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Closure operations that induce big Cohen–Macaulay modules and classification of singularities

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Geoffrey Dietz introduced a set of axioms for a closure operation on a complete local domain R so that the existence of such a closure operation is equivalent to the existence of a big Cohen–Macaulay module. These closure operations are called Dietz closures. In complete rings of characteristic $p > 0$, tight closure and plus closure satisfy the axioms.

We define module closures and discuss their properties. For many of these properties, there is a smallest closure operation satisfying the property. In particular, we discuss properties of big Cohen–Macaulay module closures, and prove that every Dietz closure is contained in a big Cohen–Macaulay module closure. Using this result, we show that under mild conditions, a ring R is regular if and only if all Dietz closures on R are trivial. Finally, we show that solid closure in equal characteristic 0, integral closure, and regular closure are not Dietz closures, and that all Dietz closures are contained in liftable integral closure.

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1. Introduction

The question of the existence of big Cohen–Macaulay modules has motivated many results in commutative algebra. While they are known to exist over rings of equal characteristic [16] and rings of mixed characteristic and dimension at most 3 [7,20], it is not known whether they exist over mixed characteristic rings of higher dimension. The existence of big Cohen–Macaulay modules (or algebras) is also sufficient to imply a large group of equivalent conjectures, including the Direct Summand Conjecture [15], Monomial Conjecture [15], and Canonical Element Conjecture [18]. The equal characteristic case of these results was achieved using tight closure methods. One obstruction to extending these techniques to rings of mixed characteristic is the lack of such a closure operation.

In [2], Dietz gave a list of axioms for a closure operation such that for a local domain R, the existence of a closure operation satisfying these properties (which we call a Dietz closure) is equivalent to the existence of a big Cohen–Macaulay module. The closure operation can be used to show that when module modifications (see Definition 2.12) are applied to R, the image of 1 in the resulting module is not contained in the image of the maximal ideal of R. When R is complete and has characteristic $p > 0$, tight closure is a Dietz closure, as are plus closure and solid closure [2]. However, Frobenius closure is not a Dietz closure [2].

In Section 3, we develop some basic properties of closure operations that are used throughout the paper, including properties of big Cohen–Macaulay module closures (see Definition 2.3). This is followed in Section 4 by a discussion of properties of closure operations for which there is a smallest closure satisfying the property. In particular, any ring that has a Dietz closure has a smallest Dietz closure, as well as a smallest big Cohen–Macaulay module closure. In certain rings of dimension 2, the smallest big Cohen–Macaulay module closure comes from the S_2-ification of R. Studying the smallest Dietz closure or big Cohen–Macaulay module closure should provide information on the properties of R.

We prove:

Theorem 1 *(Theorem 5.1). Let cl be a Dietz closure on a local domain (R, m). Then cl is contained in the module closure cl_B for some big Cohen–Macaulay module B, i.e., for any finitely-generated R-modules $N \subseteq M$, $N_M^{\mathrm{cl}} \subseteq N_M^{\mathrm{cl}_B}$.*

Using this result, we prove:

Theorem 2 *(Theorem 5.9, Theorem 5.10). Suppose that (R, m) is a local domain that has at least one Dietz closure (in particular, it suffices for R to have equal characteristic and any dimension, or mixed characteristic and dimension at most 3). Then R is regular if and only if all Dietz closures on R are trivial.*

In the proof of Theorem 5.10, we see that a particular module of syzygies gives a nontrivial closure operation, which we can compute explicitly. In Section 8, we use these results to compare Dietz closures to better understand closure operations such as integral closure.

In Theorem 6.1, we show that integral closure and regular closure are not Dietz closures using a criterion that can be applied more generally. As a corollary of the above theorems, we also conclude that solid closure is not always a Dietz closure for rings of equal characteristic 0. Studying the reasons why certain closure operations are or are not Dietz closures provides more information on the pieces that are needed to get a good enough closure operation in mixed characteristic.

We conclude with a list of further questions in Section 9. Interestingly, we do not know whether there is a largest big Cohen–Macaulay module closure, as discussed in Section 9.2.

2. Background

In this section we give the necessary definitions and some notation that will be used throughout the paper.

Definition 2.1. Let (R, m) be a local ring. An R-module B is a (balanced) *big Cohen–Macaulay module* for R if every system of parameters for R is a regular sequence on B, and $mB \neq B$. Note that B need not be finitely-generated. A *big Cohen–Macaulay algebra* for R is a big Cohen–Macaulay module for R that is also an R-algebra.

Definition 2.2. A *closure operation* cl on a ring R is a map $N \to N_M^{\mathrm{cl}}$ of submodules N of finitely-generated R-modules M such that if $N \subseteq N' \subseteq M$ are finitely-generated R-modules,

1. (Extension) $N \subseteq N_M^{\mathrm{cl}}$,
2. (Idempotence) $(N_M^{\mathrm{cl}})_M^{\mathrm{cl}} = N_M^{\mathrm{cl}}$, and
3. (Order-Preservation) $N_M^{\mathrm{cl}} \subseteq (N')_M^{\mathrm{cl}}$.

Definition 2.3. Suppose that S is an R-module (resp. R-algebra). We can define a closure operation cl_S on R by

$$u \in N_M^{\mathrm{cl}_S} \text{ if for all } s \in S,\ s \otimes u \in \mathrm{im}(S \otimes N \to S \otimes M)$$

for any $N \subseteq M$ finitely-generated R-modules and $u \in M$. This is called a *module (resp. algebra) closure.*

Remark 2.4. If S is an R-algebra, $u \in N_M^{\mathrm{cl}_S}$ if and only if

$$1 \otimes u \in \mathrm{im}(S \otimes N \to S \otimes M).$$

Definition 2.5 *([2, Definition 2.2])*. Let R be a ring with a closure operation cl, M a finitely-generated R-module, and $\alpha : R \to M$ an injective map with cokernel Q. We have a short exact sequence

$$0 \longrightarrow R \xrightarrow{\alpha} M \longrightarrow Q \longrightarrow 0.$$

Let $\epsilon \in \operatorname{Ext}^1_R(Q, R)$ be the element corresponding to this short exact sequence via the Yoneda correspondence. We say that α is a *cl-phantom extension* if a cocycle representing ϵ is in $P_1^\vee$ is in $\operatorname{im}(P_0^\vee \to P_1^\vee)^{\mathrm{cl}}_{P_1^\vee}$, where $P_\bullet$ is a projective resolution of Q and $\vee$ denotes $\operatorname{Hom}_R(-, R)$.

Remark 2.6. This definition is independent of the choice of $P_\bullet$ [2, Discussion 2.3].

A split map $\alpha : R \to M$ is cl-phantom for any closure operation cl: in this case, the cocycle representing ϵ is in $\operatorname{im}(P_0^\vee \to P_1^\vee)$. We can view cl-phantom extensions as maps that are "almost split" with respect to a particular closure operation.

Notation 2.7. We use some notation from [2]. Let R be a ring, M a finitely generated R-module, and $\alpha : R \to M$ an injective map with cokernel Q. Let $e_1 = \alpha(1), e_2, \ldots, e_n$ be generators of M such that the images of $e_2, \ldots, e_n$ in Q form a generating set for Q. We have a free presentation for Q,

$$R^m \xrightarrow{\nu} R^{n-1} \xrightarrow{\mu} Q \longrightarrow 0,$$

where μ sends the generators of R^{n-1} to $e_2, \ldots, e_n$ and ν has matrix $(b_{ij})_{2 \le i \le n, 1 \le j \le m}$ with respect to some basis for R^m. We have a corresponding presentation for M,

$$R^m \xrightarrow{\nu_1} R^n \xrightarrow{\mu_1} M \longrightarrow 0,$$

where μ_1 sends the generators of R^n to $e_1, \ldots, e_n$. Using the same basis for R^m as above, ν_1 has matrix $(b_{ij})_{1 \le i \le n, 1 \le j \le m}$ where $b_{1j}e_1 + b_{2j}e_2 + \ldots + b_{nj}e_n = 0$ in M [2, Discussion 2.4]. The top row of ν_1 gives a matrix representation of the map $\phi : R^m \to R$ in the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\alpha"] & M \arrow[r] & Q \arrow[r] & 0 \\
 & R^m \arrow[u, "\phi"] \arrow[r, "\nu"] & R^{n-1} \arrow[u, "\psi"] \arrow[r, "\mu"] & Q \arrow[u, "\mathrm{id}_Q"] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

In [2, Discussion 2.4], Dietz gives an equivalent definition of a phantom extension using the free presentations M and Q given above. While he assumes that R is a complete local domain and that cl satisfies 2 additional properties, these are not needed for all of the results. We restate some of his results in greater generality below.

Lemma 2.8 *([2, Lemma 2.10]). Let R be a ring possessing a closure operation cl. Let M be a finitely generated module, and let $\alpha : R \to M$ be an injective map. Let notation be as above. Then α is a cl-phantom extension of R if and only if the vector $(b_{11}, \ldots, b_{1m})^{\mathrm{tr}}$ is in $B^{\mathrm{cl}}_{R^m}$, where B is the R-span in R^m of the vectors $(b_{i1}, \ldots, b_{im})^{tr}$ for $2 \le i \le n$.*

Definition 2.9 *([2]).* Let (R, m) be a fixed local domain and let N, M, and W be arbitrary finitely generated R-modules with $N \subseteq M$. A closure operation cl is called a *Dietz closure* if the following axioms hold:

1. (Functoriality) Let $f : M \to W$ be a homomorphism. Then $f(N^{\mathrm{cl}}_M) \subseteq f(N)^{\mathrm{cl}}_W$.
2. (Semi-residuality) If $N^{\mathrm{cl}}_M = N$, then $0^{\mathrm{cl}}_{M/N} = 0$.
3. (Faithfulness) The ideal m is closed in R.
4. (Generalized Colon-Capturing) Let $x_1, \ldots, x_{k+1}$ be a partial system of parameters for R, and let $J = (x_1, \ldots, x_k)$. Suppose that there exists a surjective homomorphism $f : M \to R/J$ and $v \in M$ such that $f(v) = x_{k+1} + J$. Then $(Rv)^{\mathrm{cl}}_M \cap \ker f \subseteq (Jv)^{\mathrm{cl}}_M$.

Remark 2.10. The axioms originally included the assumption that $0^{\mathrm{cl}}_R = 0$, but this is implied by the other axioms [3].

A closure operation on any ring R can satisfy the Functoriality Axiom, the Semi-residuality Axiom, or both. A closure operation on any local ring R can satisfy the Faithfulness Axiom.

The proof of the next lemma requires Q to have a minimal generating set, so we assume that R is local for this generalization of [2, Lemma 2.11]:

Lemma 2.11. *Let (R, m) be a local ring possessing a closure operation* cl *that satisfies the Functoriality Axiom, the Semi-residuality Axiom, and the Faithfulness Axiom. If M is a finitely generated R-module such that $\alpha : R \to M$ is cl-phantom, then $\alpha(1) \notin mM$.*

Definition 2.12 *([13, Discussion 5.15]).* Let R be local and M an R-module. A *parameter module modification* of M is a map

$$M \to M' = \frac{M \oplus Rf_1 \oplus \ldots \oplus Rf_k}{R(u \oplus x_1 f_1 \oplus \ldots \oplus x_k f_k)},$$

where $x_1, \ldots, x_{k+1}$ is part of a system of parameters for R and $u_1, \ldots, u_k, u$ are elements of M such that

$$x_{k+1}u = x_1 u_1 + \ldots + x_k u_k.$$

Remark 2.13. Dietz proves in [2] that a local domain R has a Dietz closure if and only if it has a big Cohen–Macaulay module. In his proof that a Dietz closure can be used to construct a big Cohen–Macaulay module, one could replace the Generalized

Colon-Capturing Axiom with any axiom that implies that given a cl-phantom extension $\alpha : R \to M$ and a parameter module modification $M \to M'$, the map $R \to M'$ is still a cl-phantom extension. However, we do not know of a good candidate to replace the Axiom.

3. Properties of closure operations

We list some properties of closure operations that will be needed later.

Lemma 3.1. *Let R be a ring possessing a closure operation cl. In the following, N, N', and $N_i \subseteq M_i$ are all R-submodules of the finitely generated R-module M.*

- *(a) Suppose that cl satisfies the Functoriality Axiom and the Semi-residuality Axiom. Let $N' \subseteq N \subseteq M$. Then $u \in N_M^{\mathrm{cl}}$ if and only if $u + N' \in (N/N')^{\mathrm{cl}}_{M/N'}$.*
- *(b) Suppose that cl satisfies the Functoriality Axiom, $\mathcal{I}$ is a finite set, $N = \bigoplus_{i\in\mathcal{I}} N_i$, and $M = \bigoplus_{i\in\mathcal{I}} M_i$. Then $N_M^{\mathrm{cl}} = \bigoplus_{i\in\mathcal{I}} (N_i)^{\mathrm{cl}}_{M_i}$.*
- *(c) Let $\mathcal{I}$ be any set. If $N_i \subseteq M$ for all $i \in \mathcal{I}$, then $\left(\bigcap_{i\in\mathcal{I}} N_i\right)^{\mathrm{cl}}_M \subseteq \bigcap_{i\in\mathcal{I}} (N_i)^{\mathrm{cl}}_{M_i}$.*
- *(d) Let $\mathcal{I}$ be any set. If N_i is cl-closed in M for all $i \in \mathcal{I}$, then $\bigcap_{i\in\mathcal{I}} N_i$ is cl-closed in M.*
- *(e) $(N_1 + N_2)^{\mathrm{cl}}_M = \left((N_1)^{\mathrm{cl}}_M + (N_2)^{\mathrm{cl}}_M\right)^{\mathrm{cl}}_M$.*
- *(f) Suppose that cl satisfies the Functoriality Axiom. Let $N \subseteq N' \subseteq M$. Then $N^{\mathrm{cl}}_{N'} \subseteq N^{\mathrm{cl}}_M$.*
- *(g) Suppose that R is a domain, cl satisfies the Functoriality Axiom, $0^{\mathrm{cl}}_R = 0$, and M is a torsion-free R-module. Then $0^{\mathrm{cl}}_M = 0$.*
- *(h) Suppose that (R, m) is local and cl satisfies the Functoriality Axiom, the Semi-residuality Axiom, and the Faithfulness Axiom. Then $N^{\mathrm{cl}}_M \subseteq N + mM$.*

Proof. Parts (a) to (e) are proved in [2, Lemma 1.2].

For part (f), let $f : N' \to M$ be the inclusion map. Then by the Functoriality Axiom,

$$N^{\mathrm{cl}}_{N'} = f(N^{\mathrm{cl}}_{N'}) \subseteq f(N)^{\mathrm{cl}}_M = N^{\mathrm{cl}}_M.$$

For part (g), notice that $M \hookrightarrow R^s$ for some $s > 0$. By part (f), $0^{\mathrm{cl}}_M \subseteq 0^{\mathrm{cl}}_{R^s}$. By part (b), $0^{\mathrm{cl}}_{R^s} = \bigoplus 0^{\mathrm{cl}}_R = 0$.

For part (h), we first prove that for F a finitely-generated free module, $(mF)^{\mathrm{cl}}_F = mF$. By part (a), this is equivalent to $0^{\mathrm{cl}}_{F/mF} = 0$. Let $u \in 0^{\mathrm{cl}}_{F/mF}$ be nonzero. Then there exists a map $\phi : F/mF \to R/m$ with $\phi(u) \neq 0$. By the Functoriality Axiom, $\phi(u) \in 0^{\mathrm{cl}}_{R/m} = 0$ (since $m^{\mathrm{cl}}_R = m$), which is a contradiction. Hence $0^{\mathrm{cl}}_{F/mF} = 0$.

By part (a), it suffices to show that $0^{\mathrm{cl}}_M \subseteq mM$. Let

$$F_1 \longrightarrow F_0 \xrightarrow{\pi} M \longrightarrow 0$$

be part of a minimal free resolution of M. Then $\operatorname{im}(F_1) \subseteq mF_0$. This implies that $\operatorname{im}(F_1)^{\mathrm{cl}}_{F_0} \subseteq (mF_0)^{\mathrm{cl}}_{F_0} = mF_0$. By part (a), $0^{\mathrm{cl}}_M = \pi(\operatorname{im}(F_1)^{\mathrm{cl}}_{F_0})$. We have

$$0^{\mathrm{cl}}_M = \pi(\operatorname{im}(F_1)^{\mathrm{cl}}_{F_0}) \subseteq \pi(mF_0) = m\pi(F_0) = mM,$$

as desired. □

Lemma 3.2. *Let R be a ring and S an R-module or R-algebra. Then cl_S satisfies the Functoriality Axiom and the Semi-residuality Axiom. Hence cl_S has properties (a)–(f) of Lemma 3.1. Further, for $N \subseteq M$ finitely generated R-modules, cl_S satisfies*

$$I^{\mathrm{cl}_S} N^{\mathrm{cl}_S}_M \subseteq (IN)^{\mathrm{cl}_S}_M$$

for all $I \subseteq R$. In particular, $yN^{\mathrm{cl}_S}_M \subseteq (yN)^{\mathrm{cl}_S}_M$ for all $y \in R$.

Remark 3.3. If R is a domain, then this Lemma implies that cl_S is semi-prime as in [5].

Proof. First we show that cl_S satisfies the Functoriality Axiom and the Semi-residuality Axiom. Suppose that $N \subseteq M$ and W are finitely generated R-modules, and $f : M \to W$ is an R-module homomorphism. Let $u \in N^{\mathrm{cl}_S}_M$. We will show that $f(u) \in f(N)^{\mathrm{cl}_S}_W$. For every $s \in S$, $s \otimes u \in \operatorname{im}(S \otimes N \to S \otimes M)$. Applying $\mathrm{id}_S \otimes_R f$, we get $s \otimes f(u) \in \operatorname{im}(S \otimes f(N) \to S \otimes W)$ for every $s \in S$. So cl_S satisfies the Functoriality Axiom.

Suppose $N^{\mathrm{cl}_S}_M = N$. We will show that $0^{\mathrm{cl}_S}_{M/N} = 0$. Let $\bar{u} \in 0^{\mathrm{cl}_S}_{M/N}$. Then for every $s \in S$, $s \otimes \bar{u} = 0$ in $S \otimes M/N$. Since $S \otimes \_$ is right exact, $S \otimes M/N \cong (S \otimes M)/(S \otimes N)$. Thus $s \otimes u \in \operatorname{im}(S \otimes N \to S \otimes M)$. Since this holds for every $s \in S$, $u \in N^{\mathrm{cl}_S}_M = N$. Thus $\bar{u} = 0$ in M/N. So cl_S satisfies the Semi-residuality Axiom.

Now we prove that

$$I^{\mathrm{cl}_S} N^{\mathrm{cl}_S}_M \subseteq (IN)^{\mathrm{cl}_S}_M$$

for all $I \subseteq R$. Suppose that $u \in N^{\mathrm{cl}_S}_M$ and $y \in I^{\mathrm{cl}_S}$. Then for every $s \in S$,

$$s \otimes u \in \operatorname{im}(S \otimes N \to S \otimes M),$$

and $ys \in IS$. In particular, for every $s \in S$,

$$s \otimes yu = ys \otimes u = i_1(s_1 \otimes u) + i_2(s_2 \otimes u) + \ldots + i_n(s_n \otimes u)$$

for some $i_1, \ldots, i_n \in I$, $s_1, \ldots, s_n \in S$. But each

$$i_j(s_j \otimes u) = s_j \otimes i_j u \in \operatorname{im}(S \otimes IN \to S \otimes M).$$

Hence $yu \in (IN)^{\mathrm{cl}_S}_M$.

The last statement follows, because

$$yN_M^{\mathrm{cl}_S} \subseteq (y)^{\mathrm{cl}_S} N_M^{\mathrm{cl}_S} \subseteq (yN)_M^{\mathrm{cl}_S}$$

by the previous statement. □

The following lemma allows us to generalize the idea of an algebra closure.

Lemma 3.4. *Let $\mathcal{S}$ be a directed family of R-algebras. We can define a closure operation $\mathrm{cl}_{\mathcal{S}}$ by $u \in N_M^{\mathrm{cl}_{\mathcal{S}}}$ if for some $S \in \mathcal{S}$, $u \in N_M^{\mathrm{cl}_S}$.*

Proof. To see that $N_M^{\mathrm{cl}_{\mathcal{S}}}$ is a submodule of M, let $u, v \in N_M^{\mathrm{cl}_{\mathcal{S}}}$. It is clear that for any $r \in R$, $ru \in N_M^{\mathrm{cl}_{\mathcal{S}}}$. To see that $u + v \in N_M^{\mathrm{cl}_{\mathcal{S}}}$, note that there is some $S, S' \in \mathcal{S}$ such that $u \in N_M^{\mathrm{cl}_S}$ and $v \in N_M^{\mathrm{cl}_{S'}}$. Since $\mathcal{S}$ is a directed family, there is some $T \in \mathcal{S}$ such that S, S' both map to T. We will have $1 \otimes u, 1 \otimes v \in \mathrm{im}(T \otimes N \to T \otimes M)$, so $1 \otimes (u + v) \in \mathrm{im}(T \otimes N \to T \otimes M)$. Hence $u + v \in N_M^{\mathrm{cl}_T} \subseteq N_M^{\mathrm{cl}_{\mathcal{S}}}$.

The extension and order-preservation properties of a closure operation are not difficult to prove. We prove the idempotence property. Let $u \in (N_M^{\mathrm{cl}_{\mathcal{S}}})_M^{\mathrm{cl}_{\mathcal{S}}}$. Then for some $S \in \mathcal{S}$, $1 \otimes u \in \mathrm{im}(S \otimes N_M^{\mathrm{cl}_{\mathcal{S}}} \to S \otimes M)$, say $1 \otimes u = \sum_{i=1}^n s_i \otimes u_i$ with the $u_i \in N_M^{\mathrm{cl}_{\mathcal{S}}}$. For each i, there is some $S_i \in \mathcal{S}$ such that $u_i \in N_M^{\mathrm{cl}_{S_i}}$. There is some $T \in \mathcal{S}$ such that each S_i maps to T. Hence $1 \otimes u \in \mathrm{im}(T \otimes N \to T \otimes M)$. □

Proposition 3.5. *Let cl be a closure operation that commutes with finite direct sums (in particular, it is enough to assume that cl satisfies the Functoriality Axiom). Suppose the map $R \to M$ that sends $1 \mapsto u$ is cl-phantom, as is the map $R \to N$ that sends $1 \mapsto v$. Then the map $f : R \to (M \oplus N)/(u \oplus -v)$ that sends $1 \mapsto (u, 0) = (0, v)$ is cl-phantom, too. Further, any phantom extension $R \to Q$ that factors through both M and N factors through $(M \oplus N)/(u \oplus -v)$ as well.*

Note: If f split, we would have $M = R \oplus M_0$, $N = R \oplus N_0$, and $(M \oplus N)/(u \oplus -v) = R \oplus (M_0 \oplus N_0)$.

Proof. The last statement is automatic from the definition of a push-out. The cokernel f is the direct sum of the cokernels of the maps $R \to M$ and $R \to N$, and the direct sum of free resolutions $P_\bullet$ and $P'_\bullet$, respectively, of these cokernels gives us a free resolution of the cokernel of f. If $\phi : P_1 \to R$ and $\phi' : P'_1 \to R$ are maps induced by the identity map on the cokernels, then the hypothesis tells us that

$$\phi \in (\mathrm{im}(\mathrm{Hom}(P_0, R) \to \mathrm{Hom}(P_1, R)))^{\mathrm{cl}}_{\mathrm{Hom}(P_1,R)}$$

and

$$\phi' \in (\mathrm{im}(\mathrm{Hom}(P'_0, R) \to \mathrm{Hom}(P'_1, R)))^{\mathrm{cl}}_{\mathrm{Hom}(P'_1,R)}.$$

Since cl commutes with direct sums, we get

$$\phi \oplus \phi' \in (\mathrm{im}(\mathrm{Hom}(P_0 \oplus P_0', R) \to \mathrm{Hom}(P_1 \oplus P_1', R)))^{\mathrm{cl}}_{\mathrm{Hom}(P_1 \oplus P_1', R)},$$

as desired. □

Proposition 3.6. *Let S and T be R-modules such that for each $t \in T$, there is a map $S \to T$ whose image contains t. Then $\mathrm{cl}_S \subseteq \mathrm{cl}_T$, i.e., for any finitely-generated R-modules $N \subseteq M$, $N_M^{\mathrm{cl}_S} \subseteq N_M^{\mathrm{cl}_T}$.*

Proof. Suppose that $N \subseteq M$ are finitely-generated R-modules, and that $u \in N_M^{\mathrm{cl}_S}$. We will show that $u \in N_M^{\mathrm{cl}_T}$. Since $u \in N_M^{\mathrm{cl}_S}$, for each $s \in S$, $s \otimes u \in \mathrm{im}(S \otimes N \to S \otimes M)$. Let $t \in T$. Then there is some map $f : S \to T$ whose image contains t, say $s' \mapsto t$. There is some element y of $S \otimes N$ that maps to $s' \otimes u$ in $S \otimes M$. The image $(f \otimes \mathrm{id})(y)$ of y in $T \otimes N$ maps to $t \otimes u$ in $T \otimes M$, by the commutativity of the following diagram:

$$\begin{tikzcd} S \otimes N \arrow[r] \arrow[d, "f\otimes\mathrm{id}"'] & S \otimes M \arrow[d, "f\otimes\mathrm{id}"'] \\ T \otimes N \arrow[r] & T \otimes M \end{tikzcd}$$

Hence $t \otimes u \in \mathrm{im}(T \otimes N \to T \otimes M)$ for every $t \in T$, which implies that $u \in N_M^{\mathrm{cl}_T}$. □

Notation 3.7. We refer to the intersection of two closure operations cl and cl′, as defined in [5]. Let $N \subseteq M$ be finitely-generated R-modules. We say that

$$u \in N_M^{\mathrm{cl}\cap\mathrm{cl}'} \text{ if } u \in N_M^{\mathrm{cl}} \cap N_M^{\mathrm{cl}'}.$$

Proposition 3.8. *Let S and T be R-modules. Then $\mathrm{cl}_{S\oplus T} = \mathrm{cl}_S \cap \mathrm{cl}_T$.*

Proof. Suppose that $N \subseteq M$ are finitely-generated R-modules, and $u \in N_M^{\mathrm{cl}_{S\oplus T}}$. Then for each $(s,t) \in S \oplus T$,

$$\begin{aligned}(s,t) \otimes u &\in \mathrm{im}((S \oplus T) \otimes N \to (S \oplus T) \otimes M) \\ &= \mathrm{im}(S \otimes N \to S \otimes M) \oplus \mathrm{im}(T \otimes N \to T \otimes M).\end{aligned}$$

So $s \otimes u$ is in the first image, and $t \otimes u$ is in the second. Thus $u \in N_M^{\mathrm{cl}_S} \cap N_M^{\mathrm{cl}_T}$. If $u \in N_M^{\mathrm{cl}_S} \cap N_M^{\mathrm{cl}_T}$, then for each $s \in S$, $s \otimes u \in \mathrm{im}(S \otimes N \to S \otimes M)$ and for each $t \in T$, $t \otimes u \in \mathrm{im}(T \otimes N \to T \otimes M)$. Hence $(s,t) \otimes u \in \mathrm{im}((S \oplus T) \otimes N \to (S \oplus T) \otimes M)$. □

3.1. Properties of big Cohen–Macaulay module closures

We give several useful properties of big Cohen–Macaulay module closures.

Definition 3.9. Let *cl* be a closure operation on a ring R.

1. We say that cl satisfies *colon-capturing* if for every partial system of parameters $x_1, \ldots, x_{k+1}$ on R,
$$(x_1, \ldots, x_k) : x_{k+1} \subseteq (x_1, \ldots, x_k)^{\mathrm{cl}}.$$
2. We say that cl satisfies *strong colon-capturing, version A*, if for every partial system of parameters $x_1, \ldots, x_k$ on R,
$$(x_1^t, x_2, \ldots, x_k)^{\mathrm{cl}} : x_1^a \subseteq (x_1^{t-a}, x_2, \ldots, x_k)^{\mathrm{cl}}$$
for all $a < t$.
3. We say that cl satisfies *strong colon-capturing, version B*, if for every partial system of parameters $x_1, \ldots, x_{k+1}$ on R,
$$(x_1, \ldots, x_k)^{\mathrm{cl}} : x_{k+1} \subseteq (x_1, \ldots, x_k)^{\mathrm{cl}}.$$

   This is a stronger condition than colon-capturing.

Proposition 3.10. *Let B be a big Cohen–Macaulay module over a local domain R. Then the module closure cl_B satisfies strong colon-capturing, version A.*

Proof. Let $x_1, \ldots, x_k$ be a partial system of parameters on R. Suppose that $a < t$, and that $u \in (x_1^t, x_2, \ldots, x_k)^{\mathrm{cl}} : x_1^a$. In other words, for each $b \in B$,
$$ux_1^a b \in (x_1^t, \ldots, x_k)B,$$
say $ux_1^a b = x_1^t b_1 + x_2 b_2 + \ldots + x_k b_k$. Then $x_1^a(ub - x_1^{t-a} b_1) \in (x_2, \ldots, x_k)B$. Since B is a big Cohen–Macaulay module, this implies that $ub - x_1^{t-a} b_1 \in (x_2, \ldots, x_k)B$. Hence $ub \in (x_1^{t-a}, x_2, \ldots, x_k)B$. Since this holds for each $b \in B$, $u \in (x_1^{t-a}, x_2, \ldots, x_k)^{\mathrm{cl}_B}$. $\square$

Proposition 3.11. *Let B be a big Cohen–Macaulay module over a local domain R. Then cl_B satisfies strong colon-capturing, version B. As a consequence, cl_B satisfies colon-capturing.*

Proof. Let $x_1, \ldots, x_{k+1}$ be a partial system of parameters on R. Suppose that $v \in R$ such that $vx_{k+1} \in (x_1, \ldots, x_k)^{\mathrm{cl}_B}$. Then for each $b \in B$, $vx_{k+1}b \in (x_1, \ldots, x_k)B$. Equivalently, $x_{k+1}(vb) \in (x_1, \ldots, x_k)B$. Since $x_1, \ldots, x_{k+1}$ form part of a system of parameters on R, they form a regular sequence on B. Hence $vb \in (x_1, \ldots, x_k)B$. As we proved this for an arbitrary $b \in B$, $v \in (x_1, \ldots, x_k)^{\mathrm{cl}_B}$, as desired. $\square$

4. Smallest closures

4.1. Intersection stable properties

Given a set $\{\mathrm{cl}_\lambda\}_{\lambda\in\Lambda}$ of closure operations, their intersection $\bigcap_{\lambda\in\lambda} \mathrm{cl}_\lambda$ is also a closure operation [5, Construction 3.1.3].

Definition 4.1. Given a property P of a closure operation, we call P *intersection stable* if whenever cl_λ satisfies P for every $\lambda \in \Lambda$, $\bigcap_{\lambda\in\Lambda} \mathrm{cl}_\lambda$ also satisfies P.

The following lemma is immediate:

Lemma 4.2. *Suppose that P is an intersection stable property of a closure operation and that R has a closure operation satisfying P. Then R has a smallest closure operation satisfying P.*

Theorem 4.3. *The Functoriality Axiom is intersection stable. The Semi-residuality Axiom is intersection stable on sets of closures that satisfy the Functoriality Axiom. When R is local, the Faithfulness Axiom and the Generalized Colon-Capturing Axiom are intersection stable.*

Proof. Let $\{\mathrm{cl}_\lambda\}_{\lambda\in\Lambda}$ be a family of closure operations, and

$$\mathrm{cl} = \bigcap_{\lambda\in\Lambda} \mathrm{cl}_\lambda.$$

If each cl_λ satisfies the Functoriality Axiom, $f : M \to W$ is an R-module map, and $N \subseteq M$ is a submodule, then $f(N_M^{\mathrm{cl}}) \subseteq f(N_M^{\mathrm{cl}_\lambda}) \subseteq f(N)_W^{\mathrm{cl}_\lambda}$ for each λ. Thus $f(N_M^{\mathrm{cl}}) \subseteq \bigcap_\lambda f(N)_W^{\mathrm{cl}_\lambda} = f(N)_W^{\mathrm{cl}}$, as desired.

Suppose that $N_M^{\mathrm{cl}} = N$, and that for each λ, cl_λ satisfies the Functoriality Axiom and the Semi-residuality Axiom. We will show that $0_{M/N}^{\mathrm{cl}} = 0$. Suppose that $\bar{u} \in 0_{M/N}^{\mathrm{cl}}$. Then for each λ, $\bar{u} \in 0_{M/N}^{\mathrm{cl}_\lambda}$. By Lemma 3.1, $u \in N_M^{\mathrm{cl}_\lambda}$ if and only if $\bar{u} \in 0_{M/N}^{\mathrm{cl}_\lambda}$. Hence $u \in N_M^{\mathrm{cl}_\lambda}$ for each λ, which implies that $u \in N_M^{\mathrm{cl}} = N$. Thus $\bar{u} = 0$, and so cl satisfies the Semi-residuality Axiom.

It is clear that the Faithfulness Axiom is intersection stable.

Suppose that cl_λ satisfies the Generalized Colon-Capturing Axiom for each λ and that $x_1, \ldots, x_{k+1}$ is part of a system of parameters for R, $J = (x_1, \ldots, x_k)$, and $f : M \twoheadrightarrow R/J$ such that there is some $v \in M$ with $f(v) = x_{k+1} + J$. We need to show that $(Rv)_M^{\mathrm{cl}} \cap \ker(f) \subseteq (Jv)_M^{\mathrm{cl}}$. Since $(Rv)_M^{\mathrm{cl}} \cap \ker(f) \subseteq (Rv)_M^{\mathrm{cl}_\lambda} \cap \ker(f) \subseteq (Jv)_M^{\mathrm{cl}_\lambda}$ for each λ, the Generalized Colon-Capturing Axiom holds for cl. $\square$

Corollary 4.4. *If a local domain R has a Dietz closure, then it has a smallest Dietz closure.*

In the case of a Cohen–Macaulay ring, the smallest Dietz closure is the trivial closure. However, we do not know what it looks like in more generality.

Remark 4.5. Colon-capturing is a useful property for a closure operation to have, but it is not enough on its own for our purposes. For example, the closure $N_M^{\mathrm{cl}} = M$ captures colons, but is too large to be useful.

Lemma 4.6. *Colon-capturing is an intersection stable property.*

Proof. This is immediate from Definition 3.9. □

Lemma 4.7. *Strong colon-capturing, version A, as in Definition 3.9 is intersection stable.*

Proof. To see this, notice that if $x_1, \dots, x_k$, t, and a are as in the definition of strong colon-capturing, version A, then

$$(x_1^t, x_2, \dots, x_k)^{\mathrm{cl}} :_R x_1^a \subseteq (x_1^t, x_2, \dots, x_k)^{\mathrm{cl}_\lambda} :_R x_1^a \subseteq (x_1^{t-a}, x_2, \dots, x_k)^{\mathrm{cl}_\lambda}$$

for each λ. Hence $(x_1^t, x_2, \dots, x_k)^{\mathrm{cl}} :_R x_1^a \subseteq (x_1^{t-a}, x_2, \dots, x_k)^{\mathrm{cl}}$. □

Remark 4.8. A similar proof works for strong colon-capturing, version B.

If cl is defined on a category of rings, then we would like to find the smallest closure operation as above (if any such exist) that captures colons and also satisfies the following property:

Definition 4.9. A closure operation satisfies *persistence for change of rings* if whenever $R \to S$ is a morphism in this category, and $N \subseteq M$ are finitely generated R-modules, then $\mathrm{im}(S \otimes_R N_M^{\mathrm{cl}} \to S \otimes_R M) \subseteq (\mathrm{im}(S \otimes_R N \to S \otimes_R M))_{S \otimes_R M}^{\mathrm{cl}}$.

Remark 4.10. Tight closure satisfies both persistence for change of rings and colon-capturing when R is a complete local domain [11].

The trivial closure always satisfies persistence for change of rings, but in the local case, it captures colons if and only if R is Cohen–Macaulay.

Proposition 4.11. *Persistence for change of rings is an intersection stable property.*

Proof. Suppose that cl_λ are closure operations, each defined on all rings in the category, that are persistent for change of rings. Let $\mathrm{cl} = \bigcap_{\lambda \in \Lambda} \mathrm{cl}_\lambda$. We will show that cl is persistent for change of rings. Let $R \to S$ be a morphism in the category, and suppose that $u \in N_M^{\mathrm{cl}}$. Our goal is to show that $1 \otimes u \in (\mathrm{im}(S \otimes_R N \to S \otimes_R M))_{S \otimes_R M}^{\mathrm{cl}}$. By definition of cl, $u \in N_M^{\mathrm{cl}_\lambda}$ for every $\lambda \in \Lambda$. Since each cl_λ is persistent with change of rings, this implies that

$$1 \otimes u \in (\operatorname{im}(S \otimes_R N \to S \otimes_R M))^{\mathrm{cl}_\lambda}_{S \otimes_R M}$$

for every $\lambda \in \Lambda$. Hence $1 \otimes u \in (\operatorname{im}(S \otimes_R N \to S \otimes_R M))^{\mathrm{cl}}_{S \otimes_R M}$. □

Corollary 4.12. *The category of all complete local domains has a smallest persistent closure operation that captures colons.*

Proof. This follows immediately from Lemma 4.6, Remark 4.10, and Proposition 4.11. □

Question 4.13. When R is a complete local domain that is not Cohen–Macaulay, what is the smallest persistent closure operation that captures colons?

4.2. Smallest big Cohen–Macaulay module closure

Given a big Cohen–Macaulay module B over a local domain R, we get a module closure cl_B. In [2], Dietz proves that cl_B is a Dietz closure. We can define a new closure operation by intersecting all of these closures. Since the property of being a Dietz closure is intersection stable, this is also a Dietz closure. As we prove below, it is also a big Cohen–Macaulay module closure.

Proposition 4.14. *Let R be a local domain, and let B be a big Cohen–Macaulay module constructed using the method of [2]. If B' is any big Cohen–Macaulay R-module, $\mathrm{cl}_B \subseteq \mathrm{cl}_{B'}$. As a consequence, cl_B is the smallest big Cohen–Macaulay module closure on R.*

Proof. Let B be a big Cohen–Macaulay module constructed as above, and B' an arbitrary big Cohen–Macaulay module. Then for each map $R \to B'$, we can construct a map $B \to B'$ that takes the image of 1 in B to the image of 1 in B' via the given map $R \to B'$. To get this map, we start with the map $R \to B'$. If we already have maps from $M_0 = R, M_1, \ldots, M_t$ to B', we extend the map to M_{t+1} as follows:

$$M_{t+1} = (M \oplus Rf_1 \oplus \ldots \oplus Rf_k)/(u \oplus x_1 f_1 \oplus \ldots x_k f_k)$$

for some $u \in M_t$ and partial system of parameters $x_1, \ldots, x_k$ for R such that

$$x_{k+1} u = x_1 m_1 + \ldots + x_k m_k$$

is a bad relation in M_t. Since B' is a big Cohen–Macaulay module, the image of u in B' under the map already constructed is in $(x_1, \ldots, x_k)B'$, say $u = x_1 b_1 + \ldots + x_k b_k$ with $b_1, \ldots, b_k \in B'$. We extend our map $M_t \to B'$ to a map from M_{t+1} to B' by sending $f_i \mapsto b_i$. Take the direct limit of this system of maps $M_t \to B'$ as $t \to \infty$ to get the desired map $B \to B'$. Since we can start with any map $R \to B'$, every element of B' is in the image of a map constructed this way. Hence Proposition 3.6 implies that $\mathrm{cl}_B \subseteq \mathrm{cl}_{B'}$. □

In certain rings of dimension 2, we know more about the smallest big Cohen–Macaulay module closure.

Definition 4.15 *([12]).* For R a local domain, the S_2*-ification* of R is the unique smallest extension of R in its fraction field that satisfies Serre's condition S_2, if such a ring exists. When it exists, it can be constructed by adding to R all elements $f \in \mathrm{Frac}(R)$ such that some height 2 ideal of R multiplies f into R.

Proposition 4.16. *Let R be a local domain of dimension 2 that has an S_2-ification S. Then the module closure cl_S is the smallest big Cohen–Macaulay module closure on R.*

Proof. Let B be a big Cohen–Macaulay module constructed by the method of [2], so that cl_B is the smallest big Cohen–Macaulay module closure on R. Since S is Cohen–Macaulay when R has dimension 2, we know that $\mathrm{cl}_B \subseteq \mathrm{cl}_S$. By Proposition 3.6, it is enough to show that for any map $R \to B$, $1 \mapsto u$, we have a map $S \to B$ whose image contains u. To do this, we need to extend the map from R to S by defining it on elements $f \in \mathrm{Frac}(R)$ such that some height 2 ideal of R multiplies f into R. Let f be such an element. Since $\dim(R) = 2$, there is some system of parameters x, y for R such that $xf, yf \in R$. Then the map is already defined on xf, yf, say $xf \mapsto v$, $yf \mapsto w$. The element xyf must map to yv, but also must map to xw, so $yv = xw$. Since x, y is a regular sequence on B, $v = xv_0$ and $w = yw_0$ for some $v_0, w_0 \in B$. Then $xyv_0 = yv = xw$, so $w = yv_0$. Hence $yv_0 = yw_0$, which implies that $v_0 = w_0$. Thus $f \mapsto v_0$ is a well-defined extension of the map $R \to B$. Further, 1_S maps to u, so this is the map we need to see that $\mathrm{cl}_S \subseteq \mathrm{cl}_B$. $\square$

Example 4.17. Let $R = k[[x^4, x^3y, xy^3, y^4]]$. The S_2-ification S of R must contain x^2y^2, since $x^4(x^2y^2) = (x^3y)^2 \in R$ and $y^4(x^2y^2) = (xy^3)^2 \in R$. In fact, S is the subring $k[[x^4, x^3y, x^2y^2, xy^3, y^4]]$ of $k[[x, y]]$. Since $(x^3y)^2 = x^4(x^2y^2)$ in S, $(x^3y)^2 \in (x^4)_R^{\mathrm{cl}_S}$. Similarly, $(xy^3)^2 \in (y^4)_R^{\mathrm{cl}_S}$. Hence $(x^3y)^2 \in (x^4)_R^{\mathrm{cl}}$ and $(xy^3)^2 \in (y^4)_R^{\mathrm{cl}}$ for every Dietz closure cl on R.

4.3. Smallest module closure containing another closure

Given a closure operation cl on R, we can construct the smallest module closure containing cl. This will be used later on to prove that every Dietz closure is contained in a big Cohen–Macaulay module closure. To construct the smallest module closure containing a given closure, we use a second type of module modification.

Definition 4.18. Let cl be a closure operation on R, $G \subseteq R^s$ a submodule of a finitely-generated free R-module generated by

$$e_1 = (e_{11}, \ldots, e_{1s}), \ldots, e_k = (e_{k1}, \ldots, e_{ks}),$$

and let $v = (v_1, \ldots, v_s) \in G^{\mathrm{cl}}_{R^s} - G$. A *containment module modification* of an R-module M relative to an element $x \in M$ is a map

$$M \to M' = \frac{M \oplus Rf_1 \oplus \ldots \oplus Rf_k}{R(v_1 x \oplus e_{11} f_1 \oplus \ldots \oplus e_{k1} f_k, \ldots, v_s x \oplus e_{1s} f_1 \oplus \ldots \oplus e_{ks} f_k)}.$$

Proposition 4.19. *Let R be a ring, W an R-module, and cl a closure operation on R satisfying the Functoriality Axiom and the Semi-residuality Axiom. Then there is an R-module S with a map $\phi : W \to S$ such that $\mathrm{cl} \subseteq \mathrm{cl}_S$, and for any R-module T such that $\mathrm{cl} \subseteq \mathrm{cl}_T$ and any map $\psi : W \to T$, we have a map $\gamma : S \to T$ such that $\psi = \gamma \circ \phi$.*

Proof. To create such an S, we apply containment module modifications to finitely-generated submodules of W. First, we show that we have a direct limit system of containment module modifications. Given a finite set of modules $G_1, \ldots, G_t$ with $G_i \subseteq R^{s_i}$, and for each i, a finite set of elements $v_{i1}, v_{i2}, \ldots, v_{i\ell_i} \in (G_i)^{\mathrm{cl}}_{R^{s_i}} - G_i$, we can apply finitely many containment module modifications to a finitely-generated submodule $W_0 \subseteq W$ to get a module W_1 such that for each $1 \le i \le t$ and $1 \le j \le \ell_i$,

$$\mathrm{im}(v_{ij} \otimes W_0 \to R^{s_i} \otimes W_1) \subseteq \mathrm{im}(G_i \otimes W_1 \to R^{s_i} \otimes W_1).$$

Then we apply finitely many containment module modifications to W_1, forcing

$$\mathrm{im}(v_{ij} \otimes W_1 \to R^{s_i} \otimes W_2) \subseteq \mathrm{im}(G_i \otimes W_2 \to R^{s_i} \otimes W_2)$$

for all i, j. Repeating this process infinitely many times, we get a module W_∞ that is the direct limit of the W_r and such that

$$\mathrm{im}(v_{ij} \otimes W_\infty \to R^s \otimes W_\infty) \subseteq \mathrm{im}(G_i \otimes W_\infty \to R^s \otimes W_\infty)$$

for all i, j. We have a map $W_0 \to W_\infty$ since each containment module modification comes with a map from W_0.

Consider all finite sets $\mathcal{G} = \{G_1, \ldots, G_t, v_{11}, v_{12}, \ldots, v_{1\ell_1}, v_{21}, v_{22}, \ldots, v_{t\ell_t}\}$ with $G_i \subseteq R^{s_i}$ and finitely many elements $v_{i1}, \ldots, v_{i\ell_i} \in (G_i)^{\mathrm{cl}}_{R^s} - G_i$ for each $1 \le i \le t$, and also all finitely-generated submodules W_0 of W. Suppose that $\mathcal{G} \subseteq \mathcal{G}'$ are two such sets, that $W_0 \subseteq W_0'$ are finitely-generated submodules of W, and that W_∞ and W'_∞ are corresponding direct limit modules constructed from W_0 using $\mathcal{G}$ and from W_0' using $\mathcal{G}'$, respectively. We build a map $W_\infty \to W'_\infty$, starting with the map $W_0 \subseteq W_0' \to W'_\infty$.

It suffices to demonstrate that the map can be extended to a single containment module modification. Let P be an intermediate module in the direct limit system of W_∞ with a map $P \to W'_\infty$, $v = v_{ij} \in \mathcal{G}$ for some i, j, $e_1, \ldots, e_k$ be the generators of $G = G_i$, and $x \in Q$ as in Definition 4.18. We need to specify the images of $f_1, \ldots, f_k$ in W'_∞. Since $v \otimes W'_\infty \subseteq G \otimes W'_\infty$, $vx = e_1 w_1 + e_2 w_2 + \ldots + e_k w_k$ for some $w_1, \ldots, w_k \in W'_\infty$. Then the map that sends $f_i \mapsto w_i$ is a well-defined extension of the map $P \to W'_\infty$. Hence we have a map $W_\infty \to W'_\infty$ for any $\mathcal{G} \subseteq \mathcal{G}'$.

The W_∞ form a partially ordered set via $W_\infty \leq W'_\infty$ if the corresponding finite sets satisfy $\mathcal{G} \subseteq \mathcal{G}'$ and $W_0 \subseteq W'_0$. This is a directed set, using the maps $W_\infty \to W'_\infty$ we constructed above. Let S be the direct limit. By the set-up above, we have a well-defined map $\phi : W \to S$. We are now done proving that for submodules G of finitely-generated free R-modules R^s, $G^{\mathrm{cl}}_{R^s} \subseteq G^{\mathrm{cl}_S}_{R^s}$.

Suppose that $N \subseteq M$ are arbitrary finitely-generated R-modules. We will show that $N^{\mathrm{cl}}_M \subseteq N^{\mathrm{cl}_S}_M$. There is some s for which $M/N \cong R^s/G$, where G is a submodule of R^s. Let $u \in N^{\mathrm{cl}}_M$. By Lemma 3.1, part (a), $\bar{u} \in 0^{\mathrm{cl}}_{M/N} \cong 0^{\mathrm{cl}}_{R^s/G}$. Applying the Lemma again, any lift v of $\mathrm{im}(\bar{u})$ to R^s is in $G^{\mathrm{cl}}_{R^s}$, which is contained in $G^{\mathrm{cl}_S}_{R^s}$ by the previous paragraph. Applying the Lemma twice more, we get $\bar{u} \in 0^{\mathrm{cl}_S}_{M/N}$, which implies that $u \in N^{\mathrm{cl}_S}_M$.

Now suppose that T is an R-module such that $\mathrm{cl} \subseteq \mathrm{cl}_T$, and we have a map $\psi : W \to T$. Let $\phi : W \to S$ be as above. For any intermediate module P in the direct limit system of S, let ϕ_P be the corresponding map $W \to P$. Suppose that we have a map $\gamma_P : P \to T$ such that $\psi = \gamma_P \circ \phi_P$. We demonstrate how to extend the map to a map $\gamma_{P'} : P' \to T$ such that $\psi = \gamma_{P'} \circ \phi_{P'}$ when P' is a containment module modification of P. We have:

$$P \to P' = \frac{P \oplus Rf_1 \oplus \ldots \oplus Rf_k}{R(v_1 x \oplus e_{11} f_1 \oplus \ldots \oplus e_{k1} f_k, \ldots, v_s x \oplus e_{1s} f_1 \oplus \ldots \oplus e_{ks} f_k)},$$

where $x \in P$, and $v, e_1, \ldots, e_k$ are as in Definition 4.18. We need to specify the images of the f_i. Since $\mathrm{cl} \subseteq \mathrm{cl}_T$, $vx \in (e_1, \ldots, e_k)T$, say $vx = e_1 t_1 + \ldots + e_k t_k$. Then sending $f_i \mapsto t_i$ gives us a well-defined extension of γ_P such that $\psi = \gamma_{P'} \circ \phi_{P'}$. Since S is a direct limit of such containment module modifications, we get a map $\gamma : S \to T$ such that $\psi = \gamma \circ \phi$. $\square$

Theorem 4.20. *Let R be a ring and cl a closure operation on R satisfying the Functoriality Axiom and the Semi-residuality Axiom. Then if we set $W = R$ and construct a module S as in Proposition 4.19, cl_S is the smallest module closure containing cl, i.e., if T is any R-module such that $\mathrm{cl} \subseteq \mathrm{cl}_T$, we have $\mathrm{cl}_S \subseteq \mathrm{cl}_T$. In particular, if cl is a module closure, then $\mathrm{cl} = \mathrm{cl}_S$ (conversely, if cl is not a module closure, then $\mathrm{cl} \subsetneq \mathrm{cl}_S$).*

Proof. By Proposition 4.19, for every R-module map $R \to T$, we have a map $S \to T$ that agrees with the original map on the image of R. So for every element $t \in T$, we have a map $S \to T$ whose image contains t. By Proposition 3.6, this implies that $\mathrm{cl}_S \subseteq \mathrm{cl}_T$. $\square$

5. A connection between Dietz closures and singularities

In this section, we show that for any local domain R that has a Dietz closure, R is regular if and only if all Dietz closures on R are trivial. First, we prove a result on the relationship between general Dietz closures and big Cohen–Macaulay module closures.

Theorem 5.1. *Let cl be a Dietz closure on a local domain (R, m). Then cl is contained in cl_B for some big Cohen–Macaulay module B.*

Proof. Let cl be a Dietz closure on R. To construct B, we use both parameter module modifications and containment module modifications. First, we construct a big Cohen–Macaulay module S_1 using parameter module modifications as in [2]. We apply containment module modifications to S_1 as in Proposition 4.19 to get a module S_2 such that $\mathrm{cl} \subseteq \mathrm{cl}_{S_2}$ and a map $S_1 \to S_2$, and then we use parameter module modifications to construct an R-module S_3 such that every system of parameters on R is a regular sequence on S_3 and a map $S_2 \to S_3$. We repeat these two constructions countably many times, getting maps

$$R = S_0 \to S_1 \to S_2 \to S_3 \to \dots$$

The direct limit B is an R-module such that $\mathrm{cl} \subseteq \mathrm{cl}_B$ and every system of parameters on R is a regular sequence on B. We need to show that $\mathrm{im}(1) \notin mB$ when we apply the map $R \to B$ that is the direct limit of the maps $R \to S_i$.

We follow the proof of [14, Proposition 3.7]. If $\mathrm{im}(1) \in mB$, then there is a finitely-generated R-module P with $1 \in mP$ such that P maps to B.

Claim: There is an R-module W constructed from R by taking finitely many module modifications (of either or both types) such that the map $P \to B$ passes through W.

Proof of Claim. Given any finitely-generated R-module P with a map $P \to B$, there is some $i > 0$ for which $\mathrm{im}(P) \subseteq S_i$. Then there is also a finite sequence of containment module modifications and parameter module modifications of S_{i-1} giving a module W_{i-1} such that the map $P \to B$ passes through W_{i-1}. We use induction on the value of i. If $i = 1$, then the result is immediate. Suppose the result holds for $i = 1, 2, \dots, k-1$, and let S be a module gotten from S_{k-1} by applying a finite sequence of module modifications, such that $\mathrm{im}(P) \subseteq S$. By induction, there is an R-module W_{k-1} that is constructed from R by taking finitely many module modifications, and such that $\mathrm{im}(P \cap S_{k-1}) \subseteq W_{k-1}$. Any element of P not in S_{k-1} must come from one of the module modifications applied to S_{i-1} to get S. So when we apply the same sequence of module modifications to W_{k-1}, we get an R-module W_k that is constructed by applying finitely many module modifications to R and such that $\mathrm{im}(P) \subseteq W_k$. □

Further, if we apply any finite sequence of module modifications to R to get a module W, we have a map $W \to B$, constructed in the same way as the maps $W_\infty \to W'_\infty$ in the proof of Proposition 4.19 and the maps $M_t \to B'$ in the proof of Proposition 4.14. Therefore, $\mathrm{im}(1) \in mB$ if and only if $\mathrm{im}(1) \in mW$, where W is an R-module obtained by applying finitely many module modifications to R. We will show that we cannot have $\mathrm{im}(1) \in mW$. To do this, we show that if we have a cl-phantom map $R \to M$, and we apply a single module modification to M to get M', the resulting map $R \to M'$ is cl-phantom. Hence $\mathrm{im}(1) \notin mM'$.

Assume $\alpha : R \to M$ is a phantom extension of R. If we apply a parameter module modification to M, we know that the resulting map $\alpha' : R \to M'$ is phantom by [2]. In the following Lemma, we show that $\alpha' : R \to M'$ is phantom when we apply a containment module modification to M. Hence by Lemma 2.11, $\alpha'(1) \notin mM'$. This guarantees that in the limit, $mB \neq B$. $\square$

Lemma 5.2. *Suppose that* (R, m) *is a local domain and cl is a Dietz closure on* R *that satisfies the Functoriality Axiom and the Semi-residuality Axiom, and such that* $0_R^{\mathrm{cl}} = 0$. *Suppose that* $\alpha : R \to M$ *is a cl-phantom extension, and let* M' *be a containment module modification of* M. *Then* $\alpha' : R \to M'$ *is a cl-phantom extension.*

Proof. Let $v = (v_1, \ldots, v_s) \in G_{R^s}^{\mathrm{cl}} - G$ for some nonzero submodule $G \subseteq R^s$ (as $0_{R^s}^{\mathrm{cl}} = 0$ by assumption), and let $x \in M$. Let u be the image of 1 in M. Taking a single module modification, we get

$$M' = \frac{M \oplus Rf_1 \oplus \ldots \oplus Rf_k}{R\,(v_1 x \oplus e_{11} f_1 \oplus \ldots \oplus e_{k1} f_k, \ldots, v_s x \oplus e_{1s} f_1 \oplus \ldots \oplus e_{ks} f_k)}.$$

First, we need to show that the composite map $\alpha' : R \to M \to M'$ is injective. Let $F = \mathrm{Frac}(R)$. Then $F \to F \otimes_R M$ is injective, and it suffices to show that $F \to F \otimes M'$ is injective, i.e. that it is nonzero (if $R \to M'$ were not injective, applying $F\otimes$ would preserve this). We claim that $v \in \mathrm{im}(F \otimes G \to F^s)$. To see that this is true, notice that by Lemma 3.1, $0_{R^s/G}^{\mathrm{cl}}$ is contained in the torsion part of R^s/G. Hence $v \in G_{R^s}^{\mathrm{cl}}$ implies that $\bar{v}$ is a torsion element of R^s/G. Hence $\bar{v} = 0$ in $F^s/(F \otimes G)$, which implies that $v \in \mathrm{im}(F \otimes G \to F^s)$. Then the relations we kill to get $F \otimes M'$ already hold in $F \otimes M$, so there is a retraction $F \otimes M' \to F \otimes M$. This implies that $F \otimes M \to F \otimes M'$ is injective, and so $F \to F \otimes M'$ is injective, as desired.

Remark 5.3. In the special case $s = 1$, we can show that the map $M \to M'$ sending each element $y \mapsto y \oplus 0 \oplus \ldots \oplus 0$ is injective. If $y \mapsto 0$, then $y \oplus 0 \oplus \ldots \oplus 0 = r(vx \oplus r_1 f_1 \oplus \ldots \oplus r_k f_k)$ in $M \oplus Rf_1 \oplus \ldots \oplus Rf_k$, for some $r \in R$. We may assume without loss of generality that some r_i is nonzero, say r_1. Then $rr_1 f_1 = 0$, so $rr_1 = 0$. Since R is a domain, $r = 0$. So $y = rvx = 0$.

Following Notation 2.7 and [2, Discussion 2.4], pick a generating set $w_1, \ldots, w_n$ for M such that $w_1 = u$ and $w_n = x$. Then the images of $w_2, \ldots, w_n$ form a generating set for Q. Let

$$R^m \xrightarrow{\ \nu\ } R^{n-1} \xrightarrow{\ \mu\ } Q \longrightarrow 0$$

be a free presentation of Q, where μ sends the generators of R^{n-1} to $w_2, \ldots, w_n$, respectively. We can choose a basis for R^m such that ν is given by the $(n-1) \times m$ matrix $(b_{ij})_{2 \leq i \leq n, 1 \leq j \leq m}$. As in [2], we construct the diagram

$$\begin{array}{ccccccc} R^m & \xrightarrow{\nu_1} & R^n & \xrightarrow{\mu_1} & M & \longrightarrow & 0 \\ \big\downarrow \mathrm{id} & & \big\downarrow \pi & & \big\downarrow & & \\ R^m & \xrightarrow{\nu} & R^{n-1} & \xrightarrow{\mu} & Q & \longrightarrow & 0, \end{array}$$

where π kills the first generator of R^n and the rows are exact. The map μ_1 sends the generators of R^n to $w_1, \ldots, w_n$, respectively, and ν_1 has matrix $(b_{ij})_{1 \le i \le n, 1 \le j \le m}$ with respect to the same basis for R^m used to give ν.

Now we construct corresponding resolutions for M' and Q'. M' has k new generators and s new relations, as does Q', so we get the following diagram:

$$\begin{array}{ccccccc} R^{m+s} & \xrightarrow{\nu_1'} & R^{n+k} & \xrightarrow{\mu_1'} & M' & \longrightarrow & 0 \\ \big\downarrow \mathrm{id} & & \big\downarrow \pi & & \big\downarrow & & \\ R^{m+s} & \xrightarrow{\nu'} & R^{n-1+k} & \xrightarrow{\mu'} & Q' & \longrightarrow & 0 \end{array}$$

The maps μ' and μ_1' take the generators of Q' and M' to $\overline{w_2}, \ldots, \overline{w_n}, f_1, \ldots, f_k$ and $w_1, \ldots, w_n, f_1, \ldots, f_k$, respectively. The map π kills the first generator of R^{n+k}. The map ν_1' can be given by the matrix

$$\left(\begin{array}{c|c} & 0 \\ \nu_1 & \vdots \\ & 0 \\ & v \\ \hline & e_1 \\ 0 & \vdots \\ & e_k \end{array}\right),$$

and ν' is this matrix with the top row removed.

The rows of this diagram are exact. We demonstrate the exactness at R^{n+k}. To see that $\mu_1' \circ \nu_1' = 0$, we observe that $\mu_1 \circ \nu_1 = 0$, and for all i, $v_i x + e_{1i} f_1 + \ldots + e_{ki} f_k = 0$ in M'. To see that $\ker(\mu_1') \subseteq \operatorname{im}(\nu_1')$, suppose that $\mu_1'(a_1, \ldots, a_{n+k})^{\mathrm{tr}} = 0$. Then

$$\begin{aligned} a_1 w_1 + \ldots + a_n w_n + a_{n+1} f_1 + \ldots + a_{n+k} f_k &= r_1(v_1 x + e_{11} f_1 + \ldots + e_{k1} f_k) \\ &\quad + r_2(v_2 x + e_{12} f_1 + \ldots + e_{k2} f_k) \\ &\quad + \ldots + r_s(v_s x + e_{1s} f_1 + \ldots + e_{ks} f_k) \end{aligned}$$

in $M \oplus Rf_1 \oplus \ldots \oplus Rf_k$, for some $r_1, \ldots, r_s \in R$. So

$$a_1 w_1 + \ldots + a_{n-1} w_{n-1} + (a_n - \sum_{i=1}^{s} r_s v)x = 0,$$

and

$$a_{n+i} = \sum_{j=1}^{s} r_j e_{ij}$$

for $1 \leq i \leq k$. This implies that $(a_1, \ldots, a_{n-1}, a_n - \sum_{i=1}^{s} r_s v, 0, \ldots, 0)^{\mathrm{tr}}$ is in the image of the first m columns of ν_1', and $(0, \ldots, 0, \sum_{i=1}^{s} r_s v, a_{n+1}, \ldots, a_{n_k})^{\mathrm{tr}}$ is in the image of the last s columns of ν_1'. Hence $(a_1, \ldots, a_{n+k})^{\mathrm{tr}}$ is in the image of ν_1', as desired.

By Lemma 2.8, α' is phantom if and only if the top row of ν_1' is in the cl-closure of the span of the other rows. Denote the top row of ν_1 by $\boldsymbol{x}$, the bottom row by $\boldsymbol{y}$, and the span of the middle rows by H. Then α' is phantom if and only if

$$\boldsymbol{x} \oplus 0 \in (R(\boldsymbol{y} \oplus v) + (H \oplus \mathbf{0}) + (\mathbf{0} \oplus G))^{\mathrm{cl}}_{R^{m+s}}.$$

But since α is phantom, $\boldsymbol{x} \in (R\boldsymbol{y} + H)^{\mathrm{cl}}_{R^m}$. Hence $\boldsymbol{x} \oplus \mathbf{0} \in (R\boldsymbol{y} + H)^{\mathrm{cl}}_{R^m} \oplus \mathbf{0}$, and we have

$$\begin{aligned}(R\boldsymbol{y} + H)^{\mathrm{cl}}_{R^m} \oplus \mathbf{0} &= (R\boldsymbol{y} + H)^{\mathrm{cl}}_{R^m} \oplus 0^{\mathrm{cl}}_{R^s} \\ &= ((R\boldsymbol{y} + H) \oplus \mathbf{0})^{\mathrm{cl}}_{R^{m+s}} \\ &= ((R\boldsymbol{y} \oplus \mathbf{0}) + (H \oplus \mathbf{0}))^{\mathrm{cl}}_{R^{m+s}}\end{aligned}$$

We want to show that this is contained in $(R(\boldsymbol{y} \oplus v) + (H \oplus \mathbf{0}) + (\mathbf{0} \oplus G))^{\mathrm{cl}}_{R^{m+s}}$. We have

$$\begin{aligned}(R\boldsymbol{y} \oplus 0) + (H \oplus \mathbf{0}) &\subseteq R(\boldsymbol{y} \oplus v) + (H \oplus \mathbf{0}) + (\mathbf{0} \oplus G^{\mathrm{cl}}_{R^s}) \\ &= (R\boldsymbol{y} \oplus v) + (H \oplus \mathbf{0}) + ((\mathbf{0} \oplus G))^{\mathrm{cl}}_{R^{m+s}} \\ &\subseteq (R(\boldsymbol{y} \oplus v) + (H \oplus \mathbf{0}))^{\mathrm{cl}}_{R^{m+s}} + (\mathbf{0} \oplus G)^{\mathrm{cl}}_{R^{m+s}}.\end{aligned}$$

Thus

$$\begin{aligned}((R\boldsymbol{y} \oplus 0) + (H \oplus \mathbf{0}))^{\mathrm{cl}}_{R^{m+s}} &\subseteq \left((R(\boldsymbol{y} \oplus v) + (H \oplus \mathbf{0}))^{\mathrm{cl}}_{R^{m+s}} + (\mathbf{0} \oplus G)^{\mathrm{cl}}_{R^{m+s}}\right)^{\mathrm{cl}}_{R^{m+s}} \\ &= (R(\boldsymbol{y} \oplus v) + (H \oplus \mathbf{0}) + (\mathbf{0} \oplus G))^{\mathrm{cl}}_{R^{m+s}}\end{aligned}$$

by Lemma 3.1. Therefore, α' is phantom. □

It turns out that the closure operation cl_B from Theorem 5.1 is the smallest big Cohen–Macaulay-module closure containing cl, the initial Dietz closure.

Lemma 5.4. *Let notation be as in Theorem 5.1. Given a big Cohen–Macaulay-module B' such that* $\mathrm{cl} \subseteq \mathrm{cl}_{B'}$, $\mathrm{cl}_B \subseteq \mathrm{cl}_{B'}$.

Proof. For any map $R \to B'$, we construct a map $B \to B'$. We already know from the proof of Proposition 4.14 how to extend the map $M \to B'$ to a map $M' \to B'$, where

M' is a parameter module modification of M. We need to know how to extend the map when

$$M' = \frac{M \oplus Rf_1 \oplus \ldots \oplus Rf_k}{R\,(v_1x \oplus e_{11}f_1 \oplus \ldots \oplus e_{k1}f_k, \ldots, v_sx \oplus e_{1s}f_1 \oplus \ldots \oplus e_{ks}f_k)}$$

is a containment module modification of M. Since $(v_1, \ldots, v_s) \in G^{\mathrm{cl}}_{R^s}$, for each $b' \in B'$, $(v_1, \ldots, v_s) \otimes b' \in \operatorname{im}(G \otimes B' \to R^s \otimes B')$. In particular, for each $1 \le i \le s$, $v_ix = e_{1i}b_1 + e_{2i}b_2 + \ldots + e_{ki}b_k$ where $b_1, \ldots, b_k \in B'$. Define the map $M' \to B'$ by sending $f_i \mapsto b_i$.

Now for every map $R \to B'$ sending $1 \mapsto u$, we have a map $B \to B'$ whose image contains u. So by Proposition 3.6, $\mathrm{cl}_B \subseteq \mathrm{cl}_{B'}$. □

Question 5.5.

1. Are all Dietz closures big Cohen–Macaulay module closures, or any kind of module closure? If not, is there a nice way of characterizing the difference between Dietz closures that are big Cohen–Macaulay module closures and those that are not?
2. If we use only containment module modifications as in Proposition 4.19, are there useful hypotheses that guarantee that the constructed module S is a big Cohen–Macaulay module?

We use the following definition in our proof that Dietz closures are trivial on regular rings.

Definition 5.6. Given a closure operation cl, a ring R is *weakly cl-regular* if for $N \subseteq M$ finitely generated R-modules, $N^{\mathrm{cl}}_M = N$.

Remark 5.7. It is equivalent to say that $I^{\mathrm{cl}}_R = I$ for all ideals I of R. This follows from an argument in [8].

Proposition 5.8. *Let cl be a closure operation on a regular local ring (R, m) that satisfies*

1. *strong colon-capturing, version A,*
2. *$m^{\mathrm{cl}} = m$, and*
3. *if $N' \subseteq N \subseteq M$ are finitely-generated R-modules, then $(N')^{\mathrm{cl}}_N \subseteq (N')^{\mathrm{cl}}_M$.*

Then R is weakly cl-regular.

Proof. Let $N \subseteq M$ be finitely-generated R-modules, and let $x_1, \ldots, x_d$ be regular parameters for R (i.e., $(x_1, \ldots, x_d) = m$). Since $N = \bigcap_s (N + m^sM)$, by Lemma 3.1 it suffices to show that $N + m^sM$ is cl-closed in M for each s. Fix a value of s. By the same Lemma, we may replace M by $M/(N + m^sM)$ and show that 0 is cl-closed in this module instead. Since M now has finite length, for some t, $I_t = (x_1^{t+1}, x_2^{t+1}, \ldots, x_d^{t+1})$

kills M, and so M is an R/I_t-module. Now I_t is m-primary, so R/I_t is 0-dimensional. Additionally, R is regular and $x_1, \dots, x_d$ form a system of parameters, so R/I_t is Gorenstein. Hence R/I_t is injective as a module over itself and is also the only indecomposable injective R/I_t-module. This implies that $M \hookrightarrow (R/I_t)^h$ for some $h \geq 0$. Now it suffices to show that I_t is cl-closed in R, as then 0 is cl-closed in $(R/I_t)^h$. Since $0 \subseteq M \subseteq (R/I_t)^h$, this implies that $0_M^{\mathrm{cl}} \subseteq 0_{(R/I_t)^h}^{\mathrm{cl}} = 0$.

We show that I_t is cl-closed in R for all t. Let $x = x_1 x_2 \cdots x_d$. Since $(x_1, \dots, x_d) = m$, 1 generates the socle in $R/I_0 = R/m$. Then x^t generates the socle in R/I_t for $t \geq 1$. So if I_t is not cl-closed, we must have $x^t \in (I_t)_R^{\mathrm{cl}}$. Thus it suffices to show that $x^t \notin (I_t)_R^{\mathrm{cl}}$.

Suppose that $x^t \in (I_t)_R^{\mathrm{cl}}$. Then

$$x_1^t (x_2^t \cdots x_d^t) \in (x_1^{t+1}, \dots, x_d^{t+1})_R^{\mathrm{cl}}.$$

By hypothesis (1) on cl,

$$x_2^t \cdots x_d^t \in (x_1, x_2^{t+1}, \dots, x_d^{t+1})_R^{\mathrm{cl}}.$$

Using this hypothesis again,

$$x_3^t \cdots x_d^t \in (x_1, x_2, x_3^{t+1}, \dots, x_d^{t+1})_R^{\mathrm{cl}}.$$

Continuing in this manner, we see that

$$x_d^t \in (x_1, x_2, \dots, x_{d-1}, x_d^{t+1})_R^{\mathrm{cl}},$$

and taking one more step, $1 \in (x_1, \dots, x_d)_R^{\mathrm{cl}}$. However, $m_R^{\mathrm{cl}} = m$, so this is a contradiction. Therefore, $(I_t)_R^{\mathrm{cl}} = I_t$ for all t, which finishes the proof that $N_M^{\mathrm{cl}} = N$ for all submodules N of finitely-generated R-modules M. □

Theorem 5.9. *Dietz closures are trivial on regular local rings.*

Proof. Earlier, we showed that any Dietz closure is contained in a big Cohen–Macaulay module closure and that big Cohen–Macaulay module closures satisfy strong colon-capturing. Since they are Dietz closures, they satisfy the other two properties required to use Proposition 5.8. Therefore, Dietz closures are trivial on regular rings. □

It is also possible to show that big Cohen–Macaulay module closures are trivial on regular rings by noting that a big Cohen–Macaulay module B over a regular ring is faithfully flat [9], so that ideals and submodules of finitely-generated modules are "contracted" from B.

Theorem 5.10. *Suppose that (R, m, K) is a local domain that has at least one Dietz closure (in particular, it suffices for R to have equal characteristic and any dimension, or mixed characteristic and dimension at most 3), and that all Dietz closures on R are trivial. Then R is regular.*

Proof. Since R has a big Cohen–Macaulay module B that gives a trivial Dietz closure cl_B, R is Cohen–Macaulay. We show that R is also approximately Gorenstein. If $\dim(R) \geq 2$, then $\mathrm{depth}(R) \geq 2$, so this follows from [17]. If $\dim(R) = 0$, then R is a field, which is approximately Gorenstein. If $\dim(R) = 1$, then the integral closure S of R is a big Cohen–Macaulay algebra for R. Let $b/a \in S$. We have $b \in (a)^{\mathrm{cl}_S}$, but cl_S must be trivial on R, so $b \in (a)$. Hence $S = R$, and so R is normal. By [17], R is approximately Gorenstein.

Let $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_t \supseteq \ldots$ be a sequence of m-primary ideals such that each R/I_t is Gorenstein and the I_t are cofinal with the powers of m. Let $E = E_R(K)$, the injective hull of K over R. Then E is equal to the increasing union $\bigcup_t \mathrm{Ann}_E(I_t)$. Further, each $\mathrm{Ann}_E(I_t)$ is isomorphic to $E_{R/I_t}(K) \cong R/I_t$, so we have injective maps $R/I_t \to R/I_{t+1}$ for each $t \geq 1$. Let u_1 be a generator of the socle in R/I_1. For $t \geq 1$, let u_{t+1} be the image of u_t in R/I_{t+1}, which will generate the socle in R/I_{t+1}.

Suppose that M is a finitely-generated Cohen–Macaulay module with no free summand. We will show that M is equal to the increasing union of $I_tM : u_t$, so that $u_tM \subseteq I_tM$ for $t \gg 1$. This will imply that cl_M is a nontrivial Dietz closure. To see that the union is increasing, suppose that $v \in I_tM : u_t$. Then $u_tv \in I_tM$. Applying the map $I_tM \to I_{t+1}M$ induced by the map $R/I_t \to R/I_{t+1}$, we see that $u_{t+1}v \in I_{t+1}M$.

Suppose that $M \neq \bigcup_t I_tM : u_t$. Then we can pick $v \in M - \bigcup_t I_tM : u_t$. For every $t \geq 1$, $u_tv \notin I_tM$. Consider the map $R \to M$ given by multiplication by v. Since R is local and M is finitely-generated, this splits if and only if $E \to E \otimes M$ is injective. But this is true if and only if $R/I_t \to M/I_tM$ is injective for all $t \gg 1$. For any t, $u_t \mapsto u_tv \notin I_{t+1}M$, so the socle of R/I_t is not contained in the kernel of the map $R/I_t \to M/I_tM$. Hence $R/I_t \to M/I_tM$ is injective, which implies that $R \to M$ splits. This contradicts our assumption that M had no free summand.

If R is not regular, then since R is Cohen–Macaulay, $\mathrm{syz}^d(k)$ is a finitely-generated Cohen–Macaulay module that is not free. Then it has some minimal direct summand (that can't be written as a nontrivial direct sum) that is not free. This gives us a nontrivial Dietz closure on R. Therefore, R must be regular. $\square$

Remark 5.11. By a result of [4], $\mathrm{syz}^d(k)$ has no free summand when R is not regular, so we can use $\mathrm{syz}^d(k)$ instead of a minimal direct summand of it.

The following is a corollary to the proof of Theorem 5.10.

Corollary 5.12. *Let R be a local domain with at least one Dietz closure. Suppose that R has a finitely-generated Cohen–Macaulay module B with no free summands and that R is approximately Gorenstein but not regular. Then R has a nontrivial Dietz closure, cl_B.*

R satisfies these hypotheses when it is Cohen–Macaulay, $\dim(R) \neq 1$, and R is not regular. Alternatively, it suffices for R to be complete but not regular. If R is Cohen–Macaulay of dimension not equal to 1 but is not regular, $syz^d(k)$ gives a nontrivial closure on R. In particular, if R has equal characteristic and is weakly F-regular (or F-regular or strongly F-regular) but not regular, $\mathrm{cl}_{syz^d(k)}$ is nontrivial on R.

6. Proofs that certain closures are not Dietz closures

Dietz gives some examples of Dietz closures, as well as some closures that fail to be Dietz closures. Understanding why certain closure operations fail to be Dietz closures adds to our understanding of Dietz closures, and may help us find a good closure operation for rings of mixed characteristic. The following result gives one way for a closure operation to be “too big” to be a Dietz closure.

Theorem 6.1. *Let R be a local domain with $x_1, \ldots, x_k$ part of a system of parameters for R and $(x_1 \cdots x_k)^t \in (x_1^{t+1}, x_2^{t+1}, \ldots, x_k^{t+1})^{\mathrm{cl}}$ for some $t \geq 0$ and closure operation cl. Then cl is not a Dietz closure.*

Proof. Suppose that cl is a Dietz closure. Then by Theorem 5.1, there is a big Cohen–Macaulay module B such that $\mathrm{cl} \subseteq \mathrm{cl}_B$. Then we have

$$(x_1 \cdots x_k)^t \in (x_1^{t+1}, \ldots, x_k^{t+1})^{\mathrm{cl}} \subseteq (x_1^{t+1}, \ldots, x_k^{t+1})^{\mathrm{cl}_B}.$$

By Proposition 3.10, this implies that

$$(x_2 \cdots x_k)^t \in (x_1, x_2^{t+1}, \ldots, x_k^{t+1})^{\mathrm{cl}_B},$$

which implies that

$$(x_3 \cdots x_k)^t \in (x_1, x_2, x_3^{t+1}, \ldots, x_k^{t+1})^{\mathrm{cl}_B},$$

and so on until

$$1 \in (x_1, \ldots, x_k)^{\mathrm{cl}_B}.$$

But $(x_1, \ldots, x_k)^{\mathrm{cl}_B} \subseteq m^{\mathrm{cl}_B} = m$, which is a contradiction. Therefore, cl is not a Dietz closure. □

Corollary 6.2. *Integral closure is not a Dietz closure on R if* $\dim(R) \geq 2$.

Proof. Let x, y be part of a system of parameters for R. We always have $xy \in \overline{(x^2, y^2)}$, so by Theorem 6.1, integral closure is not a Dietz closure. □

Definition 6.3 *([22]).* We define *regular closure* on a ring R by $u \in N_M^{\mathrm{reg}}$ if for every regular R-algebra S, $u \in N_M^{\mathrm{cl}_S}$, where $N \subseteq M$ are finitely-generated R-modules and $u \in M$.

Lemma 6.4. *Let $R = k[[x, y, z]]/(x^3 + y^3 + z^3)$, where char$(k) \neq 3$. Then $(x, y)^t \subseteq (x^t, y^t)^{\mathrm{reg}}$.*

Proof. In [10], Hochster and Huneke show that $z \in (x,y)^{\text{reg}}$ but $z \notin (x,y)^*$, where $*$ denotes tight closure. To do this, they reduce to the case of maps to complete regular local rings with algebraically closed residue field and show that any solution (a,b,c) of $u^3+v^3+w^3=0$ in S has the form $(\alpha d, \beta d, \gamma d)$, where $d \in S$ and (α,β,γ) is a solution of the same equation such that either at least two of α, β, and γ are units, or all three are 0.

If all three are 0, then clearly $(x,y)^t S \subseteq (x^t, y^t)S$. If at least one of α or β is a unit, then $(x^t,y^t)S = (d^t)$, which must contain $(x,y)^t S$. Since these are the only possible cases, we have $(x,y)^t S \subseteq (x^t,y^t)S$ for any regular R-algebra S. Hence $(x,y)^t \subseteq (x^t,y^t)^{\text{reg}}$. □

Corollary 6.5. *Regular closure may fail to be a Dietz closure.*

Proof. Consider the ring $R = k[[x,y,z]]/(x^3+y^3+z^3)$, where $\text{char}(k) \neq 3$. In this ring, $xy \in (x^2,y^2)^{\text{reg}}$ by Lemma 6.4. □

Remark 6.6. By the exact argument used in Lemma 6.4, UFD closure (consider all R-algebras that are UFD's, rather than the regular R-algebras) may fail to be a Dietz closure.

Theorem 6.7. *For rings of equal characteristic 0, solid closure is not always a Dietz closure. In particular, solid closure is not a Dietz closure on regular local rings containing the rationals with dimension at least 3.*

Proof. By Theorem 5.9, Dietz closures are trivial on regular rings. By [19, Corollary 7.24] and [23], if R is a regular local ring containing the rationals with dimension at least 3, then solid closure is not trivial on R. Hence solid closure is not a Dietz closure on these rings. □

7. Full extended plus closure

We do not know whether Heitmann's mixed characteristic plus closure, full extended plus closure, and full rank one closure [7] are Dietz closures, even in dimension 3. To discuss this question, we first extend the definition of full extended plus closure (epf) to finitely generated modules. The other definitions can be extended similarly.

Definition 7.1. Let R be a mixed characteristic local domain, whose residue field has characteristic p. Let $N \subseteq M$ be finitely generated modules over R. We define the *full extended plus closure* of N in M by $u \in M$ is in N_M^{epf} if there is some $c \neq 0 \in R$ such that for all $n \in \mathbb{Z}_+$,

$$c^{1/n} \otimes u \in \text{im}(R^+ \otimes N + R^+ \otimes p^n M \to R^+ \otimes M).$$

Proposition 7.2. *For R a local domain of mixed characteristic p, full extended plus closure is a closure operation that satisfies the Functoriality Axiom, the Semi-residuality Axiom, and the Faithfulness Axiom, and $0_R^{\mathrm{epf}} = 0$.*

Proof. It is easy to prove the extension and order-preservation properties. To see that epf is idempotent, making it a closure operation, let $u \in (N_M^{\mathrm{epf}})_M^{\mathrm{epf}}$. Then there is some $c \neq 0$ in R such that

$$c^{1/n} \otimes u \in \mathrm{im}(R^+ \otimes N_M^{\mathrm{epf}} + R^+ \otimes p^n M \to R^+ \otimes M)$$

for all n, say

$$c^{1/n} \otimes u = \sum_i r_i \otimes y_i + \sum_j s_j \otimes p^n m_j,$$

with $r_i, s_j \in R^+$, $y_i \in N_M^{\mathrm{epf}}$, and $m_j \in M$. For each i, there is some nonzero $d_i \in R$ such that

$$d_i^{1/n} \otimes y_i \in \mathrm{im}(R^+ \otimes N + R^+ \otimes p^n M \to R^+ \otimes M).$$

Then

$$\begin{aligned} c^{1/n} \cdot \Pi_i d_i^{1/n} \otimes u &= \Pi_i d_i^{1/n} \left(\sum_i r_i \otimes y_i + \sum_j s_j \otimes p^n m_j \right) \\ &\in \mathrm{im}(R^+ \otimes N + R^+ \otimes p^n M \to R^+ \otimes M). \end{aligned}$$

Since $c \cdot \Pi_i d_i$ is a nonzero element of R, this proves that $u \in N_M^{\mathrm{epf}}$.

For the Functoriality Axiom, let $f : M \to W$ be an R-module homomorphism and $N \subseteq M$. Let $u \in N_M^{\mathrm{epf}}$. Then there is some nonzero $c \in R$ such that

$$c^{1/n} \otimes u \in \mathrm{im}(R^+ \otimes N + R^+ \otimes p^n M \to R^+ \otimes M)$$

for every $n > 0$. Apply f. This tells us that

$$c^{1/n} \otimes f(u) \in \mathrm{im}(R^+ \otimes f(N) + R^+ \otimes p^n W \to R^+ \otimes W)$$

for every $n > 0$, which implies that $f(u) \in f(N)_W^{\mathrm{epf}}$.

Next, suppose that $N_M^{\mathrm{epf}} = N$. We will show that $0_{M/N}^{\mathrm{epf}} = 0$. Let $\bar{u} \in 0_{M/N}^{\mathrm{epf}}$, where $u \in M$. Then there is some nonzero $c \in R$ with

$$c^{1/n} \otimes \bar{u} \in \mathrm{im}(R^+ \otimes p^n(M/N) \to R^+ \otimes M).$$

But $R^+ \otimes p^n(M/N)$ is isomorphic to $p^n(R^+ \otimes M)/(R^+ \otimes N)$, which tells us that

$$c^{1/n} \otimes u \in \operatorname{im}(R^+ \otimes p^n M + R^+ \otimes N \to R^+ \otimes M).$$

This implies that $u \in N_M^{\text{epf}} = N$, so $\bar{u} = 0$ in M/N.

To see that $m_R^{\text{epf}} = m$, let $u \in m_R^{\text{epf}}$. Then

$$c^{1/n} u \in (m, p^n)R^+$$

for some nonzero $c \in R$ (using the ideal version of the definition of epf) and for all n. Since $p^n \in m$, $c^{1/n}u \in mR^+$ for all n. If $u \notin m$, then $c^{1/n} \in mR^+$ for all n. But we can extend the m-adic valuation on R to a $\mathbb{Q}$-valued valuation on R^+. The order of $c^{1/n}$ will be $\frac{1}{n}\operatorname{ord}(c)$. So this is impossible.

Now let $u \in 0_R^{\text{epf}}$. Then $c^{1/n}u \in p^n R^+$ for some $c \neq 0$ in R and for all n. Let ord denote a $\mathbb{Q}$-valued valuation on R^+ that extends the m-adic valuation on R. Let $s = \operatorname{ord}(c)$ and $t = \operatorname{ord}(p)$. Then we must have $s/n + \operatorname{ord}(u) \geq nt$ for all n. This implies that $u = 0$. $\square$

A similar argument works for mixed characteristic plus closure and for full rank one closure.

If at least one of these closures is a Dietz closure in dimension 3, this would tie the results of [7,20] in to the results of this paper. If they are not Dietz closures in dimension 3, this would imply that the Dietz axioms are stronger than they need to be–there could be a weaker set of axioms that would be sufficient for the proof of the Direct Summand Conjecture in mixed characteristic rings.

8. Connections between Dietz closures and other closure operations

We show that Dietz closures are contained in (liftable) integral closure. This is proved for ideals in [1] with the added assumption that the closures are persistent for change of rings, but we do not need this assumption here.

Theorem 8.1. *Let R be a domain and $\mathrm{cl} = \mathrm{cl}_M$ where M is a solid module over R. Then $I^{\mathrm{cl}} \subseteq \bar{I}$ for every ideal I of R.*

Proof. Since M is solid, there is some nonzero map $f : M \to R$, with image $\mathfrak{a}$, a nonzero ideal of R. Suppose that $I \subseteq J \subseteq I^{\mathrm{cl}}$. Then $JM = IM$. Applying f, we get $J\mathfrak{a} = I\mathfrak{a}$. Since R is a domain, $\mathfrak{a}$ is a finitely-generated, torsion-free R-module. By the lemma below, $J \subseteq \bar{I}$. $\square$

Lemma 8.2 *([21]). Suppose that $I \subseteq J$ are ideals of a domain R such that $IM = JM$ for some finitely-generated, torsion-free R-module M. Then $J \subseteq \bar{I}$.*

Corollary 8.3. *Let R be a complete local domain and B a big Cohen–Macaulay module over R. Then $I^{\mathrm{cl}_B} \subseteq \bar{I}$ for all ideals I of R.*

Proof. By [19, Proposition 10.5], B is a solid module over R. Hence by Theorem 8.1, $I^{\mathrm{cl}_B} \subseteq \bar{I}$ for every ideal I of R. $\square$

There are several ways to extend integral closure to modules. Here we use liftable integral closure, denoted $\vdash$, as defined by Epstein and Ulrich.

Definition 8.4 *([6]).* Let G be a submodule of a finitely-generated free R-module R^s, let S be the symmetric algebra over R defined by R^s, and let T be the subring of S induced by the inclusion $G \subseteq R^s$. Observe that S is $\mathbb{N}$-graded and generated in degree 1 over R, and that T is an $\mathbb{N}$-graded subring of S, also generated in degree 1 over R. We define the *integral closure* $G^-_{R^s}$ of G in R^s to be the degree 1 part of the integral closure of the subring T of S.

Now let $N \subseteq M$ be finitely-generated R-modules. Take a free module R^s and a surjection $\pi : R^s \to M$, and let $G = \pi^{-1}(N)$. We define the *liftable integral closure* of N in M by

$$N^{\vdash}_M = \pi(G^-_{R^s}).$$

Proposition 8.5. *Let R be a domain and $\mathrm{cl} = \mathrm{cl}_M$ where M is a solid R-module. Then for all finitely-generated free modules F over R and all submodules G of F, $G^{\mathrm{cl}}_F \subseteq G^{\vdash}_F$.*

Proof. Let F be a free module of rank h over R and $G \subseteq F$. Let $S = \mathrm{Sym}(F) \cong R[x_1, \dots, x_h]$, I the ideal generated by the image of G in S, and $\widetilde{M} = S \otimes_R M$. We will show that G^{cl}_F is contained in the degree one piece of $I_S^{\mathrm{cl}_{\widetilde{M}}}$.

Suppose that $u \in G^{\mathrm{cl}}_F$. Then for every $m \in M$, $m \otimes u \in \mathrm{im}(M \otimes G \to M^h)$. This implies that $m \otimes u \otimes 1 \in \mathrm{im}(M \otimes_R G \otimes_R S \to M^h \otimes_R S)$. By associativity and commutativity of tensor, $M \otimes_R G \otimes_R S \cong \widetilde{M} \otimes_S I \cong I\widetilde{M}$. This isomorphism takes $m \otimes u \otimes 1 \mapsto u(1 \otimes m)$. Then $u(s \otimes m) \in I\widetilde{M}$ for all $s \in S$, $m \in M$, which implies that $u \in I_S^{\mathrm{cl}_{\widetilde{M}}}$. Since $u \in G$, its image in S is of degree 1.

Since S is a domain and $\widetilde{M}$ is solid over S, $I^{\mathrm{cl}_{\widetilde{M}}} \subseteq \bar{I}$ for all ideals I of S. This implies that u is contained in the degree 1 piece of $\bar{I}$, and hence $u \in G^{\vdash}_F$. $\square$

Theorem 8.6. *Let R be a domain and $\mathrm{cl} = \mathrm{cl}_M$ where M is a solid module over R. Then* cl *is contained in liftable integral closure. In particular, if R is a complete local domain, all big Cohen–Macaulay modules closures on R are contained in liftable integral closure. This implies that all Dietz closures on R are contained in liftable integral closure.*

Proof. Let $L \subseteq N$ be finitely-generated modules over R, and let $\pi : F \to N$ be a surjection of a finitely-generated free module F onto M. Let $K = \pi^{-1}(L)$. Let $u \in L^{\mathrm{cl}}_N$. Then by Lemma 3.1, any lift $\tilde{u}$ of u to F is contained in K^{cl}_F. By Proposition 8.5, $\tilde{u} \in K^{\vdash}_F$. Hence $u \in L^{\vdash}_N$. $\square$

Recall that a family of closure operations cl on a class of rings and maps between them is persistent for change of rings if given any $R \to S$ in the class, and $N \subseteq M$ finitely-generated R-modules,

$$S \otimes_R N_M^{\mathrm{cl}} \subseteq (S \otimes_R N)^{\mathrm{cl}}_{S \otimes_R M}.$$

Proposition 8.7. *Any persistent family of Dietz closures is contained in regular closure.*

Proof. Suppose that $u \in I^{\mathrm{cl}}$. Then in any map to a regular ring S, $u \in (IS)^{\mathrm{cl}} = IS$ by persistence. So $u \in I^{\mathrm{reg}}$. $\square$

9. Further questions

9.1. Examples of Cohen–Macaulay module closures

In the proof of Theorem 5.10, we showed that if a local domain (R, m, k) is Cohen–Macaulay of dimension not equal to 1 but is not regular, $\mathrm{cl}_{\mathrm{syz}^d(k)}$ is a non-trivial Dietz closure for R. We give another class of non-trivial Dietz closures, which can only occur when R is not regular.

Example 9.1. Let $R = k[[x^2, xy, y^2]]$. Then $M = (x^2, xy)$ is a non-maximal Cohen–Macaulay module over R (it has height=depth=1). Let $I = (x^4, x^3y, xy^3, y^4)$ and $J = (x^4, x^3y, x^2y^2, xy^3, y^4)$. Then $I \subsetneq J$, but

$$I(x^2, xy) = (x^6, x^5y, x^3y^3, x^2y^4, x^5y, x^4y^2, x^2y^4, xy^5) = J(x^2, xy).$$

So $I^{\mathrm{cl}_M} = J^{\mathrm{cl}_M}$.

Example 9.2. Let $R = k[[x, y, u, v]]/(xy - uv)$. Then $M = (x, u)$ is a non-maximal Cohen–Macaulay module over R. Let $I = (y^2, v^2)$ and $J = (yv)$. Then $I \neq J$, but $IM = JM = (xyv, yuv)$, so $I^{\mathrm{cl}_M} = J^{\mathrm{cl}_M}$.

In addition, if we let $I = (x^2, u^2)$ and $J = (x^2, xu, u^2)$, then $IM = JM = (x^3, x^2u, xu^2, u^3)$, even though $I \neq J$.

This gives rise to a more general class of examples: suppose that (x, u) is a non-principal ideal that is a Cohen–Macaulay module (height 1, depth 1), and $xu \notin (x^2, u^2)$. Then $(x^2, u^2)(x, u) = (x^2, xu, u^2)(x, u)$.

Example 9.3. Let $R = k[[x, y, z]]/(x^3 + y^3 + z^3)$, with the characteristic of k not equal to 3. Then $(x, y + z)$ is a height 1 prime of depth 1. Since $x(y + z) \notin (x, y + z)^2$, we are in the case above.

All of these examples are Gorenstein rings, so in each case the canonical module (a maximal Cohen–Macaulay module) is equal to the ring.

Question 9.4. If R is not Gorenstein and has a canonical module ω, then ω is a Cohen–Macaulay module for R with no free summand. Hence by the proof of Theorem 5.10, cl_ω is a nontrivial closure operation on R. How else might we characterize this closure?

9.2. Largest big Cohen–Macaulay module closure

We do not know whether there is a largest Dietz closure. If there is one, then by Theorem 5.1 it will also be the largest big Cohen–Macaulay module closure. Hence there is a largest big Cohen–Macaulay module closure if and only if there is a largest Dietz closure.

Proposition 9.5. *If Dietz closures on a local domain R form a directed set, then the sum of all Dietz closures is equal to the largest Dietz closure.*

Proof. Let D denote the sum of all Dietz closures. To see that it is a Dietz closure (it will be a closure operation, by [5]), we use the fact [5] that since R is Noetherian, for any particular $N \subseteq M$ finitely-generated R-modules, there is some Dietz closure cl such that $N_M^{\mathrm{cl}} = N_M^{\mathrm{D}}$.

Functoriality Axiom: Let $f : M \to W$ be a map of R-modules, and $N \subseteq M$. Let cl be a Dietz closure such that $N_M^{\mathrm{cl}} = N_M^{\mathrm{D}}$. Then $f(N_M^{\mathrm{D}}) = f(N_M^{\mathrm{cl}}) \subseteq f(N)_W^{\mathrm{cl}} \subseteq f(N)_W^{\mathrm{D}}$.

Semi-residuality Axiom: Suppose that $N_M^{\mathrm{D}} = N$. Then $N_M^{\mathrm{cl}} = N$ for every Dietz closure cl. Hence $0_{M/N}^{\mathrm{cl}} = 0$ for every Dietz closure cl, which implies that $0_{M/N}^{\mathrm{D}} = 0$.

Faithfulness Axiom: We must have $m^{\mathrm{D}} = m$, since m is cl-closed for any Dietz closure cl.

Generalized Colon-Capturing Axiom: With R, v, and J as in the statement of Axiom 4, let cl be a Dietz closure such that $(Rv)_M^{\mathrm{D}} = (Rv)_M^{\mathrm{cl}}$. Then $(Rv)_M^{\mathrm{D}} \cap \ker(f) = (Rv)_M^{\mathrm{cl}} \cap \ker(f) \subseteq (Jv)_M^{\mathrm{cl}} \subseteq (Jv)_M^{\mathrm{D}}$. $\square$

So to prove that there is a largest Dietz closure, it suffices to show that Dietz closures form a directed set. To do this, it would be enough to show that given 2 Dietz closures cl and cl$'$, we can construct a big Cohen–Macaulay module B such that $\mathrm{cl}, \mathrm{cl}' \subseteq \mathrm{cl}_B$. It is not clear that if we perform a modification that is cl-phantom, then one that is cl$'$-phantom, that $\mathrm{im}(1)$ stays out of the image of m, so we do not know of a way to construct such a big Cohen–Macaulay module.

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References

[1] Geoffrey Dietz, Closure Operations in Positive Characteristic and Big Cohen–Macaulay Algebras, PhD thesis, University of Michigan, Ann Arbor, 2005.
[2] Geoffrey D. Dietz, A characterization of closure operators that induce big Cohen-Macaulay modules, Proc. Amer. Math. Soc. 138 (11) (2010) 3849–3862.
[3] Geoffrey Dietz, Axiomatic closure operations, phantom extensions, and solidity, preprint, http://arxiv.org/abs/1511.04286, 2015.
[4] S.P. Dutta, Syzygies and homological conjectures, in: Commutative Algebra, Berkeley, CA, 1987, in: Math. Sci. Res. Inst. Publ., vol. 15, Springer, New York, 1989, pp. 139–156.
[5] Neil Epstein, A guide to closure operations in commutative algebra, in: Progress in Commutative Algebra 2, Walter de Gruyter, Berlin, 2012, pp. 1–37.
[6] Neil Epstein, Bernd Ulrich, Liftable integral closure, preprint, http://arxiv.org/abs/1309.6966, 2014.
[7] Raymond C. Heitmann, The direct summand conjecture in dimension three, Ann. of Math. (2) 156 (2) (2002) 695–712.
[8] Melvin Hochster, Craig Huneke, Tight closure, invariant theory, and the Briançon–Skoda theorem, J. Amer. Math. Soc. 3 (1) (1990) 31–116.
[9] Melvin Hochster, Craig Huneke, Infinite integral extensions and big Cohen–Macaulay algebras, Ann. of Math. (2) 135 (1) (1992) 53–89.
[10] Melvin Hochster, Craig Huneke, Phantom homology, Mem. Amer. Math. Soc. 103 (490) (1993), vi+91.
[11] Melvin Hochster, Craig Huneke, F-regularity, test elements, and smooth base change, Trans. Amer. Math. Soc. 346 (1) (1994) 1–62.
[12] Melvin Hochster, Craig Huneke, Indecomposable canonical modules and connectedness, in: Commutative Algebra: Syzygies, Multiplicities, and Birational Algebra, South Hadley, MA, 1992, in: Contemp. Math., vol. 159, Amer. Math. Soc., Providence, RI, 1994, pp. 197–208.
[13] Melvin Hochster, Craig Huneke, Tight closure of parameter ideals and splitting in module-finite extensions, J. Algebraic Geom. 3 (4) (1994) 599–670.
[14] Melvin Hochster, Craig Huneke, Applications of the existence of big Cohen–Macaulay algebras, Adv. Math. 113 (1) (1995) 45–117.
[15] M. Hochster, Contracted ideals from integral extensions of regular rings, Nagoya Math. J. 51 (1973) 25–43.
[16] Melvin Hochster, Topics in the homological theory of modules over commutative rings, Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society, Providence, R.I., 1975 in: Expository Lectures from the CBMS Regional Conference Held at the University of Nebraska, Lincoln, Neb., June 24–28, 1974, in: Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, vol. 24, 1975.
[17] Melvin Hochster, Cyclic purity versus purity in excellent Noetherian rings, Trans. Amer. Math. Soc. 231 (2) (1977) 463–488.
[18] Melvin Hochster, Canonical elements in local cohomology modules and the direct summand conjecture, J. Algebra 84 (2) (1983) 503–553.
[19] Melvin Hochster, Solid closure, in: Commutative Algebra: Syzygies, Multiplicities, and Birational Algebra, South Hadley, MA, 1992, in: Contemp. Math., vol. 159, Amer. Math. Soc., Providence, RI, 1994, pp. 103–172.
[20] Melvin Hochster, Big Cohen–Macaulay algebras in dimension three via Heitmann's theorem, J. Algebra 254 (2) (2002) 395–408.
[21] Craig Huneke, Irena Swanson, Integral Closure of Ideals, Rings, and Modules, London Math. Soc. Lecture Note Ser., vol. 336, Cambridge University Press, Cambridge, 2006.
[22] Moira A. McDermott, Regular closure, Proc. Amer. Math. Soc. 127 (7) (1999) 1975–1979.
[23] Paul C. Roberts, A computation of local cohomology, in: Commutative Algebra: Syzygies, Multiplicities, and Birational Algebra, South Hadley, MA, 1992, in: Contemp. Math., vol. 159, Amer. Math. Soc., Providence, RI, 1994, pp. 351–356.