Journal of Algebra 492 (2017) 44–51

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Ideal containments under flat extensions

Solomon Akesseh

ARTICLE INFO

Article history:
Received 22 December 2016
Available online 18 August 2017
Communicated by Steven Dale Cutkosky

Keywords:
Symbolic powers
Fat points
Points in the projective plane
Fermat configuration
Regular sequences
Cohen–Macaulay
Algebraic geometry
Commutative algebra

ABSTRACT

Let $\varphi : S = k[y_0, ..., y_n] \to R = k[y_0, ..., y_n]$ be given by $y_i \to f_i$ where $f_0, ..., f_n$ is an R-regular sequence of homogeneous elements of the same degree. A recent paper shows for ideals, $I_\Delta \subsetneq S$, of matroids, Δ, that $I_\Delta^{(m)} \subseteq I^r$ if and only if $\varphi_*(I_\Delta)^{(m)} \subseteq \varphi_*(I_\Delta)^r$ where $\varphi_*(I_\Delta)$ is the ideal generated in R by $\varphi(I_\Delta)$. We prove this result for saturated homogeneous ideals I of configurations of points in $\mathbb{P}^n$ and use it to obtain many new counterexamples to $I^{(rn-n+1)} \subseteq I^r$ from previously known counterexamples.

1. Introduction

Let R be a commutative Noetherian domain. Let I be an ideal in R. We define the mth symbolic power of I to be the ideal

$$I^{(m)} = R \cap \bigcap_{P \in Ass_R(I)} I^m R_P \subseteq R_{(0)}.$$

E-mail address: sakesseh2@math.unl.edu.

http://dx.doi.org/10.1016/j.jalgebra.2017.07.026

In this note we shall be interested in symbolic powers of homogeneous ideals of 0-dimensional subschemes in $\mathbb{P}^n$. In the case that the subscheme is reduced, the definition of the symbolic power takes a rather simple form by a theorem of Zariski and Nagata [11] and does not require passing to the localizations at various associated primes. Let $I \subseteq k[\mathbb{P}^n]$ be a homogeneous ideal of reduced points, $p_1, ..., p_l$, in $\mathbb{P}^n$ with k a field of any characteristic. Then $I = I(p_1) \cap \cdots \cap I(p_l)$ where $I(p_i) \subseteq k[\mathbb{P}^n]$ is the ideal generated by all forms vanishing at p_i, and the mth symbolic power of I is simply $I^{(m)} = I(p_1)^m \cap \cdots \cap I(p_l)^m$.

In [10], Ein, Lazarsfeld and Smith proved that if $I \subseteq k[\mathbb{P}^n]$ is the radical ideal of a 0-dimensional subscheme of $\mathbb{P}^n$, where k is an algebraically closed field of characteristic 0, then $I^{(mr)} \subseteq (I^{(r+1-n)})^m$ for all $m \in \mathbb{N}$ and $r \geq n$. Letting $r = n$, we get that $I^{(mn)} \subseteq I^m$ for all $m \in \mathbb{N}$. Hochster and Huneke in [15] extended this result to all ideals $I \subseteq k[\mathbb{P}^n]$ over any field k of arbitrary characteristic.

In [5] Bocci and Harbourne introduced a quantity $\rho(I)$, called the resurgence, associated to a nontrivial homogeneous ideal I in $k[\mathbb{P}^n]$, defined to be $\sup\{s/t : I^{(s)} \nsubseteq I^t\}$. It is seen immediately that if $\rho(I)$ exists, then for $s > \rho(I)t$, $I^{(s)} \subseteq I^t$. The results of [10,15] guarantee that $\rho(I)$ exists since $I^{(mn)} \subseteq I^m$ implies that $\rho(I) \leq n$ for an ideal I in $k[\mathbb{P}^n]$. For an ideal I of points in $\mathbb{P}^2$, $I^{(mn)} \subseteq I^m$ gives $I^{(4)} \subseteq I^2$. According to [5] Huneke asked if $I^{(3)} \subseteq I^2$ for a homogeneous ideal I of points in $\mathbb{P}^2$. More generally Harbourne conjectured in [3] that if $I \subseteq k[\mathbb{P}^n]$ is a homogeneous ideal, then $I^{(rn-(n-1))} \subseteq I^r$ for all r. This led to the conjectures by Harbourne and Huneke in [13] for ideals I of points that $I^{(mn-n+1)} \subseteq \mathfrak{m}^{(m-1)(n-1)} I^m$ and $I^{(mn)} \subseteq \mathfrak{m}^{m(n-1)} I^m$ for $m \in \mathbb{N}$.

The second conjecture remains open. Cooper, Embree, Ha and Hoeful give a counterexample in [7] to the first for $n = 2 = m$ for a homogeneous ideal $I \subseteq k[\mathbb{P}^2]$. The ideal I in this case is $I = (xy^2, yz^2, zx^2, xyz) = (x^2, y) \cap (y^2, z) \cap (z^2, x)$ whose zero locus in $\mathbb{P}^2$ is the 3 coordinate vertices of $\mathbb{P}^2$, $[0 : 0 : 1]$, $[0 : 1 : 0]$ and $[1 : 0 : 0]$ together with 3 infinitely near points, one at each of the vertices, for a total of 6 points. Clearly the monomial $x^2y^2z^2 \in (x^2, y)^3 \cap (y^2, z)^3 \cap (z^2, x)^3$ so $x^2y^2z^2$ is in $I^{(3)}$. Note $xyz \in I$ so $x^2y^2z^2 \in I^2$, but $x^2y^2z^2 \notin \mathfrak{m}I^2$.

Shortly thereafter a counterexample to the containment $I^{(3)} \nsubseteq I^2$ was given by Dumnicki, Szemberg and Tutaj-Gasinska in [9]. In this case I is the ideal of the 12 points dual to the 12 lines of the Hesse configuration. The Hesse configuration consists of the 9 flex points of a smooth cubic and the 12 lines through pairs of flexes. Thus I defines 12 points lying on 9 lines. Each of the lines goes through 4 of the points, and each point has 3 of the lines going through it. Specifically I is the saturated radical homogeneous ideal $I = (x(y^3 - z^3), y(x^3 - z^3), z(x^3 - y^3)) \subset \mathbb{C}[\mathbb{P}^2]$. Its zero locus is the 3 coordinate vertices of $\mathbb{P}^2$ together with the 9 intersection points of any 2 of the forms $x^3 - y^3$, $x^3 - z^3$ and $y^3 - z^3$. The form $F = (x^3 - y^3)(x^3 - z^3)(y^3 - z^3)$ defining the 9 lines belongs to $I^{(3)}$ since for each point in the configuration, 3 of the lines in the zero locus of F pass through the point, but $F \notin I^2$ and hence $I^{(3)} \nsubseteq I^2$. (Of course this also means that $I^{(3)} \nsubseteq \mathfrak{m}I^2$.) More generally, $I = (x(y^n - z^n), y(x^n - z^n), z(x^n - y^n))$ defines a configuration of $n^2 + 3$ points called a Fermat configuration [1]. For $n \geq 3$, we again have $I^{(3)} \nsubseteq I^2$ [14,17] over any field of characteristic not 2 or 3 containing n distinct nth roots of 1.

Subsequent counterexamples to $I^3 \subseteq I^2$ were given in [4], [2], [14], [8] and [17] including related counterexamples to $I^{(nr-n+1)} \subseteq I^r$ for ideals of points in $\mathbb{P}^n$ in positive characteristic given in [14]. All of the counterexamples to $I^3 \subseteq I^2$ are ideals of points where the points are singular points of multiplicity at least 3 of a configuration of lines. By considering flat morphisms $\mathbb{P}^n \to \mathbb{P}^n$, we obtain many new counterexamples to $I^{(rn-n+1)} \subseteq I^r$, taking I to be the ideal of the fibers over the points of previously known counterexamples.

The idea for this comes from [12]. Suppose Δ is a matroid on $[s] = \{1, ..., s\}$ of dimension $s-1-c$ and let $f_1, ..., f_s \in R = k[y_0, ..., y_n]$ be homogeneous polynomials that form an R-regular sequence, $n \geq c$. Suppose now that $\varphi : S = k[y_1, ..., y_s] \to R$ is a k-algebra map defined by $y_i \to f_i$. Then [12] shows that if $I_\Delta \subseteq S$ is the ideal of the matroid and m and r are positive integers, then $I_\Delta^{(m)} \subseteq I_\Delta^r$ if and only if $\varphi_*(I_\Delta)^{(m)} \subseteq \varphi_*(I_\Delta)^r$ where $\varphi_*(I_\Delta)$ denotes the ideal generated by $\varphi(I_\Delta)$ in R. Of course a natural question is whether $I^{(m)} \subseteq I^r$ if and only if $\varphi_*(I)^{(m)} \subseteq \varphi_*(I)^r$ for any saturated homogeneous ideal. The current note answers this question in the affirmative for ideals I of points in $\mathbb{P}^n$, relying on the ideas in [12].

I would like to thank Brian Harbourne, my adviser, for suggesting the idea of this note and providing guidance throughout its writing. I would like to thank Tom Marley, Alexandra Seceleanu and Mark Walker for helpful conversations.

2. Results

Throughout this note, let $R = S = k[y_0, ..., y_n]$ and let $\{f_0, ..., f_n\} \subseteq R$ be an R-regular sequence of homogeneous elements of R of the same degree. Let $\varphi : S \to R$ be the k-algebra map given by $y_i \mapsto f_i$. For an ideal $I \subseteq S$, let $\varphi_*(I) \subseteq R$ denote the ideal generated by $\varphi(I)$.

Lemma 1. *Let $\varphi : S \to R$ be as above. Then R is a free graded S-module, hence R is faithfully flat as an S-module.*

Proof. It suffices to show that R is free over S since free modules are faithfully flat modules. Note that φ is injective since $\{f_0, ..., f_n\}$ is a regular sequence. It follows that $S \cong k[f_0, ..., f_n] \subseteq R$. So we identify S with $k[f_0, ..., f_n]$ and show that R is free over $k[f_0, ..., f_n]$. Since $\{f_0, ..., f_n\}$ is a maximal homogeneous R-regular sequence, it is a homogeneous system of parameters (sop). The reason is that every regular sequence is part of an sop and because R is Cohen–Macaulay (CM), every sop is a regular sequence ($\mathrm{depth} R = \dim R$) and so if $\{f_0, ..., f_n\}$ is a maximal regular sequence, then it is an sop. Since $R = k[\mathbb{P}^n]$ is a positively graded affine k-algebra, the fact that $\{f_0, ..., f_n\}$ is a homogeneous sop is equivalent to R being a finite S-module by [6, Theorem 1.5.17]. Since both R and S are CM, $\mathrm{depth} R = \dim R = n+1 = \dim S = \mathrm{depth} S$. By the Auslander–Buchsbaum formula [11, Exercise 19.8] [16, Theorem 15.3], $\mathrm{pd}_S R + \mathrm{depth}\, R = \mathrm{depth}\, S$. It follows that $\mathrm{pd}_S R = 0$. So looking at the minimal free resolution of R as an S-module, we see that R is a free S-module. Therefore R is a faithfully flat S-module. $\square$

Lemma 2. *Let $I \subseteq S$ be a homogeneous saturated ideal defining a 0-dimensional subscheme of $\mathbb{P}^n$. Then $\varphi_*(I) \subseteq R$ also defines a 0-dimensional subscheme of $\mathbb{P}^n$.*

Proof. We start by showing that $R/\varphi_*(I)$ has the same Krull dimension as S/I. By the graded Auslander–Buchsbaum formula, $\mathrm{pd}_S(R/\varphi_*(I)) + \mathrm{depth}(R/\varphi_*(I)) = \mathrm{depth}(S) = \mathrm{pd}_S(S/I) + \mathrm{depth}(S/I)$. By 3.1 in [12], S/I and $R/\varphi_*(I)$ have the same graded Betti numbers so $\mathrm{pd}_S(S/I) = \mathrm{pd}_S(R/\varphi_*(I))$. Therefore $\mathrm{depth}(S/I) = \mathrm{depth}(R/\varphi_*(I))$. By 3.1 in [12] again, S/I is Cohen–Macaulay (CM) if and only if $R/\varphi_*(I)$ is CM. Since I defines an ideal of points and is saturated, we have that S/I is CM. It follows that $R/\varphi_*(I)$ is CM. For CM modules, the depth is the dimension so that $\dim S/I = \dim R/\varphi_*(I)$. Now since S/I and $R/\varphi_*(I)$ are both CM, $\mathrm{Ass}(R/\varphi_*(I))$ and $\mathrm{Ass}(S/I)$ are both unmixed with their elements having height $\mathrm{ht}(\varphi_*(I))$ and $\mathrm{ht}(I)$ respectively. But $\mathrm{ht}(\varphi_*(I)) = \mathrm{ht}(I)$ since $\dim S/I = \dim R/\varphi_*(I)$. It follows that the elements of $\mathrm{Ass}(R/\varphi_*(I))$ are all ideals of points. It follows that $\varphi_*(I)$ defines a 0-dimensional subscheme of $\mathbb{P}^n$. $\square$

Lemma 3. *Let $I \subseteq S$ be a saturated homogeneous ideal such that the zero locus of I in $\mathbb{P}^n$ is 0-dimensional. Let $\varphi : S \to R$ be as above. Then $\varphi_*(I^{(m)}) = \varphi_*(I)^{(m)}$.*

Proof. By Lemma 2, $\varphi_*(I)$ is the defining ideal of a 0-dimensional subscheme so that $(\varphi_*(I))^{(m)} = \mathrm{Sat}((\varphi_*(I))^m)$ where $\mathrm{Sat}((\varphi_*(I))^m)$ denotes the saturation of the ideal $(\varphi_*(I))^m$. An ideal and its saturation have the same graded homogeneous components for high enough degree so that for $t \gg 0$, $((\varphi_*(I))^{(m)})_t = ((\varphi_*(I))^m)_t$.

Using again that the symbolic power of an ideal of a 0-dimensional subscheme in $\mathbb{P}^n$ is the saturation of the ordinary power, $I^{(m)} = \mathrm{Sat}(I^m)$, we have that $(I^{(m)})_t = (I^m)_t$ for $t \gg 0$. Therefore $(\varphi_*(I^{(m)}))_t = (I^{(m)} \otimes_S R)_t = (I^m \otimes_S R)_t = (\varphi_*(I^m))_t$ for $t \gg 0$. Since φ is a ring map, $\varphi_*(I^m) = (\varphi_*(I))^m$. This gives that $(\varphi_*(I^{(m)}))_t = ((\varphi_*(I))^m)_t$ for $t \gg 0$.

The last two paragraphs imply that $((\varphi_*(I))^{(m)})_t = \varphi_*(I^{(m)})_t$ for $t \gg 0$. Recall that $(\varphi_*(I))^{(m)}$ is saturated since it is the saturation of $(\varphi_*(I))^m$ and $\varphi_*(I^{(m)})$ is saturated by Lemma 3.1 in [12]. Two saturated graded homogeneous ideals that agree in degree t for $t \gg 0$, agree in all degrees. Hence $(\varphi_*(I))^{(m)} = \varphi_*(I^{(m)})$. $\square$

Theorem 4. *Let $I \subseteq S$ be a saturated homogeneous ideal such that $V(I) \subseteq \mathbb{P}^n$ is a 0-dimensional subscheme. Let $\varphi : S \to R$ be given by $y_i \to f_i$, $0 \leq i \leq n$, where $\{f_0, ..., f_n\}$ is an R-regular sequence of homogeneous elements of R of the same degree. Let $\varphi_*(I)$ denote the ideal in R generated by $\varphi(I)$. Then $I^{(m)} \subseteq I^r$ if and only if $(\varphi_*(I))^{(m)} \subseteq (\varphi_*(I))^r$.*

Proof. ($\Longrightarrow$) Suppose that $I^{(m)} \subseteq I^r$. Then $\varphi(I^{(m)}) \subseteq \varphi(I^r)$ and so $\varphi_*(I^{(m)}) \subseteq \varphi_*(I^r)$. Since φ is a homomorphism, $\varphi(I^r) = (\varphi(I))^r$. Note that $\varphi(I^r)$ generates $\varphi_*(I^r)$ in R and $(\varphi(I))^r$ generates $(\varphi_*(I))^r$ in R. It follows that $\varphi_*(I^r) = (\varphi_*(I))^r$ since they have the same generating set. Now applying Lemma 3 we have that $(\varphi_*(I))^{(m)} = \varphi_*(I^{(m)}) \subseteq \varphi_*(I^r) = \varphi_*(I)^r$ concluding the forward direction.

($\Longleftarrow$) Suppose now that for some homogeneous ideals I and J of S, $I \nsubseteq J$ but $\varphi_*(I) \subseteq \varphi_*(J)$. Then there is a homogeneous element $f \in I \backslash J$ such that $\varphi(f) \in \varphi_*(J)$. We may assume with no loss in generality that $I = (f)$. We have the sequence

$$0 \to I \cap J \to I \oplus J \to I + J \to 0$$

with the first map given by $g \mapsto (g, -g)$ and the second map given by $(h, r) \mapsto h + r$. It is clear that the sequence is exact. Since φ is faithfully flat, we get an exact sequence

$$0 \to \varphi_*(I \cap J) \to \varphi_*(I) \oplus \varphi_*(J) \to \varphi_*(I + J) \to 0.$$

Since $\varphi_*(I) \subseteq \varphi_*(J)$, $\varphi_*(I + J) = \varphi_*(J)$. Then the map $\varphi_*(I) \oplus \varphi_*(J) \to \varphi_*(J)$ has kernel $\varphi_*(I)$. It follows that $\varphi_*(I \cap J) = \varphi_*(I)$. This is impossible since the generators of $\varphi_*(I \cap J)$ are the images of the generators of $I \cap J$ and thus have degree greater than degree f and hence greater than degree of $\varphi(f)$ which generates $\varphi_*(I) = I \otimes_S R \neq 0$.

So it is the case that $\varphi(f) \notin \varphi_*(J)$. Hence $\varphi_*(I) \nsubseteq \varphi_*(J)$. Therefore if $I^{(m)} \nsubseteq I^r$, then by Lemma 3, $(\varphi_*(I))^{(m)} = \varphi_*(I^{(m)}) \nsubseteq (\varphi_*(I))^r$. Hence $(\varphi_*(I))^{(m)} \subseteq (\varphi_*(I))^r$ if and only if $I^{(m)} \subseteq I^r$. $\square$

3. Examples

Using the above result, we obtain many new counterexamples to the containment $I^{(3)} \subseteq I^2$ of ideals in $k[\mathbb{P}^2]$ and more generally counterexamples to the containment

$$I^{(nr-n+1)} \subseteq I^r \tag{$\star$}$$

in $\mathbb{P}^n$. In particular if $I \subseteq k[\mathbb{P}^n]$ gives a counterexample to ($\star$), then $\varphi_*(I)$ is a counterexample for any choice of homogeneous regular sequence $\{f_0, ..., f_n\}$ of elements of the same degree. We illustrate this below with a few examples.

Example 1. In this example, we work over $\mathbb{C}$. In [9], the Fermat configuration, for $n = 3$, was considered and its ideal $I = (x(y^3 - z^3), y(x^3 - z^3), z(x^3 - y^3)) \subseteq \mathbb{C}[x, y, z]$ was found to be a counterexample to the containment $I^{(3)} \subseteq I^2$. Recall the configuration consists of the 3 coordinate vertices and the 9 intersection points of $y^3 - z^3$ and $x^3 - z^3$. The ideal I is radical and all of the points in the configuration are reduced points. Now let $\varphi : \mathbb{C}[\mathbb{P}^2] \to \mathbb{C}[\mathbb{P}^2]$ by $x \to f = x^2 + y^2$, $y \to g = y^2 + z^2$ and $z \to h = x^2 + z^2$. One easily checks that $\{x^2+y^2, y^2+z^2, x^2+z^2\}$ is a $\mathbb{C}[\mathbb{P}^2]$-regular sequence. Then φ induces a map of schemes $\varphi^\# : \mathbb{P}^2 \to \mathbb{P}^2$ which is faithfully flat. Consider the scheme-theoretic fibers of $\varphi^\#$ over the Fermat configuration and call it the fibered Fermat configuration. Note that the fibered Fermat configuration is 0-dimensional. Since $\varphi^\#$ has degree 4, the fibers consist of 48 points of $\mathbb{P}^2$ where we count with multiplicity. The fibered Fermat configuration gives rise to the radical ideal $\varphi_*(I) = (f(g^3 - h^3), g(f^3 - h^3), h(f^3 - g^3)) \subseteq \mathbb{C}[\mathbb{P}^2]$ and by analyzing the ideal we see that the configuration consists of 4 multiplicity 1 points over each of the 3 coordinate vertices, given by $f = 0 = g$, $f = 0 = h$ and $g = 0 = h$. The remaining 36 points, each of multiplicity 1, in the configuration are the zero locus of $f^3 - h^3$ and $f^3 - g^3$. Since $I^{(3)} \nsubseteq I^2$, we have by Theorem 4 that $\varphi_*(I)^{(3)} \nsubseteq \varphi_*(I)^2$.

Example 2. We give another example of a fibered Fermat configuration whose ideal also gives a counterexample to the containment $I^{(3)} \subseteq I^2$. The difference here is that 36 of the points in the configuration have multiplicity 1 while the remaining 3 points each has multiplicity 4. So there are still 48 points counting with multiplicity. Let $\varphi : \mathbb{C}[\mathbb{P}^2] \to \mathbb{C}[\mathbb{P}^2]$ by $x \to f = x^2$, $y \to g = y^2$ and $z \to h = z^2$. This faithfully flat ring map induces a morphism of schemes $\varphi^\# : \mathbb{P}^2 \to \mathbb{P}^2$ that is also flat. The fibers of $\varphi^\#$ over the Fermat configuration gives the fibered Fermat configuration that consists of the 36 points, each of multiplicity 1, of intersection of the degree 6 forms $f^3 - g^3$ and $g^3 - h^3$. The configuration has 3 more points each of multiplicity 4 over the 3 coordinate points. They are the zero loci of $f = 0 = g$, $f = 0 = h$ and $g = 0 = h$. So the fibered Fermat configuration here has points that are not all reduced. By Theorem 4, its nonradical ideal $\varphi_*(I)$ is a counterexample to the containment $\varphi_*(I)^{(3)} \subseteq \varphi_*(I)^2$.

Example 3. Similarly for the Fermat configurations considered in [14] for $n \geq 3$, we can construct new configurations of points, that may or may not be reduced in $\mathbb{P}^2$, that are the fibers of a morphism of schemes $\varphi^\# : \mathbb{P}^2 \to \mathbb{P}^2$. The morphism $\varphi^\#$ is induced by the ring map $\varphi : \mathbb{C}[\mathbb{P}^2] \to \mathbb{C}[\mathbb{P}^2]$ given by $x \to f$, $y \to g$ and $z \to h$ where $\{f, g, h\}$ is a homogeneous $\mathbb{C}[\mathbb{P}^2]$-regular sequence of the same degree. The Fermat configuration gives rise to a radical ideal $I = (x(y^j - z^j), y(x^j - z^j), z(x^j - y^j)) \subseteq \mathbb{C}[\mathbb{P}^2]$, $j \geq 3$, and for a choice of $\{f, g, h\}$, the fibered Fermat configuration gives rise to an ideal $\varphi_*(I) = (f(g^j - h^j), g(f^j - h^j), h(f^j - g^j))$, $j \geq 3$, not necessarily radical, that is also a counterexample to $\varphi_*(I)^{(3)} \subseteq \varphi_*(I)^2$. Here the Fermat configuration consists of the reduced j^2 points of intersection of $y^j - z^j$ and $x^j - y^j$ together with the 3 coordinate vertices for a total of $j^2 + 3$ points. If the degree of the homogeneous elements in $\{f, g, h\}$ is d, then the fibered configuration consists of the $d^2 j^2$ points of intersection of $g^j - h^j$ and $f^j - h^j$ together with the $3d^2$ fiber points over the three coordinate vertices that are the solutions of the three equations $f = 0 = g$, $f = 0 = h$ and $g = 0 = h$, counted with multiplicity. Again the points in the fibered configuration may or may not be reduced.

Example 4. Now we consider an example given in [4] that is inspired by the example of the Fermat configuration. Let $k = \mathbb{Z}/3\mathbb{Z}$ and let K be an algebraically closed field containing k. Note that $\mathbb{P}^2_K$ has 13 k-points and 13 k-lines such that each line contains 4 of the points and each point is incident to 4 of the lines. The forms $xy(x^2 - y^2)$, $xz(x^2 - z^2)$ and $yz(y^2 - z^2)$ vanish at all 13 points of $\mathbb{P}^2_k$ but the form $x(x^2 - y^2)(x^2 - z^2)$ does not vanish at the point $[1 : 0 : 0]$. One checks easily that the ideal $I = (xy(x^2 - y^2), xz(x^2 - z^2), yz(y^2 - z^2), x(x^2 - y^2)(x^2 - z^2)) \subseteq k[\mathbb{P}^2_K]$ is radical and its zero locus is the 13 k-points of $\mathbb{P}^2_K$. Then $F = x(x - z)(x + z)(x^2 - y^2)((x - z)^2 - y^2)((x + z)^2 - y^2)$ defines 9 lines meeting at 12 points with each point incident to 3 of the lines. It is not hard to see that $F \in I^{(3)}$ but $F \notin I^2$. So the reduced configuration that comes from $\mathbb{P}^2_k$ with the point $[1:0:0]$ removed together with all its incident lines gives rise to an ideal that is a counterexample to the containment $I^{(3)} \subseteq I^2$. Let $\varphi : k[\mathbb{P}^2_K] \to k[\mathbb{P}^2_K]$ be the ring map $x \to f = x^2$, $y \to g = y^2$ and $z \to h = z^2$. Applying the degree 4 morphism of schemes $\varphi^{\#} : \mathbb{P}^2_K \to \mathbb{P}^2_K$, induced by φ, and taking its fibers over the k-points, we get a configuration of 48 points. For each point in the original configuration, we get 4 points in the fibered configuration. The points in this new configuration are not all reduced. For instance over the point $[0:0:1]$, the fiber of $\varphi^{\#}$ is a point of multiplicity 4 in $\mathbb{P}^2_K$ given by the vanishing of y^2 and x^2. The ideal of the fibered configuration as schemes is the ideal $\varphi_*(I) = (fg(f^2-g^2), fh(f^2-h^2), gh(g^2-h^2), f(f^2-g^2)(f^2-h^2))$. This ideal is not radical and since $\{f, g, h\} \subset \mathbb{P}^2_K$ is a regular sequence, we have by Theorem 4 that $\varphi_*(I)^{(3)} \nsubseteq \varphi_*(I)^2$. If instead we take $f = x^2 + y^2$, $g = y^2 + z^2$ and $h = x^2 + z^2$ in the above example, then the fibered configuration we obtain is a reduced configuration and the ideal $\varphi_*(I)$ is a radical ideal satisfying $\varphi_*(I)^{(3)} \nsubseteq \varphi_*(I)^2$.

Variations of the above example are considered in $\mathbb{P}^n$ for various n in [14], giving counterexamples for the more general conjecture $I^{(nr-n+1)} \subseteq I^r$. We can apply our result to these to obtain new counterexamples to the more general containment.

References

[1] Michela Artebani, Igor Dolgachev, The Hesse pencil of plane cubic curves, Enseign. Math. (2) 55 (3–4) (2009) 235–273.

[2] Thomas Bauer, Sandra Di Rocco, Brian Harbourne, Jack Huizenga, Anders Lundman, Piotr Pokora, Tomasz Szemberg, Bounded negativity and arrangements of lines, Int. Math. Res. Not. IMRN (19) (2015) 9456–9471.

[3] Thomas Bauer, Sandra Di Rocco, Brian Harbourne, Michał Kapustka, Andreas Knutsen, Wioletta Syzdek, Tomasz Szemberg, A primer on Seshadri constants, in: Interactions of Classical and Numerical Algebraic Geometry, in: Contemp. Math., vol. 496, Amer. Math. Soc., Providence, RI, 2009, pp. 33–70.

[4] Cristiano Bocci, Susan M. Cooper, Brian Harbourne, Containment results for ideals of various configurations of points in $\mathbf{P}^N$, J. Pure Appl. Algebra 218 (1) (2014) 65–75.

[5] Cristiano Bocci, Brian Harbourne, Comparing powers and symbolic powers of ideals, J. Algebraic Geom. 19 (3) (2010) 399–417.

[6] Winfried Bruns, Jürgen Herzog, Cohen–Macaulay Rings, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993.

[7] Susan M. Cooper, Robert J.D. Embree, Huy Tài Hà, Andrew H. Hoefel, Symbolic powers of monomial ideals, Proc. Edinb. Math. Soc. (2) 60 (1) (2016) 39–55.

[8] Adam Czapliński, Agata Główka, Grzegorz Malara, Magdalena Lampa-Baczyńska, Patrycja Łuszcz Świdecka, Piotr Pokora, Justyna Szpond, A counterexample to the containment $I^{(3)} \subset I^2$ over the reals, Adv. Geom. 16 (1) (2016) 77–82.

[9] Marcin Dumnicki, Tomasz Szemberg, Halszka Tutaj-Gasińska, Counterexamples to the $I^{(3)} \subset I^2$ containment, J. Algebra 393 (2013) 24–29.

[10] Lawrence Ein, Robert Lazarsfeld, Karen E. Smith, Uniform bounds and symbolic powers on smooth varieties, Invent. Math. 144 (2) (2001) 241–252.

[11] David Eisenbud, Commutative Algebra, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, with a view toward algebraic geometry.

[12] A.V. Geramita, B. Harbourne, J. Migliore, U. Nagel, Matroid configurations and symbolic powers of their ideals, 2015.

[13] Brian Harbourne, Craig Huneke, Are symbolic powers highly evolved?, J. Ramanujan Math. Soc. 28A (2013) 247–266.

[14] Brian Harbourne, Alexandra Seceleanu, Containment counterexamples for ideals of various configurations of points in $\mathbf{P}^N$, J. Pure Appl. Algebra 219 (4) (2015) 1062–1072.
[15] Melvin Hochster, Craig Huneke, Comparison of symbolic and ordinary powers of ideals, Invent. Math. 147 (2) (2002) 349–369.
[16] Irena Peeva, Graded Syzygies, Algebra and Applications, vol. 14, Springer-Verlag London, Ltd., London, 2011.
[17] Alexandra Seceleanu, A homological criterion for the containment between symbolic and ordinary powers of some ideals of points in $\mathbb{P}^2$, J. Pure Appl. Algebra 219 (11) (2015) 4857–4871.