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Chudnovsky's conjecture for very general points in \mathbb{P}_k^N



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ABSTRACT

We prove a long-standing conjecture of Chudnovsky for very general and generic points in \mathbb{P}_k^N , where k is an algebraically closed field of characteristic zero, and for any finite set of points lying on a quadric, without any assumptions on k . We also prove that for any homogeneous ideal I in the homogeneous coordinate ring $R = k[x_0, \dots, x_N]$, Chudnovsky's conjecture holds for large enough symbolic powers of I .

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1. Introduction

This manuscript deals with the following general interpolation question:

Question 1.1. *Given a finite set of n distinct points $X = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ in \mathbb{P}_k^N , where k is a field, what is the minimum degree, $\alpha_m(X)$, of a hypersurface $f \neq 0$ passing through each \mathbf{p}_i with multiplicity at least m ?*

Question 1.1 has been considered in various forms for a long time. We mention a few conjectures and motivations. For instance, this question plays a crucial role in the proof of Nagata's counterexamples to Hilbert's fourteenth problem [19]. In the same paper Nagata conjectured that $\alpha_m(X) > m\sqrt{n}$ for sets of n general points in $\mathbb{P}_{\mathbb{C}}^2$ [19], and a vast number of papers in the last few decades are related to his conjecture. Another reason for the interest sparked by the above question comes from the context of complex analysis: an answer to Question 1.1 would provide information about the Schwarz exponent, which is very important in the investigation of the arithmetic nature of values of Abelian functions of several variables [4].

However, besides a few very special classes of points (e.g., if these n points lie on a single hyperplane or one has $n = \binom{\beta+N-1}{N}$ points forming a star configuration and m is a multiple of N [5],[2]), at the moment a satisfactory answer to this elusive question appears out of reach. Therefore, there has been interest in finding effective lower bounds for $\alpha_m(X)$. In fact, lower bounds for $\alpha_m(X)$ yield upper bounds for the Schwarz exponent. Using complex analytic techniques, Waldschmidt [22] and Skoda [21] in 1977 proved that for all $m \geq 1$

$$\frac{\alpha_m(X)}{m} \geq \frac{\alpha(X)}{N},$$

where $\alpha(X) = \alpha_1(X)$ is the minimum degree of a hypersurface passing through every point of X and $k = \mathbb{C}$. In 1981, Chudnovsky [4] improved the inequality in the 2-dimensional projective space. He showed that if X is a set of n points in $\mathbb{P}_{\mathbb{C}}^2$, then for all $m \geq 1$

$$\frac{\alpha_m(X)}{m} \geq \frac{\alpha(X) + 1}{2}.$$

He then raised the following conjecture for higher dimensional projective spaces:

Conjecture 1.2 (Chudnovsky [4]). *If X is a finite set of points in $\mathbb{P}_{\mathbb{C}}^N$, then for all $m \geq 1$*

$$\frac{\alpha_m(X)}{m} \geq \frac{\alpha(X) + N - 1}{N}.$$

The first improvement towards Chudnovsky's Conjecture 1.3 was achieved in [9] by Esnault and Viehweg, who employed complex projective geometry techniques to show

$\frac{\alpha_m(X)}{m} \geq \frac{\alpha(X) + 1}{N}$ for points in $\mathbb{P}_{\mathbb{C}}^N$. In fact, this inequality follows by a stronger statement, refining previous inequalities from Bombieri, Waldschmidt and Skoda, see [9].

From the algebraic point of view, Chudnovsky's [Conjecture 1.3](#) can be interpreted in terms of symbolic powers via a celebrated theorem of Nagata and Zariski. Let $R = k[x_0, \dots, x_N]$ be the homogeneous coordinate ring of \mathbb{P}_k^N and I a homogeneous ideal in R . We recall that the m -th *symbolic power* of I is defined as the ideal $I^{(m)} = \bigcap_p I^m R_p \cap R$, where p runs over all associated prime ideals of R/I , and the *initial degree* of I , $\alpha(I)$, is the least degree of a polynomial in I . Nagata and Zariski showed that if k is algebraically closed and X is a finite set of points in \mathbb{P}_k^N , then $\alpha_m(X) = \alpha(I_X^{(m)})$, where I_X is the ideal consisting of all polynomials in R that vanish on X . Thus in this setting Chudnovsky's [Conjecture 1.3](#) is equivalent to

$$\frac{\alpha(I_X^{(m)})}{m} \geq \frac{\alpha(I_X) + N - 1}{N} \quad \text{for all } m \geq 1.$$

The limit $\lim_{m \rightarrow \infty} \frac{\alpha(I_X^{(m)})}{m} = \gamma(I_X)$, called the *Waldschmidt constant* of I_X , exists and is an “inf” [2]. Thus another equivalent formulation of Chudnovsky's [Conjecture 1.3](#) is

$$\gamma(I_X) \geq \frac{\alpha(I_X) + N - 1}{N}.$$

We remark here that there is a tight connection between the Waldschmidt constant (especially for general points) and an instance of the multipoint Seshadri constant [1, Section 8].

We now state a generalized version of Chudnovsky's [Conjecture 1.2](#). When k is an algebraically closed field then the following conjecture is equivalent to Chudnovsky's [Conjecture 1.2](#).

Conjecture 1.3. *If X is a finite set of points in \mathbb{P}_k^N , where k is any field, then for all $m \geq 1$*

$$\frac{\alpha(I_X^{(m)})}{m} \geq \frac{\alpha(I_X) + N - 1}{N}.$$

In 2001, Ein, Lazarsfeld, and Smith proved a containment between ordinary powers and symbolic powers of homogeneous ideals in polynomial rings over the field of complex numbers. More precisely, for any homogeneous ideal I in $R = \mathbb{C}[x_0, \dots, x_N]$, they proved that $I^{(Nm)} \subseteq I^m$ [8]. Their result was soon generalized over any field by Hochster and Huneke using characteristic p techniques [16]. Using this result, Harbourne and Huneke observed that the Waldschmidt–Skoda inequality $\frac{\alpha(I^{(m)})}{m} \geq \frac{\alpha(I)}{N}$ actually holds for every homogeneous ideal I in R [15]. In the same article, Harbourne and Huneke posed the following conjecture:

Conjecture 1.4 (Harbourne–Huneke [15]). *If X is a finite set of points in \mathbb{P}_k^N , then for all $m \geq 1$*

$$I_X^{(Nm)} \subseteq M^{m(N-1)} I_X^m,$$

where $M = (x_0, \dots, x_N)$ is the homogeneous maximal ideal of $R = k[x_0, \dots, x_N]$.

Conjecture 1.4 strives to provide a structural reason behind Chudnovsky’s Conjecture 1.3: if it holds, then it would imply Chudnovsky’s Conjecture 1.3 in a similar way as to how the Ein–Lazarsfeld–Smith and Hochster–Huneke containment implies the Waldschmidt–Skoda inequality [15]. These results have since raised new interest in Chudnovsky’s Conjecture 1.3.

Harbourne and Huneke proved their conjecture for general points in \mathbb{P}_k^2 and when the points form a star configuration in \mathbb{P}_k^N . In 2011, Dumnicki proved the Harbourne–Huneke Conjecture 1.4 for general points in \mathbb{P}_k^3 and at most $N + 1$ points in general position in \mathbb{P}_k^N for $N \geq 2$ [6]. In summary, Chudnovsky’s Conjecture 1.3 is known in the following cases:

- any finite set of points in \mathbb{P}_k^2 [4], [15];
- any finite set of general points in \mathbb{P}_k^3 , where k is a field of characteristic 0 [6];
- any set of at most $N + 1$ points in general position in \mathbb{P}_k^N [6];
- any set of a binomial number of points in \mathbb{P}_k^N forming a star configuration [5], [2].

In the present paper, we prove that Chudnovsky’s Conjecture 1.3 holds for

- any finite set of very general points in \mathbb{P}_k^N , where k is an algebraically closed field of characteristic 0 (Theorem 2.8);
- any finite set of generic points in $\mathbb{P}_{k(\underline{z})}^N$, where k is an algebraically closed field of characteristic 0 (Theorem 2.7);
- any finite set of points in \mathbb{P}_k^N lying on a quadric, without any assumptions on k (Proposition 2.6).

As a corollary, we obtain that the Harbourne–Huneke Conjecture 1.4 holds for sets of a binomial number of very general points in \mathbb{P}_k^N (Corollary 2.9). This result also yields a new lower bound for the multipoint Seshadri constant of very general points in \mathbb{P}_k^N (Corollary 2.10).

In the final section of the paper, we prove that for any homogeneous ideal I in the homogeneous coordinate ring $R = k[x_0, \dots, x_N]$, Chudnovsky’s Conjecture 1.3 holds for sufficiently large symbolic powers $I^{(t)}$, $t \gg 0$ (Theorem 3.7). In the case of ideals of finite sets of points in \mathbb{P}_k^N , we prove a uniform bound, namely that if $t \geq N - 1$, then $I^{(t)}$ satisfies Chudnovsky’s Conjecture 1.3 (Proposition 3.10).

Very recently, Dumnicki and Tutaj-Gasińska proved the Harbourne–Huneke [Conjecture 1.4](#) for at least 2^N number of very general points in \mathbb{P}_k^N . As a corollary, they obtain Chudnovsky’s [Conjecture 1.3](#) for at least 2^N number of very general points in \mathbb{P}_k^N [7]. These results are obtained independently from ours and with different methods.

2. Generic and very general points in \mathbb{P}_k^N

We begin by discussing our general setting.

Set-up 2.1. Let $R = k[x_0, \dots, x_N]$ be the homogeneous coordinate ring of \mathbb{P}_k^N , where k is an algebraically closed field. Let n be a positive integer and let $S = k(\underline{z})[\underline{x}]$, where $k \subseteq k(\underline{z})$ is a purely transcendental extension of fields obtained by adjoining $n(N+1)$ variables $\underline{z} = (z_{ij})$, $1 \leq i \leq n$, $0 \leq j \leq N$. A set of n generic points P_1, \dots, P_n consists of points $P_i = [z_{i0} : z_{i1} : \dots : z_{iN}] \in \mathbb{P}_{k(\underline{z})}^N$. We denote the defining ideal of n generic points as

$$H = \bigcap_{i=1}^n I(P_i),$$

where $I(P_i)$ is the ideal defining the point P_i .

For any nonzero vector $\underline{\lambda} = (\lambda_{ij}) \in \mathbb{A}_k^{n(N+1)}$, where $1 \leq i \leq n$, $0 \leq j \leq N$, we define the set of points $\{p_1, \dots, p_n\} \subseteq \mathbb{P}_k^N$ as the points $p_i = P_i(\underline{\lambda}) = [\lambda_{i0} : \lambda_{i1} : \dots : \lambda_{iN}] \in \mathbb{P}_k^N$. For $1 \leq i \leq n$, let $I(p_i)$ be the ideal of R defining the point p_i and define

$$H(\underline{\lambda}) = \bigcap_{i=1}^n I(p_i).$$

For any ideal J in S , recall that Krull [17,18] defined the specialization $J_{\underline{\lambda}}$ with respect to the substitution $\underline{z} \rightarrow \underline{\lambda}$ as follows:

$$J_{\underline{\lambda}} = \{f(\underline{\lambda}, \underline{x}) \mid f(\underline{z}, \underline{x}) \in J \cap k[\underline{z}, \underline{x}]\}.$$

In general, one has that $H_{\underline{\lambda}} \subseteq H(\underline{\lambda})$, where H and $H(\underline{\lambda})$ are defined as in [Set-up 2.1](#) and $H_{\underline{\lambda}}$ is the specialization with respect to the substitution $\underline{z} \rightarrow \underline{\lambda}$ defined by Krull. Notice that equality holds if $\underline{\lambda}$ is in a dense Zariski-open subset of $\mathbb{A}_k^{n(N+1)}$ [20].

Recall the collection of all sets consisting of n points (not necessarily distinct) in \mathbb{P}_k^N is parameterized by $G(1, n, N+1)$, the *Chow variety* of algebraic 0-cycles of degree n in \mathbb{P}_k^N . It is well-known that $G(1, n, N+1)$ is isomorphic to the symmetric product $\text{Sym}^n(\mathbb{P}_k^N)$, see for instance [10]. One says that a property \mathcal{P} holds for n *general points* in \mathbb{P}_k^N if there is a dense Zariski-open subset W of $G(1, n, N+1)$ such that \mathcal{P} holds for every set $X = \{p_1, \dots, p_n\}$ of n points with $p_1 + \dots + p_n \in W$. Similarly, one says that a property \mathcal{P} holds for n *very general points* in \mathbb{P}_k^N if \mathcal{P} holds for every set of n points

in a nonempty subset W of $G(1, n, N + 1)$ of the form $W = \bigcap_{i=1}^{\infty} U_i$, where the U_i are dense Zariski-open sets (when k is uncountable, then W is actually a dense subset). We conclude this part by recalling the following well-known fact.

Remark 2.2. Let n be a positive integer. The collection of all sets consisting of n distinct points in \mathbb{P}_k^N is parameterized by a dense Zariski-open subset $W(n)$ of $G(1, n, N + 1)$.

Unless specified, for the rest of this paper by a “set of points” we mean “a set of simple points”, i.e. points whose defining ideal is radical.

Instead of working directly with the Chow variety, we will work over $\mathbb{A}_k^{n(N+1)}$ (in order to specialize from the generic situation). We first need to prove that if a property holds on a dense Zariski-open subset of $\mathbb{A}_k^{n(N+1)}$, then it also holds on a dense Zariski-open subset of the Chow variety. This is precisely the content of our first lemma.

Lemma 2.3. Assume [Set-up 2.1](#) and let $U \subseteq \mathbb{A}_k^{n(N+1)}$ be a dense Zariski-open subset such that a property \mathcal{P} holds for $H(\underline{\lambda})$ whenever $\underline{\lambda} \in U$. Then property \mathcal{P} holds for n general points in \mathbb{P}_k^N . Moreover, if a property \mathcal{P} holds for $H(\underline{\lambda})$ whenever $\underline{\lambda} \in U$, where $U = \bigcap_{i=1}^{\infty} U_i \subseteq \mathbb{A}_k^{n(N+1)}$ is nonempty and each U_i is a dense Zariski-open set, then \mathcal{P} holds for n very general points in \mathbb{P}_k^N .

Proof. For every $i = 1, \dots, n$, let $\pi_i : \mathbb{A}_k^{n(N+1)} \dashrightarrow \mathbb{P}_k^N$ be the rational map defined by projection as follows:

$$\pi_i(\underline{\lambda}) = [\lambda_{i0} : \lambda_{i1} : \dots : \lambda_{iN}],$$

where $\underline{\lambda} = (\lambda_{ij}) \in \mathbb{A}_k^{n(N+1)}$. It is clear that π_i is defined on the complement of the Zariski-closed proper subset $C_i = \{\underline{\lambda} \in \mathbb{A}_k^{n(N+1)} \mid \lambda_{i0} = \dots = \lambda_{iN} = 0\}$.

Taking products of these rational maps, we obtain the rational map

$$\pi = (\pi_1 \times \pi_2 \times \dots \times \pi_n) : \mathbb{A}_k^{n(N+1)} \dashrightarrow \mathbb{P}_k^N \times_k \mathbb{P}_k^N \times_k \dots \times_k \mathbb{P}_k^N.$$

The map π is defined on the complement of the closed proper subset $C = \bigcup_{i=1}^n C_i$, where C_i is as above. Note that $U \setminus C$ is still open in $\mathbb{A}_k^{n(N+1)}$, and since π is surjective and thus dominant, then $\pi(U \setminus C)$ contains a non-empty Zariski-open subset $W' \subseteq \mathbb{P}_k^N \times_k \mathbb{P}_k^N \times_k \dots \times_k \mathbb{P}_k^N$ (see for instance [\[13, II. Ex. 3.19 \(b\)\]](#)).

Now, since the symmetric group S_n on n elements is finite, the image W of W' in $(\mathbb{P}_k^N \times_k \mathbb{P}_k^N \times_k \dots \times_k \mathbb{P}_k^N)/S_n \cong \text{Sym}^n(\mathbb{P}_k^N) \cong G(1, n, N + 1)$ contains a non-empty Zariski-open subset of $G(1, n, N + 1)$. \square

Let H be as in [Set-up 2.1](#). We now prove that the initial degree of any symbolic power of H is no smaller than the initial degree of any ideal of a set with the same number of points. Equivalently, if I is the defining ideal of a set of n points in \mathbb{P}_k^N , then $\alpha(H^{(m)}) \geq \alpha(I^{(m)})$ for all $m \geq 1$.

Theorem 2.4. *Let $m \geq 1$. Assume [Set-up 2.1](#) and that k has characteristic 0. Then $\alpha(H^{(m)}) \geq \alpha(I_X^{(m)})$, for every set X of n distinct points in \mathbb{P}_k^N . Moreover, for every $m \geq 1$, there is a dense Zariski-open subset $U_m \subseteq \mathbb{A}_k^{n(N+1)}$ for which equality holds.*

Proof. Let $t \geq 0$. We define $V_t = \{\underline{\lambda} = (\lambda_{ij}) \in \mathbb{A}_k^{n(N+1)} \mid \alpha(H(\underline{\lambda})^{(m)}) \leq t\}$. We first prove that V_t is a closed subset of $\mathbb{A}_k^{n(N+1)}$. Indeed, notice that

$$V_t = \{\underline{\lambda} = (\lambda_{ij}) \in \mathbb{A}_k^{n(N+1)} \mid \text{there exists } 0 \neq f \in H(\underline{\lambda})^{(m)} \text{ of degree } t\}.$$

Let $f \in R$ be a homogeneous polynomial with $\deg f = t$ and write $f = \sum_{|\underline{\alpha}|=t} C_{\underline{\alpha}} x^{\underline{\alpha}}$. Since

k is algebraically closed of characteristic 0, the statement $f \in H(\underline{\lambda})^{(m)}$ is equivalent to $\partial_{\underline{\beta}} f(p_i) = 0$ for all $\underline{\beta}$ with $|\underline{\beta}| \leq m-1$ and all points p_1, \dots, p_n . Since $P_i = [\underline{z}_i] = [z_{i0} : z_{i1} : \dots : z_{iN}]$ and $p_i = [\underline{\lambda}_i] = [\lambda_{i0} : \lambda_{i1} : \dots : \lambda_{iN}]$, we write $\partial_{\underline{\beta}} f(P_i) = \partial_{\underline{\beta}} f(\underline{z}_i) = \sum_{|\underline{\alpha}|=t} C_{\underline{\alpha}} \partial_{\underline{\beta}} z_i^{\underline{\alpha}}$ and $\partial_{\underline{\beta}} f(p_i) = \sum_{|\underline{\alpha}|=t} C_{\underline{\alpha}} \partial_{\underline{\beta}} \lambda_i^{\underline{\alpha}}$. (For instance, $\partial_{(2,0,1)} \underline{z}_i^{(3,3,2)} = \partial_{x_0 x_0 x_2} x_0^3 x_1^3 x_2^2|_{\underline{x}=\underline{z}_i} = 12x_0 x_1^3 x_2^2|_{\underline{x}=\underline{z}_i} = 12z_{i0} z_{i1}^3 z_{i2}$ and $\partial_{(2,0,1)} \underline{\lambda}_i^{(3,3,2)} = 12\lambda_{i0} \lambda_{i1}^3 \lambda_{i2}$.)

To order these equations we use, for instance, the natural deglex order in \mathbb{N}_0^{N+1} , i.e., $\underline{\alpha} = (\alpha_0, \dots, \alpha_N) > \underline{\beta} = (\beta_0, \dots, \beta_N)$ if and only if $|\underline{\alpha}| > |\underline{\beta}|$ or $|\underline{\alpha}| = |\underline{\beta}|$ and there exists j such that $\alpha_i = \beta_i$ for $i \leq j$ and $\alpha_{j+1} > \beta_{j+1}$. Then the system of equations $\{\partial_{\underline{\beta}} f(P_i) = 0\}_{|\underline{\beta}| \leq m-1, 1 \leq i \leq n}$ can be written in the following form

$$\mathbb{B}_{m,t} [C_{(t,\dots,0)} \dots C_{\underline{\alpha}} \dots C_{(0,\dots,t)}]^T = \underline{0},$$

where the rows of $\mathbb{B}_{m,t}$ are

$$\left[\partial_{\underline{\beta}} z_{i0}^t \quad \dots \quad \partial_{\underline{\beta}} z_{i1}^{\underline{\alpha}} \quad \dots \quad \partial_{\underline{\beta}} z_{iN}^t \right], \quad \text{where } 1 \leq i \leq n \text{ and } |\underline{\beta}| \leq m-1.$$

By construction, the existence of a nonzero element $f \in H(\underline{\lambda})^{(m)}$ of degree t is equivalent to the existence of a non-trivial solution for the homogeneous system

$$[\mathbb{B}_{m,t}]_{\underline{\lambda}} [C_{(t,\dots,0)} \dots C_{\underline{\alpha}} \dots C_{(0,\dots,t)}]^T = \underline{0}.$$

Observe that the matrix $\mathbb{B}_{m,t}$ has size $n \binom{m+N}{m-1} \times \binom{t+N}{N}$. If $n \binom{m+N}{m-1} < \binom{t+N}{N}$, then for every $\underline{\lambda} \in \mathbb{A}_k^{n(N+1)}$ the homogeneous system $[\mathbb{B}_{m,t}]_{\underline{\lambda}} [C_{(t,\dots,0)} \dots C_{\underline{\alpha}} \dots C_{(0,\dots,t)}]^T = \underline{0}$ has non-trivial solutions. Therefore $V_t = \mathbb{A}_k^{n(N+1)}$, which is closed in $\mathbb{A}_k^{n(N+1)}$.

If instead, $n \binom{m+N}{m-1} \geq \binom{t+N}{N}$, then the system $[\mathbb{B}_{m,t}]_{\underline{\lambda}} [C_{(t,\dots,0)} \dots C_{\underline{\alpha}} \dots C_{(0,\dots,t)}]^T = \underline{0}$ has non-trivial solutions if and only if $\text{rank} [\mathbb{B}_{m,t}]_{\underline{\lambda}} < \binom{t+N}{N}$. This is a closed condition on $\underline{\lambda}$ as it requires the vanishing of finitely many minors, and therefore V_t is closed in $\mathbb{A}_k^{n(N+1)}$.

Next, let $t_0 = \alpha(H^{(m)})$. The set $V_{t_0} = \{\underline{\lambda} = (\lambda_{ij}) \in \mathbb{A}_k^{n(N+1)} \mid \alpha(H(\underline{\lambda})^{(m)}) \leq t_0\}$ contains a dense Zariski-open subset of $\mathbb{A}_k^{n(N+1)}$. Indeed, let $0 \neq f \in \bigcap_{i=1}^n I(P_i)^m$ be such that $\deg f = t_0$. We may assume that $f(\underline{z}, \underline{x}) \in k[\underline{z}][x_0, \dots, x_N]$. Then there exists a dense Zariski-open subset U_m of $\mathbb{A}_k^{n(N+1)}$ such that the polynomial $0 \neq f(\underline{\lambda}, \underline{x}) \in (H^{(m)})_{\underline{\lambda}} = (H(\underline{\lambda}))^{(m)}$ and $\deg f = t_0$ (since $f(\underline{z}, \underline{x}) \neq 0$, there is a non-empty Zariski-open subset of specializations $\underline{z} \rightarrow \underline{\lambda}$ such that $f(\underline{\lambda}, \underline{x}) \neq 0$).

Finally, since V_{t_0} is a Zariski-closed subset which also contains a dense Zariski-open subset of $\mathbb{A}_k^{n(N+1)}$, then $V_{t_0} = \mathbb{A}_k^{n(N+1)}$, which proves the statement. The second part of the statement also follows from the above argument. \square

Following [12, Definition 2.4], we say that a set X of n points in \mathbb{P}_k^N is in *generic position* if it has the “generic Hilbert function”, i.e., if $h_{R/I_X}(d) = \min\{\dim_k(R_d), n\}$ for every $d \geq 0$. Being in generic position is an open condition; indeed any set of generic (or general) points (see Set-up 2.1) is in generic position. We now prove a reduction argument, which will allow us to concentrate on certain binomial numbers of points.

Proposition 2.5.

- (a) Chudnovsky’s Conjecture 1.3 holds for any finite set of generic points if it holds for sets of $\binom{\beta+N-1}{N}$ generic points for all $\beta \geq 1$.
- (b) Chudnovsky’s Conjecture 1.3 holds for any finite set of points if it holds for sets of $\binom{\beta+N-1}{N}$ points in generic position for all $\beta \geq 1$.

Proof. (a): Let $H = \bigcap_{i=1}^n I(P_i)$ be the defining ideal of the n generic points P_1, \dots, P_n as in Set-up 2.1. Let $\beta \geq 1$ be the unique integer such that

$$\binom{\beta + N - 1}{N} \leq n < \binom{\beta + N}{N}.$$

Let $t = \binom{\beta+N-1}{N}$ and let $J = \bigcap_{i=1}^t I(P_i)$ be the ideal defining t of these generic points. Since the set $Y = \{P_1, \dots, P_t\}$ is in generic position, in particular we have $\alpha(J) = \alpha(H) = \beta$.

Now assume Chudnovsky’s Conjecture 1.3 holds for $\binom{\beta+N-1}{N}$ generic points. Then for all $m \geq 1$

$$\frac{\alpha(J^{(m)})}{m} \geq \frac{\alpha(J) + N - 1}{N}.$$

Since $H^{(m)} \subseteq J^{(m)}$, one has that $\alpha(H^{(m)}) \geq \alpha(J^{(m)})$ and

$$\frac{\alpha(H^{(m)})}{m} \geq \frac{\alpha(J^{(m)})}{m} \geq \frac{\alpha(J) + N - 1}{N} = \frac{\alpha(H) + N - 1}{N}.$$

(b): The proof of (b) is similar in spirit to (a). Let X be any finite set of points in \mathbb{P}_k^N , let I_X be its defining ideal, and let $t = \binom{(\alpha-1)+N-1}{N}$, where $\alpha = \alpha(I_X)$. By linear independence (e.g., by [12, Theorem 2.5 (c)]), there is a subset $Y \subseteq X$ of t points with the property that $H_{R/I_Y}(i) = H_{R/I_X}(i) = \dim_k R_i$ for every $i = 0, \dots, \alpha - 1$; in particular $H_{R/I_Y}(\alpha - 1) = t$. Since $|Y| = t$, it follows that $H_{R/I_Y}(i) = t$ for all $i \geq \alpha$, proving that Y is in generic position. Similar to (a), assume Chudnovsky's Conjecture 1.3 holds for $t = \binom{(\alpha-1)+N-1}{N}$ points in generic position. Then

$$\frac{\alpha(I_Y^{(m)})}{m} \geq \frac{\alpha(I_Y) + N - 1}{N}$$

for all $m \geq 1$. Since $\alpha(I_X) = \alpha = \alpha(I_Y)$ and $I_X^{(m)} \subseteq I_Y^{(m)}$, then for all $m \geq 1$ we obtain

$$\frac{\alpha(I_X^{(m)})}{m} \geq \frac{\alpha(I_Y^{(m)})}{m} \geq \frac{\alpha(I_Y) + N - 1}{N} = \frac{\alpha(I_X) + N - 1}{N}. \quad \square$$

Dumnicki proved Chudnovsky's Conjecture 1.3 for at most $N + 1$ points in general position \mathbb{P}_k^N [6] (this specific result does not need any assumptions on the characteristic of k). The idea is that one can take them to be coordinate points so that the ideal of the points is monomial and one can compute explicitly its symbolic powers. If one has more than $N + 1$ points, the ideal of the points is almost never monomial and explicit computations of a generating set of any of its symbolic powers are nearly impossible to perform. We extend the result of Dumnicki to the case of up to $\binom{N+2}{2} - 1$ points in \mathbb{P}_k^N .

Proposition 2.6. *Chudnovsky's Conjecture 1.3 holds for any finite set of points lying on a quadric in \mathbb{P}_k^N , where k is any field. In particular, any set of $n \leq \binom{N+2}{2} - 1$ points in \mathbb{P}_k^N satisfies Chudnovsky's Conjecture 1.3.*

Proof. Let X be a set of n points in \mathbb{P}_k^N . If they all lie on a hyperplane, then Chudnovsky's Conjecture 1.3 is clearly satisfied, since $\alpha(I_X^{(m)}) = m$ for every $m \geq 1$. We may then assume there is no hyperplane containing all the points. Thus we can find a set $Y \subseteq X$ of $N + 1$ points not on a hyperplane, i.e. in general position. Then for all $m \geq 1$

$$\frac{\alpha(I_X^{(m)})}{m} \geq \frac{\alpha(I_Y^{(m)})}{m} \geq \frac{N + 1}{N} = \frac{\alpha(I_X) + N - 1}{N},$$

where the second inequality follows by [6] and the equality holds because $\alpha(I_X) = \alpha(I_Y) = 2$. \square

Let us recall that a set X of $\binom{N+s}{N}$ points in \mathbb{P}_k^N form a *star configuration* if there are $N + s$ hyperplanes L_1, \dots, L_{N+s} meeting properly such that X consists precisely of the points obtained by intersecting any N of the L_i 's. Star configurations (in \mathbb{P}_k^2) were already considered by Nagata and they have been deeply studied, see for instance [11] and references within. We employ them to show that Chudnovsky's [Conjecture 1.3](#) holds for any number of generic points.

Theorem 2.7. *Let $H = \bigcap_{i=1}^n I(P_i)$, where P_1, \dots, P_n are n generic points in $\mathbb{P}_{k(z)}^N$ defined as in [Set-up 2.1](#). Suppose k has characteristic 0. Then Chudnovsky's [Conjecture 1.3](#) holds for H .*

Proof. By [Proposition 2.5](#) (a), we may assume $n = \binom{\beta+N-1}{N}$ for some $\beta \geq 1$. Let $\underline{\lambda} \in \mathbb{A}_k^{n(N+1)}$ be such that $H(\underline{\lambda})$ is the defining ideal of n points in \mathbb{P}_k^N forming a star configuration. It is well-known that $\alpha(H(\underline{\lambda})) = \alpha(H) = \beta$ [[11, Proposition 2.9](#)]. Now by [Theorem 2.4](#) and [[15, Corollary 3.9](#)] we have that for all $m \geq 1$

$$\frac{\alpha(H^{(m)})}{m} \geq \frac{\alpha(H(\underline{\lambda})^{(m)})}{m} \geq \frac{\alpha(H(\underline{\lambda})) + N - 1}{N} = \frac{\alpha(H) + N - 1}{N}. \quad \square$$

We are now ready to prove our main result that Chudnovsky's [Conjecture 1.3](#) holds for any finite set of *very general* points in \mathbb{P}_k^N .

Theorem 2.8. *Let I be the defining ideal of n very general points in \mathbb{P}_k^N , where k is an algebraically closed field of characteristic 0. Then I satisfies Chudnovsky's [Conjecture 1.3](#).*

Proof. Being in generic position is an open condition and therefore we may assume the points are in generic position. Then, as in [Proposition 2.5](#) (a), we may assume that $n = \binom{\beta+N-1}{N}$ for some $\beta \geq 1$. It suffices to show that $\gamma(I) \geq \frac{\alpha(I)+N-1}{N}$. Let $R, S, \underline{z}, \underline{\lambda}$, and H be as in [Set-up 2.1](#). Consider the decreasing chain of ideals

$$I^{(N)} \supseteq I^{(2N)} \supseteq I^{(2^2N)} \supseteq \dots \supseteq I^{(2^sN)} \supseteq \dots$$

For each $s \geq 0$, define

$$U_s = \{\underline{\lambda} = (\lambda_{ij}) \in \mathbb{A}_k^{n(N+1)} \mid \alpha(H(\underline{\lambda})^{(2^sN)}) \geq 2^s (\alpha(H(\underline{\lambda})) + N - 1)\}.$$

By the proof of [Theorem 2.4](#), U_s is a Zariski-open subset of $\mathbb{A}_k^{n(N+1)}$. We claim that U_s is not empty. Indeed, by [Theorem 2.4](#), for every $s \geq 0$, there is a dense Zariski-open subset $W_s \subseteq \mathbb{A}_k^{n(N+1)}$ for which $\alpha(H(\underline{\lambda})^{(2^sN)}) = \alpha(H(\underline{\lambda}))$ for every $\underline{\lambda} \in W_s$. By [Theorem 2.7](#), one also has that for every $\underline{\lambda} \in W_s$,

$$\frac{\alpha(H(\underline{\lambda})^{(2^s N)})}{2^s N} = \frac{\alpha(H^{(2^s N)})}{2^s N} \geq \frac{\alpha(H) + N - 1}{N} = \frac{\alpha(H(\underline{\lambda})) + N - 1}{N}.$$

Hence $W_s \subset U_s$.

Set $U = \bigcap_{s=0}^{\infty} U_s$ and notice U is non-empty because a star configuration of n points lies in U [2, Lemma 2.4.2]. By construction, if $\underline{\lambda} \in U$ we have

$$\gamma(H(\underline{\lambda})) = \lim_{s \rightarrow \infty} \frac{\alpha(H(\underline{\lambda})^{(2^s N)})}{2^s N} \geq \frac{\alpha(H(\underline{\lambda})) + N - 1}{N}.$$

Finally, apply Lemma 2.3. \square

As a corollary, we show that the Harbourne–Huneke Conjecture 1.4 holds for sets of binomial numbers of very general points or generic points.

Corollary 2.9. *Let I be the defining ideal of either $\binom{\beta+N-1}{N}$ very general points in \mathbb{P}_k^N or $\binom{\beta+N-1}{N}$ generic points in $\mathbb{P}_{k(\underline{z})}^N$ for some $\beta \geq 1$, where k is an algebraically closed field of characteristic 0. Then I satisfies the Harbourne–Huneke Conjecture 1.4.*

Proof. The proof follows by Theorem 2.8 and [15, Proposition 3.3 and Remark 3.4]. \square

For an unmixed ideal I in R , the *Waldschmidt constant* is defined by $\gamma(I) = \lim_{m \rightarrow \infty} \frac{\alpha(I^{(m)})}{m}$, see [2] for more details. Recall that for a finite set of points $X = \{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ in \mathbb{P}_k^N , the Waldschmidt constant is tightly related to the *(multipoint) Seshadri constant* defined as

$$\epsilon(N, X) = \sqrt[N-1]{\inf \left\{ \frac{\deg(F)}{\sum_{i=1}^n \text{mult}_{\mathbf{p}_i}(F)} \right\}},$$

where the infimum is taken with respect to all hypersurfaces F passing through at least one of the \mathbf{p}_i . The study of Seshadri constants has been an active area of research for the last twenty years, see for instance the survey [1] and references within. Here we only note that one has $\gamma(I_X) \geq n\epsilon(N, X)^{N-1}$ and equality holds if X consists of n general simple points in \mathbb{P}_k^N . In particular, equality also holds if X consists of n very general simple points \mathbb{P}_k^N . Therefore, our estimate for the Waldschmidt constant also yields an estimate for the (multipoint) Seshadri constant for very general simple points of \mathbb{P}_k^N .

Corollary 2.10. *For any set X of n very general points in \mathbb{P}_k^N , where k is an algebraically closed field of characteristic 0, one has*

$$\epsilon(N, X) \geq \sqrt[N-1]{\frac{\alpha(X) + N - 1}{nN}}.$$

3. Homogeneous ideals in $k[x_0, \dots, x_N]$

Let $R = k[x_0, \dots, x_N]$ be the homogeneous coordinate ring of \mathbb{P}_k^N , where k is any field, and I a homogeneous ideal. For an ideal I which may have embedded components, there are multiple potential definitions of symbolic powers. Following [8] and [16], we define the m -th symbolic power of I to be

$$I^{(m)} = \bigcap_{p \in \text{Ass}(R/I)} (I^m R_p \cap R).$$

Since $I^{(Nm)} \subseteq I^m$ (see [8] and [16]), one can prove that the Waldschmidt–Skoda inequality $\frac{\alpha(I^{(m)})}{m} \geq \frac{\alpha(I)}{N}$ holds for every homogeneous ideal I in R [15]. Therefore one has that $\gamma(I) \geq \frac{\alpha(I)}{N}$. One can also prove that $\frac{\alpha(I^{(m)})}{m} \geq \gamma(I)$ for every $m \geq 1$, see for instance [2].

It is then natural to ask whether Chudnovsky’s Conjecture 1.3 holds for any homogeneous ideal. We pose it here as an optimistic conjecture, for which we provide some evidence below:

Conjecture 3.1. *Let $R = k[x_0, \dots, x_N]$, where k is any field. For any nonzero homogeneous ideal I in R , one has*

$$\frac{\alpha(I^{(m)})}{m} \geq \frac{\alpha(I) + N - 1}{N}$$

for every $m \geq 1$.

It is easy to see that if an ideal I satisfies Conjecture 3.1 then also $I^{(t)}$ does for any $t \geq 1$. Thus in search for evidence for a positive answer to Conjecture 3.1, one may ask whether for every homogeneous ideal $I \subseteq k[x_0, \dots, x_N]$ there is an exponent t_0 such that $I^{(t)}$ satisfies Conjecture 3.1 for every $t \geq t_0$. We give a positive answer to this question in Theorem 3.7.

We state a few lemmas before stating the main result of this section, Theorem 3.7. The following lemma and its proof can be found in the proof of [2, Lemma 2.3.1].

Lemma 3.2. *Let I be a homogeneous ideal in R and let $m \geq t$ be two positive integers. Write $m = qt + r$ for some integers q and r such that $0 \leq r < t$. Then*

$$\frac{\alpha(I^{(m)})}{m} \leq \frac{\alpha(I^{(t)})}{t} + \frac{\alpha(I^{(r)})}{m}.$$

In particular, if $r = 0$ then we have $\frac{\alpha(I^{(tq)})}{tq} \leq \frac{\alpha(I^{(t)})}{t}$.

For ideals J with $\text{Ass}(R/J) = \text{Min}(J)$ it is easily verified that $\text{Ass}(R/J^{(m)}) = \text{Ass}(R/J)$ and $(J^{(m)})^{(t)} = J^{(mt)}$ for all $m \geq 1$ and $t \geq 1$. However, when J has embedded components we found examples of ideals J (even monomial ideals) and exponents $m \geq 2$, $t \geq 2$ with $(J^{(m)})^{(t)} \neq J^{(mt)}$. Borrowing techniques from a very recent paper by Hà, Nguyen, Trung and Trung [14] we present here an example where $(J^{(2)})^{(2)} \neq J^{(4)}$.

Example 3.3. Let $R = k[x, t, u, v]$ and $J = J_1 J_2$, where

$$J_1 = (x^4, x^3u, xu^3, u^4, x^2u^2v) \quad \text{and} \quad J_2 = (t^3, tuv, u^2v).$$

Then $(J^{(2)})^{(2)} \neq J^{(4)}$.

Proof. It is easy to see check that $m = (x, t, u, v) \in \text{Ass}(R/J)$, for example because $y = x^2t^2u^3v$ is a non-trivial socle element of R/J . Therefore $J^{(n)} = J^n$ for every $n \geq 1$. Then $\text{depth}(R/J^{(2)}) = \text{depth}(R/J^2) = 1$ and $\text{depth}(R/J^{(4)}) = \text{depth}(R/J^4) = 0$, for example by [14, Example 6.6]. In particular, $m \notin \text{Ass}(R/J^{(2)})$ and $m \in \text{Ass}(R/J^{(4)}) = \text{Ass}(R/J^4)$. Hence by Remark 3.4 (a) below, $m \notin \text{Ass}(R/(J^{(2)})^{(2)})$ and therefore $(J^{(2)})^{(2)} \neq J^{(4)}$. \square

Remark 3.4. Let J be an ideal in a Noetherian ring S and $m \geq 1$ be a positive integer. Then

- (a) for any $q \in \text{Ass}(S/J^{(m)})$ there exists $p \in \text{Ass}(S/J)$ with $q \subseteq p$;
- (b) for any $p \in \text{Ass}(S/J)$ one has $(J^{(m)})_p = J_p^m$;
- (c) for any $p \in V(J)$ one has $(J^{(m)})_p \subseteq (J_p)^{(m)}$.

Despite Example 3.3, we prove that for any arbitrary ideal J in a Noetherian ring S there exists an integer $m_0 = m_0(J)$ such that $(J^{(m)})^{(t)} = J^{(mt)}$ for all $m \geq m_0$ and $t \geq 1$. Of course, when $\text{Ass}(S/J) = \text{Min}(J)$ one can take $m_0 = 1$.

Proposition 3.5. Let J be an ideal in a Noetherian ring S . Then

- (a) for all $m \geq 1$ and $t \geq 1$ one has $J^{(mt)} \subseteq (J^{(m)})^{(t)}$;
- (b) there exists $m_0 \geq 1$ such that for all $m \geq m_0$ and $t \geq 1$ one has

$$(J^{(m)})^{(t)} = J^{(mt)}.$$

Proof. (a): Let $x \in J^{(mt)}$. Then by definition there exists $c \in S$ which is a non-zero divisor on S/J such that $cx \in J^{mt} \subseteq (J^{(m)})^t$. By Remark 3.4 (a) we see that c is also a non-zero divisor on $S/J^{(m)}$ and therefore $x \in (J^{(m)})^{(t)}$.

(b): Let $\text{Ass}(S/J) = \{p_1, \dots, p_r\}$; it is well-known that there exist integers m_1, \dots, m_r such that $\text{Ass}(S_{p_i}/J_{p_i}^m) = \text{Ass}(S_{p_i}/J_{p_i}^{m_i})$ for every $m \geq m_i$ [3]. Let $m_0 = \max\{m_i\}$. By (a) we only need to prove $[J^{(m)}]^{(t)} \subseteq J^{(mt)}$ for all $m \geq m_0$ and $t \geq 1$. It suffices to prove it locally at every associated prime q of $J^{(mt)}$.

By [Remark 3.4](#) (a) there exists $p \in \text{Ass}(S/J)$ such that $q \subseteq p$. By [Remark 3.4](#) (c) and (b) we have

$$\left(\left[J^{(m)} \right]^{(t)} \right)_p \subseteq \left[\left(J^{(m)} \right)_p \right]^{(t)} = \left[J_p^m \right]^{(t)}.$$

Now observe that since $q \in \text{Ass}(S/J^{(mt)})$ and $q \subseteq p$, then $q \in \text{Ass}(S_p/(J^{(mt)})_p)$ and then $q \in \text{Ass}(S_p/(J^{mt})_p)$ by [Remark 3.4](#) (b). Since $mt \geq m \geq m_0$, then $q \in \text{Ass}(S_p/J_p^m)$. Therefore, by [Remark 3.4](#) (b) one has $\left[(J_p^m)^{(t)} \right]_q = \left[J_p^{mt} \right]_q = J_q^{mt} = \left[J^{(mt)} \right]_q$. \square

We now go back to our original setting.

Lemma 3.6. *Let $R = k[x_0, \dots, x_N]$, where k is any field. Let I be a homogeneous ideal in R and assume $\gamma(I) > \frac{\alpha(I)}{N}$. Then there exists an integer $t_0 > 0$ such that $I^{(t)}$ satisfies [Conjecture 3.1](#) for all $t \geq t_0$.*

Proof. Let m_0 be as in [Proposition 3.5](#) (b) and write $\gamma(I) = \frac{\alpha(I)}{N} + \epsilon$ for some $\epsilon > 0$. Let $t_0 \geq \max\{\frac{N-1}{N\epsilon}, m_0\}$. Then if $t \geq t_0$ and $m \geq 1$ we have

$$\begin{aligned} \frac{\alpha((I^{(t)})^{(m)})}{m} &= \frac{\alpha(I^{(tm)})}{m} \geq \gamma(I) \cdot t = \frac{\alpha(I)t}{N} + \epsilon t \\ &\geq \frac{\alpha(I)t}{N} + \frac{N-1}{Nt_0}t \geq \frac{\alpha(I^{(t)}) + N-1}{N}. \quad \square \end{aligned}$$

We are ready to prove the main result of this section.

Theorem 3.7. *Let $R = k[x_0, \dots, x_N]$, where k is any field and let I be a nonzero homogeneous ideal in R . Then there exists an integer $t_0 > 0$ such that $I^{(t)}$ satisfies [Conjecture 3.1](#) for all $t \geq t_0$.*

Proof. Let m_0 be as in [Proposition 3.5](#) (b) and $m \geq 1$ be an integer. Since $I^m \subseteq I^{(m)}$, then $m\alpha(I) \geq \alpha(I^{(m)})$. First, if for every $s \geq 1$ we have $s\alpha(I) = \alpha(I^{(s)})$, then for any $t \geq m_0$ and $m \geq 1$,

$$\frac{\alpha((I^{(t)})^{(m)})}{m} = \frac{\alpha(I^{(tm)})}{m} = \frac{tm\alpha(I)}{m} = t\alpha(I) \geq \alpha(I^{(t)}) \geq \frac{\alpha(I^{(t)}) + N-1}{N}.$$

Next, suppose that there exists $T_1 > 0$ such that $T_1\alpha(I) > \alpha(I^{(T_1)})$. Hence, for every $t \geq T_1$ one has $t\alpha(I) > \alpha(I^{(t)})$. Indeed, if $t = T_1 + a$ for some $a \geq 0$, then

$$t\alpha(I) = T_1\alpha(I) + a\alpha(I) > \alpha(I^{(T_1)}) + a\alpha(I) = \alpha(I^{(T_1)} \cdot I^a) \geq \alpha(I^{(T_1+a)}) = \alpha(I^{(t)}),$$

where the last inequality follows from the inclusion $I^{(T_1)} \cdot I^a \subseteq I^{(T_1+a)}$.

Let $t_1 = \max\{T_1, m_0\}$ and notice that by the above $t_1\alpha(I) \geq \alpha(I^{(t_1)}) + 1$. Then for all $m \geq 1$

$$\frac{\alpha((I^{(t_1)})^{(m)})}{m} = \frac{\alpha(I^{(t_1 m)})}{m} = \frac{\alpha(I^{(t_1 m)})}{t_1 m} \cdot t_1 \geq \frac{\alpha(I) t_1}{N} \geq \frac{\alpha(I^{(t_1)}) + 1}{N}.$$

So $\gamma(I^{(t_1)}) \geq \frac{\alpha(I^{(t_1)})}{N} + \frac{1}{N} > \frac{\alpha(I^{(t_1)})}{N}$. By [Lemma 3.6](#), there exists $t_2 > 0$ such that for any $t \geq t_2$, the ideal $(I^{(t_1)})^{(t)} = I^{(t_1 t)}$ satisfies [Conjecture 3.1](#).

Finally, let t_0 be an integer such that $t_0 \geq t_1 t_2 + \frac{\alpha(I^{(t_1 t_2)}) t_1 t_2}{N-1}$. For any $t \geq t_0$, write $t = (t_1 t_2)q + r$, where $0 \leq r < t_1 t_2$; by [Lemma 3.2](#) and the fact that the ideal $I^{(t_1 t_2)}$ satisfies [Conjecture 3.1](#), then for all $m \geq 1$ we have

$$\begin{aligned} \frac{\alpha((I^{(t)})^{(m)})}{m} &\geq \frac{\alpha((I^{(t)})^{(t_1 t_2 m)})}{t_1 t_2 m} = \frac{\alpha((I^{(t_1 t_2)})^{(tm)})}{tm} \cdot \frac{t}{t_1 t_2} \\ &\geq \frac{\alpha(I^{(t_1 t_2)}) + N - 1}{N} \cdot \frac{t}{t_1 t_2} = \frac{\alpha(I^{(t_1 t_2)})}{t_1 t_2} \cdot \frac{t}{N} + \frac{(N-1)t}{N t_1 t_2} \\ &\geq \left(\frac{\alpha(I^{(t)})}{t} - \frac{\alpha(I^{(r)})}{t} \right) \cdot \frac{t}{N} + \frac{(N-1)t}{N t_1 t_2} = \frac{\alpha(I^{(t)})}{N} - \frac{\alpha(I^{(r)})}{N} + \frac{(N-1)t}{N t_1 t_2} \\ &= \frac{\alpha(I^{(t)}) + N - 1}{N} + \frac{(N-1)(t - t_1 t_2) - \alpha(I^{(r)}) t_1 t_2}{N t_1 t_2} \\ &\geq \frac{\alpha(I^{(t)}) + N - 1}{N} + \frac{\alpha(I^{(t_1 t_2)}) t_1 t_2 - \alpha(I^{(r)}) t_1 t_2}{N t_1 t_2} \\ &\geq \frac{\alpha(I^{(t)}) + N - 1}{N}. \quad \square \end{aligned}$$

When I has no embedded components, we have a more explicit description of t_0 .

Corollary 3.8. *If I is a homogeneous ideal with $\text{Ass}(R/I) = \text{Min}(I)$, then one can take $t_0 = (N-1)\delta$, where δ is the first positive integer s with $s\alpha(I) > \alpha(I^{(s)})$.*

Although $(N-1)\delta$ is reasonably small, in general it is not the smallest possible t_0 for which [Theorem 3.7](#) holds. For instance, when I is the ideal of three non-collinear points in \mathbb{P}_k^2 , it is easy to see that $\delta = 2$. Thus [Corollary 3.8](#) yields that for any $t \geq (N-1)\delta = 2$ the ideal $I^{(t)}$ satisfies [Conjecture 3.1](#); however, it is well-known that I satisfies [Conjecture 3.1](#). A natural question then arises.

Question 3.9. *Let I be a homogeneous ideal in R . Does there exist a number $t_0 = t_0(N)$ such that $I^{(t)}$ satisfies [Conjecture 3.1](#) for every $t \geq t_0$?*

Of course, [Conjecture 3.1](#) is true if and only if the integer $t_0 = 1$ works for any homogeneous ideal I . [Theorem 2.8](#) says that $t_0 = 1$ is sufficient for any finite set of very

general points in \mathbb{P}_k^N . The following proposition shows that $t_0 = N - 1$ is sufficient for any finite set of points in $\mathbb{P}_{\mathbb{C}}^N$.

Proposition 3.10. *Let I be the radical ideal of a finite set of points in $\mathbb{P}_{\mathbb{C}}^N$. Then $I^{(t)}$ satisfies [Conjecture 3.1](#) for every $t \geq N - 1$.*

Proof. By the result of Esnault and Viehweg [9] one has $\gamma(I) \geq \frac{\alpha(I)+1}{N} = \frac{\alpha(I)}{N} + \frac{1}{N}$. Set $\varepsilon = \frac{1}{N}$. Then by the proof of [Lemma 3.6](#) (here $m_0 = 1$ because I is radical) we can take t_0 such that $\frac{1}{N} \geq \frac{N-1}{Nt_0}$; thus we can take $t_0 = N - 1$. \square

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