mathbf{k} \cdot \varphi'_m \chi^m.$$

Letting $m \in L'$ we let $r \in \mathbb{Z}_{>0}$ be such that $rm \in L$. Then, by Lemma 5.4 (i) and (ii) we can write $\lambda \varphi_{rm} = \varphi'_{rm} = (\varphi'_m)^r$, where $\lambda \in \mathbf{k}^*$. So $\varphi'_m \chi^m$ is integral over $\ker \partial$. By normality of A and since $\ker \partial$ is algebraically closed in A one has $\varphi'_m \chi^m \in \ker \partial$. Hence $L' = L$ and so $\ker \partial = \ker \partial'$.

(iii) Up to multiplying the LFIHD ∂ by a homogeneous kernel element, we may assume that $\deg \partial = e \in \omega$. In particular, the algebra

$$A_\omega = \bigoplus_{m \in \omega_M} A_m \chi^m \quad \text{is } \partial\text{-invariant.}$$

By virtue of assertions (i) and (ii) in the lemma, we may suppose that $C = \mathbb{A}_{\mathbf{k}}^1$. Let $m \in \omega_M$. We have $A_{dm+m'} = A_{dm} \cdot A_{m'} = \varphi_{dm} A_{m'}$ for all $m' \in \omega_M$. Hence, the principal ideal $(\varphi_{dm} \chi^{dm})$ in the ring A_ω is $\partial|_{A_\omega}$ -invariant. By Proposition 2.4 (f), we have $\varphi_{dm} \chi^{dm} \in \ker \partial$ and so $dm \in \omega_L$. This yields $d \cdot \omega_M \subseteq \omega_L$ and (iii) follows. \square

The following result provides a geometrical characterization of horizontal non-hyperbolic affine \mathbb{G}_m -surfaces. See [16, Theorems 3.3 and 3.16] for the case where the base field is \mathbb{C} .

Corollary 5.6. *Assume \mathbf{k} is perfect. Let $N = \mathbb{Z}$ and $\sigma = \mathbb{R}_{\geq 0}$, so that \mathfrak{D} is uniquely determined by the \mathbb{Q} -divisor $\mathfrak{D}(1)$. If the graded algebra A admits a homogeneous LFIHD of horizontal type, then the following statements hold.*

- (i) *If $C = \mathbb{A}_{\mathbf{k}}^1$, then the fractional part $\{\mathfrak{D}(1)\}$ has at most one point in its support.*
- (ii) *If $C = \mathbb{P}_{\mathbf{k}}^1$, then $\{\mathfrak{D}(1)\}$ has at most two points in its support.*

In each case, the support of $\{\mathfrak{D}(1)\}$ consists of rational points. In particular, every horizontal non-hyperbolic affine \mathbb{G}_m -surface over \mathbf{k} is toric.

Proof. (i) We first prove the result in the case where \mathbf{k} is algebraically closed. Let d be the smallest positive integer such that $\mathfrak{D}(d)$ is an integral divisor. Letting $f \in \mathbf{k}(t)$ a

generator of A_d , i.e. $A_d = f \cdot A_0$, we let B be the integral closure of $A[\sqrt[d]{f}\chi]$ in its field of fractions. Up to a principal divisor, we may assume $\mathfrak{D}(1) < 0$ and so $f \in \mathbf{k}[t]$ is a polynomial. By Lemma 5.5 (ii), we have $f\chi^d \in \ker \partial$.

By Corollary 2.6, we obtain the existence of an LFIHD ∂' on B extending ∂ and satisfying $\sqrt[d]{f}\chi \in \ker \partial'$. Write $B = A[C', \mathfrak{D}']$ for some polyhedral divisor \mathfrak{D}' on a regular affine curve $C' = \text{Spec } B_0$. Actually, B_0 is the normalization of $\mathbf{k}[t, \sqrt[d]{f}]$ and also a polynomial algebra of one variable over \mathbf{k} (see Lemma 5.2 (iii)). The fact that $B_0^* = \mathbf{k}^*$ and that B_0 is a unique factorization domain implies that $f = (t - z)^r$ for some $z \in \mathbf{k}$ and some $r \in \mathbb{Z}_{>0}$. Since $\text{div } f + d \cdot \mathfrak{D}(1) = 0$ one concludes that $\{\mathfrak{D}(1)\}$ is supported in at most on the point z .

Assume now that \mathbf{k} is not algebraically closed and that $\{\mathfrak{D}(1)\}$ is supported in at least two points. Extending the scalar to the algebraic closure $\bar{\mathbf{k}}$ gives a contradiction by Lemma 1.9.

(ii) Multiplying ∂ by a homogeneous element in its kernel, we may assume that the degree of ∂ is non-negative. By Lemma 5.5 (ii), the LFIHD ∂ extends to a homogeneous LFIHD ∂' of horizontal type on the normalization A' of the algebra $A[t]$. Note that the graded algebra A' is given by the polyhedral divisor $\mathfrak{D}|_{\mathbb{A}_{\mathbf{k}}^1}$. Applying (i) for the non-elliptic graded algebra A' , the fractional part $\{\mathfrak{D}|_{\mathbb{A}_{\mathbf{k}}^1}(1)\}$ has at most one point in its support. So $\{\mathfrak{D}(1)\}$ is supported in at most two points. This yields (ii).

Let us show the latter claim. By a similar argument, we deduce that in any case the support of $\{\mathfrak{D}(1)\}$ consists of rational points (see Lemma 1.9). Assume that A is non-elliptic. Since $\{\mathfrak{D}(1)\}$ is supported in at most one rational point, without loss of generality, we can let

$$\mathfrak{D}(1) = -\frac{e}{d} \cdot 0, \quad \text{where } 0 \leq e < d, \quad \text{and} \quad \gcd(e, d) = 1.$$

A straightforward computation shows that

$$A = \bigoplus_{b \geq 0, ad - be \geq 0} \mathbf{k} t^a \chi^b,$$

see e.g. [16, Lemma 3.8] and [23, Example 3.20]. The algebra A admits an effective \mathbb{Z}^2 -grading endowing $X = \text{Spec } A$ with a structure of a toric surface. Assume that A is elliptic. Using the fact that every integral divisor over \mathbb{P}^1 of degree 0 is principal, we can reduce to the case where \mathfrak{D} is supported in the points 0 and ∞ . We conclude by a similar argument as in [23, Example 3.21]. \square

As a consequence of Corollary 5.6, we obtain the following result.

Corollary 5.7. *With the notation in 5.3, we let $A_\omega = \bigoplus_{m \in \omega_M} A_m \chi^m$ and let $\tau = \omega^\vee \subseteq N_{\mathbb{R}}$. Then $A_\omega \simeq A[C, \mathfrak{D}_\omega]$ as M -graded algebras, where \mathfrak{D}_ω is τ -proper polyhedral divisor over the curve C satisfying the following conditions.*

- (i) If A is non-elliptic, then $\mathfrak{D}_\omega = (v + \tau) \cdot 0$ for some $v \in N_\mathbb{Q}$.
(ii) If A is elliptic, then $\mathfrak{D}_\omega = (v + \tau) \cdot 0 + \Delta'_\infty \cdot \infty$ for some $v \in N_\mathbb{Q}$ and some $\Delta'_\infty \in \text{Pol}_\tau(N_\mathbb{R})$ satisfying $v + \Delta'_\infty \subsetneq \tau$.

Proof. (i) We will follow the argument in [23, Lemma 3.23]. Note that the degree e of ∂ belongs to ω . For $\ell \in \omega_L$ denote by ∂_ℓ the homogeneous LFIHD $\varphi_\ell \cdot \partial$. The subalgebra

$$B_{(\ell+e)} = \bigoplus_{r \geq 0} A_{r(\ell+e)} \chi^{r(\ell+e)}$$

is ∂_ℓ -invariant. Since the homogeneous LFIHD $\partial_\ell|_{B_{(\ell+e)}}$ is of horizontal type, we can apply Corollary 5.6 to conclude that $\{\mathfrak{D}(\ell + e)\}$ is supported in at most one point. By Lemma 5.4 (i), for all $\ell, \ell' \in \omega_L$ we have

$$-\text{div } \varphi_{\ell'} + \mathfrak{D}(\ell + e) = \mathfrak{D}(\ell + \ell' + e) = \mathfrak{D}(\ell' + e) - \text{div } \varphi_\ell, \quad \text{and so} \quad \{\mathfrak{D}(\ell + e)\} = \{\mathfrak{D}(\ell' + e)\}.$$

Thus, the union of the supports of the divisors $\{\mathfrak{D}(\ell + e)\}$ has at most one element, where ℓ runs over ω_L . By the linearity of \mathfrak{D} in ω and Lemma 5.4 (iv), up to a principal polyhedral divisor, the polyhedral divisor \mathfrak{D}_ω of A_ω is supported in at most one point. This point needs to be rational so (i) follows.

(ii) By multiplying ∂ with a kernel element, we may assume $e \in \omega$. Let A'_ω be the normalization of $A_\omega[t]$. By Lemma 5.5, elements of degree $m \in \omega_M$ in A'_ω correspond to the product of a global section of $\mathfrak{D}|_{\mathbb{A}^1_{\mathbf{k}}}(m)$ and the character χ^m . In addition, ∂ extends to a homogeneous LFIHD of horizontal type on A'_ω . By (i), the union of the supports of the divisors $\{\mathfrak{D}|_{\mathbb{A}^1_{\mathbf{k}}}(m)\}$, where m runs through ω_M , has at most one rational point. This concludes the proof. \square

For our next theorem, which is a key ingredient in our classification result, we introduce the following notation. Let \mathfrak{D} be a proper σ -polyhedral divisor over $\mathbb{A}^1_{\mathbf{k}}$ or $\mathbb{P}^1_{\mathbf{k}}$ such that the coefficient Δ_0 at zero is $v + \sigma$ for some $v \in N_\mathbb{Q}$. Let $\widehat{M} = M \times \mathbb{Z}$ and let $\widehat{N} = N \times \mathbb{Z}$. We also let $\widehat{\sigma}$ be the cone in $\widehat{N}_\mathbb{R}$ generated by $(v, 1)$ and $(\sigma, 0)$ if $C = \mathbb{A}^1_{\mathbf{k}}$ and by $(v, 1)$, $(\sigma, 0)$ and $(\Delta_\infty, -1)$ if $C = \mathbb{P}^1_{\mathbf{k}}$.

Theorem 5.8. *Let \mathfrak{D} be a σ -proper polyhedral divisor over a regular curve C . Assume that \mathfrak{D} satisfies one of the following conditions.*

- (i) If C is affine, then $C = \mathbb{A}^1_{\mathbf{k}} = \text{Spec } \mathbf{k}[t]$ and $\mathfrak{D} = (v + \sigma) \cdot 0$ for some $v \in N_\mathbb{Q}$.
(ii) If C is projective, then $C = \mathbb{P}^1_{\mathbf{k}}$ and $\mathfrak{D} = (v + \sigma) \cdot 0 + \Delta_\infty \cdot \infty$ for some $v \in N_\mathbb{Q}$ and for some $\Delta_\infty \in \text{Pol}_\sigma(N_\mathbb{R})$.

Let d be the smallest positive integer such that $dv \in N$. For any $m \in M$ we let $h(m) = \langle m, v \rangle$. Then there exists a homogeneous LFIHD ∂ of horizontal type on $A = A[C, \mathfrak{D}]$ with $\deg \partial = e$ if and only if the following statements hold.

- (a) If $\text{char } \mathbf{k} = p > 0$, then there exists a sequence of integers $0 \leq s_1 < s_2 < \dots < s_r$ such that for $i = 1, \dots, r$ we have $(p^{s_i}e, -1/d - h(p^{s_i}e)) \in \text{Rt } \widehat{\sigma}$.
- (b) If $\text{char } \mathbf{k} = 0$, then $(e, -1/d - h(e)) \in \text{Rt } \widehat{\sigma}$.

Under these latter conditions, the LFIHD ∂ is of following form. Let $\zeta = \sqrt[d]{t}$. Let us consider the LFIHD ∂_ζ on the algebra $\mathbf{k}[\zeta]$ with exponential map

$$e^{x\partial_\zeta}(\zeta) = \zeta + \sum_{i=1}^r \lambda_i x^{p^{s_i}}, \quad (6)$$

where $\lambda_1, \dots, \lambda_r \in \mathbf{k}^*$ (resp. with $\partial_\zeta^{(1)} = \lambda \frac{d}{d\zeta}$, where $\lambda \in \mathbf{k}^*$) whenever $\text{char } \mathbf{k} > 0$ (resp. $\text{char } \mathbf{k} = 0$). Then the i -th term of ∂ is given by the equality

$$\partial^{(i)}(t^l \chi^m) = \zeta^{-dh(m+ie)} \partial_\zeta^{(i)}(\zeta^{dh(m)} t^l) \chi^{m+ie} \quad \text{for all } t^l \chi^m \in A. \quad (7)$$

Proof. Assume that \mathfrak{D} satisfies (i) and fix an LFIHD ∂ on the algebra A of horizontal type and of degree e . Let B be the normalization of the subalgebra

$$A[\zeta^{-dh(e)} \chi^e] \subseteq \mathbf{k}(\zeta)[M].$$

Consider the affine line $C' = \text{Spec } \mathbf{k}[\zeta]$ and the polyhedral divisor $\mathfrak{D}' = (dv + \sigma) \cdot 0$ over C' . Since $d = \min\{r \in \mathbb{Z}_{>0} \mid re \in L\}$ (see Lemma 5.4 (iv)), the algebra $A[C', \mathfrak{D}']$ is precisely B (see [21, Theorem 2.5]). According to Lemma 4.1 (ii) we have $e \in \sigma^\vee$ and so $A[\zeta^{-dh(e)} \chi^e]$ is a cyclic extension of the ring A . Since $\varphi_{de} \chi^{de} \in \ker \partial$ by Corollary 2.6, ∂ extends to a unique LFIHD ∂' on B . Using further that $dv \in N$ we obtain a natural isomorphism of M -graded algebras

$$\varphi : B \rightarrow E, \quad \zeta^l \chi^m \mapsto \zeta^{dh(m)+l} \chi^m,$$

where $E = \mathbf{k}[\sigma_M^\vee][\zeta]$. Consider $\varphi_* \partial'$ the homogeneous LFIHD of horizontal type on E given by

$$\varphi_* \partial'^{(i)} = \varphi \circ \partial'^{(i)} \circ \varphi^{-1},$$

where $i \in \mathbb{Z}_{\geq 0}$. Now, Lemma 5.5 (iii) implies that $\ker \varphi_* \partial' = \mathbf{k}[\sigma_M^\vee]$ so that $\varphi_* \partial' = \chi^e \cdot \partial_\zeta$ for some non-trivial LFIHD ∂_ζ . An easy computation shows that the LFIHD $\partial = \varphi_*^{-1}(\varphi_* \partial')$ is as in (7).

Assume that $\text{char } \mathbf{k} = p > 0$ and let us show that (a) holds. By Proposition 2.4 (d), the exponential map of ∂_ζ is given as in (6) for some integers $0 \leq s_1 < \dots < s_r$. If p does not divide d , then consider $l \in \mathbb{Z}_{\geq 0} \setminus p\mathbb{Z}$ such that $dl \geq p^{s_1}$. Note that $t^l \in A$. By Lemma 2.13 and (7) we obtain the equality

$$\partial^{(p^{s_1})}(t^l) = \lambda_1 d l t^{-1/d - h(p^{s_1}e) + l} \chi^{p^{s_1}e}.$$

Since $\partial^{(p^{s_1})}(t^l) \in A \setminus \{0\}$, it follows that $-1/d - h(p^{s_1}e) \in \mathbb{Z}$.

Otherwise, assume that p divide d . By the minimality of d there exists $m \in \sigma_M^\vee$ such that $dh(m)$ is not divisible by p . Taking $l \in \mathbb{Z}_{\geq 0}$ such that $dl \geq \max\{p^{s_1}, -dh(m)\}$ we have $t^l \chi^m \in A \setminus \{0\}$ and so [Lemma 2.13](#) implies

$$\partial^{(p^{s_1})}(t^l \chi^m) = \lambda_1 dh(m) t^{-1/d-h(p^{s_1}e)+l} \chi^{m+p^{s_1}e} \in A \setminus \{0\}.$$

Hence in any case $e_1 := (p^{s_1}e, -1/d - h(p^{s_1}e)) \in \widehat{M}$, where $\widehat{M} = M \times \mathbb{Z}$.

Let us remark that

$$A[C, \mathfrak{D}] = \bigoplus_{(m,l) \in \widehat{\sigma}_M^\vee} \mathbf{k} \chi^{(m,l)} = \mathbf{k}[\widehat{\sigma}_M^\vee],$$

where $\chi^{(m,l)} = t^l \chi^m$ and $\widehat{\sigma}$ is the cone generated by $(v, 1)$ and $(\sigma, 0)$. Since $e \in \sigma^\vee$, an easy computation shows that $e_1 = (p^{s_1}e, -1/d - h(p^{s_1}e)) \in \text{Rt } \widehat{\sigma}$ for the distinguished ray $\rho = (dv, d)$. So by [Corollary 3.7](#) the \widehat{M} -graded algebra A admits rationally homogeneous LFIHDs of degree e_1/p^{s_1} coming from the root e_1 . One of such rationally homogeneous LFIHDs is given by the equality

$$e^{x\partial_1}(t^l \chi^m) = \sum_{i=0}^{\infty} \binom{d(l+h(m))}{i} \lambda_1^i t^{l-i(1/d+h(p^{s_1}e))} \chi^{m+ip^{s_1}e} x^{ip^{s_1}},$$

where $\lambda_1 \in \mathbf{k}^*$ is as [\(6\)](#). Furthermore, by [Corollary 2.6](#) we extend ∂_1 to a homogeneous LFIHD ∂'_1 on the M -graded algebra B . Assume that $r \geq 2$. One can see $e^{x\partial'}$ and $e^{x\partial'_1}$ as automorphisms of the algebra $B[x]$ by letting $e^{x\partial'}(x) = e^{x\partial'_1}(x) = x$. Hence, using this convention we have

$$e^{x\partial'} \circ (e^{x\partial'_1})^{-1} = e^{x\varphi_*^{-1}(\chi^e \partial_{\zeta,1})},$$

where $\partial_{\zeta,1}$ is the LFIHD on $\mathbf{k}[\zeta]$ defined by

$$e^{x\partial_{\zeta,1}}(\zeta) = \zeta + \sum_{i=2}^r \lambda_i x^{p^{s_i}}.$$

Consequently, the map $e^{x\partial'} \circ (e^{x\partial'_1})^{-1}$ yields a homogeneous LFIHD ∂''_1 on A . Actually, replacing $\partial_{\zeta,1}$ by ∂_ζ , the LFIHD ∂''_1 satisfies [\(7\)](#). Again, it follows that $e_2 := (p^{s_2}e, -1/d - h(p^{s_2}e)) \in \widehat{M}$ is a root of $\widehat{\sigma}$. One concludes by induction that (a) holds.

If $\text{char } \mathbf{k} = 0$, then the locally nilpotent derivation $\partial_\zeta^{(1)}$ on the algebra $\mathbf{k}[\zeta]$ is equal to $\lambda \frac{\partial}{\partial \zeta}$ for some $\lambda \in \mathbf{k}^*$. Using [\(7\)](#) we have

$$\partial^{(1)}(t) = \lambda dt^{-1/d-h(e)+1} \chi^e \in A \setminus \{0\}$$

and so assertion (b) holds. This concludes the proof in the case where condition (i) holds.

Assume now that (ii) holds. Let A' be the normalization of $A[t]$ in the field $\text{Frac } A$. By Lemma 5.5 (iii), we have $d \cdot M = h^{-1}(\mathbb{Z}) \subseteq L$, where L is the sublattice of M generated by the set of weights of $\ker \partial$. Hence, changing ∂ by $\varphi_m \cdot \partial$ for $m \in \sigma_{d \cdot M}^\vee$, without loss of generality, we may assume $e \in \sigma_M^\vee$.

More precisely, replacing e by $e + m$ for some $m \in \sigma_{d \cdot M}^\vee$ does not change assertions (a), (b) in the theorem. With this new assumption, again by Lemma 5.5, we extend ∂ to a homogeneous LFIHD $\bar{\partial}$ on A' of horizontal type. By the previous argument (the case where $C = \mathbb{A}_{\mathbf{k}}^1$) applied to $(A', \bar{\partial})$ and since $\bar{\partial}$ stabilizes $\mathbf{k}[\widehat{\sigma}^\vee \cap \widehat{M}]$ we obtain (a) and (b).

It remains to show that if a lattice vector e verifies assertions (a), (b), then one can build a homogeneous LFIHD on $A = A[C, \mathfrak{D}]$ of horizontal type and of degree e as in (7). Assume that $\text{char } \mathbf{k} > 0$ and let $e_i = (e, -1/d - h(p^{s_i}e))$. By (a) we have $e_i \in \text{Rt } \widehat{\sigma}$ and we can consider the rationally homogeneous LFIHDs $\partial_{e_1, s_1}, \dots, \partial_{e_r, s_r}$ on the semigroup algebra $\mathbf{k}[\widehat{\sigma}_M^\vee]$ (see Example 3.2). Using the isomorphism φ and considering every $e^{x\partial_{e_i, s_i}}$ as automorphism of the ring $A[x]$, a computation shows that the composition

$$e^{x\partial_{e_1, s_1}} \circ e^{x\partial_{e_2, s_2}} \circ \dots \circ e^{x\partial_{e_r, s_r}}$$

defines an LFIHD as in (7). In the case where $\text{char } \mathbf{k} = 0$, a similar argument can be applied (see also [23, Examples 3.20 and 3.21]). We leave the details to the reader. \square

For the proof of our next lemma, which is the last ingredient for our main theorem, we need the following remark.

Remark 5.9. Assume that \mathbf{k} is perfect and let $r \in \mathbb{Z}_{>0}$. Then the Frobenius map $F : \mathbf{k} \rightarrow \mathbf{k}$ mapping $\lambda \mapsto \lambda^{p^r}$ is a field automorphism. Let t be a new variable and let $x = t^{p^r}$. We will compute the ramification of the field extension $\mathbf{k}(t)/\mathbf{k}(x)$. Let $P(x) = \sum a_i x^i \in \mathbf{k}[x]$ be an irreducible polynomial. Then

$$P(x) = P(t^{p^r}) = (F^*(P)(t))^{p^r}, \quad \text{where} \quad F^*(P)(t) = \sum F^{-1}(a_i) t^i.$$

Hence $F^*(P)(t)$ is irreducible in $\mathbf{k}[t]$. Let C and C' be unique projective curves over \mathbf{k} whose function fields are $\mathbf{k}(t)$ and $\mathbf{k}(x)$, respectively (both isomorphic to $\mathbb{P}_{\mathbf{k}}^1$). The inclusion $\mathbf{k}(x) \subseteq \mathbf{k}(t)$ induces a purely inseparable morphism $\pi : C \rightarrow C'$. Our previous computation shows that for every $z \in C$ the pullback of z as Weil divisor is given by $\pi^*(z) = p^r \cdot z'$, where $z' \in C'$ lies in the schematic fiber of z .

Let $\mathfrak{D} = \sum_{z \in C} \Delta_z \cdot z$ be proper σ -polyhedral divisor over a regular curve C . Recall that h_z stands for the support function of the σ -polyhedron Δ_z for all $z \in C$, see Definition 1.5.

Lemma 5.10. Assume that \mathbf{k} is perfect. Let \mathfrak{D} be a proper σ -polyhedral divisor over $C = \mathbb{A}_{\mathbf{k}}^1$ or $C = \mathbb{P}_{\mathbf{k}}^1$, respectively. Assume that there exists a maximal cone ω on the quasifan $\Lambda(\mathfrak{D})$ or $\Lambda(\mathfrak{D}|_{\mathbb{A}_{\mathbf{k}}^1})$, respectively, such that for any $z \in C$ different from 0 and ∞

we have $h_z|_\omega = 0$. Let ∂ be an LFIHD of degree e on the algebra $A[C, \mathfrak{D}_\omega]$ given by formula (7). Let $p = \text{char } \mathbf{k}$ if $\text{char } \mathbf{k} > 0$ and $p = 1$ if $\text{char } \mathbf{k} = 0$. Then ∂ extends to an LFIHD on $A = A[C, \mathfrak{D}]$ if and only if for any $m \in \sigma_M^\vee$ such that $m + p^{s_1}e \in \sigma_M^\vee$ the following hold.

- (i) If $h_z(m + p^{s_1}e) \neq 0$, then $\lfloor p^k h_z(m + p^{s_1}e) \rfloor - \lfloor p^k h_z(m) \rfloor \geq 1, \forall z \in C, z \neq 0, \infty$.
- (ii) If $h_0(m + p^{s_1}e) \neq h(m + p^{s_1}e)$, then $\lfloor dh_0(m + p^{s_1}e) \rfloor - \lfloor dh_0(m) \rfloor \geq 1 + dh(p^{s_1}e)$.
- (iii) If $C = \mathbb{P}_{\mathbf{k}}^1$, then $\lfloor dh_\infty(m + p^{s_1}e) \rfloor - \lfloor dh_\infty(m) \rfloor \geq -1 - dh(p^{s_1}e)$.

Here h is the linear extension of $h_0|_\omega$ to $M_\mathbb{R}$, $d \in \mathbb{Z}_{>0}$ is the smallest positive integer such that dh is integral and k is the unique non-negative integer such that $d = d'p^k$ with $\gcd(d', p) = 1$.

Proof. Considering $m \in \sigma_M^\vee$ we can write $h(m) = \langle m, v \rangle$ for some $v \in N_\mathbb{Q}$. Since every h_z is upper convex, $h_z(m) \leq 0 \forall z \in C \setminus \{0, \infty\}$, and obviously $h_0(m) \leq h(m)$. Letting

$$A_M = \bigoplus_{m \in M} \mathbf{k}[t] \cdot \varphi_m \chi^m,$$

where $\varphi_m = t^{-\lfloor h(m) \rfloor}$ and localizing by a homogeneous element of $\ker \partial$, by Lemma 2.5, ∂ extends to a homogeneous LFHD on A_M . We also denote this extension by ∂ . Hence, ∂ extends to an LFIHD on A if and only if the extension ∂ on A_M stabilizes A . In addition, we may assume that $\mathbf{k} = \bar{\mathbf{k}}$ is algebraically closed since the extension $\partial_{\bar{\mathbf{k}}}$ of ∂ on $A_M \otimes_{\mathbf{k}} \bar{\mathbf{k}}$ stabilizes $A \otimes_{\mathbf{k}} \bar{\mathbf{k}}$ if and only if ∂ stabilizes A .

For the characteristic zero case, the proof is available in [23, Lemma 3.26]. In the sequel, we assume $\text{char } \mathbf{k} = p > 0$. The proof is divided into three steps, (similar to [23, Lemma 3.26]) where we assume $h = 0$, $h(m)$ integral for all m and finishing with the general case.

Case $h = 0$. In this case we have $d = 1$, $L = M$ and by Theorem 5.8, $\partial = \chi^e \partial_t$ for some LFIHD ∂_t on $\mathbf{k}[t]$. By Proposition 2.4 (d), the LFIHD ∂_t is determined by a sequence of integers $0 \leq s_1 < \dots < s_r$. Furthermore, since $h_z \leq 0$ for any $z \in \mathbb{A}_{\mathbf{k}}^1$, then $h_\infty \geq 0$ in the elliptic case. Fixing $m \in \sigma_M^\vee$ such that $m + p^{s_1}e \in \sigma_M^\vee$ the conditions of our lemma become:

- (i') If $h_z(m + p^{s_1}e) \neq 0$, then $\lfloor h_z(m + p^{s_1}e) \rfloor - \lfloor h_z(m) \rfloor \geq 1 \forall z \in \mathbb{A}_{\mathbf{k}}^1$.
- (iii') If $C = \mathbb{P}_{\mathbf{k}}^1$, then $\lfloor h_\infty(m + p^{s_1}e) \rfloor - \lfloor h_\infty(m) \rfloor \geq -1$.

Under the above assumption we have

$$A_m = H^0(C, \mathcal{O}_C(\mathfrak{D}(m))) \subseteq \mathbf{k}[t]$$

and ∂ stabilizes A if and only if

$$f(t) \in A_m \Rightarrow \partial_t^{(i)}(f(t)) \in A_{m+ie}, \forall m \in \sigma_M^\vee, \quad \forall i \in \mathbb{Z}_{\geq 0},$$

or equivalently,

$$\operatorname{div} f + [\mathfrak{D}(m)] \geq 0 \Rightarrow \operatorname{div} \partial_t^{(i)}(f) + [\mathfrak{D}(m + ie)] \geq 0, \quad \forall m \in \sigma_M^\vee, \quad \forall i \in \mathbb{Z}_{\geq 0}.$$

This is also equivalent to

$$\begin{aligned} \operatorname{ord}_z f + [h_z(m)] \geq 0 &\Rightarrow \operatorname{ord}_z \partial_t^{(i)}(f) + [h_z(m + ie)] \geq 0, \\ \forall m \in \sigma_M^\vee, \quad \forall i \in \mathbb{Z}_{\geq 0}, \quad \forall z \in C. \end{aligned} \quad (8)$$

We will first show the lemma in the case where $C = \mathbb{A}_{\mathbf{k}}^1$. Let us show first that (i') implies (8) and so ∂ stabilizes A . If $h_z(m + p^{s_1}e) \neq 0$ with $m \in \sigma_M^\vee$ such that $m + p^{s_1}e \in \sigma_M^\vee$. Then we have $h_z(m) \neq 0$ so that $f \in (t - z)\mathbf{k}[t]$.

Let $i \in \mathbb{Z}_{\geq 0}$. If $\partial_t^{(i)}(f) = 0$, then $\partial_t^{(i)}(f) \in A_{m+ie}$. Otherwise, $\partial_t^{(i)}(f) \neq 0$ and so $m + ie \in \sigma^\vee$. Letting $i = lp^{s_1}$ for some $l \in \mathbb{Z}_{\geq 0}$, we have $\operatorname{ord}_z \partial_t^{(i)}(f) \geq \operatorname{ord}_z(f) - l$. Hence it follows that

$$\operatorname{ord}_z \partial^{(i)}(f) + [h_z(m + ie)] \geq \operatorname{ord}_z(f) + [h_z(m)] + ([h_z(m + lp^{s_1}e)] - [h_z(m)] - l).$$

By convexity of σ^\vee for $1 \leq j \leq l$ we have $m + jp^{s_1}e \in \sigma^\vee$. If $h_z(m + ie) = 0$, then $\operatorname{ord}_z \partial^{(i)}(f) + [h_z(m + ie)] \geq 0$ and (8) holds. Otherwise, $h_z(m + ie) \neq 0$ and again $h_z(m + (l - j)p^{s_1}e) \neq 0$ for $1 \leq j \leq l$. Combining the previous inequality with (i'), and the fact that $\operatorname{ord}_z f + [h_z(m)] \geq 0$ we obtain

$$\begin{aligned} \operatorname{ord}_z \partial^{(i)}(f) + [h_z(m + ie)] &\geq \operatorname{ord}_z(f) + [h_z(m)] \\ &\quad + \sum_{j=1}^l ([h_z(m + (l - j)p^{s_1}e + p^{s_1}e)] \\ &\quad - [h_z(m + (l - j)p^{s_1}e)] - 1) \geq 0. \end{aligned}$$

This yields (8) in the case where $C = \mathbb{A}_{\mathbf{k}}^1$.

Now, we show the converse. Assume that $C = \mathbb{A}_{\mathbf{k}}^1$ and that ∂ stabilizes A . Recall that ∂ stabilizes A if and only if (8) holds. If ω is the unique maximal cone in $\Lambda(\mathfrak{D})$, then h_z is identically zero for all $z \in C$ and so (i') is trivially satisfied. Therefore the lemma follows in this case.

In the sequel, we assume that $\Lambda(\mathfrak{D})$ has at least two maximal cones. Let $\omega_0 \in \Lambda(\mathfrak{D})$ be a maximal cone different from ω . Then there exists a lattice vector $m \in \operatorname{rel.int} \omega_0$ such that $h_z(m) \in \mathbb{Z}$ and $\partial^{(lp^{s_1})}(\varphi_m) \neq 0$ for some $l \in \mathbb{Z}_{\geq 0}$. Note that here $\ker \partial = \bigoplus_{m \in \omega_M} \mathbf{k} \cdot \varphi_m \chi^m$. Taking m big enough we may suppose that $-h_z(m) \geq lp^{s_1}$ and by Lemma 2.13 we may suppose that

$$\operatorname{ord}_z \partial_t^{(lp^{s_1})}(\varphi_m) = -h_z(m) - l.$$

By (8) we have

$$\lfloor h_z(m + lp^{s_1}e) \rfloor - h_z(m) - l \geq 0. \quad (9)$$

Letting \bar{h}_z be the linear extension of $h_z|_{\omega_0}$ we have

$$\lfloor h_z(m + lp^{s_1}e) \rfloor = \lfloor h_z(m) + l\bar{h}_z(p^{s_1}e) \rfloor = h_z(m) + \lfloor l\bar{h}_z(p^{s_1}e) \rfloor. \quad (10)$$

Now, (9) and (10) yield

$$l\bar{h}_z(p^{s_1}e) \geq \lfloor l\bar{h}_z(p^{s_1}e) \rfloor \geq l$$

and so $\bar{h}_z(p^{s_1}e) \geq 1$. Finally, letting $m \in \sigma_M^\vee$, we obtain

$$\lfloor h_z(m + p^{s_1}e) \rfloor \geq \lfloor h_z(m) \rfloor + \lfloor \bar{h}_z(p^{s_1}e) \rfloor \geq \lfloor h_z(m) \rfloor + 1.$$

This yields (i') and so concludes the proof of the lemma in the case where $C = \mathbb{A}_{\mathbf{k}}^1$.

Assume now that $C = \mathbb{P}_{\mathbf{k}}^1$. Then for $z \in C \setminus \{\infty\}$ and for any $m \in \sigma_M^\vee$ such that $A_m \neq 0$, we can find $\varphi_{m,z} \in A_m$ satisfying $\text{ord}_z(\varphi_{m,z}) + \lfloor h_z(m) \rfloor = 0$. Replacing φ_m by $\varphi_{m,z}$ in the previous argument and using Lemma 2.13 for $z = \infty$ in an analog way as in the above proof, we obtain the equivalence between (8) and (i'), (iii').

Case h integral. Again in this case we have $d = 1$. Let $v \in N$ be such that $\langle m, v \rangle = h(m)$ for all $m \in \omega_M$. Let us consider the polyhedral divisor defined by $\mathfrak{D}' = \mathfrak{D} + (-v + \sigma) \cdot 0$ if C is affine, and by $\mathfrak{D}' = \mathfrak{D} + (-v + \sigma) \cdot 0 + (v + \sigma) \cdot \infty$ if C is projective. Now A is equivariantly isomorphic to $A[C, \mathfrak{D}']$ and $A[C, \mathfrak{D}']$ is as in the case where $h = 0$. Conjugating ∂ by the equivariant isomorphism $A \simeq A[C, \mathfrak{D}']$ (see [21, Proposition 4.5]), the algebra A is ∂ -invariant if and only if assertions (i'), (iii') hold for the polyhedral divisor \mathfrak{D}' . An easy computation shows that this is equivalent to \mathfrak{D} satisfying (i), (ii), (iii).

General case. Now, we assume that h is not integral, i.e., that $d > 1$. Let us consider the normalization B of the cyclic extension $A[\zeta^{-dh(w)}\chi^w] \subseteq \mathbf{k}(\zeta)[M]$, where $\zeta^d = t$ and $w \in \text{rel.int}(\omega) \cap M$ satisfies $\gcd(dh(w), d) = 1$. We remark that B is naturally M -graded. Furthermore,

$$K'_0 = \left\{ \frac{a}{b} \mid a, b \in B_m, m \in M, \text{ and } b \neq 0 \right\} = \mathbf{k}(\zeta).$$

Hence, $B = A[C', \mathfrak{D}']$, where $C' \simeq \mathbb{P}_{\mathbf{k}}^1$ if A is elliptic and $C' \simeq \mathbb{A}_{\mathbf{k}}^1$ otherwise. We let k and d' be the unique pair of positive integers such that $d = d'p^k$ with $\gcd(d', p) = 1$. Let $\pi : C' \rightarrow C$ be the morphism induced by the field inclusion $K_0 = \mathbf{k}(t) \subseteq \mathbf{k}(\zeta) = K'_0$. Then by Lemma 1.10, Remark 5.9 and [33, Section 3.12, Exercise 3.8], we obtain

$$\mathfrak{D}' = \begin{cases} d \cdot \Delta_0 \cdot [0] + \sum_{z' \in C' \setminus \{0\}} p^k \cdot \Delta_z \cdot z', & \text{if } C = \mathbb{A}_{\mathbf{k}}^1 \\ d \cdot \Delta_0 \cdot [0] + \sum_{z' \in C' \setminus \{0, \infty\}} p^k \cdot \Delta_z \cdot z' + d \cdot \Delta_\infty \cdot [\infty], & \text{if } C = \mathbb{P}_{\mathbf{k}}^1 \end{cases}$$

This yields $h'_0 = dh_0$, $h'_\infty = dh_\infty$ and $h'_{z'} = p^k h_z$, where $\pi(z') = z$ and $h'_{z'}$ is the support function of the coefficient $\Delta'_{z'}$ of \mathfrak{D}' at z' . Moreover, $h'_0|_\omega$ is integral and so the algebra B satisfies the conditions of the previous case (h integral). We let $h' : M_{\mathbb{R}} \rightarrow \mathbb{R}$ be the linear extension of $h'_0|_\omega$.

Let

$$B_M = \bigoplus_{m \in M} \varphi'_m \cdot \mathbf{k}[\zeta] \cdot \chi^m, \quad \text{where} \quad \varphi'_m = \zeta^{-dh(m)}.$$

Since $A_M \subseteq B_M$ is a cyclic extension, by Corollary 2.6 the LFIHD ∂ on A_M extends to an LFIHD ∂' on B_M . Furthermore, ∂ stabilizes A if and only if ∂' stabilizes B (see the argument in [23, Lemma 3.26]).

By the previous case, B is stabilized by ∂' if and only if for every $m \in \sigma_M^\vee$ such that $m + p^{s_1}e \in \sigma_M^\vee$, the following conditions are satisfied.

- (i'') If $h'_{z'}(m + p^{s_1}e) \neq 0$, then $\lfloor h'_{z'}(m + p^{s_1}e) \rfloor - \lfloor h'_{z'}(m) \rfloor \geq 1$, $\forall z' \in C'$, $z' \neq 0, \infty$.
- (ii'') If $h'_0(m + p^{s_1}e) \neq h'(m + p^{s_1}e)$, then $\lfloor h'_0(m + p^{s_1}e) \rfloor - \lfloor h'_0(m) \rfloor \geq 1 + dh'(p^{s_1}e)$.
- (iii'') If $C = \mathbb{P}_{\mathbf{k}}^1$, then $\lfloor h'_\infty(m + p^{s_1}e) \rfloor - \lfloor h'_\infty(m) \rfloor \geq -1 - h'(p^{s_1}e)$.

Now, the lemma follows replacing h' by dh , h'_0 by dh_0 , h'_∞ by dh_∞ and $h'_{z'}$ by $p^k h_z$ for all $z' \in C'$, $z \neq 0, \infty$. \square

The following is our main result in this section. It gives a classification of horizontal LFIHDs on affine \mathbb{T} -varieties of complexity one over a perfect field. It is a direct consequence of the results in this section.

Theorem 5.11. *Assume that the base field \mathbf{k} is perfect. Let $p = \text{char } \mathbf{k}$ if $\text{char } \mathbf{k} > 0$ and $p = 1$ if $\text{char } \mathbf{k} = 0$. Let \mathfrak{D} be a proper σ -polyhedral divisor over a regular curve C and let $A = A[C, \mathfrak{D}]$. Let $\omega \subseteq M_{\mathbb{R}}$ be a rational cone and let $e \in M$ be a lattice vector.*

Then there exists a homogeneous LFIHD on A of horizontal type with $\deg \partial = e$ and with ω as weight cone of $\ker \partial$ if and only if the following conditions hold.

- (i) $C = \mathbb{A}_{\mathbf{k}}^1$ or $C = \mathbb{P}_{\mathbf{k}}^1$.
- (ii) If $C = \mathbb{A}_{\mathbf{k}}^1$, then ω is a maximal cone in the quasifan $\Lambda(\mathfrak{D})$, and there exists a rational point $z_0 \in C$ such that $h_z|_\omega$ is integral $\forall z \in C$, $z \neq z_0$.
- (ii') If $C = \mathbb{P}_{\mathbf{k}}^1$, then there exists a rational point z_∞ such that (ii) holds for $C_0 := \mathbb{P}_{\mathbf{k}}^1 \setminus \{z_\infty\}$.

Without loss of generality, we may suppose that $z_0 = 0$, $z_\infty = \infty$, and $h_z|_\omega = 0 \forall z \in C$, $z \neq 0, \infty$. Let also h be the linear extension of $h_0|_\omega$ to $M_{\mathbb{R}}$ given by $h(m) = \langle m, v \rangle$ for some $v \in N_{\mathbb{Q}}$, let $d > 0$ be the smallest integer such that dh is integral and let k be the unique non-negative integer such that $d = d'p^k$, with $\gcd(d', p) = 1$. Let $\tau = \omega^\vee$ and

denote by $\hat{\tau}$ the cone in $\hat{N}_{\mathbb{R}}$ generated by $(v, 1)$ and $(\tau, 0)$ if $C = \mathbb{A}_{\mathbf{k}}^1$ and by $(v, 1)$, $(\tau, 0)$ and $(\Delta_{\infty}, -1)$ if $C = \mathbb{P}_{\mathbf{k}}^1$.

(iii) There exists $s_1 \in \mathbb{Z}_{\geq 0}$ such that $(p^{s_1}e, -1/d - h(p^{s_1}e)) \in \text{Rt } \hat{\tau}$.

For any $m \in \sigma_M^{\vee}$ such that $m + p^{s_1}e \in \sigma_M^{\vee}$ the following hold.

(iv) If $h_z(m + p^{s_1}e) \neq 0$, then $\lfloor p^k h_z(m + p^{s_1}e) \rfloor - \lfloor p^k h_z(m) \rfloor \geq 1$, $\forall z \in C, z \neq 0, \infty$.

(v) If $h_0(m + p^{s_1}e) \neq h(m + p^{s_1}e)$, then $\lfloor dh_0(m + p^{s_1}e) \rfloor - \lfloor dh_0(m) \rfloor \geq 1 + dh(p^{s_1}e)$.

(vi) If $C = \mathbb{P}_{\mathbf{k}}^1$, then $\lfloor dh_{\infty}(m + p^{s_1}e) \rfloor - \lfloor dh_{\infty}(m) \rfloor \geq -1 - dh(p^{s_1}e)$.

More precisely, all possible homogeneous LFIHD ∂ on A of horizontal type with e, ω satisfying (i)–(iv) are given by the formula (7) in Theorem 5.8. If $\text{char } \mathbf{k} > 0$, then ∂ is described by a sequence of integers $0 \leq s_1 < s_2 < \dots < s_r$, where every $(p^{s_i}e, -1/d - h(p^{s_i}e))$ belongs to $\text{Rt } \hat{\tau}$. Moreover,

$$\ker \partial = \bigoplus_{m \in \omega_L} \mathbf{k} \varphi_m \chi^m,$$

where $L = h^{-1}(\mathbb{Z})$ and $\varphi_m \in A_m$ satisfies the relation

$$\text{div } \varphi_m + \mathfrak{D}(m) = 0 \quad \text{if } C = \mathbb{A}_{\mathbf{k}}^1; \quad \text{or} \quad (\text{div } \varphi_m)|_{C_0} + \mathfrak{D}(m)|_{C_0} = 0 \quad \text{if } C = \mathbb{P}_{\mathbf{k}}^1.$$

Example 5.12. Let the notation be as in Example 1.8. By Theorem 5.11, there exists a homogeneous LFIHD on A with degree $\deg \partial = e = (1, 2)$ and with weight cone ω of $\ker \partial$ equal to the cone generated by $(0, 1)$ and $(1, 1)$ in $M_{\mathbb{R}}$. Indeed, (i) holds since $C = \mathbb{P}_{\mathbf{k}}^1$ and (ii') holds with $z_0 = 0$ and $z_{\infty} = \infty$. With this choice, $h_z|_{\omega} = 0$ for all $z \in C$, $z \neq 0, \infty$. The vector $v \in N_{\mathbb{R}}$ such that $h(m) = \langle m, v \rangle$ corresponds to $v = (1/2, 0)$. The cone τ is generated in $N_{\mathbb{R}}$ by $(1, 0)$ and $(-1, 1)$ and the cone $\hat{\tau}$ in $\hat{N}_{\mathbb{R}}$ is generated by $(1, 0, 2)$, $(-1, 1, 0)$ and $(1, 0, -2)$. Taking $s_1 = 0$, we have that $(e, -1) = (1, 2, -1) \in \text{Rt } \hat{\tau}$ so that (iii) holds. Furthermore, a straightforward verification shows that (iv), (v) and (vi) hold.

Example 5.13. We assume in this example that the ground field \mathbf{k} is algebraically closed of characteristic 2. Let us consider the Bertin surface

$$W_{2,5} = \{x^2y = x + z^5\} \subseteq \mathbb{A}_{\mathbf{k}}^3$$

of type $(2, 5)$. This is a smooth affine surface endowed with the \mathbb{G}_m -action

$$\lambda \cdot (x, y, z) = (\lambda^5 x, \lambda^{-5} y, \lambda z),$$

where $\lambda \in \mathbb{G}_m$ and $(x, y, z) \in W_{2,5}$. Consider the polyhedral divisor

$$\mathfrak{D} = \left\{ \frac{1}{5} \right\} \cdot [0] + \left[0, \frac{1}{5} \right] \cdot [1]$$

over the affine line $\mathbb{A}^1 = \mathbb{A}_{\mathbf{k}}^1$. Here we have $N = M = \mathbb{Z}$. The elements

$$x = t^{-1}\chi^5, \quad y = (t+1)t\chi^{-5}, \quad z = \chi^1$$

generate the \mathbb{Z} -graded algebra $A = A[\mathbb{A}^1, \mathfrak{D}]$ and satisfy the equation of $W_{2,5}$. Hence we may identify the \mathbb{G}_m -surface $X = \operatorname{Spec} A$ with $W_{2,5}$. The quotient map by the \mathbb{G}_m -action is

$$\pi : (x, y, z) \mapsto t = xy + 1.$$

The fiber $\pi^{-1}(1)$ consists in two distinct toric curves which intersect only at the origin:

$$\pi^{-1}(1) = \{(0, y, 0) \mid y \in \mathbf{k}\} \cup \{(z^5, 0, z) \mid z \in \mathbf{k}\}.$$

In the setting of [Theorem 5.11](#), we may take $z_0 = 0$ so that $\tau = \mathbb{R}_{\geq 0}$ and

$$\hat{\tau} = \mathbb{R}_{\geq 0}(1, 0) + \mathbb{R}_{\geq 0}(1, 5).$$

If $e = 1$ and $s := s_1 = 2$, then $(2^s e, -\frac{1}{5} - \frac{2^s e}{5}) = (4, -1)$ is a Demazure root of $\hat{\tau}$ with distinguished ray $(1, 5)$. Condition (iv) of [Theorem 5.11](#) is not fulfilled. The corresponding homogeneous iterative higher derivation ∂ verifies the formula

$$e^{\alpha\partial}(t^l\chi^m) = \sum_{i=0}^{\infty} \binom{5l+m}{i} t^{l-i} \chi^{m+4i} \alpha^{4i}$$

for any $(m, l) \in \mathbb{Z}^2$. This implies directly that

$$e^{\alpha\partial}(x) = x \text{ and } e^{\alpha\partial}(z) = z + \alpha^4 x,$$

and so the subalgebra $\mathbf{k}[x, z] \subseteq A$ is ∂ -stable. However, we have $\partial^{(4)}(y) = t\chi^{-1} \notin A$.

Now let us take $e = 1$ and $s = 6$. Then $(2^s e, -\frac{1}{5} - \frac{2^s e}{5}) = (64, -13)$ is a Demazure root of $\hat{\tau}$. The conditions of [Theorem 5.11](#) are satisfied and the associated LFIHD ∂' has exponential map

$$e^{\alpha\partial'}(t^l\chi^m) = \sum_{i=0}^{\infty} \binom{5l+m}{i} t^{l-13i} \chi^{m+64i} \alpha^{64i}.$$

Therefore

$$e^{\alpha\partial'}(x) = x, \quad e^{\alpha\partial'}(z) = z + \alpha^{64} x^{13},$$

and

$$e^{\alpha\partial'}(y) = x^{-1}(1 + e^{\alpha\partial'}(t)) = y + \alpha^{64}x^{11}z^4 + \alpha^{256}x^{50}z + \alpha^{320}x^{63}.$$

The kernel of ∂' is the subalgebra $\mathbf{k}[x] \subseteq A$.

Remark 5.14. A generalization of [23, Section 4.1] allows to define and compute the homogeneous Makar-Limanov invariant of an affine \mathbb{T} -variety of complexity one of arbitrary characteristic. Due to lack of space, we omit this straightforward generalization.

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