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Ordered structures and large conjugacy classes [☆]



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ABSTRACT

This article is a contribution to the following problem: does there exist a Polish non-archimedean group (equivalently: automorphism group of a Fraïssé limit) that is extremely amenable, and has ample generics. As Fraïssé limits whose automorphism groups are extremely amenable must be ordered, i.e., equipped with a linear ordering, we focus on ordered Fraïssé limits. We prove that automorphism groups of the universal ordered boron tree, and the universal ordered poset have a comeager conjugacy class but no comeager 2-dimensional diagonal conjugacy class. We formulate general conditions implying that there is no comeager conjugacy class, comeager 2-dimensional diagonal conjugacy class or non-meager 2-dimensional topological similarity class in the automorphism group of an ordered Fraïssé limit. We also provide a number of applications of these results.

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1. Introduction

This article is a contribution to the following question: does there exist a Polish non-archimedean group, i.e., a Polish group with a neighborhood basis at the identity consisting of open subgroups, that simultaneously satisfies two frequently studied properties: it is extremely amenable, and it has ample generics.

A Polish group G is *extremely amenable* if every continuous action of G on a compact space has a fixed point. The group G has *ample generics* if, for every $n \geq 1$, there exists an n -dimensional diagonal conjugacy class in G , i.e., a set of the form

$$\{(gg_1g^{-1}, \dots, gg_ng^{-1}) \in G^n : g \in G\},$$

for some $g_1, \dots, g_n \in G$, which is comeager in G^n . Such a group admits only one Polish group topology, and all of its (abstract) homomorphisms into separable groups are continuous (Kechris-Rosendal [8]). In particular, *every* action by homeomorphisms of an extremely amenable group with ample generics on a compact separable space has a fixed point.

It is known that there exist Polish groups sharing both of these features. Pestov-Schneider [15] proved that, for any Polish group G , the group $L_0(G)$, i.e., the group of measurable functions with values in G , is extremely amenable, provided that G is amenable, while Kaïchouh-Le Maître [9] proved that $L_0(G)$ has ample generics whenever G does. As S_∞ , i.e., the group of all permutations of natural numbers, is amenable, and has ample generics, $L_0(S_\infty)$ is extremely amenable and it has ample generics. However, it is still an open problem whether there are such groups in the non-archimedean realm.

It is well known that every Polish non-archimedean group is isomorphic to the automorphism group $\text{Aut}(M)$ of a structure M (i.e., a set equipped with relations and functions) equal to the Fraïssé limit of a Fraïssé class \mathcal{F} of finite structures (see the next section for precise definitions of notions used in the introduction). The group $\text{Aut}(M)$ naturally acts on the compact space of linear orderings of M , viewed as a subspace of $\{0, 1\}^{M \times M}$. This implies that when $\text{Aut}(M)$ is extremely amenable, then there is a linear ordering of M preserved by $\text{Aut}(M)$, see also [11]. Therefore if $\text{Aut}(M)$ is extremely amenable, we can actually assume that \mathcal{F} is an order class, i.e., that each structure in \mathcal{F} is equipped with a linear ordering $<$ of its elements. Thus, we can pose a more general question: does there exist a Fraïssé limit M of an order class \mathcal{F} such that the automorphism group $\text{Aut}(M)$ has ample generics. This article gives some partial answers as to when such a situation cannot happen.

Curiously enough, there are no known examples of Polish groups that do not have ample generics but they have a comeager diagonal conjugacy class for some $n \geq 2$. In other words, our article can be also viewed as a study of the question whether comeager diagonal conjugacy classes resemble weak mixing in topological dynamics, which implies weak mixing of all orders (see [4]).

One of our main tools is a theorem of Kechris-Rosendal, connecting the structure of diagonal conjugacy classes in the automorphism group of the Fraïssé limit M of a Fraïssé class \mathcal{F} with the joint embedding property (JEP), and the weak amalgamation property (WAP) in classes \mathcal{F}_n of n -tuples of partial automorphisms of elements of \mathcal{F} . They prove (see also [5]) that $\text{Aut}(M)$ has a comeager n -dimensional diagonal conjugacy class if and only if \mathcal{F}_n has JEP and WAP. Thus, showing that $\text{Aut}(M)$ does not have a comeager n -dimensional diagonal conjugacy class reduces to verifying that \mathcal{F}_n has no JEP or WAP.

First, we study the one-dimensional case. A class of structures \mathcal{F} has the 1-Hrushovski property if every partial automorphism of an $A \in \mathcal{F}$ can be extended to an automorphism of some $B \in \mathcal{F}$. Clearly, if \mathcal{F} is an order class of finite structures, then \mathcal{F} does not have the 1-Hrushovski property because in this case non-trivial orbits are necessarily infinite. We introduce the notions of strong splitting and always strong splitting in a Fraïssé class, which capture the idea of ‘flexible’ amalgamation. Then we prove (Theorem 3.5) that \mathcal{F}_1 has no WAP, provided that one of following holds: \mathcal{F} is a Fraïssé class that does not have the 1-Hrushovski property, and it always strongly splits, or \mathcal{F} is a full order expansion of \mathcal{K} (i.e., \mathcal{F} is the class of *all* linear orderings on elements of \mathcal{K}), where \mathcal{K} is a Fraïssé class that strongly splits. On the other hand, we show (Theorem 3.10) that the class \mathcal{SB}_1 of partial automorphisms of ordered boron trees, and (Theorem 3.13) the class \mathcal{P}_1 of partial automorphisms of ordered partial orders, have CAP, so, in particular, they have WAP. It seems that these are, except for $\text{Aut}(\mathbb{Q})$ (see Truss [17]), the only known order classes such that the automorphism group of the limit has a comeager conjugacy class. We also give (Theorem 3.8) a short and elementary proof of a result of Slutsky [16] who showed that the class $(\mathcal{QU}_\prec)_1$ of partial automorphisms of ordered metric spaces with rational distances has no WAP.

Next, we turn to the two-dimensional case. For a Fraïssé class \mathcal{F} , we formulate a simple but efficient condition (Proposition 4.1) implying that \mathcal{F}_2 has no WAP, and we verify it for a number of cases such as precompact Ramsey expansions of ultrahomogeneous directed graphs, in particular for \mathcal{P}_2 . Using a similar approach, we also show (Theorem 4.4) that \mathcal{SB}_2 does not have WAP. Then we investigate topological similarity classes. For a Polish group G , $n \geq 1$, and an n -tuple (f_1, \dots, f_n) in G , the n -dimensional topological similarity class of (f_1, \dots, f_n) is the family of all n -tuples (g_1, \dots, g_n) in G such that the mapping $f_i \mapsto g_i$ (uniquely) extends to a topological group isomorphism. Clearly, this is a generalization of the notion of the diagonal conjugacy class, and it is still not known whether there exists a Polish group G such that for some $n \geq 2$ there is a non-meager n -dimensional topological similarity class, but all n -dimensional diagonal conjugacy classes are meager. Generalizing methods and results of Slutsky [16], we show (Theorem 5.5) that if M is the Fraïssé limit of a Fraïssé class \mathcal{F} that is a full order expansion and that satisfies certain additional conditions, then all 2-dimensional topological similarity classes in $\text{Aut}(M)$ are meager. In particular, this is true if \mathcal{K} is a class with free amalgamation, or the class of ordered tournaments (Theorem 5.7).

2. Definitions

A topological group is *Polish* if its topology is separable, and completely metrizable. A Polish group is *non-archimedean* if it has a neighborhood basis at the identity consisting of open subgroups, or, equivalently, it is topologically isomorphic to the automorphism group $\text{Aut}(M)$ of a countable structure, equipped with the product topology (i.e., $\text{Aut}(M) \subseteq M^M$, where M is regarded as a discrete space).

By a structure we always mean a relational structure (i.e., a set equipped with relations), and we consider only classes of finite structures. Let A be a structure, and let $B, C \subseteq A$. By $\text{qftp}_A(B/C)$, we denote the quantifier-free type of B over C in A . Let p be a partial automorphism of A . We write $\text{def}(p) = \text{dom}(p) \cup \text{rng}(p)$, and $\text{supp}(p) = \{x \in \text{def}(p) : p(x) \neq x\}$. An *orbit* of p is a maximal subset $\mathcal{O} \subseteq A$ that can be enumerated into $\{a_0, \dots, a_m\}$ so that $p(a_i) = a_{i+1}$ for $i < m$. If $p(a_m) = a_0$, we say that \mathcal{O} is a *cyclic orbit*. An orbit is *trivial* if it consists of a single element.

Let \mathcal{F} be a class of structures in a given signature. We say that \mathcal{F} has *JEP* (the joint embedding property) if any two $A, B \in \mathcal{F}$ can be embedded in a single $C \in \mathcal{F}$. We say that \mathcal{F} has *AP* (the amalgamation property) if for every $A, B, C \in \mathcal{F}$ and embeddings $\alpha: A \rightarrow B$, $\beta: A \rightarrow C$ there is $D \in \mathcal{F}$ and embeddings $\gamma: B \rightarrow D$, $\delta: C \rightarrow D$ such that $\gamma \circ \alpha = \delta \circ \beta$. In that case, we say that B and C *amalgamate* over A . We say that \mathcal{F} has *SAP* (the strong amalgamation property) if, additionally, $\gamma[B] \cap \delta[C] = \gamma \circ \alpha[A]$. We say that \mathcal{F} has *CAP* (the cofinal amalgamation property) if there is a cofinal (with respect to inclusion) subclass of \mathcal{F} with AP. We say that \mathcal{F} has *WAP* (the weak amalgamation property) if for every $A \in \mathcal{F}$ there is $A' \in \mathcal{F}$ and an embedding $\phi: A \rightarrow A'$ such that for every $B, C \in \mathcal{F}$ and embeddings $\alpha: A' \rightarrow B$, $\beta: A' \rightarrow C$ there is $D \in \mathcal{F}$ and embeddings $\gamma: B \rightarrow D$, $\delta: C \rightarrow D$ such that $\gamma \circ \alpha \circ \phi = \delta \circ \beta \circ \phi$. Clearly, if \mathcal{F} has CAP, then it has WAP. Actually, in the definition of AP (CAP and WAP), it suffices to consider B, C such that $B \cap C = A$ ($B \cap C = A'$), and only trivial embeddings, i.e., inclusions.

A class of finite structures \mathcal{F} is a *Fraïssé class*, if it is countable up to isomorphism, closed under isomorphism, closed under taking substructures, and has JEP and AP. A countable first-order structure M is *ultrahomogeneous* if every isomorphism between finite substructures of M can be extended to an automorphism of the whole M . Then $\text{Age}(M)$ – the class of all finite substructures embeddable in M – is a *Fraïssé class*. Conversely, by the classical theorem due to Fraïssé, for every Fraïssé class \mathcal{F} of finite structures, there is a unique up to isomorphism countable ultrahomogeneous structure M such that $\mathcal{F} = \text{Age}(M)$. We call this M the *Fraïssé limit* of \mathcal{F} .

A Fraïssé class \mathcal{F} is called an *order class* if its signature includes a binary relation defining a linear ordering on each element of \mathcal{F} . If \mathcal{F} is an order class, \mathcal{F}^- denotes the reduct of \mathcal{F} obtained by removing the order relation $<$ from the signature of \mathcal{F} . We call \mathcal{F} a *full order expansion* if $\mathcal{F} = \mathcal{F}^- * \mathcal{LO}$, i.e., it is a class of elements of the form $(A, <)$, where $A \in \mathcal{F}^-$, and $<$ is *any* linear ordering of A . We will frequently use the observation that if M is the Fraïssé limit of a full order expansion with SAP, then for any finite

$A \subset M$, $b \in M$, and any $<$ -interval I in M , there exists $b' \in I$ such that $A \cup \{b\}$ is isomorphic to $A \cup \{b'\}$ via an isomorphism that pointwise fixes A .

We will be mostly interested in classes of tuples of partial automorphisms of structures coming from a given class \mathcal{F} . Formally, for $n \geq 1$, denote

$$\mathcal{F}_n = \{(A, p_1, \dots, p_n) : A \in \mathcal{F}, p_i \text{ is a partial automorphism of } A, i \leq n\}.$$

Often, we will think of elements of \mathcal{F}_n simply as tuples of partial automorphisms. Then (p_1, \dots, p_n) is identified with $(\bigcup_i \text{def}(p_i), p_1, \dots, p_n)$. A map $\phi : (A, p_1, \dots, p_n) \rightarrow (B, q_1, \dots, q_n)$ will be called an *embedding* if it is an embedding of A into B , and $\phi \circ p_i = q_i \circ \phi$ for $i \leq n$. Using this notion of embedding, we can also define properties JEP, AP, CAP, and WAP for classes \mathcal{F}_n . Then we have:

Theorem 2.1 (Kechris-Rosendal [8]). *Let \mathcal{F} be a Fraïssé class, and let M be the Fraïssé limit of \mathcal{F} .*

- (1) *There exists a dense diagonal n -conjugacy class in $\text{Aut}(M)$ iff \mathcal{F}_n has JEP,*
- (2) *there exists a comeager diagonal n -conjugacy class in $\text{Aut}(M)$ iff \mathcal{F}_n has JEP and WAP.*

In particular, it follows that if \mathcal{F}_n has JEP but no WAP, then $\text{Aut}(M)$ has all n -dimensional diagonal conjugacy classes meager.

Let $F_n = F_n(s_1, \dots, s_n)$ denote the free group on n generators s_1, \dots, s_n . For a word $w \in F_n$, and an n -tuple $\bar{f} = (f_1, \dots, f_n)$ in a group G , the evaluation $w(\bar{f})$ denotes the element of G obtained from w by substituting f_i for s_i , and performing the group operations on the resulting sequence. By a word, we will always mean a reduced word.

3. The one-dimensional case. Conjugacy classes

3.1. A condition that implies the failure of WAP

Recall that a family \mathcal{F} of finite structures in a given signature has the *Hrushovski property* if for every $n \in \mathbb{N}$, $A \in \mathcal{F}$ and a tuple (f_1, \dots, f_n) of partial automorphisms of A , there exists $B \in \mathcal{F}$ such that $A \subseteq B$, and every f_i can be extended to an automorphism of B . We say that \mathcal{F} has the *n -Hrushovski property* if the above holds for a given n .

In [10, Theorem 4.7], we proved the following trichotomy.

Theorem 3.1. *Let M be a Fraïssé limit of a Fraïssé family \mathcal{F} such that algebraic closures of finite subsets of M are finite. Then one of the following holds:*

- (1) *\mathcal{F} has the Hrushovski property,*
- (2) *\mathcal{F} does not have the 1-Hrushovski property,*

- (3) *there exists n such that none of n -dimensional topological similarity classes in $\text{Aut}(M)$ is comeager. In particular, $\text{Aut}(M)$ does not have ample generics.*

It is well known that if a Fraïssé class \mathcal{F} has the Hrushovski property, and sufficiently free amalgamation, then the automorphism group $\text{Aut}(M)$ of its limit M has ample generics. By the above trichotomy, if \mathcal{F} does not have the Hrushovski property, ample generics may be present in $\text{Aut}(M)$ only if \mathcal{F} does not even have the 1-Hrushovski property – which is true, in particular, for order classes. In this section, we prove (in Theorem 3.5) that such situations always presuppose a very rigid form of amalgamation in \mathcal{F} . In order to specify what ‘rigid’ is about in this context, let us introduce two definitions that capture what ‘flexible’ amalgamation means for us.

In [14], Panagiotopolus studies extensions of automorphisms of generic substructures of a given structure. He introduces the notion of splitting in a Fraïssé class \mathcal{F} . An element $C \in \mathcal{F}$ *splits* \mathcal{F} if for every $D \in \mathcal{F}$ with $C \subseteq D$ there exist $D_1, D_2 \in \mathcal{F}$ with $D \subsetneq D_1, D_2$, and a bijection $f: D_1 \rightarrow D_2$ such that

- (1) f pointwise fixes D ,
- (2) $f \upharpoonright (D_1 \setminus C)$ is an isomorphism between $D_1 \setminus C$ and $D_2 \setminus C$,
- (3) f is not an isomorphism between D_1 and D_2 .

Analogously, we will say that $C \in \mathcal{F}$ *strongly splits* \mathcal{F} if for all $D, D_1 \in \mathcal{F}$ with $C \subseteq D \subsetneq D_1$ there exists $D_2 \in \mathcal{F}$ with $D \subsetneq D_2$, and a bijection $f: D_1 \rightarrow D_2$ such that Conditions (1)-(3) above hold. We will say that \mathcal{F} *strongly splits* if there exists $C \in \mathcal{F}$ that strongly splits \mathcal{F} , and that \mathcal{F} *always strongly splits* if every $C \in \mathcal{F}$ strongly splits \mathcal{F} .

We can think of C in the above definitions as one ‘ear’ of an amalgamation diagram $U \subsetneq V, W$, i.e., $C = V \setminus U$ and $D = V$. Then C strongly splits if for any other ‘ear’ $W \setminus U$ (i.e., $D_1 \setminus D$), there are at least two non-equivalent ways in which we can define relations involving elements from the ‘ears’ $V \setminus U$ and $W \setminus U$ to form an amalgam of V and W over U : one represented by D_1 (where $W = D_1 \setminus C$), the other one by D_2 (where $D_2 \setminus C$ is an isomorphic copy of W).

In particular, the property of always strong splitting can be also expressed as a variant of the amalgamation property: \mathcal{F} always strongly splits if amalgamation in \mathcal{F} is not too rigid, that is, if there is always more than one way of amalgamating structures. To be more precise, fix C, D and D_1 as above, and think of D_1 as an amalgam of D and $D_1 \setminus C$ over $D \setminus C$. Then any other non-isomorphic amalgam with the same underlying sets gives a required D_2 . In other words, a class \mathcal{F} always strongly splits if for any $A, U, V \in \mathcal{F}$ with $A < U, V$ there exist two non-isomorphic amalgams W_1, W_2 of U and V over A , with $U \cup V$ as the underlying set, and such that U and V embed in both W_1 and W_2 by the identity mapping.

Proving the next two lemmas is straightforward, and left to the reader.

Lemma 3.2. *If \mathcal{F} is a full order expansion of a class that (always) strongly splits, then \mathcal{F} also (always) strongly splits.*

Lemma 3.3. *Let p be a partial automorphism of a structure A , and let $x \in \text{rng}(p) \setminus \text{dom}(p)$. Suppose that $y, y' \in A$ are such that $p \cup (x, y)$ is a partial automorphism, and $\text{qftp}_A(y/\text{rng}(p)) = \text{qftp}_A(y'/\text{rng}(p))$. Then $p \cup (x, y')$ is also a partial automorphism of A .*

Lemma 3.4. *Let \mathcal{F} be a full order expansion with SAP. Then for every $p \in \mathcal{F}_1$, there exists $q \in \mathcal{F}_1$ extending p such that for every $r \in \mathcal{F}_1$ extending q , distinct orbits of p are contained in distinct orbits of r .*

Proof. Fix $p \in \mathcal{F}_1$, and let $\mathcal{O}_0, \dots, \mathcal{O}_n$ be orbits of p . Fix $i < j \leq n$, and suppose that there exists an extension q' of p such that \mathcal{O}_i and \mathcal{O}_j are in the same orbit of q' . Then there must exist a partial automorphism q'' extending p , and $x, y \in \text{rng}(q'') \Delta \text{dom}(q'')$, x in the orbit of q'' determined by \mathcal{O}_i , y in the orbit of q'' determined by \mathcal{O}_j , and we can extend q'' by putting $q''(x) = y$ or $(q'')^{-1}(x) = y$. Without loss of generality, we can assume that $x < y$, and $q'' \cup (x, y)$ extends q'' . Since \mathcal{F} is a full order expansion with SAP, there exists $C \in \mathcal{F}$ with $\text{def}(q'') \subseteq C$, and $y' \in C \setminus \text{def}(q'')$ with $y' > y$, and such that $\text{qftp}_C(y/\text{rng}(q'')) = \text{qftp}_C(y'/\text{rng}(q''))$. But then, by Lemma 3.3, $q = q'' \cup (x, y')$ also extends q'' , and $x < y < q(x)$. Clearly, \mathcal{O}_i and \mathcal{O}_j stay distinct in any extension of q . By iterating this construction, we can find q that works for all $i < j \leq n$. \square

Theorem 3.5. *Let \mathcal{F} be a Fraïssé class. Suppose that*

- (1) \mathcal{F} does not have the 1-Hrushovski property, and it always strongly splits, or
- (2) \mathcal{F} is a full order expansion with SAP, and it strongly splits.

Then \mathcal{F}_1 has no WAP.

Proof. Assume that Condition (1) holds. Fix $p \in \mathcal{F}_1$ witnessing that \mathcal{F} does not have the 1-Hrushovski property. We show that p also witnesses that \mathcal{F}_1 does not have WAP.

Fix $q \in \mathcal{F}_1$ that extends p . Clearly, there must exist an orbit \mathcal{O} of q intersecting $\text{dom}(p)$ that is non-cyclic – otherwise the union of such orbits would be a structure in \mathcal{F} invariant under q . Let $\mathcal{O} = \{o_0, \dots, o_n\}$ with $q(o_i) = o_{i+1}$ for $i < n$. As \mathcal{O} is not cyclic, q is not defined on o_n . Fix $y_0 \notin \text{def}(q)$ such that $q_0 = q \cup (o_n, y_0)$ is a partial automorphism of $D_1 = \text{def}(q) \cup \{y_0\}$. Since \mathcal{F} always strongly splits, by putting $C = \text{dom}(q) \setminus \text{rng}(q)$ (note that $o_0 \in C$, so $C \neq \emptyset$), $D = \text{def}(q)$ (and $D_1 = \text{def}(q) \cup \{y_0\}$), we can find y_1 such that $D_2 = \text{def}(q) \cup \{y_1\}$ witnesses that C strongly splits. However, this means that

$$\text{qftp}_C(y_0/\text{rng}(q)) = \text{qftp}_C(y_1/\text{rng}(q)) \tag{1}$$

but

$$\text{qftp}_C(y_0/(\text{def}(q))) \neq \text{qftp}_C(y_1/(\text{def}(q))). \tag{2}$$

Then (1) together with Lemma 3.3 implies that $q_1 = q \cup (o_n, y_1)$ is also a partial automorphism. On the other hand, for every $r \in \mathcal{F}_1$ such that q_0 and q_1 can be embedded into r by embeddings e_0 and e_1 , respectively, that agree on $\text{def}(q)$, we must have $e_0(y_0) = e_1(y_1)$, and this is impossible because of (2). Moreover, the same argument can be applied to any extension of p , so, in fact, if e_0, e_1 agreed on $\text{def}(p)$, they would agree on $\text{def}(q)$ as well. Thus, q_0, q_1 cannot be amalgamated over p . As q was arbitrary, p witnesses that \mathcal{F}_1 does not have WAP.

Assume now that Condition (2) holds. Fix $A \in \mathcal{F}$ witnessing that \mathcal{F} strongly splits. Fix $p \in \mathcal{F}_1$ such that $A = \text{dom}(p) \setminus \text{rng}(p)$ (this can be easily done using the assumption that \mathcal{F} is a full order expansion with SAP), and extend p to a partial automorphism q as in Lemma 3.4. Then for every extension $r \in \mathcal{F}_1$ of q , there exists $A' \subseteq \text{dom}(r) \setminus \text{rng}(r)$ that is isomorphic with A , and so A' also witnesses that \mathcal{F} strongly splits. Now we proceed as in the proof of (1). Fix a (non-cyclic) orbit $\mathcal{O} = \{o_0, \dots, o_n\}$ of r intersecting $\text{dom}(p)$, find y_0 such that $r_0 = r \cup \{(o_n, y_0)\}$ is a partial automorphism, and put $C = A', D = \text{def}(r), D_1 = \text{def}(r) \cup \{y_0\}$ to obtain y_1 such that r_0 and $r_1 = r \cup (o_n, y_1)$ cannot be amalgamated over p . \square

Proposition 3.6. *The classes of ordered graphs, and ordered tournaments always strongly split.*

Proof. Let \mathcal{F} be either the class of ordered graphs or ordered tournaments. Fix $C, D, D_1 \in \mathcal{F}$ with $C \subseteq D \subsetneq D_1$. Fix $c \in C$ and $d_1 \in D_1 \setminus D$. For graphs, define D_2 to be the graph that differs from D_1 only in that $\{c, d_1\}$ is an edge in D_2 if and only if $\{c, d_1\}$ is not an edge in D_1 . Similarly, for ordered tournaments, define D_2 to be the ordered tournament that differs from D_1 only in that (d_1, c) is an arrow in D_2 if and only if (c, d_1) is an arrow in D_1 . \square

Corollary 3.7.

- (1) *The class of partial automorphisms of finite ordered tournaments has no WAP.*
- (2) *(Slutsky) The class of partial automorphisms of finite ordered graphs has no WAP.*

Proof. Finite ordered graphs and finite ordered tournaments are full order expansions with SAP. By Proposition 3.6, both of them always strongly split, so, by Theorem 3.5, they have no WAP. \square

On the other hand, it is easy to see that both of these classes have JEP, which means (by Theorem 2.1), that automorphism groups of the universal ordered tournament and the universal ordered graph (i.e., Fraïssé limits of the above classes) have all conjugacy classes meager.

Slutsky [16] also proved that the automorphism group $\text{Aut}(\mathbb{Q}\mathbb{U}_{\prec})$ of the ordered rational Urysohn space $\mathbb{Q}\mathbb{U}_{\prec}$, i.e., the full order expansion of finite ordered metric spaces with rational distances, has meager conjugacy classes. One of the ingredients of his proof is a deep theorem of Solecki saying that the class of finite metric spaces has the Hrushovski property. Theorem 3.5 cannot be used to recover Slutsky’s result because the class of ordered finite metric spaces with rational distances does not strongly split. However, a similar approach gives rise to a more elementary argument. We sketch it below. Note that $(\mathbb{Q}\mathcal{U}_{\prec})_1$ has JEP, therefore it will suffice to prove that it has no WAP.

Theorem 3.8 (Slutsky). *The class $(\mathbb{Q}\mathcal{U}_{\prec})_1$ has no WAP.*

Proof. First fix $A \in \mathbb{Q}\mathcal{U}$, and $x \in A$. Without loss of generality, we can assume that $A \subseteq \mathbb{Q}\mathbb{U}$. Suppose that $y \in \mathbb{Q}\mathbb{U}$ is such that the type $\text{qftp}_A(y/(A \setminus \{x\}))$ determines $d(x, y)$. By the triangle inequality, this is possible only when there are $a, a' \in A$ such that

$$\begin{aligned} d(y, x) &= d(y, a) + d(a, x), \\ d(a', x) &= d(a', y) + d(y, x). \end{aligned}$$

But then, in particular, $d(x, y) \leq \text{diam}(A)$. Thus, if for some partial automorphism p there was an automorphism q extending p such that any two extensions r_0, r_1 of q could be amalgamated over p , then for every automorphism ϕ of $\mathbb{Q}\mathbb{U}_{\prec}$ extending q , orbits of ϕ determined by p would be bounded by $\text{diam}(\text{def}(q))$. But it is well known (see, e.g., Section 3.1 in [3]) that every partial isometry q of $\mathbb{Q}\mathbb{U}$ with no cyclic orbits can be extended to an isometry of $\mathbb{Q}\mathbb{U}$ with unbounded orbits. And since $\mathbb{Q}\mathcal{U}_{\prec}$ is a full order expansion of $\mathbb{Q}\mathcal{U}$, every partial automorphism of a finite subspace of $\mathbb{U}\mathbb{Q}_{\prec}$ also can be extended to an automorphism of $\mathbb{Q}\mathbb{U}_{\prec}$ with unbounded orbits. \square

Remark 3.9. Using a similar approach, and a construction as in the proof of Lemma 5.6, one can also prove that the class of partial automorphisms of ordered K_n -free graphs does not have WAP, for every $n \geq 3$. It is not hard to see that this class does not strongly split.

It is known that the class of partial automorphisms of finite linear orderings has JEP and WAP. In the next two sections, we present two others such order classes: ordered boron trees, and ordered posets.

3.2. Ordered boron trees - a comeager conjugacy class

In this section, we prove that the automorphism group of the universal ordered boron tree has a comeager conjugacy class. The class of boron trees was introduced by Cameron [1].



Fig. 1. Expanding $(A = \{a, b, c, d, e\}, R^A)$ by S^A and $<_{lex}^A$.

Let \mathcal{T} denote the class of all graph-theoretic trees such that the valency of each vertex is equal to 1 or 3. If $T \in \mathcal{T}$ and $a, b, c, d \in T$, we let $ab|cd$ iff arcs ab and cd do not intersect. To each $T \in \mathcal{T}$ we assign a structure $(B(T), R^{B(T)})$ such that $B(T)$ is the set of endpoints of T , and $R^{B(T)}(a, b, c, d)$ iff a, b, c, d are pairwise different and $ab|cd$. Structures $(B(T), R^{B(T)})$, together with the one point structure, we call *boron trees*, and we denote the class of all these structures by \mathcal{B} . The *universal boron tree* is the Fraïssé limit of \mathcal{B} .

Let $2^{<n}$ denote the set of binary sequences of the length $< n$, including the empty sequence. Let T'_n denote the binary tree, that is, a graph with the set of vertices equal to $2^{<n}$ and edges exactly between vertices s and si , $i = 0, 1$, $s \in 2^{<n}$. Let T_n be the graph obtained by removing the vertex \emptyset from T'_n and replacing edges $[\emptyset, 0]$ and $[\emptyset, 1]$ by the edge $[0, 1]$, and denote $B_n = B(T_n)$. Let \leq^n be the lexicographical order on B_n , i.e. we let $s \leq^n t$ iff $s = t$ or $s(i) < t(i)$, where i is the least such that $s(i) \neq t(i)$. For $s \in 2^{<n}$ we define the *height* as the length of s , i.e. $ht(s) = |s|$. For a fixed n and $s, t \in T'_n$, we let $s < t$ if s is an initial segment of t , and let for $s, t \in T'_n$ the *meet* of s and t , denoted $meet(s, t)$, be the least upper bound of s and t with respect to the partial order $<$. Let $(A, R^A) \in \mathcal{B}$ and let $\phi: (A, R^A) \rightarrow (B_n, R^{B_n})$ be an embedding (with respect to R , this is not necessarily a graph embedding). We let \leq_{lex}^A to be the order inherited from \leq^n and define a ternary relation S^A on A as follows:

$$S^A(a, b, c) \iff \phi(a), \phi(b) <_{lex}^n \phi(c) \text{ and } ht(meet(\phi(a), \phi(b))) > ht(meet(\phi(b), \phi(c))).$$

There are multiple ways to expand an $(A, R^A) \in \mathcal{B}$ by adding relations S^A and $<_{lex}^A$. Intuitively, the relation S^A adds a root at an edge of the tree T such that $B(T) = A$, viewed as a graph. We illustrate the construction of an expansion in Fig. 1. Structures $(A, R^A, S^A, \leq_{lex}^A)$, we call *ordered boron trees*, and we denote the class of all these structures by \mathcal{SB} . The *universal ordered boron tree* is the Fraïssé limit of \mathcal{SB} . From the work of Jasiński [6], it follows that the automorphism group of the universal ordered boron tree is extremely amenable.

Note that for $a <_{lex} b <_{lex} c <_{lex} d$ we have $R(a, b, c, d)$ iff $S(a, b, c)$ or $(\neg S(a, b, c)$ and $\neg S(b, c, d))$. Therefore, we can recover R from S and $<_{lex}$.

Theorem 3.10. *The family \mathcal{SB}_1 has CAP.*

For $A \in \mathcal{SB}$, denote by T_A the binary tree such that $A = B(T_A)$. For every A there exists unique such a T_A . The *root* of T_A , denoted by ρ_A , is the $<$ -least element of T_A . By $<_A$ we denote the usual tree partial ordering of being an initial segment on elements of T_A . In the sequel, the structure A in symbols \leq_{lex}^A , $<^A$, R^A , and S^A will be always clear from the context, so, in order to simplify notation, we will simply write \leq_{lex} , $<$, R , and S , respectively.

Let $(A, p) \in \mathcal{SB}_1$. We say that a non-trivial orbit $O = \{a_0, \dots, a_n\}$ of p is *increasing* if $a_0 <_{lex} \dots <_{lex} a_n$; analogously, we define a *decreasing* orbit. Clearly, every orbit is either increasing, decreasing, or trivial. Note that, setting $t_i = \text{meet}(a_i, a_{i+1})$, we either have $t_0 < t_1 < t_2 < \dots < t_{n-1}$ or $t_{n-1} < \dots < t_2 < t_1 < t_0$. In the first case, we say that O is *meet-increasing*, and in the second that it is *meet-decreasing*. If $(B, p) \in \mathcal{SB}_1$ extends (A, p) , then we will denote by O_B the extension of O in B .

Let $A = (A, p) \in \mathcal{SB}_1$. We will call two orbits $O = \{a_0, \dots, a_m\}$ and $P = \{b_0, \dots, b_n\}$ of p *intertwining* if the \leq_{lex} -intervals $(\min\{a_0, a_m\}, \max\{a_0, a_m\})_{lex}$ and $(\min\{b_0, b_n\}, \max\{b_0, b_n\})_{lex}$ intersect. Say that O is \leq_{lex} -*contained* in P if $(\min\{a_0, a_m\}, \max\{a_0, a_m\})_{lex}$ is contained in $(\min\{b_0, b_n\}, \max\{b_0, b_n\})_{lex}$. A point $x \in A$ is *meet-locked* by O if for every extension (B, q) of (A, p) such that $a_{-1} = q^{-1}(a_0)$ and $a_{m+1} = q(a_m)$ are defined, denoting $t_i = \text{meet}(a_i, a_{i+1})$, we have the following. (1) If O is increasing and meet-increasing, then $a_{m+1} <_{lex} x$ and $t_{-1} < \text{meet}(x, a_{m+1}) < t_m$, (2) if O is decreasing and meet-increasing, then $x <_{lex} a_{m+1}$ and $t_{-1} < \text{meet}(x, a_{m+1}) < t_m$, (3) if O is increasing and meet-decreasing, then $x <_{lex} a_{-1}$ and $t_{-1} < \text{meet}(x, a_{-1}) < t_m$, (4) if O is decreasing and meet-decreasing, then $a_{-1} <_{lex} x$ and $t_{-1} < \text{meet}(x, a_{-1}) < t_m$. Two orbits O and P are *meet-intertwining* if there is $x \in O$ meet-locked by P or there is $x \in P$ meet-locked by O . Note that if O and P are meet-intertwining then one of them is increasing and the other one is decreasing. Moreover, either both are meet-increasing or both are meet-decreasing.

We call a *cone* any set $\text{Cone}_t = \{s \in A : t \leq s\}$, for some $t \in T_A$. The *root* of the orbit O is the *meet* $t_O \in T_A$ of all points a_0, \dots, a_n (which, in fact is the *meet* of two first elements in the orbit, if the orbit is meet-increasing, or the last two, if it is meet-decreasing), and the cone Cone_O of O is defined as Cone_{t_O} . Note that any two cones are either disjoint or one is contained in the other. Denote by $\text{Cone}(p)$ the collection of all cones of orbits of p .

For $A \in \mathcal{SB}$, by a *segment* we mean an ordered pair (x, y) with $x, y \in T_A$ such that $x < y$ and there is no $z \in T_A$ satisfying $x < z < y$. For $A, E \in \mathcal{SB}$ and a segment (x, y) in A , let $K = A(x, y, E, \emptyset) \in \mathcal{SB}$ be the result of attaching E to A on (x, y) on the left. Specifically, think that elements of each of T_A and T_E are binary sequences, in particular, if $x = s$ and $y = t$, then $t = s0$ or $t = s1$. We let T_K to consist of the following binary sequences. If $r \in T_A$ does not extend properly $x = s$, we let $r \in T_K$. We let $t \in T_K$. If $tr \in T_A$ for some r , we let $t1r \in T_K$ and if $r \in T_E$, let $t0r \in T_K$. This defines $K \in \mathcal{SB}$. Analogously, define $K = A(x, y, \emptyset, F) \in \mathcal{SB}$ as the result of attaching E to A on (x, y) on the right. More generally, for $A \in \mathcal{SB}$, $\mathcal{E} = (E_1, \dots, E_m)$ and $\mathcal{F} = (F_1, \dots, F_n)$ and a segment (x, y) in A , we define $K = A(x, y, \mathcal{E}, \mathcal{F}) \in \mathcal{SB}$ as the result of attaching

E_1, \dots, E_m to the segment (x, y) on the left in a way that $E_1 <_{lex} \dots <_{lex} E_m$ and attaching F_1, \dots, F_n to the segment (x, y) on the right in a way that $F_1 <_{lex} \dots <_{lex} F_n$ and the root of each E_i is below the root of each F_j . In that case, we may also write $(x, y, \mathcal{E}, \mathcal{F})$ for $\{z \in T_K : x < \text{meet}(y, z) < y\}$.

We say that $(A, p) \in \mathcal{SB}_1$ is in a *simple normal form* if (1) there are orbits $P = \{a_0, \dots, a_n\} <_{lex} Q = \{b_0, \dots, b_n\}$, $n \geq 2$, of p , such that for every $i = 0, \dots, n - 1$, $\text{meet}(a_i, a_{i+1}) < \text{meet}(b_i, b_{i+1})$ and for every $i = 0, \dots, n - 2$, $\text{meet}(b_i, b_{i+1}) < \text{meet}(a_{i+1}, a_{i+2})$, (2) any non-trivial orbit $O = \{c_0, \dots, c_l\}$ in A is \leq_{lex} -contained in P or in Q and it holds $l = n - 1$, (3) for any x with $p(x) = x$ it holds $\max\{a_0, a_n\} <_{lex} x <_{lex} \min\{b_0, b_n\}$, where \min and \max are taken with respect to the $<_{lex}$ order.

We say that (A, p) is in a *normal form* if (1) $A = \text{def}(p)$, and any non-constant orbit of p has at least 3 elements, (2) p cannot be extended to a partial automorphism q such that some two orbits that did not intertwine (or meet-intertwine) in p now they intertwine (or meet-intertwine, respectively) or they form one orbit in q , and (3) there is a partition \mathcal{P}^A of A into singletons $\{x\}$ and closed \leq_{lex} -intervals that will be grouped into pairs $([a, b], [c, d])$ so that if $\{x\} \in \mathcal{P}^A$, then $p(x) = x$, and if $([a, b], [c, d]) \in \mathcal{P}^A$, then the structure $p \upharpoonright ([a, b] \cup [c, d])$ is in a simple normal form without a non-trivial orbit, witnessed by some $P = \{a_0, \dots, a_n\}$ and $Q = \{b_0, \dots, b_n\}$ with $a = a_0, b = a_n, c = b_n, d = b_0$ (or $b = a_0, a = a_n, d = b_n, c = b_0$). We will sometimes identify $([a, b], [c, d]) \in \mathcal{P}^A$ with the set $[a, b] \cup [c, d]$.

Lemma 3.11. *Any $(A, p) \in \mathcal{SB}_1$ can be extended to $(A', p') \in \mathcal{SB}_1$, which is in a normal form.*

Proof. Conditions (1) and (2) can be easily satisfied. To have (3), consider the equivalence relation on orbits: O and P are equivalent iff there is a sequence of orbits $O = O_1, \dots, P = O_n$ such that for each i , O_i and O_{i+1} intertwine or meet intertwine. An equivalence class E either is a singleton containing a constant orbit, or it does not contain a constant orbit. In the second case, after extending A if necessary, the class E contains two meet-intertwining orbits $P_0 <_{lex} Q_0$ (there are usually many such choices). Extend P_0 to P and Q_0 to Q so that every orbit in E is contained in P or in Q . Finally, extend the remaining orbits in E so that (2) in the definition of simple normal form is satisfied. We do the induction on the number of equivalence classes E . \square

Let $(A, p) \in \mathcal{SB}_1$ be in a normal form and let $X \in \mathcal{P}^A$. Then $(X, p_X) \in \mathcal{SB}_1$, where $p_X = p \upharpoonright X$, is in a simple normal form. The tree T_X can be naturally identified with a subtree of T_A (in fact, T_X is the closure of X in T_A under taking the meet), let ρ_X be the root of T_X , and let $\text{Cone}_X = \text{Cone}_{\rho_X}$. To X as above we associate $X^* \in \mathcal{SB}$, and $(X^*, p_X^*) \in \mathcal{SB}_1$ in a simple normal form as follows. If $X = \{x\}$, let $p^*(x) = x$. Otherwise, if $X = ([a, b], [c, d])$, we let

$$X^* = X \cup \{z \in \text{Cone}_Y : z \text{ is maximal in } (\{\text{Cone}_Z \subsetneq \text{Cone}_X : Z \in \mathcal{P}^A\}, \subseteq)\}$$

and we let p_X^* to be the extension of p_X that is equal to the identity on $X^* \setminus X$. The set of all X^* obtained in this way we denote by $(\mathcal{P}^A)^*$. Definitions introduced above are illustrated in Example 3.12.

Proof of Theorem 3.10. Let $(A, p) \in \mathcal{SB}_1$ and consider extensions $(B, q), (C, r) \in \mathcal{SB}_1$ of (A, p) . By ϕ, ψ , we denote the identity embeddings of (A, p) into $(B, q), (C, r)$, respectively. We show that if (A, p) is in a normal form, then we can amalgamate (B, q) and (C, r) over (A, p) . Since the family of all elements in a normal form is cofinal in \mathcal{SB}_1 , this will finish the proof.

(1) Suppose that (A, p) is in a simple normal form.

Let $P = \{a_0, \dots, a_n\}$ and $Q = \{b_0, \dots, b_n\}$ be as in the definition of the simple normal form. Without loss of generality, P is increasing, and hence Q is decreasing. Set $t_i = \text{meet}(a_i, a_{i+1})$ and let $s_i = \text{meet}(b_i, b_{i+1})$. Note that all trees T_{X_i} with $X_i = [a_i, a_{i+1}]_{lex}$ are isomorphic, and all trees T_{Y_i} with $Y_i = [b_{i+1}, b_i]_{lex}$ are isomorphic as well.

Pick some N such that each of $q^{-N}(a_0), q^N(a_n), q^{-N}(b_0), q^N(b_n), r^{-N}(a_0), r^N(a_n), r^{-N}(b_0), r^N(b_n)$ is undefined. Let $\bar{\phi}: T_A \rightarrow T_B$ and $\bar{\psi}: T_A \rightarrow T_C$ be the unique meet-preserving extensions of ϕ and ψ .

For every k , let (A_k, p_k) be defined as follows. Take an extension p' of p such that for every $x \in [a_0, a_1]_{lex} \cup (b_1, b_0]_{lex}$, the values $q^{n+k}(x)$ and $q^{-k}(x)$ are defined and every orbit in p' extends an orbit in p . Then let

$$p_k = p' \upharpoonright [a_{-k}, a_{n+k}]_{lex} \cup \{(c, c) : c \in A, a_n <_{lex} c <_{lex} b_n\} \cup p' \upharpoonright [b_{-k}, b_{n+k}]_{lex},$$

where $a_i = q^i(a_0)$ and $b_i = q^i(b_0)$, and let $A_k = \text{def}(p_k)$. Note that $p = p_0$ and that each p_k is in a simple normal form as witnessed by $P_k = \{a_{-k}, \dots, a_{n+k}\}$ and $Q_k = \{b_{-k}, \dots, b_{n+k}\}$. Consider $D_0 = A_N, s_0 = p_N$ for N as above.

Let

$$A_B = \{y \in B : (\exists x \in A, k \in \mathbb{Z}) q^k(x) = y\},$$

and define A_C similarly. Let $\alpha: A_B \rightarrow D_0$ and $\beta: A_C \rightarrow D_0$ be the unique embeddings that agree on A . Denote by $\bar{\alpha}: T_{A_B} \rightarrow T_{D_0}$ and $\bar{\beta}: T_{A_C} \rightarrow T_{D_0}$ the tree embeddings corresponding to α and β . We clearly have $\rho_B \leq \bar{\phi}(\rho_A)$ or $\rho_C \leq \bar{\psi}(\rho_A)$, where ρ_A, ρ_B, ρ_C are roots of T_A, T_B, T_C , and both inequalities can be strict. Then, to obtain the required amalgam (D, s) , first consider \widehat{T}_{D_0} obtained from T_{D_0} by adding a new point v , which satisfies $v < \rho_{D_0}$, where ρ_{D_0} is the root of T_{D_0} . Next, fix a segment (x, y) in \widehat{T}_{D_0} , and let $(x^B, y^B) = (\bar{\alpha}^{-1}(x), \bar{\alpha}^{-1}(y))$, if defined, and $(x^C, y^C) = (\bar{\beta}^{-1}(x), \bar{\beta}^{-1}(y))$, if defined. Suppose that $\mathcal{E}^B, \mathcal{E}^C, \mathcal{F}^B, \mathcal{F}^C$ are such that $\{z \in T_B : x^B < \text{meet}(y^B, z) < y^B\}$ can be identified with $(x^B, y^B, \mathcal{E}^B, \mathcal{F}^B)$ and $\{z \in T_C : x^C < \text{meet}(y^C, z) < y^C\}$ can be identified with $(x^C, y^C, \mathcal{E}^C, \mathcal{F}^C)$. Then replace (x, y) in \widehat{T}_{D_0} by $(x, y, (\mathcal{E}^B \mathcal{E}^C), (\mathcal{F}^C \mathcal{F}^B))$, where $(\mathcal{E}^B \mathcal{E}^C)$ and $(\mathcal{F}^C \mathcal{F}^B)$ are concatenations of sequences \mathcal{E}^B with \mathcal{E}^C and \mathcal{F}^C with \mathcal{F}^B , respectively. We additionally require that the root of every $T_{E_i^C}$ is above the root

of every $T_{F_j^B}$ and the root of every $T_{F_i^C}$ is above the root of every $T_{E_j^B}$, where $E_i^C \in \mathcal{E}^C$, etc. The obtained tree T defines $D \in \mathcal{SB}$ such that $T_D = T$, and it defines embeddings $\alpha: B \rightarrow D$ and $\beta: C \rightarrow D$ of structures in \mathcal{SB} . We let $s(a) = b$ iff $\alpha^{-1}(a), \alpha^{-1}(b)$ are defined and $q(\alpha^{-1}(a)) = \alpha^{-1}(b)$, or $\beta^{-1}(a), \beta^{-1}(b)$ are defined and $r(\beta^{-1}(a)) = \beta^{-1}(b)$, or $p^N(a) = b$. For every segment (x, y) in \widehat{T}_{D_0} and $z \in T_D$, $x < z < y$, we will call the subtree Cone_z of T_D a *triangle* (coming from B or from C).

We have to show that s is a partial automorphism of D . Clearly $<_{lex}$ is preserved. Let \bar{s}_0 be the meet-preserving extension of s_0 to \widehat{T}_{D_0} . Then, clearly, a triangle attached to a segment (x, y) is mapped to a triangle attached to a segment $(\bar{s}_0(x), \bar{s}_0(y))$. The key observation is that for every k , and in particular for $k = N$, and every $X \subseteq \text{dom}(p_k)$, the tree T_X are isomorphic with the tree $T_{s_0(X)}$ via the tree isomorphism extending the bijection $x \mapsto s_0(x)$, $x \in X$. Moreover, for $a, b \in D$ that lie in different triangles, $\text{meet}(a, b)$ is equal to the meet of the roots of the triangles to which a and b belong. Note also that if $\rho \leq \rho'$ are roots of two triangles then $\bar{s}(\rho) \leq \bar{s}(\rho')$. Therefore, if $x, y, z \in D$ lie in different triangles, we have $S(x, y, z)$ iff $S(s(x), s(y), s(z))$. Clearly, if all x, y, z lie in the same triangle, then the conclusion holds. If $x \leq_{lex} y \leq_{lex} z$ and x and y lie in a triangle T and z lies in a different triangle S , if ρ_T and ρ_S denote roots of S and T , then $S(x, y, z)$ holds iff $\rho_S < \rho_T$ and $S(s(x), s(y), s(z))$ holds iff $\bar{s}_0(\rho_S) < \bar{s}_0(\rho_T)$, hence $S(x, y, z)$ iff $S(s(x), s(y), s(z))$. The case when y and z lies in the same triangle and x lies in a different one is analogous.

(2) Suppose that (A, p) is in a normal form.

Without loss of generality, (B, q) and (C, r) are also in the normal form. Let $\widehat{\mathcal{P}}^B$ be the partition of B into points and pairs of closed \leq_{lex} -intervals, which is a coarsening of \mathcal{P}^B , and has the following property: for every $([a, b], [c, d]) \in \widehat{\mathcal{P}}^B$ there is exactly one $([a_0, b_0], [c_0, d_0]) \in \mathcal{P}^A$ such that $[a_0, b_0] \subseteq [a, b]$ and $[c_0, d_0] \subseteq [c, d]$. We define $(\mathcal{P}^A)^*$ out of \mathcal{P}^A , and the corresponding partial automorphisms p_X^* , in a way explained earlier. We define $(\widehat{\mathcal{P}}^B)^*$ and $(\widehat{\mathcal{P}}^C)^*$ similarly, but with respect to $\widehat{\mathcal{P}}^B$ and $\widehat{\mathcal{P}}^C$. Let

$$X^* = X \cup \{z = \text{Cone}_Y : z \text{ is maximal in } (\{\text{Cone}_Z \subsetneq \text{Cone}_X : Z \in \widehat{\mathcal{P}}^B\}, \subseteq)\},$$

and we let p_X^* to be the extension of p_X that is equal to the identity on $X^* \setminus X$. The set of all X^* obtained in this way we denote by $(\widehat{\mathcal{P}}^B)^*$. We similarly define $(\widehat{\mathcal{P}}^C)^*$. For a given $X \in \mathcal{P}^A$, let $X_B \in \widehat{\mathcal{P}}^B$ and $X_C \in \widehat{\mathcal{P}}^C$ be the unique sets containing X . Amalgamate $(X_B^*, q_{X_B}^*)$ and $(X_C^*, r_{X_C}^*)$ over (X^*, p_X^*) , as we did in (1). We obtain $D_X \in \mathcal{SB}$, a partial automorphism s_X of D_X , and a pair of embeddings $\alpha_X: X_B^* \rightarrow (D_X, s_X)$ and $\beta_X: X_C^* \rightarrow (D_X, s_X)$ such that $\alpha_X \upharpoonright X^* = \beta_X \upharpoonright X^*$. Let ρ_X be the root of T_X , and let $\rho_1^X, \dots, \rho_{l_X}^X$ be an enumeration of all $z = \text{Cone}_Y$ from the definition of X^* . To the tuple $(X^*, \rho_X, \rho_1^X, \dots, \rho_{l_X}^X)$ we associate the tuple $(D_X, \rho_{D_X}, \rho_1^{D_X}, \dots, \rho_{l_X}^{D_X})$, where ρ_{D_X} is the root of T_{D_X} , and $\rho_i^{D_X} = \alpha_X(\rho_i^X) = \beta_X(\rho_i^X)$ for each i . By the definition of $(\mathcal{P}^A)^*$, the tree T_A is the disjoint union of $\{T_{X^*} : X \in \mathcal{P}^A\}$ after the identification of each ρ_i^X with a ρ_Y for some $Y \in (\mathcal{P}^A)$. Finally, let T_D be obtained by taking the disjoint union of $\{T_{D_X} : X \in \mathcal{P}^A\}$, and then identifying a $\rho_i^{D_X}$ with a ρ_{D_Y} if and only if ρ_i^X was identified

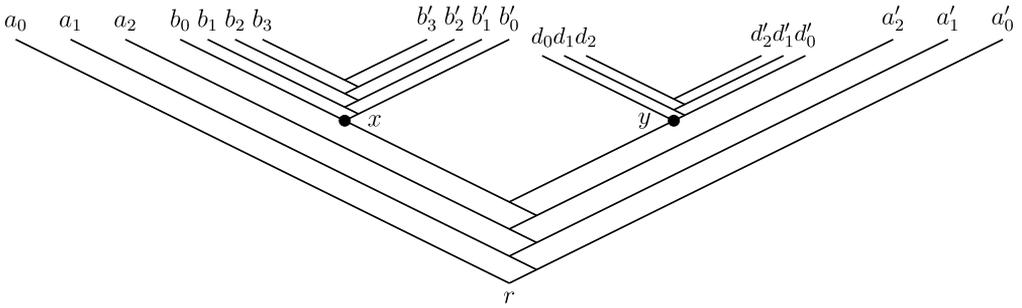


Fig. 2. A structure A.

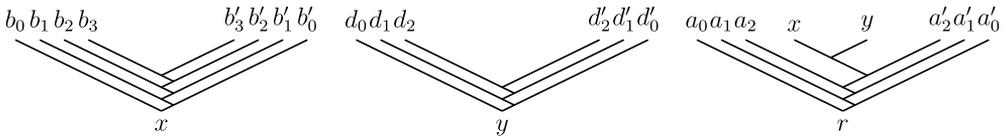


Fig. 3. From left to right: X^*, Y^*, Z^* .

with a $\rho_Y, X, Y \in \mathcal{P}^A$. This T_D defines $D \in \mathcal{SB}$ and let $s = \bigcup_{X \in \mathcal{P}^A} s_X \upharpoonright X$. Then $(D, s) \in \mathcal{SB}_1$. Clearly, s preserves $<_{lex}$. To show that it preserves S , observe first the following.

(i) If $x \in D_X$ and $y \in D_Y$ with $X \neq Y$, and ρ_{D_X} and ρ_{D_Y} are incomparable in (T_D, \leq) , then $\text{meet}(x, y) = \text{meet}(\rho_{D_X}, \rho_{D_Y}) = \text{meet}(s(x), s(y))$.

(ii) If $x \in D_X$ and $y \in D_Y$ with $X \neq Y$ and $\rho_{D_X} \leq \rho_{D_Y}$, then $\text{meet}(x, y) = \text{meet}(x, \rho_{D_Y})$ and $\text{meet}(s(x), s(y)) = \text{meet}(s(x), \rho_{D_Y})$. Moreover, in that case, if $y' \in D_Y$, then $\text{meet}(x, y), \text{meet}(x, y') < \text{meet}(y, y')$.

Let (x, y, z) be a $<_{lex}$ ordered triple of points in D . There are several cases to consider. Clearly, if there is some $X \in \mathcal{P}^A$ such that $x, y, z \in D_X$, then $S(x, y, z)$ iff $S(s(x), s(y), s(z))$. If $x, y \in D_X, z \in D_Y$ with $X \neq Y$ and ρ_{D_X} and ρ_{D_Y} are incomparable, then both $S(x, y, z)$ and $S(s(x), s(y), s(z))$ hold. Similarly, if $x \in D_X$ and $y, z \in D_Y$ with $X \neq Y$ and ρ_{D_X} and ρ_{D_Y} are incomparable then none of $S(x, y, z), S(s(x), s(y), s(z))$ holds. In the case when $\rho_{D_X} \leq \rho_{D_Y}$, two of the x, y, z belong to D_X and the remaining point belongs to $D_Y, X \neq Y$, then using (ii) we get that s preserves S on (x, y, z) because p_X preserves S (the point belonging to D_Y we can replace with an appropriate $\rho_i^{D_X}$). If two of the x, y, z belong to D_Y and the remaining point belongs to D_X , use the second sentence of (ii) to get the conclusion. Finally, suppose that $x \in D_X, y \in D_Y, z \in D_Z$ with X, Y, Z pairwise different. There are a few cases to consider, ρ_{D_X} and ρ_{D_Y} can be comparable or not, and the same for the other two pairs. Each time, reasoning similarly as above and using (i) and (ii), we get the required conclusion. \square

Example 3.12. Let $(A, p) \in \mathcal{SB}_1$ be as in Fig. 2 with $p(a_0) = a_1, p(a_1) = a_2, p(a'_0) = a'_1, p(a'_1) = a'_2$, etc. Then $\mathcal{P}^A = \{X = ([a_0, a_2], [a'_2, a'_0]), Y = ([b_0, b_3], [b'_3, b'_0]), Z = ([d_0, d_2], [d'_2, d'_0])\}$. Fig. 3 illustrates X^*, Y^*, Z^* , where $p_Z^*(x) = x$ and $p_Z^*(y) = y$.

3.3. Ordered posets - a comeager conjugacy class

In this part, we will show that the automorphism group of the universal ordered poset has a comeager conjugacy class.

A *poset* is a shortcut for a partially ordered set. By an *ordered poset*, we mean a structure of the form $(P, \prec, <)$, where (P, \prec) is a finite poset, and $<$ is a linear ordering of P extending \prec . We denote the class of all finite ordered posets by \mathcal{P} . Then the *universal ordered poset* is the Fraïssé limit of \mathcal{P} . Kuske-Truss [12] showed that the class of partial automorphisms of finite posets has CAP, and hence the corresponding automorphism group has a comeager conjugacy class. The same turns out to be true for the class \mathcal{P}_1 .

We will see that the proof of Kuske-Truss generalizes to our context.

Theorem 3.13. *The class \mathcal{P}_1 has CAP.*

Below, we will always use the symbol \prec for the poset relation, and $<$ for the linear order. For $A = (A, p) \in \mathcal{P}_1$, and an orbit $O = \{a_0, a_1, \dots, a_n\}$ of p , we say that O is *$<$ -increasing* if $a_0 < \dots < a_n$, otherwise, it is *$<$ -decreasing*. As with boron trees, if $(B, q) \in \mathcal{P}_1$ extends (A, p) , the orbit of q extending O will be denoted by O_B .

We say that a pair of orbits (O, N) is *determined* if for any extensions $(B, q), (C, r)$ of (A, p) , such that for every $k \in \mathbb{Z}$ and $x \in O \cup N$ we have that $q^k(x)$ is defined iff $r^k(x)$ is defined, the following holds: $(O_B \cup N_B, q \upharpoonright (O_B \cup N_B))$ and $(O_C \cup N_C, r \upharpoonright (O_C \cup N_C))$ are isomorphic via a mapping which is the identity on $O \cup N$. Clearly, (O, N) is determined iff (N, O) is determined. An orbit O is *determined* if the pair (O, O) is determined.

Let (A, p) be given. Let O be an orbit in (A, p) and $x \in O$. Denote

$$t(O, x) = \{n \in \mathbb{Z} : p^n(x) \in O \text{ and } x \prec p^n(x)\}.$$

Note that if O is $<$ -increasing then $t(O, x)$ consists of non-negative integers and if O is $<$ -decreasing then $t(O, x)$ consists of non-positive integers. Moreover, if $n \in t(O, x)$ then by the transitivity of \prec , for every positive integer k , we have that if $p^{kn}(x) \in O$ then $kn \in t(O, x)$. More generally, if (O, N) are orbits in (A, p) and $x \in O, y \in N$, then let

$$t(O, N, x, y) = \{n \in \mathbb{Z} : p^n(y) \in N \text{ and } x \prec p^n(y)\}.$$

We say that (O, N, x, y) is *positive determined* if orbits O and N are determined and for any extensions $(B, q), (C, r)$ of (A, p) , such that for every $k \in \mathbb{Z}$ we have that $q^k(x)$ is defined iff $r^k(x)$ is defined and $q^k(y)$ is defined iff $r^k(y)$ is defined, we have that $t(O_B, N_B, x, y) \cap \mathbb{N} = t(O_C, N_C, x, y) \cap \mathbb{N}$. We similarly define when (O, N, x, y) is *negative determined*. We let (O, N, x, y) to be *determined* iff (O, N, x, y) is both positive and negative determined. The (O, x) is *determined* if for any extensions $(B, q), (C, r)$ of (A, p) , such that for every $k \in \mathbb{Z}$ we have that $q^k(x)$ is defined iff $r^k(x)$ is defined, it holds $t(O_B, x) = t(O_C, x)$. Note that a pair of orbits (O, N) is determined iff for every $x \in O, y \in N$, the (O, N, x, y) and (N, O, y, x) are determined iff for some $x \in O, y \in N$, the

(O, N, x, y) and (N, O, y, x) are determined. Similarly, O is determined iff for some/every $x \in O$, the (O, x) is determined.

An orbit O will be called an *antichain* if for every extension (B, q) of (A, p) and for every/some $x \in O$, we have $t(O_B, x) = \emptyset$. An $X \subset \mathbb{Z}$ is called *positive eventually periodic* if there exist $N, k \geq 0$ such that $X \cap [N, \infty) = \{N + kn : n \geq 0\}$. The number k we will call the *positive period* of X . We similarly define a *negative eventually periodic* set and the *negative period*. We call a set *eventually periodic* if it is both positive and negative periodic. We will call two orbits $O = \{a_0, \dots, a_m\}$ and $N = \{b_0, \dots, b_n\}$ of p *inter-twining* if the $<$ -intervals $(\min\{a_0, a_m\}, \max\{a_0, a_m\})_<$ and $(\min\{b_0, b_n\}, \max\{b_0, b_n\})_<$ intersect.

Proof of Theorem 3.13. The proof is similar to the proof of Kuske-Truss [12].

Step 1. Every $(A, p) \in \mathcal{P}_1$ can be extended to some $(B, q) \in \mathcal{P}_1$, in which every orbit is determined. Moreover, we can do it so that we do not add new orbits.

Take an orbit O and $x \in O$, and without loss of generality suppose that O is $<$ -increasing. If O is an antichain then it is already determined. Otherwise, let (perhaps after passing to an extension) $k \in t(O, x)$. Now for every positive integer n , we have that if $p^{kn}(x) \in O$ then $kn \in t(O, x)$. Using this remark, we obtain (B, q) , an extension of (A, p) , such that for every $i = 1, \dots, k - 1$, if there is an extension (B_1, q_1) of (B, q) with $n_i k + i \in t(O_{B_1}, x)$, for some $n_i \geq 0$, then, in fact, for every extension (B_2, q_2) of (B, q) , it holds $n_i k + i \in t(O_{B_2}, x)$, as long as $n_i k + i \in O_{B_2}$. To obtain such a (B, q) , we construct a sequence of extensions $(A_1, p_1), \dots, (A_{k-1}, p_{k-1})$ of (A, p) such that (A_i, p_i) has the required property for i , and we take $(B, q) = (A_{k-1}, p_{k-1})$. Then clearly O_B is determined.

Step 2. Every $(B, q) \in \mathcal{P}_1$ can be extended to some $(D, s) \in \mathcal{P}_1$, in which every pair of orbits is determined. For this we find a (D, s) such that for every pair of orbits (O, N) , $x \in O, y \in N$, there is an almost periodic set $X \subseteq \mathbb{Z}$ such that for every extension (D_1, s_1) of (D, s) , if m_1, m_2 are the least such that $s_1^{m_1}(y)$ and $s_1^{-m_2}(y)$ are undefined, we have $X \cap (-m_1, m_2) = t(O_D, x)$.

Step 2a. The (B, q) obtained in Step 1 can be extended to some $(C, r) \in \mathcal{P}_1$ such that all pairs of orbits in which both orbits are antichains, are determined. Moreover, we can do it in a way that for each such a pair we add four new orbits, none of which is an antichain.

We fix a pair (O, N) of such orbits and let $x \in O$. As in Kuske-Truss, after possibly extending N , find $y \in N$ and $0 < n$ with $q^n(y) \in \text{rng}(q) \setminus \text{dom}(q)$ such that for every $z \in \text{def}(q) \setminus N$, it holds: $z \prec y$ iff $z \prec q^n(y)$, $y \prec z$ iff $q^n(y) \prec z$, z is incomparable with y iff z is incomparable with $q^n(y)$. For this pick some $y_0 \in N$ and, possibly extending q , using the pigeonhole principle, choose $k_1 < k_2$ sufficiently large so that $q^{k_2}(y_0) \in \text{rng}(q) \setminus \text{dom}(q)$, $z \prec q^{k_1}(y_0)$ iff $z \prec q^{k_2}(y_0)$, $q^{k_1}(y_0) \prec z$ iff $q^{k_2}(y_0) \prec z$, z is incomparable with $q^{k_1}(y_0)$ iff z is incomparable with $q^{k_2}(y_0)$. Take $y = q^{k_1}(y_0)$ and $n = k_2 - k_1$.

Let us proceed to the construction. Take $\{a_i : 0 \leq i \leq n\}$ and $\{b_i : 0 \leq i \leq n\}$ disjoint from each other and from $\text{def}(q)$ and such that for all $0 \leq i < j \leq n$, $z \in \text{def}(q) \setminus N$:

- (1) a_i and a_j are incomparable except for $a_0 \prec a_n$,
- (2) b_i and b_j are incomparable except for $b_n \prec b_0$,
- (3) $a_i \prec q^i(y) \prec b_i$,
- (4) $z \prec b_i$ iff $z \prec q^i(y)$,
- (5) $a_i \prec z$ iff $q^i(y) \prec z$,
- (6) $a_i \prec b_i, a_0 \prec b_n, a_n \prec b_0$,
- (7) $a_0 < \dots < a_n < \text{def}(q) < b_n < \dots < b_0$.

Denote the obtained structure by $(B_1, \prec, <)$. Let q_1 extend q by $q_1(a_i) = a_{i+1}$ and $q_1(b_{i+1}) = b_i, i = 0, \dots, n - 1$. It is straightforward to see that \prec is transitive, $<$ extends \prec, p preserves \prec and $<$. As in Kuske-Truss, we have that in any extension (B_2, q_2) of (B_1, q_1) ,

$$t(O_{B_2}, N_{B_2}, x, y) = \{kn + i : k \in \mathbb{N}, 0 \leq i < n, x \prec q^i(y)\} \cap B_2,$$

hence (O, N, x, y) is positive determined. Indeed, let (B_2, q_2) be an extension of (B_1, q_1) and denote $X = \{kn + i : k \in \mathbb{N}, 0 \leq i < n, x \prec q^i(y)\} \cap B_2$. Since $a_0 \prec a_n$ and $a_i \prec q^i(y)$, we get $t(O_{B_2}, N_{B_2}, x, y) \supseteq X$. Since $b_n \prec b_0$ and $q^i(y) \prec b_i$, using (4), we get $t(O_{B_2}, N_{B_2}, x, y) \subseteq X$.

By considering (B_1, q_1^{-1}) and repeating the argument above, we further extend (B_1, q_1) to obtain (C, r) in which (O, N) is determined.

Step 2b. The (C, r) obtained in Step 2a can be extended to some $(D, s) \in \mathcal{P}_1$ such that all pairs of orbits such that at least one of them is not an antichain are determined. Moreover, we can do it in a way that we do not add new orbits.

We fix a pair (O, N) of such orbits and let $x \in O$ and $y \in N$. If for every extension (C_1, r_1) of (C, r) , $t(O, N, x, y) = \emptyset$ and $t(N, O, y, x) = \emptyset$, then (O, N) is already determined. Therefore, by passing to an extension if necessary, we assume that $t(O, N, x, y) \neq \emptyset$ or $t(N, O, y, x) \neq \emptyset$. Note that at least one of the sets $t(O, N, x, y)$ and $t(N, O, y, x)$ is empty. Without loss of generality, let us assume that $x \prec y$ and $t(N, O, y, x) = \emptyset$.

If $k \in t(O, x)$ and $l \in t(N, y)$, then for every $n_1, n_2 \geq 0$, if $q^{kn_1+ln_2}(y) \in N$, then $kn_1 + ln_2 \in t(O, N, x, y)$. Hence if there is a positive number $k \in t(O, x)$ or a positive number $l \in t(N, y)$, reasoning as in Step 1, we can find an extension in which (O, N, x, y) is positive determined. Similarly, if there is a negative number $k \in t(O, x)$ or a negative number $l \in t(N, y)$, we can find an extension in which (O, N, x, y) is negative determined. Therefore, without loss of generality, what is left to be shown is the following. Suppose that O and N are $<$ -increasing, at least one of them is not an antichain, and (O, N, x, y) is positive determined. Then we can extend (C, r) to (C_1, r_1) so that (O_{C_1}, N_{C_1}, x, y) is negative determined.

Without loss of generality, O is not an antichain. Take some k such that $x \prec r^k(x)$. Then, clearly, for every $n > 0$, it holds $x \prec r^{nk}(x)$. Take an extension (C_1, r_1) such that for every $0 \leq i < k$ either for some n_i , we have $r_1^{-(n_i k+i)}(y) \in C_1$ and $x \prec$

$r_1^{-(n_1k+i)}(y)$ does not hold, or, for every extension (C_2, r_2) of (C_1, r_1) and every $n > 0$, $x \prec r_1^{-(n_1k+i)}(y)$ holds. Then (O_{C_1}, N_{C_1}, x, y) is negative determined. Indeed, note that if i and n_0 are such that $x \prec r_1^{-(n_0k+i)}(y)$ does not hold, then in every extension (C_2, r_2) of (C_1, r_1) and $n_1 > n_0$, $x \prec r_1^{-(n_1k+i)}(y)$, does not hold either by the choice of k .

We apply this procedure to every pair of orbits such that at least one of them is not an antichain. The resulting extension, which we denote by (D, s) , is as required.

We will show that \mathcal{P}_1 has CAP, i.e. we will show that for every $(A_0, p_0) \in \mathcal{P}_1$ there is $(A, p) \in \mathcal{P}_1$ extending (A_0, p_0) such that for any $(B, q), (C, r) \in \mathcal{P}_1$ extending (A, p) there exists $(D, s) \in \mathcal{P}_1$, which is an amalgam of (B, q) and (C, r) over (A, p) . For this fix $(A_0, p_0) \in \mathcal{P}_1$ and extend it to an $(A, p) \in \mathcal{P}_1$ such that any pair of orbits in A is determined, and there is no extension (A_1, p_1) of (A, p) in which some two orbits in A that did not intertwine, become one orbit or they intertwine in A_1 . Fix $(B, q), (C, r) \in \mathcal{P}_1$ extending (A, p) . Without loss of generality, we have that $A \subseteq \text{dom}(q) \cap \text{rng}(q)$, and similarly, $A \subseteq \text{dom}(r) \cap \text{rng}(r)$, as well as that for every $a \in A$ and $n \in \mathbb{Z}$, $q^n(a)$ is defined iff $r^n(a)$ is defined.

Enumerate the set $\{b \in B : (\exists a \in A, n \in \mathbb{Z}) b = r^n(a)\}$ into a $<$ -increasing sequence $a_1^B, a_2^B, \dots, a_k^B$, and similarly, enumerate the set $\{c \in C : (\exists a \in A, n \in \mathbb{Z}) c = r^n(a)\}$ into $a_1^C, a_2^C, \dots, a_k^C$ so that it is $<$ -increasing. By the assumptions on A , for any $a, b \in A$ and $m, n \in \mathbb{Z}$, if $x_B = q^m(a), y_B = q^n(b), x_C = r^m(a), y_C = q^n(b)$ are defined, then $x_B < y_B$ iff $x_C < y_C$. Denote $B_i = \{b \in B : a_i^B < b < a_{i+1}^B\}$, $i = 1, \dots, k - 1$, $B_0 = \{b \in B : b < a_1^B\}$, $B_k = \{b \in B : a_k^B < b\}$, and similarly define C_i 's.

We first amalgamate B and C over A in \mathcal{P} . Let D be the disjoint union of B and C with a_i^B and a_i^C identified. Set $a_i = a_i^B = a_i^C$, and let

$$B_0^D < C_0^D < a_1 < B_1^D < C_1^D < a_2 < \dots < a_k < B_k^D < C_k^D.$$

Denote by \prec^B the partial ordering relation on B , by \prec^C the partial ordering relation on C , and let \prec^D be the transitive closure of $\prec^B \cup \prec^C$. Then (D, \prec^D) is a partial ordering such that the linear ordering $<^D$ extends \prec^D , and so D is an amalgam of B and C over A .

We finally let $s(x) = y$ iff $x = b_1, y = b_2$ for some $b_1, b_2 \in B$ and $q(b_1) = b_2$, or $x = c_1, y = c_2$ for some $c_1, c_2 \in C$ and $r(c_1) = c_2$. This is a partial automorphism. In particular, if for some $b \in B, c \in C$, it holds $b \prec^D c$ and $s(b), s(c)$ are defined, then there is $a \in \{a_1, \dots, a_k\}$ such that $b \prec^B a$ and $a \prec^C c$, or $b \prec^C a$ and $a \prec^B c$. Without loss of generality, it holds $b \prec^B a$ and $a \prec^C c$. Note that $q(a)$ and $r(a)$ are defined and hence we have $q(b) \prec^B q(a)$ and $r(a) \prec^C r(c)$. This implies $s(b) \prec^D s(c)$. Hence (D, s) is the required amalgam. \square

4. The two-dimensional case. Conjugacy classes

We provide a condition, which we will use to obtain many examples of ordered structures M such that $\text{Aut}(M)$ has no comeager 2-dimensional diagonal conjugacy class.

Given a partial automorphism p of a structure A , and $a \in A$, say that $a \in A$ is *locked* by p if there are $x \leq a \leq y$, $x, y \in A$ such that $p(x) = y$ or $p(y) = x$.

Proposition 4.1. *Let \mathcal{F} be an Fraïssé order class. Suppose that for every $(A, p) \in \mathcal{F}_1$ and $a \in A$ not locked by p , there are extensions $(B, r), (C, s) \in \mathcal{F}_1$ of (A, p) such that $r(a) < a$ and $a < s(a)$. Then \mathcal{F}_2 has no WAP.*

Proof. It suffices to show that for a given $(A', p', q') \in \mathcal{F}_2$ and $x \in A$ such that $x < p'(x)$ there exists $(A'', p'', q'') \in \mathcal{F}_2$ which is an extension of (A', p', q') , and a word $w(s, t) \in F_2$, such that $w(p'', q'')(x)$ is defined and not locked by p'' or q'' . Indeed, let $(A, p, q) \in \mathcal{F}_2$ and $x \in A$ be such that $x < p(x)$ (wlog there is such an x in A), let $(A', p', q') \in \mathcal{F}_2$ be an arbitrary extension of (A, p, q) , and let $(A'', p'', q'') \in \mathcal{F}_2$ and $w(s, t)$ be as above. Then $y = w(p'', q'')(x)$ is not locked (wlog) by p'' . Apply the assumptions of the proposition to (A'', p'') and y to find the corresponding $(B, r), (C, s) \in \mathcal{F}_1$. Then $(B, p'', r), (C, p'', s)$ cannot be amalgamated over (A, p, q) .

We construct the required (A'', p'', q'') and w inductively. Let $(A''_0, p''_0, q''_0) = (A', p', q')$, and $w_0 = 1$. Suppose that we already constructed (A''_n, p''_n, q''_n) and $w_n(s, t)$, and suppose that $w_{n-1}(p''_{n-1}, q''_{n-1})(x) = w_{n-1}(p''_n, q''_n)(x) < w_n(p''_n, q''_n)(x)$. Denote $y = w_n(p''_n, q''_n)(x)$ and, if y is locked by p''_n or q''_n , proceed as follows. Let $z_1 \leq y \leq z_2$, and $f \in \{p''_n, (p''_n)^{-1}, q''_n, (q''_n)^{-1}\}$ be such that $f(z_1) = z_2$. Take $(A''_{n+1}, p''_{n+1}, q''_{n+1})$ to be an extension of (A''_n, p''_n, q''_n) such that $|A''_{n+1} \setminus A''_n| \leq 1$ and $f(y)$ is defined; clearly, $z_2 < f(y)$. Let $u = s$ if $f = p''_n$, let $u = s^{-1}$ if $f = (p''_n)^{-1}$, let $u = t$ if $f = q''_n$, and let $u = q^{-1}$ if $f = (q''_n)^{-1}$, and set $w_{n+1}(s, t) = uw_n(s, t)$. Clearly, $w_n(p''_n, q''_n)(x) = w_n(p''_{n+1}, q''_{n+1})(x) < w_{n+1}(p''_{n+1}, q''_{n+1})(x)$. Since p' and q' are finite and the sequence $(w_n(p''_n, q''_n)(x))_n$ is increasing, after finitely many steps we will obtain N such that $w_N(p''_N, q''_N)(x)$ is not locked by p''_N or is not locked by q''_N . Set $(A'', p'', q'') = (A''_N, p''_N, q''_N)$, $w = w_N$, and $y = w_N(p'', q'')(x)$. \square

Corollary 4.2. *Suppose that \mathcal{K} is a Fraïssé class, and let \mathcal{F} be a full order expansion of \mathcal{K} . Then \mathcal{F}_2 has no WAP.*

Corollary 4.3. *The class \mathcal{P}_2 has no WAP.*

Proof. Let $P = (P, \prec^P, <^P)$ be an ordered poset, and let p be a partial automorphism of P . Let $(Q, <^Q)$ be an extension of $(P, <^P)$ such that $|Q \setminus P| = 1$. Let $x \in P, y \in Q \setminus P$, and suppose that $q = p \cup \{(x, y)\}$ is a partial automorphism of $(Q, <^Q)$. Then there is \prec^Q extending \prec^P such that $(Q, \prec^Q, <^Q)$ is an ordered poset, and q is a partial automorphism of $(Q, \prec^Q, <^Q)$. Indeed, define \prec^Q as follows: for $a <^Q y, a \in P$, we set $a \prec^Q y$ iff there is $a \leq^Q b <^Q y, b \in \text{rng}(p)$, such that (1) $a \prec^P b$ if $a \neq b$ and (2) $\alpha^{-1}(b) \prec^P x$.

Similarly, for $y <^Q d, d \in P$, we set $y \prec^Q d$ iff there is $y <^Q c \leq^Q d, c \in \text{rng}(p)$, such that (1) $x \prec^P \alpha^{-1}(c)$ and (2) $c \prec^P d$ if $c \neq d$.

Thus, assumptions of Proposition 4.1 are satisfied. \square

It is straightforward to verify that \mathcal{P}_2 has JEP. Therefore Corollary 4.3 implies that the automorphism group of the universal ordered poset has all 2-dimensional diagonal conjugacy classes meager.

We show that there is no comeager 2-dimensional diagonal conjugacy class in the automorphism group of the universal ordered boron tree. In fact, since \mathcal{SB}_2 has JEP, which is not hard to check, this will imply that all 2-dimensional diagonal conjugacy classes of the group are meager.

Theorem 4.4. *The class \mathcal{SB}_2 has no WAP.*

Let $(A, p) \in \mathcal{SB}_1$, let $x \in A$, and let $O = \{a_0, \dots, a_n\}$ be an orbit of p , $n \geq 2$. Suppose that O is increasing and meet-increasing, other 3 cases being similar. We have therefore $a_0 <_{lex} \dots <_{lex} a_n$, and if $t_i = \text{meet}(a_i, a_{i+1})$, then $t_0 < t_1 < \dots < t_{n-1}$. Now the following two claims easily follow from the definition of the relation S . We will use them frequently.

Claim 1. *If $a_n <_{lex} x$ with $x \in \text{Cone}_O$ and $(B, q) \in \mathcal{SB}_1$ is an extension of (A, p) such that $q(x)$ is defined, then $t_{i-1} < \text{meet}(x, a_n) < t_i$ implies $t_i < \text{meet}(q(x), a_n) < t_{i+1}$, for $i = 1, \dots, n - 2$, $t_{n-2} < \text{meet}(x, a_n) < t_{n-1}$ implies $t_{n-1} < \text{meet}(q(x), a_n)$, and $t_{n-1} < \text{meet}(x, a_n)$ implies $t_{n-1} < \text{meet}(q(x), a_n)$.*

Claim 2. *If $x \in \text{Cone}_O$ is such that $x <_{lex} a_0$ and $(B, q) \in \mathcal{SB}_1$ is an extension of (A, p) such that $q^{-1}(x)$ is defined, then $\text{meet}(q^{-1}(x), a_0) < t_0$ (in particular, $q^{-1}(x) \notin \text{Cone}_O$).*

A point $x \in A$ is *locked* by O if for every extension (B, q) of (A, p) such that $q^{-1}(a_0)$ and $q(a_m)$ are defined, x belongs to the \leq_{lex} -interval with endpoints $q^{-1}(a_0)$ and $q(a_m)$. It is *locked* by p if it is locked by some orbit of p . This definition of locked, which we will use only in this section to discuss \mathcal{SB} , is slightly different than the one we used earlier in this section. A point $x \in A$ is *cone-locked* by the cone C_O , if it is contained in C_O , and it is locked or meet-locked by O_A . Finally, say that a point $x \in A$ is *cone-locked* by p if it is cone-locked by some $C \in \text{Cone}(p)$.

Proof of Theorem 4.4. Let us start with some observations. Take $(A, p) \in \mathcal{SB}_1$ such that every orbit has at least 3 points and let $a \in A$. If a is not cone-locked by a cone from $\text{Cone}(p)$, then there are extensions $(B, q), (C, r) \in \mathcal{SB}_1$ of (A, p) such that $q(a) <_{lex} a$ and $a <_{lex} r(a)$. To see this, simply take v , the immediate predecessor of a in T_A with respect to $<$, add a new point b to obtain B such that b is the immediate predecessor of a in B with respect to $<_{lex}$, $v < \text{meet}(b, a) < a$, and $q(a) = b$. The claim below shows that this gives the required (B, q) . We will then similarly define (C, r) .

Claim. *The map q preserves S .*

Proof of the Claim. Since for every $x \in A$, $\text{meet}(b, x) = \text{meet}(a, x)$, equivalently, we have to show the following.

(Δ) For every $a, b \in A$, we have $S(a, b, c)$ iff $S(p(a), p(b), c)$, and similarly, $S(a, c, b)$ iff $S(p(a), c, p(b))$ and $S(c, a, b)$ iff $S(c, p(a), p(b))$.

We have to consider a number of cases. Denote $m = \text{meet}(a, p(a))$, $n = \text{meet}(b, p(b))$, $m_1 = \text{meet}(p(a), p^2(a))$ and $n_1 = \text{meet}(p(b), p^2(b))$ (if necessary, extend (A, p) so that $p^2(a)$ and $p^2(b)$ are defined). Let O_a and O_b be orbits to which a and b belong, respectively. Without loss of generality, suppose that O_a is increasing and meet-increasing. We will frequently use the following simple observations:

- (i) If $x <_{lex} a$ and $m < \text{meet}(x, a)$ or $a <_{lex} x <_{lex} p^2(a)$, then x is O_a locked.
- (ii) If $p(a) <_{lex} x$ and $m < \text{meet}(x, p(a)) < m_1$, then x is meet-locked by O_a .
- (iii) If $c < \text{meet}(m, n)$ then (Δ) holds.

We have to consider the following cases.

- (1) It holds $b <_{lex} a$ and $\text{meet}(b, a) < m$. In that case, we can have (a) $p(b) <_{lex} b$, or (b) $b <_{lex} p(b)$, and $\text{meet}(p(b), a) < m$, or (c) $b <_{lex} p(b)$, and $m < \text{meet}(p(b), a)$, in which case, by (i), O_a and O_b intertwine.
- (2) It holds $b <_{lex} a$ and $m < \text{meet}(b, a)$, or $a <_{lex} b <_{lex} p(a)$, in which case, by (i), O_a and O_b intertwine.
- (3) It holds $p(a) <_{lex} b$ and $m < \text{meet}(p(a), b)$. In this case, $p^2(a) <_{lex} p(b)$. We can have that:
 - (a) It holds $p(a) <_{lex} b <_{lex} p^2(a)$, in which case O_a and O_b intertwine.
 - (b) It holds $p^2(a) <_{lex} b$ and $m_1 < \text{meet}(b, p^2(a))$, in which case $p^2(a) <_{lex} p(b)$ and $m < \text{meet}(p(b), p^2(a))$. In fact, we have $m_1 < \text{meet}(p(b), p^2(a))$, as otherwise O_b would be an increasing orbit meet intertwining with O_a , which is impossible.
 - (c) It holds $p^2(a) <_{lex} b$ and $m < \text{meet}(b, p^2(a)) < m_1$, in which case, by (ii), O_a and O_b meet intertwine and hence $m_1 < \text{meet}(p(b), p^2(a))$.
- (4) It holds $p^2(a) <_{lex} b$ and $\text{meet}(b, p^2(a)) < m$. Then $p^2(a) <_{lex} p(b)$ and either $\text{meet}(p(b), p^2(a)) < m$ (with $b <_{lex} p(b)$ or $p(b) <_{lex} b$) or $m < \text{meet}(p(b), p^2(a)) < m_1$, in which case O_a and O_b meet intertwine.

This reduces checking to the following cases.

Case 1: O_a and O_b intertwine. Without loss of generality, $a <_{lex} b <_{lex} p(a)$ (meaning that, if instead $a <_{lex} p^n(b) <_{lex} p(a)$, for $n = 1$ or $n = 2$, then the reasoning will be essentially the same). This has two subcases: (1a) $m < \text{meet}(a, b)$, in which case $m_1 < \text{meet}(p(a), p(b))$ and (1b) $m < \text{meet}(b, p(a))$, in which case $m_1 < \text{meet}(p(b), p^2(a))$.

Taking into account (i), (ii) and (iii), all we have to do is to directly verify that (Δ) holds in (1a) and (1b) for a c such that $p^2(a) <_{lex} c$ and $m_1 < c$.

Case 2: O_a and O_b meet intertwine. Without loss of generality, $m < b < m_1$ (again, meaning that, if instead $m < p(b) < m_1$, then the reasoning will be essentially the same).

Taking into account (i), (ii) and (iii), all we have to do is to directly verify that (Δ) holds when $p^2(a) <_{lex} c <_{lex} p(b)$ and either $m_1 < \text{meet}(c, p^2(a))$ or $m_1 < \text{meet}(c, p(b))$.

Case 3: $b, p(b) <_{lex} a$ and $\text{meet}(b, a), \text{meet}(p(b), a) < m$.

Then possibilities on c are: (3a) $b, p(b) <_{lex} c <_{lex} a$ and $\text{meet}(c, a) < m$, (3b) $p(a) <_{lex} c$ and $m < \text{meet}(p(a), c)$, (3c) $p(a) <_{lex} c$ and $\text{meet}(b, a), \text{meet}(p(b), a) < \text{meet}(p(a), c) < m$, (3d) $p(a) <_{lex} c$ and $\text{meet}(p(a), c)$ is between $\text{meet}(b, a)$ and $\text{meet}(p(b), a)$ (this cannot happen though, otherwise c would be meet-locked by O_b).

Case 4: $p(a) <_{lex} b, p(b)$ and $\text{meet}(b, p(a)), \text{meet}(p(b), p(a)) < m$. This is very similar to Case 3.

Case 5: $p^2(a) <_{lex} b, p(b)$ and $m_1 < \text{meet}(b, p^2(a)), \text{meet}(p(b), p^2(a))$. Let $p = \max\{\text{meet}(b, p^2(a)), \text{meet}(p(b), p^2(a))\}$. Then possibilities on c are: (5a) $p^2(a) <_{lex} c <_{lex} b$ and $p < \text{meet}(c, p^2(a))$, (5b) $p^2(a) <_{lex} c <_{lex} b$ and $\text{meet}(c, p^2(a)) \leq p$, (5c) $b, p(b) <_{lex} c$ and $m_1 < \text{meet}(c, p(b)) < n$. \square

Now we show that for a given $(A, p, q) \in \mathcal{SB}_2$ and $x \in A$ such that $x < p(x)$ there exists $(A', p', q') \in \mathcal{SB}_2$, an extension of (A, p, q) , and a word $w(s, t) \in F_2$, such that $w(p', q')(x)$ is defined and not cone-locked by p' or q' . Then an argument presented in the first paragraph of the proof of Proposition 4.1, will finish the proof.

Without loss of generality, every non-trivial orbit of p and q consists of at least three points. As for any $(A, p) \in \mathcal{SB}_1$, A is a substructure of the Fraïssé limit M of \mathcal{SB} , we consider

$$cl_p = \{x \in M : x \text{ is cone-locked by an orbit of } p\}.$$

Note that for every orbit O of p , the set $\{x \in M : x \text{ is cone-locked by } O\}$ is the union of two \leq_{lex} -intervals, one of them constitutes of points locked by p , and the other one of points meet-locked by p . This implies that cl_p is the union of at most $2m_p$ disjoint \leq_{lex} -intervals, where m_p is the number of orbits in p . Denote this collection of \leq_{lex} -intervals by \mathcal{I}_p , and its cardinality by n_p . Observe that the following hold:

(*) For every $I \in \mathcal{I}_p$ and $x \in I$, there is an extension (A', p') of (A, p) so that $(p')^m(x) <_{lex} I <_{lex} (p')^n(x)$ for some $m, n \in \mathbb{Z}$.

(**) For every $(A', p') \in \mathcal{SB}_1$ extending (A, p) with $A' \setminus A = \{(p')^\epsilon(a), \dots, (p')^{\epsilon k}(a)\}$ for some $a \in A$, $\epsilon \in \{-1, 1\}$, and $k \in \mathbb{N}$, and for every $I \in \mathcal{I}_{p'}$, there is $J \in \mathcal{I}_p$ such that $J \subseteq I$. In particular, $n_{p'} \leq n_p$.

We construct the required (A', p', q') and w inductively. Let $(A'_0, p'_0, q'_0) = (A, p, q)$ and $w_0 = 1$. Suppose that we already constructed (A'_n, p'_n, q'_n) and $w_n(s, t)$ and suppose that $w_{n-1}(p'_{n-1}, q'_{n-1})(x) = w_{n-1}(p'_n, q'_n)(x) <_{lex} w_n(p'_n, q'_n)(x)$. Denote $y = w_n(p'_n, q'_n)(x)$ and if y is cone-locked by p'_n and q'_n , proceed as follows. Let $I_{p,y} \in \mathcal{I}_{p'_n}$ be the \leq_{lex} -interval containing y , and similarly define $I_{q,y}$. If the right endpoint of $I_{p,y}$ is $<_{lex}$ -greater or equal than the right endpoint of $I_{q,y}$, there must exist $k \in \mathbb{Z}$ such that $y <_{lex} (q'')^k(y) \notin I_{p,y}$ in some extension (A'', p'', q'') of (A'_n, p'_n, q'_n) . Take the smallest such k , set $w_{n+1}(s, t) = t^k w_n(s, t)$ and let $(A'_{n+1}, p'_{n+1}, q'_{n+1})$ be a minimal such extension. This can be done by observation (*). Similarly, if the right endpoint of $I_{q,y}$ is $<_{lex}$ -greater than the right endpoint of $I_{p,y}$, there must exist $k \in \mathbb{Z}$ such that, in some extension (A'', p'', q'') , we

have $y <_{lex} (p'')^k(y) \notin I_{q,y}$. Take the smallest such k , set $w_{n+1}(s, t) = s^k w_n(s, t)$ and let $(A'_{n+1}, p'_{n+1}, q'_{n+1})$ be a minimal such extension.

Observation (**) implies that after at most $n_p + n_q$ many steps, this construction will stop, i.e., that for some $n \leq n_p + n_q$, we will have that $w_n(p'_n, q'_n)(x)$ is not cone-locked by p'_n or by q'_n . \square

Directed ultrahomogeneous graphs were classified by Cherlin [2] and their precompact Ramsey expansions were described by Jasiński-Laflamme-Nguyen van Thé-Woodrow [7]. See page 74 in [2] for the list of them, see also page 5 in [7]. By Kechris-Pestov-Todorćević [11] and Nguyen Van Thé [13], automorphism groups of those expansions are extremely amenable. The notation we will use comes from [7].

Theorem 4.5. *Let M be a precompact Ramsey expansion of a directed ultrahomogeneous graph, and let $\mathcal{F} = \text{Age}(M)$. Then \mathcal{F}_2 does not have WAP.*

Sketch of the proof. Proposition 4.1 applies to the age of each of these structures. A number of those structures are directly taken care of by Corollary 4.2. These are rational numbers and precompact Ramsey expansions of: the random tournament T^ω , Γ_n – the random directed graph that does not embed the edgeless graph on n vertices, $n \leq \omega$, \mathcal{T} – the random directed graph that does not embed finite tournaments from some fixed set \mathcal{T} . Moreover, structures $S(2)^*$ and $S(3)^*$, that is, precompact Ramsey expansions of $S(2)$ and $S(3)$, are first-order simply bi-definable with structures Q_2 and Q_3 , discussed by Nguyen van Thé [13], whose age is of the form as in Corollary 4.2.

Furthermore, the proofs for the precompact Ramsey expansion of the structures of the form $T[I_n]$, $I_n[T]$, where I_n is the edgeless graph on n vertices, $n \leq \omega$, and T is a homogeneous tournament, as well as of \widehat{Q} , \widehat{T}^ω and of the complete n -partite random directed graph, $n \leq \omega$, are essentially the same as those for the structures taken care of by Corollary 4.2 (perhaps there are some additional unary predicates that do not change the proof in an essential way). Let us discuss here one of these structures. We describe $\text{Age}(T[I_n]^*)$ of the expansion $T[I_n]^*$ of $T[I_n]$, where T is a generic tournament and I_n is the edgeless directed graph on $n < \omega$ vertices with the usual ordering (inherited from the natural numbers). Consider the language $L = \{E, <, L_1, \dots, L_n\}$, where $E, <$ are binary predicates and L_1, \dots, L_n are unary predicates. We will use E for the edge relation and $<$ for the linear order. We let the $\text{Age}(T[I_n]^*)$ consist of substructures of structures whose universe is of the form $S \times I_n$, where S is a linearly ordered tournament (the choice of a linear ordering is arbitrary). A pair $((x, i), (y, j))$ is an edge in $S \times I_n$ iff the pair (x, y) is an edge in S . The ordering we put on $S \times I_n$ is lexicographic with respect to the order on S and on I_n . Finally, we set $L_i(x, j)$ iff $i = j$. It is clear how to modify the proof from Corollary 4.2 to prove that assumptions of Proposition 4.1 are satisfied for $\text{Age}(T[I_n]^*)$ as well.

We have already discussed the universal ordered poset in Corollary 4.3. The precompact Ramsey expansion $\mathcal{P}(3)^*$ of the ‘twisted’ universal ordered poset $\mathcal{P}(3)$ is first-order

simply bi-definable with the Fraïssé limit of the family \mathcal{K}_0 of ordered posets, additionally equipped with 3 subsets (described using unary predicates) forming a partition of the universe of the ordered poset, see the bottom of the page 21 in [7].

Proposition 4.1 also applies to the age of \mathcal{S}^* , the precompact Ramsey expansion of the semigeneric directed graph \mathcal{S} , which is rather straightforward to check. In fact, for given (A, p) and $a \in A$ not locked by p , the required $q(a)$ and $r(a)$ (notation taken from the statement of Proposition 4.1) can be chosen in the same equivalence class with respect to the non-edge equivalence relation in which a is. \square

5. The two-dimensional case. Similarity classes

Slutsky [16] showed that all 2-dimensional topological similarity classes in $\text{Aut}(\mathbb{Q})$ are meager. In this section, we extract from Slutsky’s arguments a general condition on a structure M that implies that all 2-dimensional topological similarity classes in $\text{Aut}(M)$ are meager (Theorem 5.5).

Let \mathcal{F} be a Fraïssé class. Generalizing the terminology introduced in [16], for $A, B \in \mathcal{F}$ with $B \subseteq A$, and a partial automorphism q of A such that $\text{def}(q) \cap B = \emptyset$, we say that B is *free from q* if for every n , every relation symbol R in the signature of \mathcal{F} of arity n , for all $x_1, \dots, x_n \in B \cup \text{dom}(q)$ we have $R(x_1, \dots, x_n)$ iff $R(y_1, \dots, y_n)$, where $y_i = x_i$ if $x_i \in B$, and $y_i = q(x_i)$ if $x_i \in \text{dom}(q)$. In other words, we can extend q so that $q(x) = x$ for every $x \in B$.

We say that \mathcal{F} has *liberating automorphisms* if for any partial automorphisms p, q of $A \in \mathcal{F}$ with no cyclic orbits there exists $N \in \mathbb{N}$ such that, for every $N' > N$, p can be extended to a partial automorphism p' of an element of \mathcal{F} so that $(p')^n[A]$ is free from q for all n with $N \leq n \leq N'$.

Let \mathcal{F} be an Fraïssé order class with an order relation $<$, and let M be the limit of \mathcal{F} . Let p be a partial automorphism of M . For a convex $A \subseteq M$ (i.e., $x, y \in A$, and $x < z < y$ entails that $z \in A$), we say that p is *increasing on A* if for every $x \in A$, p can be extended so that $p(x) > x$; it is *decreasing on A* if for every $x \in A$, p can be extended so that $p(x) < x$; it is *monotone on A* if it is increasing or decreasing on A . We say that an extension p' of p *does not change monotonicity* of p if there are no new fixed points in p' , and p' is increasing (decreasing) on (a, b) iff p is increasing (decreasing) on (a, b) , for $a, b \in \text{def}(p)$. We say that p is *eventually increasing* if there exist $x, y \in \text{supp}(p)$ such that $z < p(z)$ for every $z \in \text{supp}(p)$ such that either $z \leq x$ or $y \leq z$ (i.e., the first and the last orbits of p are increasing).

Let (p, q) be a pair of partial automorphisms of M . We say that x is in a *final segment* (or *initial segment*) of (p, q) if there exists a common fixed point y of p and q such that p is monotone on $[x, y]$ (or $(y, x]$). We say that (p, q) is *elementary* if both p and q are eventually increasing, and the only common fixed points of p and q are the minimum $\min(\text{dom}(p)) = \min(\text{dom}(q))$, and the maximum $\max(\text{dom}(p)) = \max(\text{dom}(q))$ of their domains. A pair (p, q) is *piecewise elementary* if we can find $E_0 \leq \dots \leq E_n$, called *ele-*

mentary components of (p, q) , such that $\bigcup_i E_i = \text{def}(p) \cup \text{def}(q)$, and (p, q) is elementary when restricted to each E_i .

Lemma 5.1. *Let \mathcal{F} be a full order expansion with SAP. Let (p, q) be an elementary pair. Then there exists an extension (p', q') of (p, q) , and $w \in F(s, t)$, such that p' does not change monotonicity of p , and $w(p', q')[\text{def}(q')]$ is in the unique final segment of (p', q') .*

Proof. Without loss of generality, we can assume that $\min(\text{supp}(p)) < \text{supp}(q)$. Let $a_0 < \dots < a_m$ be the enumeration of $\text{def}(p) \setminus \{\min(\text{def}(p)), \max(\text{def}(p))\}$. We can assume that a_0 is not a fixed point of p .

We construct $w_i \in F(s, t)$, $i \leq m$, and extensions (p_i, q_i) of (p_{i-1}, q_{i-1}) such that $w_i \dots w_0(p_i, q_i)(a_0) > a_i$. Moreover, we require that the only new element in $\text{def}(p_i)$ or $\text{def}(q_i)$ above a_i is $w_i \dots w_0(p_i, q_i)(a_0)$.

Put $w_{-1} = \emptyset$, $p_{-1} = p$, $q_{-1} = q$. Fix $0 \leq i \leq m$, and suppose that w_j, p_j, q_j have been already constructed for $j < i$. Set $b = w_{i-1} \dots w_0(p_{i-1}, q_{i-1})(a_0)$. If there is an extension p_i of p_{i-1} such that $(p_i)^{\epsilon k}(b) > a_i$ for some $k \in \mathbb{N}$, and $\epsilon \in \{-1, 1\}$, we take the least such k , and put $w_i = s^{\epsilon k}$, $q_i = q_{i-1}$.

Suppose otherwise. If a_{i+1} is not a fixed point of q , put $c = a_i$, and $l = 0$. Otherwise, as (p_{i-1}, q_{i-1}) is elementary, it is not a fixed point of p_{i-1} , and so we can extend p_{i-1} to some p_i by adding only elements below a_i , so that, for some $l \in \mathbb{Z}$, $b < (p_i)^l(a_i) < a_i$, and $(b, (p_i)^l(a_i))$ has empty intersection with both $\text{def}(p_{i-1})$ and $\text{def}(q_{i-1})$. Put $c = (p_i)^l(a_i)$.

Let $\epsilon \in \{-1, 1\}$ be such that there exists an extension q_i of q_{i-1} with $(q_i)^\epsilon(c) < c$. Because \mathcal{F} is a full order expansion with SAP, there is an extension of p_i , which we will also denote by p_i , such that $p_i(b) \in ((q_i)^\epsilon(c), c)$. But then $(q_i)^{-\epsilon}(p_i(b)) > c$. Thus, for $w_i = s^{-l}t^{-\epsilon}s$, we have that $w_i \dots w_0(p_i, q_i)(a_0) > a_i$ and $w_i \dots w_0(p_i, q_i)(a_0)$ is the only new element in $\text{def}(p_i)$ or $\text{def}(q_i)$ above a_i . \square

Lemma 5.2. *Let \mathcal{F} be a full order expansion with SAP, and such that \mathcal{F}^- has liberating automorphisms. Let M be the limit of \mathcal{F} . Let (p, q) be a piecewise elementary pair of partial automorphisms of M such that, for some $w \in F(s, t)$, $w(p, q)(x)$ is in a final segment of (p, q) for every $x \in \text{def}(q)$. Then for any $v \in F(s, t)$, and $N \in \mathbb{N}$, there is $N' \geq N$, and a pair (p', q') extending (p, q) such that $vs^{N'}w(p', q')(x)$ is in a final segment of (p, q) for $x \in \text{def}(q')$.*

Proof. Let $E_0 \leq \dots \leq E_n \subseteq M$ be elementary components of (p, q) . Because $w(p, q)(x)$ is in a final segment of (p, q) for every $x \in \text{def}(q)$, and p is eventually increasing on each E_i , we can assume that there is $N_0 \in \mathbb{N}$ such that

$$s^{N_0}w(p, q)(x) > \text{supp}(q) \cap E_i$$

for every $i \leq n$, and $x \in \text{supp}(q) \cap E_i$. Then

$$x \leq s^{N_0}w(p, q)(y) \text{ if and only if } q(x) \leq s^{N_0}w(p, q)(y) \tag{3}$$

for every $x, y \in \text{def}(q)$.

Because \mathcal{F}^- has liberating automorphisms, and \mathcal{F} is a full order expansion with SAP, we can find $N_1 \in \mathbb{N}$, and an extension p' of p such that $s^{N_0+N_1+n}w(p', q)[\text{def}(q)]$ is free from q in \mathcal{F} for $n \leq 2|v| + N$, and (3) still holds for $\text{def}(q)$. But this means that $s^{N_0+N_1+n}w(p', q)[\text{def}(q)]$ is free from q in \mathcal{F} , and we can extend q to q' so that

$$q'(s^{N_0+N_1+n}w(p', q')(x)) = s^{N_0+N_1+n}w(p', q')(x)$$

for $n \leq 2|v| + N$ and $x \in \text{def}(q)$. It is easy to see that then

$$vs^{N_0+N_1+|v|+N}w(p', q')(x) \geq w(p', q')(x)$$

for $n \leq N$, and so $vs^{N_0+N_1+|v|+N}w(p', q')(x)$ is in a final segment of p' , for $x \in \text{def}(q')$. Therefore $N' = N_0 + N_1 + |v| + N$ is as required. \square

Lemma 5.3. *Let \mathcal{F} be a full order expansion with SAP, and such that \mathcal{F}^- has liberating automorphisms. Then for every piecewise elementary pair (p, q) there exists a piecewise elementary pair (p', q') extending (p, q) , and $w \in F(s, t)$ such that $w(p', q')(x)$ is in a final segment of (p', q') for $x \in \text{def}(q')$.*

Proof. We prove the lemma by induction on the number r of elementary components of (p, q) . For $r = 1$, this follows from Lemma 5.1. Suppose that the lemma is true for some r , and fix a piecewise elementary pair (p, q) with $r + 1$ elementary components.

Let us write $E = E_0 \cup E_1$ so that $E_0 \leq E_1$, (p, q) is elementary when restricted to E_0 , and there are r elementary components in (p, q) when restricted to E_1 . Using Lemma 5.1, and the inductive assumption, we can fix an extension (p', q') of (p, q) , so that, for (p_0, q_0) denoting the restriction of (p', q') to E_0 , and (p_1, q_1) denoting the restriction of (p', q') to E_1 , the following holds. The mapping p_0 does not change monotonicity of p restricted to E_0 , the pair (p_1, q_1) is piecewise elementary, and there exist $w_0, w_1 \in F(s, t)$ such that $w_0(p_0, q_0)(x)$ is in the unique final segment of (p_0, q_0) for $x \in \text{def}(q_0)$, and $w_1(p_1, q_1)(x)$ is in a final segment of (p_1, q_1) for $x \in \text{def}(q_1)$.

Let $d_0 = \min(\text{supp}(p_0))$. As p_0 does not change monotonicity of p restricted to E_0 , and so p_0 is increasing on the initial segment of p_0 , in the case that $w_1(p_0, q_0)(d_0) < d_0$, we can assume that there exists $N \in \mathbb{N}$ such that $s^N w_1(p_0, q_0)(d_0) > d_0$. Then $w_0 s^N w_1(p_0, q_0)(x)$ is in the final segment of p_0 for $x \in \text{def}(q_0)$.

Moreover, applying Lemma 5.2, we can assume that N is large enough so that $w_0 s^N w_1(p_1, q_1)(x)$ is in a final segment of p_1 for $x \in \text{def}(q_1)$. Thus, the pair (p', q') extends (p, q) , and, $w(p', q')(x)$ is in a final segment of p' for $x \in \text{def}(q')$, if $w = w_0 s^N w_1$. \square

Theorem 5.4. *Let \mathcal{F} be a full order expansion with SAP, and such that \mathcal{F}^- has liberating automorphisms. Let M be the limit of \mathcal{F} . Then there are comeagerly many pairs in $\text{Aut}(M)^2$ generating a non-discrete group.*

Proof. Consider the following condition: for every pair (p, q) of partial automorphisms of M there exists an extension (p', q') of (p, q) , and $w \in F_2$ such that $w(p', q')(x) = x$ for $x \in \text{def}(q)$. It is easy to verify that if it holds, then the set of pairs $(f, g) \in \text{Aut}(M)^2$ generating a non-discrete group contains a dense G_δ subset of $\text{Aut}(M)^2$, that is, it is comeager in $\text{Aut}(M)^2$.

We verify this condition. Fix a pair (p, q) of partial automorphisms of M . Without loss of generality, we can assume that it is piecewise elementary. By Lemma 5.3, we can also assume that there exists $w' \in F(s, t)$ such that $w'(p, q)(x)$ is in a final segment of p for $x \in \text{def}(q)$. But then, using our assumptions on \mathcal{F} , and the fact that p is eventually increasing on each elementary component of (p, q) , we can find $N \in \mathbb{N}$, and an extension (p', q') of (p, q) such that

$$x \leq s^N w(p', q')(y) \text{ if and only if } q(x) \leq s^N w(p', q')(y) \tag{4}$$

for every $x, y \in \text{def}(q')$, and $s^N w'(p', q')[\text{def}(q')]$ is free from q' in \mathcal{F} . Therefore we can put

$$q'(s^N w(p', q')(x)) = s^N w(p', q')(x)$$

for $x \in \text{def}(q')$. Then for $w = (s^N w)^{-1} t (s^N w)$, we have $w(p', q')(x) = x$ for $x \in \text{def}(q')$. \square

Theorem 5.5. *Let \mathcal{F} be a full order expansion with SAP, and such that \mathcal{F}^- has liberating automorphisms. Let M be the limit of \mathcal{F} . Then all 2-dimensional topological similarity classes in $\text{Aut}(M)$ are meager.*

Proof. By the above theorem, there are comeagerly many pairs in $\text{Aut}(M)^2$ generating a non-discrete group. As automorphisms of order structures have only infinite non-trivial orbits, in fact, there are comeagerly many pairs in $\text{Aut}(M)^2$ generating a non-discrete and non-precompact group. By [10, Theorem 4.4], all 2-dimensional topological similarity classes in $\text{Aut}(M)$ are meager. \square

Recall that a Fraïssé class \mathcal{F} has *free amalgamation* if for every $A, B, C \in \mathcal{F}$ with $A \subseteq B, C$, the structure $D = B \cup C$ is an amalgam of B and C over A . In other words, no tuple in D involving at the same time elements from $B \setminus A$ and from $C \setminus A$ is related in D . A typical example of a class with free amalgamation is the class of finite graphs.

Lemma 5.6. *If \mathcal{F} is a Fraïssé class with free amalgamation or the class of finite tournaments, then \mathcal{F} has liberating automorphisms.*

Proof. Suppose that \mathcal{F} has free amalgamation. Let (p, q) be a pair of partial automorphisms of $A \in \mathcal{F}$ with no cyclic orbits. Without loss of generality, we can assume that $A = \text{def}(p)$. Let $N \in \mathbb{N}$ be such that orbits in p have size at most $N/2$. Set

$C = \text{dom}(p) \setminus \text{rng}(p)$, and fix $N' > N$. We put $p_0 = p$, and construct partial automorphisms p_i and sets D_i , $0 < i \leq N'$ in the following manner. Assuming that p_i is already constructed, with an aid of free amalgamation, we extend p_i to a partial automorphism p_{i+1} by defining it on $D' = \text{rng}(p_i) \setminus \text{dom}(p_i)$ in such a way that no relation involves at the same time elements from C and $D = p_{i+1}[D']$ (where D is disjoint from $\text{def}(p_i)$). To be more precise, put $B = \text{rng}(p_i)$. Then, for every relation R of arity n , and every n -tuple \bar{b} in $B \cup D$, whether $R(\bar{a})$ holds or not, is entirely determined by the requirement that p_{i+1} is supposed to be a partial automorphism. Moreover, regardless of how we amalgamate $B \cup C$ and $B \cup D$ over B , to get a structure E with underlying set $B \cup C \cup D$, p_{i+1} will be a partial automorphism of E . Thus, E obtained by freely amalgamating these structures works as $\text{def}(p_{i+1})$. Finally, we put $D_{i+1} = D$.

Observe that no relation involves at the same time elements from $p_{N'}^n[A]$ and A for $N \leq n \leq N'$, which means, because $\text{def}(q) \subseteq A$, that each $p_{N'}^n[A]$ is free from q . Indeed, by the construction of p_i , for $i > 0$ and $x \in D_i$, no relation involves x and elements from C . And then the same is true about any $p_{i+1}^n(x)$ and $p_{i+1}^i[C]$, $i \leq n \leq N'$. As $p_{N'}^n[A] \subseteq \bigcup_{N \leq i \leq N'} D_i$, and $A \subseteq \bigcup_{i \leq N} p^i[C]$, this means that no relation involves at the same time elements from $p_{N'}^n[A]$ and A , for $N \leq n \leq N'$.

For finite tournaments, we proceed almost exactly as above. The only difference is that for every $x \in \text{dom}(p_i) \setminus \text{rng}(p_i)$, $y \in \text{rng}(p_i) \setminus \text{dom}(p_i)$, we choose (x, y) as the arrow between x and y . Then (x, y) is an arrow for every $x \in \text{def}(q)$ and $y \in p_{N'}^n[\text{def}(q)]$, $N \leq n \leq N'$. \square

Corollary 5.7. *Suppose that \mathcal{F} is a full order expansion such that \mathcal{F}^- has free amalgamation, or the class of finite ordered tournaments. Let M be the Fraïssé limit of \mathcal{F} . Then all 2-dimensional topological similarity classes in $\text{Aut}(M)$ are meager.*

Corollary 5.8. *Suppose that \mathcal{F} is a full order expansion of a class with free amalgamation, and let M be the Fraïssé limit of \mathcal{F} . Then all 2-dimensional topological similarity classes in $\text{Aut}(M)$ are meager.*

Remark 5.9. Compare the above corollary with Theorem 3.5 and Corollary 4.2.

References

- [1] P.J. Cameron, Some treelike objects, *Q. J. Math. Oxford Ser. (2)* 38 (150) (1987) 155–183.
- [2] G.L. Cherlin, The classification of countable homogeneous directed graphs and countable homogeneous n -tournaments, *Mem. Am. Math. Soc.* 131 (621) (1998).
- [3] M. Doucha, M. Malicki, Generic representations of countable groups, *Trans. Am. Math. Soc.* 372 (2019) 8249–8277.
- [4] H. Furstenberg, Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions, *J. Anal. Math.* 31 (1977) 204–256.
- [5] A. Ivanov, Generic expansions of ω -categorical structures and semantics of generalized quantifiers, *J. Symb. Log.* 64 (2) (1999) 775–789.
- [6] J. Jasiński, Ramsey degrees of boron tree structures, *Combinatorica* 33 (1) (2013) 23–44.
- [7] J. Jasiński, C. Laflamme, L. Nguyen Van Thé, R. Woodrow, Ramsey precompact expansions of homogeneous directed graphs, *Electron. J. Comb.* 21 (4) (2014), 31 pp.

- [8] A.S. Kechris, C. Rosendal, Turbulence, amalgamation, and generic automorphisms of homogeneous structures, *Proc. Lond. Math. Soc.* (3) 94 (2) (2007) 302–350.
- [9] A. Kaïchouh, F. Le Maître, Connected Polish groups with ample generics, *Bull. Lond. Math. Soc.* 47 (2015) 996–1009.
- [10] A. Kwiatkowska, M. Malicki, Automorphism groups of countable structures and groups of measurable functions, *Isr. J. Math.* (2018) 1–26.
- [11] A.S. Kechris, V.G. Pestov, S. Todorcevic, Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups, *Geom. Funct. Anal.* 15 (1) (2005) 106–189.
- [12] D. Kuske, J.K. Truss, Generic automorphisms of the universal partial order, *Proc. Am. Math. Soc.* 129 (7) (2001) 1939–1948.
- [13] L. Nguyen Van Thé, More on the Kechris-Pestov-Todorcevic correspondence: precompact expansions, *Fundam. Math.* 222 (2013) 19–47.
- [14] A. Panagiotopoulos, Extendability of automorphisms of generic substructures, *Isr. J. Math.* 208 (2015) 483–508.
- [15] V. Pestov, F.M. Schneider, On amenability and groups of measurable maps, *J. Funct. Anal.* 273 (2017) 3859–3874.
- [16] K. Slutsky, Non-genericity phenomenon in some ordered Fraïssé classes, *J. Symb. Log.* 77 (2012) 987–1010.
- [17] J.K. Truss, Generic automorphisms of homogeneous structures, *Proc. Lond. Math. Soc.* 65 (3) (1992) 121–141.