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Equality of elementary linear and symplectic orbits with respect to an alternating form

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ABSTRACT

An elementary symplectic group w.r.t. an invertible alternating matrix is defined. It is shown that the group of symplectic transvections of a symplectic module coincides with this elementary symplectic group in the free case. Equality of orbit spaces of a unimodular element under the action of the linear group, symplectic group, and symplectic group w.r.t. an invertible alternating matrix is established.

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1. Introduction

Let R denote a commutative ring with 1 in the sequel. This article and the articles [3,4] are all inspired by the article [9] and the famous lemma of L.N. Vaserstein ([9], Lemma 5.5), that

$$e_1\mathrm{E}_{2n}(R) = e_1\{\mathrm{Sp}_{2n}(R)\cap \mathrm{E}_{2n}(R)\} = e_1\{\mathrm{Sp}_{\varphi}(R)\cap \mathrm{E}_{2n}(R)\},$$

where φ is an invertible alternating matrix of size $2n$, and $\mathrm{Sp}_\varphi(R)$ is the isotropy group of φ, i.e. $\mathrm{Sp}_\varphi(R) = \{\alpha \in \mathrm{SL}_{2n}(R) \mid \alpha^t\varphi\alpha = \varphi\}$.

This is the last in a series of 3 articles inspired by the above equality of L.N. Vaserstein, regarding the equality of elementary linear and symplectic orbits of a unimodular row. The first article appeared in the Journal of K-theory [3] and we showed that if v is a unimodular row of even length $2n$, $n \geq 2$, over a commutative ring R, then $v\mathrm{E}_{2n}(R, I) = v\mathrm{ESp}_{2n}(R, I)$. The second article appeared in the Journal of Pure and Applied Algebra [4] and we showed that if $(Q, \langle,\rangle_{\psi_n})$ is a symplectic module, and $\mathbb{H}(R)$ is the usual hyperbolic space, with rank $(P) \geq 1$, then for a unimodular element $(a, b, p) \in \mathbb{H}(R) \perp P$ the elementary linear and symplectic transvection orbits coincide.

In this article, we touch upon the second equality of L.N. Vaserstein. Let $(Q, \langle,\rangle_\varphi)$ be a symplectic module w.r.t. an invertible alternating form φ. L.N. Vaserstein gave examples of symplectic matrices w.r.t. an invertible alternating matrix in ([9], Lemma 5.4). We denoted by $\mathrm{ESp}_\varphi(R)$ the subgroup of $\mathrm{Sp}_\varphi(R)$ generated by these matrices. This will be the elementary symplectic group w.r.t. an invertible alternating matrix φ, when we are dealing with the case where Q is a free module. In the general case, we will define the elementary symplectic group as follows:

Let $(Q, \langle,\rangle)$ be a symplectic R-module with Q finitely generated projective module of even rank. Recall that $\mathrm{Sp}(Q, \langle,\rangle)$ is the group of isometries. We define $V(Q, \langle,\rangle)$ to be the collection of all

$$\begin{aligned}&\{\alpha(1) : \alpha(X) \in \mathrm{Sp}(Q[X], \langle,\rangle_{\otimes R[X]}), \alpha(0) = id., \text{ and}\\ &\alpha(X)_\mathfrak{p} \in \mathrm{ESp}_{\varphi\otimes R_\mathfrak{p}[X]}(R_\mathfrak{p}[X]), \text{ for all } \mathfrak{p} \in \mathit{Spec}(R)\},\end{aligned}$$

where $\langle,\rangle$ correspond to an alternating matrix φ (w.r.t. some basis) of Pfaffian 1 over the local ring $R_\mathfrak{p}$ at the prime ideal $\mathfrak{p}$. For a relative version of this definition one can see Definition 6.2. This group can be considered as the generalisation of $\mathrm{ESp}_\varphi(R)$ in the case of projective modules. This realisation follows from Lemma 5.18 and Theorem 6.4.

We have recalled the definition of $\mathrm{Um}_n(R, I)$ and $\mathrm{Um}(Q, IQ)$ in the beginning of the next section. For definitions of $\mathrm{E}_n(R, I)$, $\mathrm{ESp}_{2n}(R, I)$, $\mathrm{ESp}_\varphi(R, I)$, $\mathrm{ETrans}(Q, IQ)$, and $\mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle,\rangle)$ one can see Definitions 2.2, 2.7, 3.4, 5.4, and 5.13 respectively. Our main results regarding equality of orbit spaces in the case of free modules and in the case of projective modules are as follows:

Theorem 1 *(Theorem 7.2). Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $n \geq 2$ and φ be an alternating matrix of size $2n$ of Pfaffian 1 such that $\varphi \equiv \psi_n \pmod I$. Then the orbit spaces $\mathrm{Um}_{2n}(R, I)/\mathrm{E}_{2n}(R, I)$, $\mathrm{Um}_{2n}(R, I)/\mathrm{ESp}_{2n}(R, I)$, and $\mathrm{Um}_{2n}(R, I)/\mathrm{ESp}_{\varphi}(R, I)$ are bijective.*

Theorem 2 *(Theorem 7.3). Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $(P, \langle , \rangle)$ be a symplectic R-module with P finitely generated projective module of rank $2n$, $n \geq 1$. Let $Q = R^2 \oplus P$ with induced form on $(\mathbb{H}(R) \oplus P)$. We also assume for each maximal ideal $\mathfrak{m}$ of R, the alternating form $\langle , \rangle$ over the local ring $R_{\mathfrak{m}}$ corresponds to an alternating matrix $\varphi_{\mathfrak{m}}$ (w.r.t. some basis), such that $\varphi_{\mathfrak{m}} \equiv \psi_n (\mathrm{mod}\ I)$. Then the orbit spaces $\mathrm{Um}(Q, IQ)/\mathrm{ETrans}(Q, IQ)$, $\mathrm{Um}(Q, IQ)/\mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle)$, and $\mathrm{Um}(Q, IQ)/V(Q, IQ, \langle , \rangle)$ are bijective.*

2. Preliminaries

A row $v = (v_1, \ldots, v_n) \in R^n$ is said to be *unimodular* if there are elements $w_1, \ldots, w_n$ in R such that $v_1w_1 + \cdots + v_nw_n = 1$. $\mathrm{Um}_n(R)$ will denote the set of all unimodular rows $v \in R^n$. Let I be an ideal in R. We denote by $\mathrm{Um}_n(R, I)$ the set of all unimodular rows of length n which are congruent to $e_1 = (1, 0, \ldots, 0)$ modulo I. (If $I = R$, then $\mathrm{Um}_n(R, I)$ is $\mathrm{Um}_n(R)$.)

Definition 2.1. Let P be a finitely generated projective R-module. An element $p \in P$ is said to be *unimodular* if there exists a R-linear map $\phi : P \to R$ such that $\phi(p) = 1$. The collection of unimodular elements of P is denoted by $\mathrm{Um}(P)$.

Let P be of the form $R \oplus Q$ and have an element of the form $(1, 0)$ which correspond to the unimodular element. An element $(a, q) \in P$ is said to be *relative unimodular* w.r.t. an ideal I of R if (a, q) is unimodular and (a, q) is congruent to $(1, 0)$ modulo IP. The collection of all relative unimodular elements w.r.t. an ideal I is denoted by $\mathrm{Um}(P, IP)$.

Let us recall that if M is a finitely presented R-module and S is a multiplicative set of R, then $S^{-1}\mathrm{Hom}_R(M, R) \cong \mathrm{Hom}_{R_S}(M_S, R_S)$. Also recall that if $f = (f_1, \ldots, f_n) \in R^n := M$, then $\Theta_M(f) = \{\phi(f) : \phi \in \mathrm{Hom}(M, R)\} = \sum_{i=1}^{n} Rf_i$. Therefore, if P is a finitely generated projective R-module of rank n, $\mathfrak{m}$ is a maximal ideal of R and $v \in \mathrm{Um}(P)$, then $v_{\mathfrak{m}} \in \mathrm{Um}_n(R_{\mathfrak{m}})$. Similarly if $v \in \mathrm{Um}(P, IP)$ then $v_{\mathfrak{m}} \in \mathrm{Um}_n(R_{\mathfrak{m}}, I_{\mathfrak{m}})$.

The group $\mathrm{GL}_n(R)$ of invertible matrices acts on R^n in a natural way: $v \longrightarrow v\sigma$, if $v \in R^n$, $\sigma \in \mathrm{GL}_n(R)$. This map preserves $\mathrm{Um}_n(R)$, so $\mathrm{GL}_n(R)$ acts on $\mathrm{Um}_n(R)$. Note that any subgroup G of $\mathrm{GL}_n(R)$ also acts on $\mathrm{Um}_n(R)$. Let $v, w \in \mathrm{Um}_n(R)$, we denote $v \sim_{\mathrm{G}} w$ or $v \in w\mathrm{G}$ if there is a $g \in \mathrm{G}$ such that $v = wg$.

Let $\mathrm{E}_n(R)$ denote the subgroup of $\mathrm{SL}_n(R)$ consisting of all *elementary* matrices, i.e. those matrices which are a finite product of the *elementary generators* $\mathrm{E}_{ij}(\lambda) = I_n + e_{ij}(\lambda)$, $1 \leq i \neq j \leq n$, $\lambda \in R$, where $e_{ij}(\lambda) \in \mathrm{M}_n(R)$ has an entry λ in its (i, j)-th position and zeros elsewhere. Here I_n denote the $n \times n$ identity matrix.

In the sequel, if α denotes an $m \times n$ matrix, then we let α^t denote its *transpose* matrix. This is of course an $n \times m$ matrix. However, we will mostly be working with square matrices, or rows and columns.

Definition 2.2 *(The relative groups* $\mathrm{E}_n(I),\ \mathrm{E}_n(R,I)$*).* Let I be an ideal of R. The *relative elementary group* $\mathrm{E}_n(I)$ is the subgroup of $\mathrm{E}_n(R)$ generated as a group by the elements $\mathrm{E}_{ij}(x)$, $x \in I$, $1 \le i \ne j \le n$.

The *relative elementary group* $\mathrm{E}_n(R,I)$ is the normal closure of $\mathrm{E}_n(I)$ in $\mathrm{E}_n(R)$. (Equivalently, $\mathrm{E}_n(R,I)$ is generated as a group by $\mathrm{E}_{ij}(a)\mathrm{E}_{ji}(x)\mathrm{E}_{ij}(-a)$, with $a \in R$, $x \in I$, $i \ne j$, provided $n \ge 3$ (see [12], §2).)

Following is an important lemma of A.A. Suslin.

Lemma 2.3. *(See [10], Corollary 1.2, Lemma 1.3.) Let $v, w \in R^n$, with $n \ge 3$ and $\langle v, w\rangle = v \cdot w^t = 0$. Assume that v is unimodular and $w \in I^{2n-1} (\subseteq R^{2n-1})$. Then $I_n + v^t w \in \mathrm{E}_n(R,I)$.*

As a consequence of the above lemma Suslin proved that $\mathrm{E}_n(R,I)$ is a normal subgroup of $\mathrm{GL}_n(R)$, for $n \ge 3$ (see [10], Corollary 1.4).

Remark 2.4. It is easy to check that if $v \in \mathrm{Um}_n(R,I)$, where $(R,\mathfrak{m})$ is a local ring and I be an ideal of R, then $v = e_1\beta$, for some $\beta \in \mathrm{E}_n(R,I)$.

Definition 2.5 *(Symplectic group* $\mathrm{Sp}_{2n}(R)$*).* The *symplectic group* $\mathrm{Sp}_{2n}(R) = \{\alpha \in \mathrm{GL}_{2n}(R) \mid \alpha^t\psi_n\alpha = \psi_n\}$, where $\psi_n = \sum_{i=1}^{n} e_{2i-1,2i} - \sum_{i=1}^{n} e_{2i,2i-1}$, the standard symplectic form.

Let σ denote the permutation of the natural numbers given by $\sigma(2i) = 2i-1$ and $\sigma(2i-1) = 2i$.

Definition 2.6 *(Elementary symplectic group* $\mathrm{ESp}_{2n}(R)$*).* We define for $z \in R$, $1 \le i \ne j \le 2n$,

$$se_{ij}(z) = \begin{cases} 1_{2n} + e_{ij}(z) & \text{if } i = \sigma(j), \\ 1_{2n} + e_{ij}(z) - (-1)^{i+j} e_{\sigma(j)\sigma(i)}(z) & \text{if } i \ne \sigma(j). \end{cases}$$

It is easy to check that all these elements belong to $\mathrm{Sp}_{2n}(R)$. We call them elementary symplectic matrices over R and the subgroup of $\mathrm{Sp}_{2n}(R)$ generated by them is called the elementary symplectic group $\mathrm{ESp}_{2n}(R)$.

Definition 2.7 *(The relative group* $\mathrm{ESp}_{2n}(I),\ \mathrm{ESp}_{2n}(R,I)$*).* Let I be an ideal of R. The relative elementary group $\mathrm{ESp}_{2n}(I)$ is the subgroup $\mathrm{ESp}_{2n}(R)$ generated as a group by the elements $se_{ij}(x)$, $x \in I$ and $1 \le i \ne j \le 2n$.

The relative elementary group $\mathrm{ESp}_{2n}(R, I)$ is the normal closure of $\mathrm{ESp}_{2n}(I)$ in $\mathrm{ESp}_{2n}(R)$.

Lemma 2.8. *(See [6], Lemma 1.5.) Let $n \geq 2$, and I be an ideal of R. Let $a \in I, v \in R^{2n}$, or $a \in R$, $v \in I^{2n}$ $(\subseteq R^{2n})$. Then $I_{2n} + av^t\widetilde{v} \in \mathrm{ESp}_{2n}(R, I)$, where $\widetilde{v} = v\psi_n$.* $\square$

Lemma 2.9. *(See [6], Lemma 1.10.) Let $n \geq 2$, and I be an ideal in R. Let $w \in \mathrm{Um}_{2n}(R)$, and $v \in I^{2n}$ $(\subseteq R^{2n})$ be such that $\widetilde{v}w^t = 0$. Then $I_{2n} + v^t\widetilde{w} + w^t\widetilde{v} \in \mathrm{ESp}_{2n}(R, I)$.* $\square$

3. Elementary symplectic group $\mathrm{ESp}_{\varphi}(R)$

Definition 3.1 *(Alternating matrix).* A matrix from $\mathrm{M}_n(R)$ is said to be *alternating* if it has the form $\nu - \nu^t$, where $\nu \in \mathrm{M}_n(R)$. (It follows that its diagonal elements are zeros.)

Definition 3.2. The group of all $2n \times 2n$ matrices $\{\alpha \in \mathrm{GL}_{2n}(R) \mid \alpha^t\varphi\alpha = \varphi\}$, where φ is an invertible alternating matrix is called *symplectic group* $\mathrm{Sp}_{\varphi}(R)$ *w.r.t.* φ.

Definition 3.3. Let $v \in R^{2n-1}$. Let φ be an invertible alternating matrix of size $2n$ of the form $\left(\begin{smallmatrix} 0 & -c \\ c^t & \nu \end{smallmatrix}\right)$, and φ^{-1} be of the form $\left(\begin{smallmatrix} 0 & d \\ -d^t & \mu \end{smallmatrix}\right)$, where $c, d \in R^{2n-1}$. In ([9], Lemma 5.4) L.N. Vaserstein considered the matrices (related to φ and $v \in R^{2n-1}$):

$$\alpha := \alpha_{\varphi}(v) := I_{2n-1} + d^t v\nu,$$
$$\beta := \beta_{\varphi}(v) := I_{2n-1} + \mu v^t c.$$

Note that $\alpha_{\varphi}(v), \beta_{\varphi}(v) \in \mathrm{E}_{2n-1}(R)$ via Lemma 2.3.

From these matrices he constructed in ([9], Lemma 5.4)

$$C_{\varphi}(v) = \begin{pmatrix} 1 & 0 \\ \alpha v^t & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v^t & \alpha \end{pmatrix} \quad \text{and}$$
$$R_{\varphi}(v) = \begin{pmatrix} 1 & v \\ 0 & \beta \end{pmatrix}.$$

(The notation $C_{\varphi}(v)$, $R_{\varphi}(v)$ is due to us.) In ([9], Lemma 5.4) it is mentioned that these matrices belong to $\mathrm{Sp}_{\varphi}(R)$. We call the subgroup of $\mathrm{Sp}_{\varphi}(R)$ generated by $C_{\varphi}(v)$ and $R_{\varphi}(v)$, for $v \in R^{2n-1}$ as the *elementary symplectic group* $\mathrm{ESp}_{\varphi}(R)$ *with respect to the invertible alternating matrix* φ.

Definition 3.4. Let I be an ideal of R. The *relative elementary group* $\mathrm{ESp}_{\varphi}(I)$ is a subgroup of $\mathrm{ESp}_{\varphi}(R)$ generated as a group by the elements $C_{\varphi}(v)$ and $R_{\varphi}(v)$, where $v \in I^{2n-1}$.

The *relative elementary group* $\mathrm{ESp}_{\varphi}(R, I)$ is the normal closure of $\mathrm{ESp}_{\varphi}(I)$ in $\mathrm{ESp}_{\varphi}(R)$.

Lemma 3.5. *Let R be a ring with $R = 2R$ and $n \geq 2$. For the standard alternating matrix ψ_n we have,*

$$\mathrm{ESp}_{\psi_n}(R) = \mathrm{ESp}_{2n}(R),$$
$$\mathrm{ESp}_{\psi_n}(R, I) = \mathrm{ESp}_{2n}(R, I).$$

Proof. For the standard alternating matrix ψ_n we have

$$C_{\psi_n}(v) = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v^t & I \end{pmatrix} = \prod_{i=2}^{2n} se_{i1}(a_{i-1}), \tag{1}$$

$$R_{\psi_n}(v) = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & I \end{pmatrix} = \prod_{i=2}^{2n} se_{1i}(a_{i-1}), \tag{2}$$

where $v = (a_1, \ldots, a_{2n-1}) \in R^{2n-1}$. Therefore $\mathrm{ESp}_{\psi_n}(R) \subseteq \mathrm{ESp}_{2n}(R)$.

Note that $se_{1i}(a), se_{j1}(b) \in \mathrm{ESp}_{\psi_n}(R)$. For all integers i, j with $i \neq j, \sigma(j)$, and for all $a, b \in R$ we have the following commutator identities for the generators of the elementary symplectic group

$$[se_{i\sigma(i)}(a), se_{\sigma(i)j}(b)] = se_{ij}(ab)se_{\sigma(j)j}((-1)^{i+j}ab^2), \tag{3}$$

$$[se_{ik}(a), se_{kj}(b)] = se_{ij}(ab), \text{ if } k \neq \sigma(i), \sigma(j), \tag{4}$$

$$[se_{ik}(a), se_{k\sigma(i)}(b)] = se_{i\sigma(i)}(2ab), \text{ if } k \neq i, \sigma(i). \tag{5}$$

Using these identities we can show that $se_{ij}(a)$, for $i, j \neq 1$ can be written as product of elements of the form $se_{1i}(x)$ and $se_{j1}(y)$, for $x, y \in R$. Hence $\mathrm{ESp}_{2n}(R) \subseteq \mathrm{ESp}_{\psi_n}(R)$.

For the second equality we first show that $\mathrm{ESp}_{\psi_n}(R, I) \subseteq \mathrm{ESp}_{2n}(R, I)$. An element of $\mathrm{ESp}_{\psi_n}(R, I)$ looks like $\gamma g_{\psi_n}(w)\gamma^{-1}$, where $\gamma \in \mathrm{ESp}_{\psi_n}(R)$, g_{ψ_n} could be either C_{ψ_n} or R_{ψ_n}, and $w \in I^{2n-1}$. By equations (1) and (2), $g_{\psi_n}(w) \in \mathrm{ESp}_{2n}(I)$. Note that $\gamma \in \mathrm{ESp}_{\psi_n}(R) = \mathrm{ESp}_{2n}(R)$. Hence $\gamma g_{\psi_n}(w)\gamma^{-1} \in \mathrm{ESp}_{2n}(R, I)$.

To show the other inclusion we recall the equivalent definition of the relative group which says that $\mathrm{ESp}_{2n}(R, I)$ is the smallest normal subgroup of $\mathrm{ESp}_{2n}(R)$ containing $se_{21}(x)$, where $x \in I$ (see [5], Lemma 2.2 for the proof in the linear case, the proof in the symplectic case is similar). We need to show $gse_{21}(x)g^{-1} \in \mathrm{ESp}_{\psi_n}(R, I)$, where $g \in \mathrm{ESp}_{2n}(R) = \mathrm{ESp}_{\psi_n}(R)$. Hence $gse_{21}(x)g^{-1} \in \mathrm{ESp}_{\psi_n}(R, I)$ and $\mathrm{ESp}_{2n}(R, I) \subseteq \mathrm{ESp}_{\psi_n}(R, I)$. Therefore the second equality is established. □

Lemma 3.6. *Let φ and φ^* be two invertible alternating matrices such that $\varphi = (1 \perp \varepsilon)^t\ \varphi^*\ (1 \perp \varepsilon)$, for some $\varepsilon \in \mathrm{E}_{2n-1}(R)$. Then we have*

$$\mathrm{Sp}_{\varphi}(R) = (1 \perp \varepsilon)^{-1}\ \mathrm{Sp}_{\varphi^*}(R)\ (1 \perp \varepsilon).$$

Proof. Let $\eta \in \mathrm{Sp}_{\varphi^*}(R)$. By definition $\eta^t \varphi^* \eta = \varphi^*$. Note that

$$\begin{aligned}
&\big((1 \perp \varepsilon)^{-1}\eta(1 \perp \varepsilon)\big)^t \ \varphi \ \big((1 \perp \varepsilon)^{-1}\eta(1 \perp \varepsilon)\big) \\
&\quad = (1 \perp \varepsilon)^t \eta^t \big((1 \perp \varepsilon)^{-1}\big)^t \ \varphi \ (1 \perp \varepsilon)^{-1}\eta(1 \perp \varepsilon) \\
&\quad = (1 \perp \varepsilon)^t \eta^t \ \varphi^* \ \eta(1 \perp \varepsilon) \\
&\quad = (1 \perp \varepsilon)^t \ \varphi^* \ (1 \perp \varepsilon) \\
&\quad = \varphi
\end{aligned}$$

hence the equality follows. □

Lemma 3.7. *Let φ and φ^* be two invertible alternating matrices such that $\varphi = (1 \perp \varepsilon)^t \ \varphi^* \ (1 \perp \varepsilon)$, for some $\varepsilon \in \mathrm{E}_{2n-1}(R)$. Then we have*

$$\mathrm{ESp}_{\varphi}(R) = (1 \perp \varepsilon)^{-1} \ \mathrm{ESp}_{\varphi^*}(R) \ (1 \perp \varepsilon).$$

Proof. Note that if φ^* is of the form $\left(\begin{smallmatrix} 0 & -c \\ c^t & \nu \end{smallmatrix}\right)$, and φ^{*-1} is of the form $\left(\begin{smallmatrix} 0 & d \\ -d^t & \mu \end{smallmatrix}\right)$, where $c, d \in R^{2n-1}$, then

$$\varphi = \begin{pmatrix} 0 & -c\varepsilon \\ \varepsilon^t c^t & \varepsilon^t \nu \varepsilon \end{pmatrix} \text{ and } \varphi^{-1} = \begin{pmatrix} 0 & d(\varepsilon^t)^{-1} \\ -\varepsilon^{-1} d^t & \varepsilon^{-1}\mu(\varepsilon^t)^{-1} \end{pmatrix}.$$

We have

$$\begin{aligned}
&(1 \perp \varepsilon)^{-1} \ C_{\varphi^*}(v) \ (1 \perp \varepsilon) \\
&\quad = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varepsilon \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 1 & 0 \\ v^t & \alpha_{\varphi^*}(v) \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varepsilon \end{smallmatrix}\right) \\
&\quad = \left(\begin{smallmatrix} 1 & 0 \\ \varepsilon^{-1} v^t & \varepsilon^{-1}\alpha_{\varphi^*}(v)\varepsilon \end{smallmatrix}\right),
\end{aligned}$$

and

$$\begin{aligned}
&(1 \perp \varepsilon)^{-1} \ R_{\varphi^*}(v) \ (1 \perp \varepsilon) \\
&\quad = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varepsilon \end{smallmatrix}\right)^{-1} \left(\begin{smallmatrix} 1 & v \\ 0 & \beta_{\varphi^*}(v) \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & \varepsilon \end{smallmatrix}\right) \\
&\quad = \left(\begin{smallmatrix} 1 & v\varepsilon \\ 0 & \varepsilon^{-1}\beta_{\varphi^*}(v)\varepsilon \end{smallmatrix}\right).
\end{aligned}$$

Note that $\varepsilon^{-1}\alpha_{\varphi^*}(v)\varepsilon = \alpha_{\varphi}(v(\varepsilon^{-1})^t)$ and $\varepsilon^{-1}\beta_{\varphi^*}(v)\varepsilon = \beta_{\varphi}(v\varepsilon)$. Therefore, $(1 \perp \varepsilon)^{-1} \ C_{\varphi^*}(v) \ (1 \perp \varepsilon) = C_{\varphi}(v(\varepsilon^{-1})^t)$ and $(1 \perp \varepsilon)^{-1} \ R_{\varphi^*}(v) \ (1 \perp \varepsilon) = R_{\varphi}(v\varepsilon)$. Hence the equality follows. □

Lemma 3.8. *Let φ and φ^* be two invertible alternating matrices such that $\varphi = (1 \perp \varepsilon)^t \ \varphi^* \ (1 \perp \varepsilon)$, for some $\varepsilon \in \mathrm{E}_{2n-1}(R, I)$. Then we have*

$$\mathrm{ESp}_{\varphi}(R, I) = (1 \perp \varepsilon)^{-1} \ \mathrm{ESp}_{\varphi^*}(R, I) \ (1 \perp \varepsilon).$$

Proof. The proof follows from the definition of $\mathrm{ESp}_\varphi(R, I)$ and the equalities $(1 \perp \varepsilon)^{-1}\ C_{\varphi^*}(v)\ (1 \perp \varepsilon) = C_\varphi(v(\varepsilon^{-1})^t)$ and $(1 \perp \varepsilon)^{-1}\ R_{\varphi^*}(v)\ (1 \perp \varepsilon) = R_\varphi(v\varepsilon)$. $\square$

4. Dilation and LG principle for $\mathrm{ESp}_{\varphi\otimes R[X]}(R[X])$

Here we establish dilation principle and Local–Global principle for $\mathrm{ESp}_{\varphi\otimes R[X]}(R[X])$. The following lemma is well known. We will use it in the proof of dilation principle.

Lemma 4.1. *Let G be a group, and $a_i, b_i \in G$, for $i = 1, \ldots, n$. Then $\prod_{i=1}^n a_i b_i = \prod_{i=1}^n r_i b_i r_i^{-1} \prod_{i=1}^n a_i$, where $r_i = \prod_{j=1}^i a_j$.* $\square$

Lemma 4.2 *(Dilation principle). Let R be a ring with $R = 2R$. Let φ be an invertible alternating matrix of size $2n$, with $n \geq 1$. Let $a \in R$ be a non-nilpotent element, and let $\varphi = (1 \perp \varepsilon)^t\ \psi_n\ (1 \perp \varepsilon)$, for some $\varepsilon \in \mathrm{E}_{2n-1}(R_a)$ over the ring R_a. Let $\theta(X) \in \mathrm{ESp}_{\varphi\otimes R_a[X]}(R_a[X])$, with $\theta(0) = Id$. Then there exists $\theta^*(X) \in \mathrm{ESp}_{\varphi\otimes R[X]}(R[X])$ such that $\theta^*(X)$ localises to $\theta(bX)$, for some $b \in (a^d)$, $d \gg 0$, and $\theta^*(0) = Id$.*

Proof. We are given that $\theta(X) \in \mathrm{ESp}_{\varphi\otimes R_a[X]}(R_a[X])$, where $\varphi = (1 \perp \varepsilon)^t\ \psi_n\ (1 \perp \varepsilon)$, for some $\varepsilon \in \mathrm{E}_{2n-1}(R_a)$ over the ring R_a. Therefore by Lemma 3.5 and Lemma 3.7 we have $\theta(X) = (1 \perp \varepsilon)^{-1}\eta(X)(1 \perp \varepsilon)$, for some $\eta(X) \in \mathrm{ESp}_{2n}(R_a[X])$. Since $\eta(0) = Id$. we can write $\eta(X) = \prod \gamma_l se_{i_l j_l}(Xf_l(X))\ \gamma_l^{-1}$, where $\gamma_l \in \mathrm{ESp}_{2n}(R_a)$, and $f_l(X) \in R_a[X]$ (see Lemma 4.1). Using commutator identities for the generators of the elementary symplectic group we get $\eta(Y^r X) = \prod se_{i_k j_k}(Yh_k(X, Y)/a^s)$, for large integer r. Here $h_k(X, Y) \in R[X, Y]$ and either $i_k = 1$ or $j_k = 1$. Let e_i denote a row vector of length $2n-1$ which has 1 in the ith position and zeros elsewhere. Using equations (1) and (2) appearing in the Lemma 3.5 it is clear that $\eta(Y^r X)$ is product of the elements of the form $C_{\psi_n}((Yh_k(X, Y)/a^s).e_i)$ or $R_{\psi_n}((Yh_k(X, Y)/a^s).e_j)$, where $1 \leq i, j \leq 2n-1$.

Note that $C_{\psi_n}((Yh_k(X, Y)/a^s).e_i) = (1 \perp \varepsilon)C_\varphi((Yh_k(X, Y)/a^s).e_i(\varepsilon^{-1})^t)(1 \perp \varepsilon)^{-1}$ and $R_{\psi_n}((Yh_k(X, Y)/a^s).e_j) = (1 \perp \varepsilon)R_\varphi((Yh_k(X, Y)/a^s).e_j\varepsilon)(1 \perp \varepsilon)^{-1}$. Therefore $\theta(Y^r X)$ is product of elements of the form $C_\varphi((Yh_k(X, Y)/a^s).e_i(\varepsilon^t)^{-1})$ or $R_\varphi((Yh_k(X, Y)/a^s).e_j\varepsilon)$. Let t be the maximum power of a appearing in the denominators of ε and $(\varepsilon^t)^{-1}$. Set $d = s + t$. Define $\theta^*(X, Y)$ as product of elements of the form $C_\varphi(Yh_k(X, a^d Y).a^t e_i(\varepsilon^t)^{-1})$ and $R_\varphi(Yh_k(X, a^d Y)a^t.e_j\varepsilon)$. Note that $\theta^*(X, Y) \in \mathrm{E}_{\varphi\otimes R[X,Y]}(R[X, Y])$. We obtain $\theta^*(X)$ substituting $Y = 1$ in $\theta^*(X, Y)$. Clearly $\theta^*(X)$ localises to $\theta(bX)$ for some $b \in (a^d)$, and $\theta^*(0) = Id$. $\square$

Remark 4.3. Let $(R, \mathfrak{m})$ be a local ring and φ be an alternating matrix of Pfaffian 1 over R of size $2n$. Then $\varphi = \varepsilon^t\psi_n\varepsilon$, for some $\varepsilon \in \mathrm{E}_{2n}(R)$.

We recollect an observation of Rao–Swan stated in the introduction of [8]. We make a contextual observation which the proof shows and include it for completeness.

Lemma 4.4 *(Rao–Swan). Let $n \geq 2$ and $\varepsilon \in \mathrm{E}_{2n}(R)$. Then there exists $\rho \in \mathrm{E}_{2n-1}(R)$ such that $(1 \perp \rho)\varepsilon \in \mathrm{ESp}_{2n}(R)$.*

Proof. Let $\varepsilon = \varepsilon_r \dots \varepsilon_1$, where each ε_i is of the form $\left(\begin{smallmatrix} 1 & v \\ 0 & I \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 0 \\ v^t & I \end{smallmatrix}\right)$, where $v = (a_1, \dots, a_{2n-1}) \in R^{2n-1}$ (see [9], Lemma 2.7). We prove the result using induction on r. It is clear when $r = 0$. Let $r \geq 1$. Let us assume the result is true for $r - 1$, i.e., when $\varepsilon = \varepsilon_{r-1} \dots \varepsilon_1$, then there exists a $\delta \in \mathrm{E}_{2n-1}(R)$ such that $(1 \perp \delta)\varepsilon \in \mathrm{ESp}_{2n}(R)$. We will prove the result when number of generators of ε is r. We have

$$C_{\psi_n}(v) = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v^t & I_{2n-1} \end{pmatrix} = \prod_{i=2}^{2n} se_{i1}(a_{i-1}),$$

$$R_{\psi_n}(v) = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & I_{2n-1} \end{pmatrix} = \prod_{i=2}^{2n} se_{1i}(a_{i-1}).$$

Note that $\alpha = \alpha_{\psi_n}(v)$, $\beta = \beta_{\psi_n}(v) \in \mathrm{E}_{2n-1}(R)$. Let us set γ equal to either α or β depending on the form of ε_1. Now, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \gamma \end{smallmatrix}\right)\varepsilon_1 \in \mathrm{ESp}_{2n}(R)$, and each $\eta_i = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \gamma \end{smallmatrix}\right)\varepsilon_i\left(\begin{smallmatrix} 1 & 0 \\ 0 & \gamma^{-1} \end{smallmatrix}\right)$ is of the form $\left(\begin{smallmatrix} 1 & v \\ 0 & I \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 0 \\ v^t & I \end{smallmatrix}\right)$. Now we have

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & \gamma^{-1} \end{pmatrix} \eta_r \dots \eta_2 \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \varepsilon_1.$$

By induction hypothesis $(1 \perp \delta)\eta_r \dots \eta_2 \in \mathrm{ESp}_{2n}(R)$, for some $\delta \in \mathrm{E}_{2n-1}(R)$. Hence $(1 \perp \rho)\varepsilon \in \mathrm{ESp}_{2n}(R)$, where $\rho = \delta^{-1}\gamma \in \mathrm{E}_{2n-1}(R)$. $\square$

Corollary 4.5 *(Rao–Swan). For $n \geq 2$ and $\varepsilon \in \mathrm{E}_{2n}(R)$, we have an $\varepsilon_0 \in \mathrm{E}_{2n-1}(R)$ such that $\varepsilon^t\psi_n\varepsilon = (1 \perp \varepsilon_0)^t\psi_n(1 \perp \varepsilon_0)$.*

Proof. Using Lemma 4.4 we get $\varepsilon_0 \in \mathrm{E}_{2n-1}(R)$ such that $(1 \perp \varepsilon_0)\varepsilon^{-1} \in \mathrm{ESp}_{2n}(R)$, and hence ${\varepsilon^{-1}}^t(1 \perp \varepsilon_0)^t\ \psi_n\ (1 \perp \varepsilon_0)\varepsilon^{-1} = \psi_n$.

Therefore we have

$$\begin{aligned} \varepsilon^t\psi_n\varepsilon &= \varepsilon^t\ \{{\varepsilon^{-1}}^t(1 \perp \varepsilon_0)^t\}\ \psi_n\ \{(1 \perp \varepsilon_0)\varepsilon^{-1}\}\ \varepsilon \\ &= (1 \perp \varepsilon_0)^t\psi_n(1 \perp \varepsilon_0). \qquad \square \end{aligned}$$

Using dilation principle we prove the following variant of D. Quillen's Local–Global principle (see [7]). The argument is standard. We include the proof for completeness.

Theorem 4.6 *(Local–Global principle). Let φ be an alternating matrix of Pfaffian 1 of size $2n$, with $n \geq 2$. Let $\theta(X) \in \mathrm{Sp}_{\varphi\otimes R[X]}(R[X])$, with $\theta(0) = Id$. If $\theta(X)_{\mathfrak{m}} \in \mathrm{ESp}_{\varphi\otimes R_{\mathfrak{m}}[X]}(R_{\mathfrak{m}}[X])$, for all maximal ideal $\mathfrak{m}$ of R, then $\theta(X) \in \mathrm{ESp}_{\varphi\otimes R[X]}(R[X])$.*

Proof. For each maximal ideal $\mathfrak{m}$ of R one can suitably choose an element $a_\mathfrak{m}$ from $R \setminus \mathfrak{m}$ such that $\theta(X)_{a_\mathfrak{m}} \in \mathrm{ESp}_{\varphi\otimes R_{a_\mathfrak{m}}[X]}(R_{a_\mathfrak{m}}[X])$. Note that over the local ring $R_\mathfrak{m}$ one has $\varphi = (1 \perp \varepsilon)^t\ \psi_n\ (1 \perp \varepsilon)$, for some $\varepsilon \in \mathrm{E}_{2n-1}(R_{a_\mathfrak{m}})$ (see Remark 4.3 and Corollary 4.5). Define $\gamma(X,Y) = \theta(X+Y)_{a_\mathfrak{m}}\theta(Y)^{-1}_{a_\mathfrak{m}}$. It is clear that

$$\gamma(X,Y) \in \mathrm{ESp}_{\varphi\otimes R_{a_\mathfrak{m}}[X,Y]}(R_{a_\mathfrak{m}}[X,Y])$$

and $\gamma(0,Y) = Id$. Therefore $\gamma(b_\mathfrak{m}X,Y) \in \mathrm{ESp}_{\varphi\otimes R[X,Y]}(R[X,Y])$, where $b_\mathfrak{m} \in (a^d_m)$ for $d \gg 0$ (see Lemma 4.2). Note that the ideal generated by $a^d_\mathfrak{m}$'s is the whole ring R. Therefore, $c_1a^d_{\mathfrak{m}_1} + \cdots + c_ka^d_{\mathfrak{m}_k} = 1$, for some $c_i \in R$. Let $b_{m_i} = c_ia^d_{m_i} \in (a^d_{m_i})$. It is easy to see that $\theta(X) = \prod_{i=1}^{k-1}\gamma(b_{m_i}X,T_i)\gamma(b_{m_k}X,0)$, where $T_i = b_{m_{i+1}}X + \cdots + b_{m_k}X$. Each term in the right hand side of this expression belongs to $\mathrm{ESp}_{\varphi\otimes R[X]}(R[X])$, and hence $\theta(X) \in \mathrm{ESp}_{\varphi\otimes R[X]}(R[X])$. □

Theorem 4.7 *(Action version of Local–Global principle). Let φ be an alternating matrix of Pfaffian 1 of size $2n$, with $n \geq 2$. Let $v(X) \in \mathrm{Um}_{2n}(R[X])$. If $v(X) \in v(0)\mathrm{ESp}_{\varphi\otimes R_\mathfrak{m}[X]}(R_\mathfrak{m}[X])$, for all maximal ideal $\mathfrak{m}$ of R, then we show $v(X) \in v(0)\mathrm{ESp}_{\varphi\otimes R[X]}(R[X])$.*

Proof. For each maximal ideal $\mathfrak{m}$ of R, we get $\beta_{(\mathfrak{m})}(X)$ in $\mathrm{ESp}_{\varphi\otimes R_\mathfrak{m}[X]}(R_\mathfrak{m}[X])$ such that $v(X)\beta_{(\mathfrak{m})}(X) = v(0)$. Let us set $\gamma(X,Y) = \beta_{(\mathfrak{m})}(X+Y)\beta_{(\mathfrak{m})}(X)^{-1}$. Clearly $\gamma(X,Y) \in \mathrm{ESp}_{\varphi\otimes R_\mathfrak{m}[X,Y]}(R_\mathfrak{m}[X,Y])$. Since there are only finitely many denominators involved in the expression of $\gamma(X,Y)$, there exists $a_\mathfrak{m} \in R \setminus \mathfrak{m}$ such that $\gamma(X,Y)$ belongs to $\mathrm{ESp}_{\varphi\otimes R_{a_\mathfrak{m}}[X,Y]}(R_{a_\mathfrak{m}}[X,Y])$ and $\gamma(X,0) = Id$. Using Lemma 4.2 it follows that $\gamma(X,b_\mathfrak{m}Y) \in \mathrm{ESp}_{\varphi\otimes R[X,Y]}(R[X,Y])$ for $b_\mathfrak{m} \in (a^d_\mathfrak{m})$, $d \gg 0$. We have $v(X+b_\mathfrak{m}Y)\gamma(X,b_\mathfrak{m}Y) = v(X+b_\mathfrak{m}Y)\beta_{(\mathfrak{m})}(X+b_\mathfrak{m}Y)\beta_{(\mathfrak{m})}(X)^{-1} = v(0)\beta_{(\mathfrak{m})}(X)^{-1} = v(X)$.

Note that the ideal generated by $a^d_\mathfrak{m}$'s is the whole ring R. Therefore $c_1a^d_{\mathfrak{m}_1} + \cdots + c_ka^d_{\mathfrak{m}_k} = 1$, for some $c_i \in R$. Let $b_{m_i} = c_ia^d_{m_i} \in (a^d_{m_i})$. In the above equation replacing X by $b_{\mathfrak{m}_2}X + \cdots + b_{\mathfrak{m}_k}X$ and replacing b_mY by $b_{m_1}X$ we get,

$$v(X) = v(b_{\mathfrak{m}_1}X + b_{\mathfrak{m}_2}X + \cdots + b_{\mathfrak{m}_k}X) \in v(b_{\mathfrak{m}_2}X + \cdots + b_{\mathfrak{m}_k}X)\ \mathrm{ESp}_{\varphi\otimes R[X]}(R[X]).$$

Again, in the above equation replacing X by $b_{\mathfrak{m}_3}X + \cdots + b_{\mathfrak{m}_k}X$ and replacing $b_\mathfrak{m}Y$ by $b_{\mathfrak{m}_2}X$ we get, $v(b_{\mathfrak{m}_2}X + \cdots + b_{\mathfrak{m}_k}X) \in v(b_{\mathfrak{m}_3}X + \cdots + b_{\mathfrak{m}_k}X)\mathrm{ESp}_{\varphi\otimes R[X]}(R[X])$. Continuing in this way we get $v(b_{m_k}X + 0) \in v(0)\mathrm{ESp}_{\varphi\otimes R[X]}(R[X])$. Combining all these we get $v(X)\prod_{i=1}^{k-1}\gamma(b_{\mathfrak{m}_{i+1}}X + \cdots + b_{\mathfrak{m}_k}X, b_{\mathfrak{m}_i}X)\gamma(0, b_{\mathfrak{m}_k}X) = v(0)$, where $\prod_{i=1}^{k-1}\gamma(b_{\mathfrak{m}_{i+1}}X + \cdots + b_{\mathfrak{m}_k}X, b_{\mathfrak{m}_i}X)\gamma(0, b_{\mathfrak{m}_k}X) \in \mathrm{ESp}_{\varphi\otimes R[X]}(R[X])$. □

We recall Swan–Weibel's trick to establish the Local–Global principle in the graded case.

Theorem 4.8 *(Graded case of action version of Local–Global principle). Let φ be an alternating matrix of Pfaffian 1 of size $2n$, with $n \geq 2$. Let us set $\mathbf{X} = (X_1, \ldots, X_t)$*

and $\mathbf{0} = (0, \dots, 0)$. *Let* $v(\mathbf{X}) \in \mathrm{Um}_{2n}(R[\mathbf{X}])$. *If* $v(\mathbf{X}) \in v(\mathbf{0})\mathrm{ESp}_{\varphi \otimes R_{\mathfrak{m}}[\mathbf{X}]}(R_{\mathfrak{m}}[\mathbf{X}])$, *for all maximal ideal* $\mathfrak{m}$ *of* R, *then* $v(\mathbf{X}) \in v(\mathbf{0})\mathrm{ESp}_{\varphi \otimes R[\mathbf{X}]}(R[\mathbf{X}])$.

Proof. Let us denote $S = R[X_1, \dots, X_t]$. Note that S is a graded ring with the grading $S = S_0 \oplus S_1 \oplus S_2 \oplus \cdots$, and $S_0 = R$. Consider the ring homomorphism $f : S \longrightarrow S[T]$ given by $f(a_0 + a_1 + a_2 + \cdots) = a_0 + a_1 T + a_2 T^2 + \cdots$, where each a_i is a homogeneous component belongs to S_i. Let us denote $v(X_1, \dots, X_t) = (v_1, \dots, v_{2n})$, where $v_i \in S$. We set $\tilde{v}(T) = (f(v_1), \dots, f(v_{2n}))$. Note that $\tilde{v}(1) = (v_1, \dots, v_{2n}) = v(X_1, \dots, X_t)$, and $\tilde{v}(0) = v(0, \dots, 0)$.

Let $\mathfrak{m}_0$ be a maximal ideal of R and let $M_0 = R \setminus \mathfrak{m}_0$. Since $v(X_1, \dots, X_t) \in v(0, \dots, 0)\mathrm{ESp}_{\varphi}(S_{M_0})$, we have $\tilde{v}(T) \in \tilde{v}(0)\mathrm{ESp}_{\varphi \otimes S_{M_0}[T]}(S_{M_0}[T])$. Therefore, there is a $s_{m_0} \in M_0$ such that $\tilde{v}(T) \in \tilde{v}(0)\mathrm{ESp}_{\varphi \otimes S_{s_{m_0}}[T]}(S_{s_{m_0}}[T])$. If $\mathfrak{m}$ is a maximal ideal of S then $s_{m_0} \notin \mathfrak{m}$ for some $\mathfrak{m}_0$. Therefore, $\tilde{v}(T) \in \tilde{v}(0)\mathrm{ESp}_{\varphi \otimes S_{\mathfrak{m}}[T]}(S_{\mathfrak{m}}[T])$, for all maximal ideals $\mathfrak{m}$ of S. Moreover, the ideal generated by all s_{m_0}, for all maximal ideals $\mathfrak{m}_0$ of R, is the whole ring R. Hence, $\tilde{v}(T) \in \tilde{v}(0)\mathrm{ESp}_{\varphi \otimes S[T]}(S[T])$. Substituting $T = 1$ we get $\tilde{v}(1) = (v_1, \dots, v_{2n}) = v(X_1, \dots, X_t) \in v(0, \dots, 0)\mathrm{ESp}_{\varphi \otimes R[X_1, \dots, X_t]}(R[X_1, \dots, X_t])$. □

5. Transvection groups

Following H. Bass in [1] one defines a transvection of a finitely generated R-module as follows:

Definition 5.1. Let M be a finitely generated R-module. Let $q \in M$ and $\pi \in M^* = \mathrm{Hom}(M, R)$, with $\pi(q) = 0$. Let $\pi_q(p) := \pi(p)q$. An automorphism of the form $1 + \pi_q$ is called a *transvection* of M, if either $q \in \mathrm{Um}(M)$ or $\pi \in \mathrm{Um}(M^*)$. Collection of transvections of M is denoted by $\mathrm{Trans}(M)$. This forms a subgroup of $\mathrm{Aut}(M)$.

Definition 5.2. Let M be a finitely generated R module. The automorphisms of $N = (R \oplus M)$ of the form

$$(a, p) \mapsto (a, p + ax),$$

or of the form

$$(a, p) \mapsto (a + \tau(p), p),$$

where $x \in M$ and $\tau \in M^*$ are called *elementary transvections* of N. Let us denote the first automorphism by E_x and the second one by E^*_τ. It can be verified that these are transvections of N. Indeed, let us consider $\pi(t, y) = t$, $q = (0, x)$ to get E_x, and consider $\pi(a, p) = \tau(p)$, where $\tau \in M^*$, $q = (1, 0)$ to get E^*_τ. The subgroup of $\mathrm{Trans}(N)$ generated by elementary transvections is denoted by $\mathrm{ETrans}(N)$.

Definition 5.3. Let I be an ideal of R. The group of *relative transvections* w.r.t. an ideal I is generated by the transvections of the form $1+\pi_q$, where either $q \in IM$ or $\pi \in IM^*$. The group generated by relative transvections is denoted by $\mathrm{Trans}(M, IM)$.

Definition 5.4. Let I be an ideal of R. The elementary transvections of $N = (R \oplus M)$ of the form E_x, E^*_τ, where $x \in IM$ and $\tau \in IM^*$ are called *relative elementary transvections* w.r.t. an ideal I, and the group generated by them is denoted by $\mathrm{ETrans}(IN)$. The normal closure of $\mathrm{ETrans}(IN)$ in $\mathrm{ETrans}(N)$ is denoted by $\mathrm{ETrans}(N, IN)$.

Lemma 5.5. *Let M be a free R module of rank $n \geq 2$, and $N = (R \oplus M)$. Then $\mathrm{ETrans}(N) = \mathrm{Trans}(N) = \mathrm{E}_{n+1}(R)$.*

Proof. Let I_{n+1} denote the identity matrix of size $n+1$. Note that when M is a free R module, an element of $\mathrm{Trans}(N)$ looks like $I_{n+1} + v^t w$, for some $v, w \in R^{n+1}$ and one of v or w is unimodular. Also, note that $\langle v, w\rangle = v \cdot w^t = 0$. Therefore $\mathrm{Trans}(N) \subseteq \mathrm{E}_{n+1}(R)$ (see Lemma 2.3).

Given that N is free R-module. E_x and E^*_τ, the generators of $\mathrm{ETrans}(N)$, are of the form $\left(\begin{smallmatrix} 1 & x \\ 0 & I_n \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 1 & 0 \\ y^t & I_n \end{smallmatrix}\right)$, respectively for some $x, y \in R^n$. Hence $\mathrm{ETrans}(N) \subseteq \mathrm{E}_{n+1}(R)$. By ([9], Lemma 2.7(a)) $\mathrm{E}_{n+1}(R)$ is generated by elements of the for $\mathrm{E}_{1i}(a)$ and $\mathrm{E}_{j1}(b)$, for $a, b \in R$ and $2 \leq i, j \leq n+1$. Therefore, $\mathrm{E}_{n+1}(R) \subseteq \mathrm{ETrans}(N)$, hence $\mathrm{ETrans}(N) = \mathrm{E}_{n+1}(R)$.

By definition $\mathrm{ETrans}(N) \subseteq \mathrm{Trans}(N)$. Therefore, $\mathrm{ETrans}(N) \subseteq \mathrm{Trans}(N) \subseteq \mathrm{E}_{n+1}(R) = \mathrm{ETrans}(N)$, and hence the result follows. □

Remark 5.6. A relative version of the above result w.r.t. an ideal I of the ring R is also true, i.e., $\mathrm{ETrans}(N, IN) = \mathrm{Trans}(N, IN) = \mathrm{E}_{n+1}(R, I)$. For details one can see ([4], Lemma 4.5).

Definition 5.7. A *symplectic R-module* is a pair $(P, \langle,\rangle)$, where P is a finitely generated projective R-module of even rank and $\langle,\rangle : P \times P \longrightarrow R$ is a non-degenerate (i.e., $P \cong P^*$ by $x \longrightarrow \langle x, -\rangle$) alternating bilinear form.

Definition 5.8. Let $(P_1, \langle,\rangle_1)$ and $(P_2, \langle,\rangle_2)$ be two symplectic R-modules. Their *orthogonal sum* is the pair $(P, \langle,\rangle)$, where $P = P_1 \oplus P_2$ and the inner product is defined by $\langle (p_1, p_2), (q_1, q_2)\rangle = \langle p_1, q_1\rangle_1 + \langle p_2, q_2\rangle_2$.

There is a non-degenerate bilinear form $\langle,\rangle$ on the R-module $\mathbb{H}(R) = R \oplus R^*$, namely $\langle (a_1, f_1), (a_2, f_2)\rangle = f_2(a_1) - f_1(a_2)$.

Definition 5.9. An *isometry* of a symplectic module $(P, \langle,\rangle)$ is an automorphism of P which fixes the bilinear form. The group of isometries of $(P, \langle,\rangle)$ is denoted by $\mathrm{Sp}(P, \langle,\rangle)$.

Definition 5.10. In [2] Bass has defined a *symplectic transvection* of a symplectic module P to be an automorphism of the form

$$\sigma(p) := p + \langle u, p\rangle v + \langle v, p\rangle u + \alpha\langle u, p\rangle u,$$

where $\alpha \in R$, $u, v \in P$ are fixed elements with $\langle u, v\rangle = 0$, and either u or v is unimodular. It is easy to check that $\langle\sigma(p), \sigma(q)\rangle = \langle p, q\rangle$ and σ has an inverse $\tau(p) = p - \langle u, p\rangle v - \langle v, p\rangle u - \alpha\langle u, p\rangle u$.

The subgroup of $\mathrm{Sp}(P, \langle,\rangle)$ generated by the symplectic transvections is denoted by $\mathrm{Trans}_{\mathrm{Sp}}(P, \langle,\rangle)$ (see [11], page 35).

Now onwards Q will denote $(R^2 \oplus P)$ with induced form on $(\mathbb{H}(R) \oplus P)$, and $Q[X]$ will denote $(R[X]^2 \oplus P[X])$ with induced form on $(\mathbb{H}(R[X]) \oplus P[X])$.

Definition 5.11. The symplectic transvections of $Q = (R^2 \oplus P)$ of the form

$$(a, b, p) \mapsto (a, b + \langle p, q\rangle + \alpha a, p + aq),$$

or of the form

$$(a, b, p) \mapsto (a - \langle p, q\rangle + \beta b, b, p + bq),$$

where $\alpha, \beta \in R$ and $q \in P$, are called *elementary symplectic transvections*. Let us denote the first isometry by $\rho(q, \alpha)$ and the second one by $\mu(q, \beta)$. It can be verified that the elementary symplectic transvections are symplectic transvections on Q. Indeed, consider $(u, v) = ((0, -1, 0), (0, 0, q))$ to get $\rho(q, -\alpha)$ and consider $(u, v) = ((-1, 0, 0), (0, 0, -q))$ to get $\mu(q, \beta)$.

The subgroup of $\mathrm{Trans}_{\mathrm{Sp}}(Q, \langle,\rangle)$ generated by elementary symplectic transvections is denoted by $\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle,\rangle)$.

Definition 5.12. Let I be an ideal of R. The group of *relative symplectic transvections* w.r.t. an ideal I is generated by the symplectic transvections of the form $\sigma(p) = p + \langle u, p\rangle v + \langle v, p\rangle u + \alpha\langle u, p\rangle u$, where $\alpha \in I$ and $u \in P$, $v \in IP$ are fixed elements with $\langle u, v\rangle = 0$. The group generated by relative symplectic transvections is denoted by $\mathrm{Trans}_{\mathrm{Sp}}(P, IP, \langle,\rangle)$.

Definition 5.13. Let I be an ideal of R. The elementary symplectic transactions of Q of the form $\rho(q, \alpha)$, $\mu(q, \beta)$, where $q \in IP$ and $\alpha, \beta \in I$ are called *relative elementary symplectic transvections* w.r.t. an ideal I.

The subgroup of $\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle,\rangle)$ generated by relative elementary symplectic transvections is denoted by $\mathrm{ETrans}_{\mathrm{Sp}}(IQ, \langle,\rangle)$. The normal closure of $\mathrm{ETrans}_{\mathrm{Sp}}(IQ, \langle,\rangle)$ in $\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle,\rangle)$ is denoted by $\mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle,\rangle)$.

Remark 5.14. Let $P = \oplus_{i=1}^{2n} Re_i$ be a free R-module. The non-degenerate alternating bilinear form $\langle,\rangle$ on P corresponds to an alternating matrix φ with Pfaffian 1 with respect to the basis $\{e_1, e_2, \ldots, e_{2n}\}$ of P and we write $\langle p, q\rangle = p\varphi q^t$.

In this case the symplectic transvection $\sigma(p) = p + \langle u, p\rangle v + \langle v, p\rangle u + \alpha\langle u, p\rangle u$ corresponds to the matrix $(I_{2n} + v^t u\varphi + u^t v\varphi)(I_{2n} + \alpha u^t u\varphi)$ and the group generated by them is denoted by $\mathrm{Trans}_{\mathrm{Sp}}(P, \langle,\rangle_\varphi)$.

Also, in this case $\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle,\rangle_{\psi_1 \perp \varphi})$ will be generated by the matrices of the form $\rho_{\psi_1\perp\varphi}(q, a) := \left(\begin{smallmatrix} 1 & 0 & 0 \\ a & 1 & -q\varphi \\ q^t & 0 & I_{2n} \end{smallmatrix}\right)$, and $\mu_{\psi_1\perp\varphi}(q, b) := \left(\begin{smallmatrix} 1 & b & q\varphi \\ 0 & 1 & 0 \\ 0 & q^t & I_{2n} \end{smallmatrix}\right)$.

Note that for $q = (q_1, \ldots, q_{2n}) \in R^{2n}$, and for the standard alternating matrix ψ_n, we have

$$\rho_{\psi_{n+1}}(q, a) = se_{21}(a) \prod_{i=3}^{2n+2} se_{i1}(q_{i-2}), \tag{6}$$

$$\mu_{\psi_{n+1}}(q, b) = se_{12}(b) \prod_{i=3}^{2n+2} se_{1i}((-1)^{i+1} q_{\sigma(i-2)}). \tag{7}$$

The following has been stated by L.N. Vaserstein (see pg. 650 of [13]). We prove it for the sake of completeness.

Lemma 5.15. *Let $(P, \langle,\rangle)$ be a symplectic R-module with P be a free module of rank $2n$, $n \geq 2$. Let us assume that the bilinear form $\langle,\rangle$ corresponds to the alternating matrix φ (w.r.t. some basis). If $\varphi = \psi_n$, the standard alternating matrix, then* $\mathrm{Trans}_{\mathrm{Sp}}(P, \langle,\rangle_{\psi_n}) = \mathrm{ESp}_{2n}(R)$.

Proof. When P is a free module and $\varphi = \psi_n$, an element of $\mathrm{Trans}_{\mathrm{Sp}}(P, \langle,\rangle_{\psi_n})$ looks like $(I_{2n} + v^t u\psi_{n+1} + u^t v\psi_{n+1})(I_{2n+2} + \alpha u^t u\psi_{n+1})$, for some $u, v \in R^{2n}$, and one of u or v is unimodular. Also, $\langle u, v\rangle = u\psi_n v^t = 0$. Hence using Lemma 2.8 and Lemma 2.9 we get $\mathrm{Trans}_{\mathrm{Sp}}(P, \langle,\rangle_{\psi_n}) \subseteq \mathrm{ESp}_{2n}(R)$.

Generators of $\mathrm{ESp}_{2n}(R)$ can be expressed as

$$\begin{aligned} se_{i\sigma(i)}(a) &= I + (-1)^{i+1} a e_i^t e_i \psi_{n+1}, \\ se_{ij}(b) &= I + (-1)^j b e_i^t e_{\sigma(j)} \psi_{n+1} + (-1)^j b e_{\sigma(j)}^t e_i \psi_{n+1}, \ j \neq \sigma(i) \end{aligned}$$

and hence $\mathrm{ESp}_{2n}(R) \subseteq \mathrm{Trans}_{\mathrm{Sp}}(P, \langle,\rangle_{\psi_n})$. Therefore, the equality follows. □

Lemma 5.16. *Let $(P, \langle,\rangle)$ be a symplectic R-module with P be a free module of rank $2n$, $n \geq 1$. Let Q denote $(R^2 \oplus P)$ with the induced form on $(\mathbb{H}(R) \oplus P)$. Let us assume that the bilinear form $\langle,\rangle$ corresponds to the alternating matrix φ (w.r.t. some basis). If $\varphi = \psi_n$, the standard alternating matrix, then* $\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle,\rangle_{\psi_{n+1}}) = \mathrm{ESp}_{2n+2}(R)$.

Proof. Using equations (6) and (7) we get $\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle,\rangle_{\psi_{n+1}}) \subseteq \mathrm{ESp}_{2n+2}(R)$. Note that using equations (3), (4), and (5) we can show that $\mathrm{ESp}_{2n}(R)$ is generated by elements of the form $se_{1i}(a)$ and $se_{j1}(b)$, for $a, b \in R$. Hence $\mathrm{ESp}_{2n}(R) \subseteq \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle,\rangle_{\psi_{n+1}})$. Therefore, the equality follows. □

Remark 5.17. A relative version of the above result w.r.t. an ideal I of the ring R is also true, i.e., $\mathrm{Trans}_{\mathrm{Sp}}(Q, IQ, \langle,\rangle_{\psi_{n+1}}) = \mathrm{ESp}_{2n+2}(R, I)$ and $\mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle,\rangle_{\psi_{n+1}}) = \mathrm{ESp}_{2n+2}(R, I)$. For details one can see ([4], Lemma 5.13 and Lemma 5.14).

Lemma 5.18. *Let $(P, \langle,\rangle)$ be a symplectic R-module with P be a free module of rank $2n$, $n \geq 1$. Let Q denote $(R^2 \oplus P)$ with the induced form on $(\mathbb{H}(R) \oplus P)$. Let us assume that the bilinear form $\langle,\rangle$ corresponds to the alternating matrix φ (w.r.t. some basis). Then* $\mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle,\rangle_{\psi_1 \perp \varphi}) = \mathrm{ESp}_{\psi_1 \perp \varphi}(R)$.

Proof. Let us fix $\varphi' = \psi_1 \perp \varphi$. Note that $\varphi' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \varphi \end{pmatrix}$, and $\varphi'^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \varphi^{-1} \end{pmatrix}$. Set $v = (a, q)$ where $a \in R$ and $q = (q_1, \ldots, q_{2n}) \in R^{2n}$. We have $C_{\varphi'}(v) = \begin{pmatrix} 1 & 0 \\ v^t & \alpha_{\varphi'}(v) \end{pmatrix}$, where $\alpha_{\varphi'}(v) = \begin{pmatrix} 1 & -q\varphi \\ 0 & I_{2n} \end{pmatrix}$. Therefore,

$$C_{\varphi'}(v) = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & -q\varphi \\ q^t & 0 & I_{2n} \end{pmatrix} = \rho_{\varphi'}(q, a). \tag{8}$$

Now we set $w = (b, q\varphi)$. We have $R_{\varphi'}(w) = \begin{pmatrix} 1 & w \\ 0 & \beta_{\varphi'}(w) \end{pmatrix}$, where $\beta_{\varphi'}(w) = \begin{pmatrix} 1 & 0 \\ q^t & I_{2n} \end{pmatrix}$. Therefore,

$$R_{\varphi'}(w) = \begin{pmatrix} 1 & b & q\varphi \\ 0 & 1 & 0 \\ 0 & q^t & I_{2n} \end{pmatrix} = \mu_{\varphi'}(q, b), \tag{9}$$

and hence the equality follows. $\square$

Lemma 5.19. *Let $(P, \langle,\rangle)$ be a symplectic R-module with P be a free module of rank $2n$, $n \geq 1$. Let I be an ideal of R. Let Q denote $(R^2 \oplus P)$ with the induced form on $(\mathbb{H}(R) \oplus P)$. Let us assume that the bilinear form $\langle,\rangle$ corresponds to the alternating matrix φ (w.r.t. some basis). Then* $\mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle,\rangle_{\psi_1 \perp \varphi}) = \mathrm{ESp}_{\psi_1 \perp \varphi}(R, I)$.

Proof. Follows from the definitions of $\mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle,\rangle_{\psi_1 \perp \varphi})$, $\mathrm{ESp}_{\psi_1 \perp \varphi}(R, I)$, and equations (8), (9). $\square$

6. $\mathrm{ESp}_{\varphi}(R)$ in the non-free case

We have recalled definitions of elementary linear group $\mathrm{E}_n(R)$, elementary symplectic group $\mathrm{ESp}_{2n}(R)$, transvection group of a finitely generated R-module $\mathrm{Trans}(M)$, and symplectic transvection group of a symplectic R-module of even rank $\mathrm{Trans}_{\mathrm{Sp}}(P, \langle,\rangle)$. Note that when P is a free module, the bilinear form $\langle,\rangle$ corresponds to an invertible alternating matrix, say φ (w.r.t. some basis). In this case we denote the symplectic transvection group as $\mathrm{Trans}_{\mathrm{Sp}}(P, \langle , \rangle_{\varphi})$. In [4] we have observed that when M is a free R-module of rank bigger than or equal to 3, the transvection group $\mathrm{Trans}(M)$ coincide with the elementary linear group $\mathrm{E}_n(R)$ (see Lemma 5.5). We have also observed when P is a free R-module of rank $2n$, where n is bigger than or equal to 2, and when the bilinear form corresponds to the standard alternating matrix ψ_n of size $2n$, then the symplectic transvection group $\mathrm{Trans}_{\mathrm{Sp}}(P, \langle , \rangle_{\psi_n})$ coincide with the elementary symplectic group $\mathrm{ESp}_{2n}(R)$ (see Lemma 5.15). A relative version of these two results with respect to an ideal of the ring R have also been established.

Due to the above mentioned results $\mathrm{Trans}(M)$ and $\mathrm{Trans}_{\mathrm{Sp}}(P, \langle , \rangle)$ can be considered as generalisation of $\mathrm{E}_n(R)$ and $\mathrm{ESp}_{2n}(R)$ respectively in the case of projective modules. Here we define a group which can be considered as generalisation of $\mathrm{ESp}_{\varphi}(R)$ in the case of projective modules.

Definition 6.1. Let $(P, \langle , \rangle)$ be a symplectic R-module with P finitely generated projective module of even rank. Recall that $\mathrm{Sp}(P, \langle , \rangle)$ is the group of isometries. We define $V(P, \langle , \rangle)$ to be the collection of all

$$\{\alpha(1) : \alpha(X) \in \mathrm{Sp}(P[X], \langle , \rangle_{\otimes R[X]}), \alpha(0) = id., \text{ and}$$
$$\alpha(X)_{\mathfrak{p}} \in \mathrm{ESp}_{\varphi \otimes R_{\mathfrak{p}}[X]}(R_{\mathfrak{p}}[X]), \text{ for } \quad \text{all } \mathfrak{p} \in \mathit{Spec}(R)\},$$

where $\langle , \rangle$ corresponds to an alternating matrix φ (w.r.t. some basis) of Pfaffian 1 over the local ring $R_{\mathfrak{p}}$ at the prime ideal $\mathfrak{p}$. *Note that this is independent of the choice of local basis chosen in view of the normality results proved in [6] and Lemma 3.7.*

Definition 6.2. Let $(P, \langle , \rangle)$ be a symplectic R-module with P finitely generated projective module of even rank. Let I be an ideal of R. We define $V(P, IP, \langle , \rangle)$ to be the collection of all

$$\{\alpha(1) : \alpha(X) \in \mathrm{Sp}(P[X], \langle , \rangle_{\otimes R[X]}), \alpha(0) = id., \text{ and}$$
$$\alpha(X)_{\mathfrak{p}} \in \mathrm{ESp}_{\varphi \otimes R_{\mathfrak{p}}[X]}(R_{\mathfrak{p}}[X], I_{\mathfrak{p}}[X]), \text{ for } \quad \text{all } \mathfrak{p} \in \mathit{Spec}(R)\},$$

where $\langle , \rangle$ correspond to an alternating matrix φ (w.r.t. some basis) of Pfaffian 1 over the local ring $R_{\mathfrak{p}}$ at the prime ideal $\mathfrak{p}$ and $\varphi \equiv \psi_n (\mathrm{mod}\ I)$. *Note that this is independent of the choice of local basis chosen in view of the normality results proved in [6] and Lemma 3.8.*

Remark 6.3. Let $P = \oplus_{i=1}^{2n} Re_i$ be a free R-module. The non-degenerate alternating bilinear form $\langle , \rangle$ on P corresponds to an alternating matrix φ with Pfaffian 1 with respect to the basis $\{e_1, e_2, \ldots, e_{2n}\}$ of P and we write $\langle p, q \rangle = p\varphi q^t$. In this case the above two groups are denoted by $V(P, \langle , \rangle_{\varphi})$ and $V(P, IP, \langle , \rangle_{\varphi})$ respectively.

Next we will prove that the above defined group is same as the group of elementary symplectic transvections defined by Bass.

Theorem 6.4. *Let $(P, \langle,\rangle)$ be a symplectic R-module with P finitely generated projective module of even rank $2n$, $n \geq 1$. Let Q denote $(R^2 \oplus P)$ with the induced form on $(\mathbb{H}(R) \oplus P)$. Then* $V(Q, \langle,\rangle) = \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle,\rangle)$.

Next theorem is a relative version of the above theorem w.r.t. an ideal I of the ring R. The above theorem can be deduced as a particular case of the below when $I = R$.

Theorem 6.5. *Let $(P, \langle,\rangle)$ be a symplectic R-module with P finitely generated projective module of even rank $2n$, $n \geq 1$. Let I be an ideal of R. Let Q denote $(R^2 \oplus P)$ with the induced form on $(\mathbb{H}(R) \oplus P)$. Then* $V(Q, IQ, \langle,\rangle) = \mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle,\rangle)$.

Proof. Let us choose a $\theta \in V(Q, IQ, \langle,\rangle)$. By definition there exists a $\theta(X)$ in $\mathrm{Sp}(Q[X], \langle,\rangle_{\otimes R[X]})$ such that $\theta(1) = \theta$. Also, $\theta(0) = id.$, and for all $\mathfrak{p} \in Spec(R)$ we have $\theta(X)_\mathfrak{p} \in \mathrm{ESp}_{(\psi_1 \perp \varphi) \otimes R_\mathfrak{p}[X]}(R_\mathfrak{p}[X], I_\mathfrak{p}[X])$. Note $\mathrm{ESp}_{(\psi_1 \perp \varphi) \otimes R_\mathfrak{p}[X]}(R_\mathfrak{p}[X], I_\mathfrak{p}[X]) = \mathrm{ETrans}_{\mathrm{Sp}}(Q[X], IQ[X], \langle,\rangle_{(\psi_1 \perp \varphi) \otimes R_\mathfrak{p}[X]})$ by Lemma 5.19. Hence for all $\mathfrak{p} \in Spec(R)$, $\theta(X)_\mathfrak{p} \in \mathrm{ETrans}_{\mathrm{Sp}}(Q[X], IQ[X], \langle,\rangle_{(\psi_1 \perp \varphi) \otimes R_\mathfrak{p}[X]})$. Therefore, by the local–global principle $\theta(X)$ belongs to $\mathrm{ETrans}_{\mathrm{Sp}}(Q[X], IQ[X], \langle,\rangle)$ (see [4], Lemma 5.20). Substituting $X = 1$ we get $\theta = \theta(1) \in \mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle,\rangle)$, hence $V(Q, IQ, \langle,\rangle) \subseteq \mathrm{ETrans}_{\mathrm{Sp}}(Q, \langle,\rangle)$.

To show the other inclusion let us choose a δ from $\mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle,\rangle)$. We can find an element $\delta(X) \in \mathrm{ETrans}_{\mathrm{Sp}}(Q[X], IQ[X], \langle,\rangle_{\otimes R[X]})$ such that $\delta = \delta(1)$ and $\delta(0) = id.$ (see [4], Lemma 5.22). Note that $R_\mathfrak{p}$ is a local ring for each $\mathfrak{p} \in Spec(R)$. Over the local ring $Q_\mathfrak{p} \cong R_\mathfrak{p}^{2n+2}$ and the bilinear form $\langle,\rangle$ correspond to $\psi_1 \perp \varphi$ (w.r.t. some basis) of Pfaffian 1. Hence, $\delta(X)_\mathfrak{p} \in \mathrm{ETrans}_{\mathrm{Sp}}(Q_\mathfrak{p}[X], IQ[X], \langle,\rangle_{(\psi_1 \perp \varphi) \otimes R_\mathfrak{p}[X]}) = \mathrm{ESp}_{(\psi_1 \perp \varphi) \otimes R_\mathfrak{p}[X]}(R_\mathfrak{p}[X], I_\mathfrak{p}[X])$, for all $\mathfrak{p} \in Spec(R)$. Therefore, by definition $\delta(X) \in V(Q[X], IQ[X], \langle,\rangle)$. Substituting $X = 1$ we get, $\delta = \delta(1) \in V(Q, IQ, \langle,\rangle)$, and the equality follows. $\square$

7. Equality of orbits

Theorem 7.1. *Let $n \geq 2$ and φ be an alternating matrix of size $2n$ of Pfaffian 1. Then the orbit spaces* $\mathrm{Um}_{2n}(R)/\mathrm{E}_{2n}(R)$, $\mathrm{Um}_{2n}(R)/\mathrm{ESp}_{2n}(R)$, *and* $\mathrm{Um}_{2n}(R)/\mathrm{ESp}_\varphi(R)$ *are bijective.*

Next we will prove a relative version of the above theorem w.r.t. an ideal I of the ring R. The above theorem can be deduced as a particular case of the below.

Theorem 7.2. *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $n \geq 2$ and φ be an alternating matrix of size $2n$ of Pfaffian 1 such that $\varphi \equiv \psi_n \pmod{I}$. Then the orbit spaces* $\mathrm{Um}_{2n}(R, I)/\mathrm{E}_{2n}(R, I)$, $\mathrm{Um}_{2n}(R, I)/\mathrm{ESp}_{2n}(R, I)$, *and* $\mathrm{Um}_{2n}(R, I)/\mathrm{ESp}_\varphi(R, I)$ *are bijective.*

Proof. In ([3], Theorem 5.6) it is shown that the natural map between the orbit spaces $\mathrm{Um}_{2n}(R,I)/\mathrm{ESp}_{2n}(R,I) \longrightarrow \mathrm{Um}_{2n}(R,I)/\mathrm{E}_{2n}(R,I)$ is bijective, for $n \geq 3$. The case when $n = 2$ was done in ([4], Theorem 7.11).

We will show the natural map $\mathrm{Um}_{2n}(R,I)/\mathrm{ESp}_{\varphi}(R,I) \longrightarrow \mathrm{Um}_{2n}(R,I)/\mathrm{E}_{2n}(R,I)$ is bijective. It is easy to see that the map is surjective. Let us choose $v, w \in \mathrm{Um}_{2n}(R,I)$ and $g \in \mathrm{E}_{2n}(R,I)$ such that $vg = w$, i.e., w is in the same $\mathrm{E}_{2n}(R,I)$-orbit of v. To show injectivity of the map we need to show that w is in the same $\mathrm{ESp}_{\varphi}(R,I)$-orbit of v.

Let $\mathfrak{m}$ be a maximal ideal on R and $R_{\mathfrak{m}}$ be the local ring at $\mathfrak{m}$. Given that $\varphi \equiv \psi_n \pmod{I}$. Hence over the local ring $R_{\mathfrak{m}}$ we have $\varphi = (1 \perp \varepsilon)^t \psi_n (1 \perp \varepsilon)$, for some $\varepsilon \in \mathrm{E}_{2n-1}(R_{\mathfrak{m}}, I_{\mathfrak{m}})$ (see [4], Lemma 5.2). Therefore, $\mathrm{ESp}_{\varphi}(R_{\mathfrak{m}}, I_{\mathfrak{m}}) = (1 \perp \varepsilon)^{-1}\mathrm{ESp}_{2n}(R_{\mathfrak{m}}, I_{\mathfrak{m}})(1 \perp \varepsilon)$ (see Lemma 3.5 and Lemma 3.7) Note that $g \in \mathrm{E}_{2n}(R,I)$. Therefore, we can write $g = \prod_{k=1}^{t} \gamma_k E_{i_k j_k}(a_k)\gamma_k^{-1}$, where $\gamma_k \in \mathrm{E}_{2n}(R)$ and $a_k \in I$. Let us set $V(X_1, \ldots, X_t) = v \prod_{k=1}^{t} \gamma_k E_{i_k j_k}(X_k)\gamma_k^{-1}$ and $W(X_1, \ldots, X_t) = V(X_1, \ldots, X_t)(1 \perp \varepsilon)^{-1}$. Note that

$$\begin{aligned} W(X_1, \ldots, X_t) &\in W(0, \ldots, 0)\ \mathrm{E}_{2n}(R_{\mathfrak{m}}[X_1, \ldots, X_t], I_{\mathfrak{m}}[X_1, \ldots, X_t]) \\ &= W(0, \ldots, 0)\ \mathrm{ESp}_{2n}(R_{\mathfrak{m}}[X_1, \ldots, X_t], I_{\mathfrak{m}}[X_1, \ldots, X_t]), \end{aligned}$$

for all maximal ideals $\mathfrak{m}$ of R (see [3], Theorem 5.5). This implies

$$\begin{aligned} V(X_1, \ldots, X_t) &\in V(0, \ldots, 0)(1 \perp \varepsilon)^{-1}\mathrm{ESp}_{2n}(R_{\mathfrak{m}}[X_1, \ldots, X_t], I_{\mathfrak{m}}[X_1, \ldots, X_t])(1 \perp \varepsilon) \\ &= V(0, \ldots, 0)\mathrm{ESp}_{\varphi}(R_{\mathfrak{m}}[X_1, \ldots, X_t], I_{\mathfrak{m}}[X_1, \ldots, X_t]), \end{aligned}$$

for all maximal ideals $\mathfrak{m}$ of R (see Lemma 3.8). Therefore,

$$V(X_1, \ldots, X_t) \in V(0, \ldots, 0)\mathrm{ESp}_{\varphi}(R[X_1, \ldots, X_t], I[X_1, \ldots, X_t]),$$

by Lemma 4.8. Substituting $(X_1, \ldots, X_t) = (a_1, \ldots, a_t)$ we get $vg \in v\mathrm{ESp}_{\varphi}(R,I)$, and hence w belongs to the same $\mathrm{ESp}_{\varphi}(R,I)$-orbit of v. □

Next theorem is an analogue of the above theorem in the case of projective modules.

Theorem 7.3. *Let R be a commutative ring with $R = 2R$, and let I be an ideal of R. Let $(P, \langle , \rangle)$ be a symplectic R-module with P finitely generated projective module of rank $2n$, $n \geq 1$. Let $Q = R^2 \oplus P$ with induced form on $(\mathbb{H}(R) \oplus P)$. We also assume for each maximal ideal $\mathfrak{m}$ of R, the alternating form $\langle , \rangle$ over the local ring $R_{\mathfrak{m}}$ corresponds to an alternating matrix $\varphi_{\mathfrak{m}}$ (w.r.t. some basis), such that $\varphi_{\mathfrak{m}} \equiv \psi_n (\mathrm{mod}\ I)$. Then the orbit spaces $\mathrm{Um}(Q, IQ)/\mathrm{ETrans}(Q, IQ)$, $\mathrm{Um}(Q, IQ)/\mathrm{ETrans}_{\mathrm{Sp}}(Q, IQ, \langle , \rangle)$, and $\mathrm{Um}(Q, IQ)/V(Q, IQ, \langle , \rangle)$ are bijective.*

Proof. Follows from ([4], Theorem 6.1) and Theorem 6.5. □

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