

Tackling the generation problem in condensation

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Abstract

A major drawback of applying condensation in computational representation theory is the so-called generation problem, i.e. given a set of elements for the group algebra we generally do not know whether the corresponding condensed elements generate the condensed algebra. In this note we present two new methods to bridge this gap. Firstly we introduce generating sets which are in practice often small enough to be of computational use. Secondly we give a criterion which allows us to verify that a subset of the condensed algebra is in fact a generating set.

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1. Introduction

The advent of computational methods in the representation theory of finite groups particularly made the problems involved in the search for the irreducible representations of sporadic simple groups amenable. While in theory it is possible to classify all irreducible representations by repeatedly applying the MEATAXE (confer [10]), i.e. by explicitly constructing them, the available computational resources seriously limit this approach and render it mostly infeasible for the open questions in the Modular Atlas project [6]. To overcome this limitation Parker and Thackray introduced the fixed-point condensation method (see [12, Chapter 6]). In general with the help of condensation it is possible to restrict our attention to subspaces of the considered modules and thus regain computational tractability.

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To make this transition precise let F be a field of characteristic $p > 0$, A a finite-dimensional F -algebra and V an A -module. In this note we will always tacitly assume any module to be a right module. Once we deviate from this rule we shall always specify the action. Furthermore let $e \in A$ be an idempotent. Then we consider the *condensation functor*

$$- \otimes_A Ae : \text{mod-}A \rightarrow \text{mod-}eAe,$$

under which V is mapped to $V \otimes_A Ae$, which we may identify with the subspace $Ve \subseteq V$. Similarly a homomorphism $\varphi \in \text{Hom}_A(V, W)$ is mapped to its restriction $\varphi|_{Ve} \in \text{Hom}_{eAe}(Ve, We)$. We refer to Ve as the *condensed module* of V and e as the *condensation idempotent*.

We will state some important facts about the condensation functor here. For a more thorough treatment and proofs see [2, Section 6]. As the tensor functor $- \otimes_A Ae$ is naturally isomorphic to the homomorphism functor $\text{Hom}_A(eA, -)$ it is exact. Furthermore if for every isomorphism class of simple A -modules we have $Se \neq 0$ for a representative S , condensation induces a Morita equivalence between the category of A -modules and the category of eAe -modules and we ideally lose no relevant information in this transition. In the less favorable case that some simple A -modules condense to 0, those which do not condense to simple eAe -modules. Hence composition series of A -modules are mapped to composition series of eAe -modules and may still retain enough information about the original module. The condensation of all simple A -modules which do not condense to 0 yields a full set of representatives of the isomorphism classes of simple eAe -modules.

In the present work we will take A to be a group algebra FG for some finite group G such that p divides $|G|$. In this case fixed-point condensation has been applied successfully to many problems in the past. To this end we choose a *condensation subgroup* K of order coprime to p , and e takes the form $1/|K| \sum_{k \in K} k$. Then the image under the projection with e is precisely the subspace of elements which are fixed by the action of K . Here we will consider a more general class of condensation idempotents, following a suggestion by Jon Thackray. We take λ to be a linear character of K and let e_λ be the idempotent of the form

$$\frac{1}{|K|} \sum_{k \in K} \lambda(k^{-1})k,$$

which we call a *linear idempotent*. For a previous application of such an idempotent in the condensation of a permutation module see [11].

The drawback of applying condensation is the problem that in general we do not know how to generate the *condensed algebra* eAe with sufficiently few elements that still adhere to a computational treatment. In particular it is in general not true that a generating set \mathcal{E} of A condenses to a generating set of eAe , i.e. the set $e\mathcal{E}e := \{eae \mid a \in \mathcal{E}\}$ generally only generates a subalgebra $\mathcal{C} \leq eAe$.

This gap in our knowledge, how to generate the condensed algebra, is referred to as the *generation problem*. It is of paramount importance to our computational approach. We can only study a module V for a particular algebra A with the MEATAXE by providing the latter with a set of matrices, which generate an algebra isomorphic to the image of A under the representation afforded by V . Hence if we cannot assert that the proposed generators (the input to the MEATAXE) generate the whole condensed algebra, we have to assume that the information given by the MEATAXE refers to the \mathcal{C} -module $V \downarrow_{\mathcal{C}}$. Gaining results for the condensed module Ve from

this can be tedious and in the past techniques have been developed to bridge this gap (see, for example, [4]).

In the following we will present two new approaches to overcome the generation problem. In Section 2, which is based on the author's PhD thesis (see [8, Section IV.2]), we will introduce a generating set for the condensed algebra which is often small enough to be of practical use. For the cases in which the cardinality of such a generating set is too large we give a method in Section 3 (see also [8, Section IV.3]) which allows us to sort out generators that are redundant. In Section 4, which generalizes [8, Sections IV.4 and IV.5], we propose a criterion with which it is possible to verify an arbitrary subset of the condensed algebra as a set of generators.

2. Inert generating sets

We begin by fixing a linear character λ of the condensation subgroup and denote the FK -module affording it by Λ . Let e_λ be the linear idempotent corresponding to λ . Furthermore let N be the normalizer of K in G . The conjugation action of N on K induces an action on the Brauercharacters of K . For this action we let ${}^n\lambda$ denote the image of λ under some $n \in N$, i.e. ${}^n\lambda(k) = \lambda(n^{-1}kn)$ for all $k \in K$.

In this situation we can make the following simple observation, which will be crucial for our later results.

Remark 2.1. We have $ne_\lambda n^{-1} = e_{{}^n\lambda}$ for all $n \in N$. In particular for $n \in K$ we have $e_\lambda n = ne_\lambda = \lambda(n)e_\lambda$.

Proof. This follows immediately from the definition of e_λ and the fact that K is a normal subgroup of N . \square

Let us denote the ordinary character theoretic scalar product by (\cdot, \cdot) . For the convenience of the reader we will now restate an established result giving a character theoretic description of the group elements which condense to zero and which also provides a basis for the condensed algebra (see [1, Proposition (11.30)]).

Lemma 2.2. Let $g \in G$. We have

$$e_\lambda g e_\lambda = e_\lambda g e_\lambda (\lambda^g \downarrow_{K^g \cap K}, \lambda \downarrow_{K^g \cap K}).$$

Proof. We write $e_\lambda g e_\lambda = |K|^{-1} \sum_{l \in K} \lambda(l)^{-1} e_\lambda g l$ and substitute each $l \in K$ by the product of an element in $K^g \cap K$ and one in the transversal $(K^g \cap K) \backslash K$. For the $k' \in K^g \cap K$ we have by Remark 2.1 that $e_\lambda g k' = e_\lambda \cdot g k' g^{-1} \cdot g = \lambda(g k' g^{-1}) e_\lambda g$. This yields

$$\frac{|K^g \cap K|}{|K|} \sum_{k \in (K^g \cap K) \backslash K} \lambda(k)^{-1} \left(\frac{1}{|K^g \cap K|} \sum_{k' \in K^g \cap K} \lambda(k')^{-1} \lambda^g(k') \right) e_\lambda g k$$

for $e_\lambda g e_\lambda$. The factor in brackets is simply the scalar product $(\lambda \downarrow_{K^g \cap K}, \lambda^g \downarrow_{K^g \cap K})$. The claim follows by multiplying this equation from the right by e_λ . \square

Theorem 2.3. *Let*

$$\mathcal{D} := \{g \in K \backslash G / K : (\lambda^g \downarrow_{K^g \cap K}, \lambda \downarrow_{K^g \cap K}) = 1\}.$$

Then $e_\lambda \mathcal{D} e_\lambda$ is a basis of $e_\lambda F G e_\lambda$.

Proof. By Frobenius reciprocity and Mackey's Theorem we get the isomorphism

$$\begin{aligned} e_\lambda F G e_\lambda &\cong \operatorname{Hom}_{FG}(e_\lambda F G, \Lambda \uparrow^G) \\ &\cong \operatorname{Hom}_{FK}(\Lambda \uparrow^G \downarrow_K, \Lambda) \\ &\cong \bigoplus_{g \in K \backslash G / K} \operatorname{Hom}_{FK}(\Lambda^g \downarrow_{K^g \cap K} \uparrow^K, \Lambda) \\ &\cong \bigoplus_{g \in K \backslash G / K} \operatorname{Hom}_{F(K^g \cap K)}(\Lambda^g \downarrow_{K^g \cap K}, \Lambda \downarrow_{K^g \cap K}) \end{aligned}$$

of vector spaces. Since Λ^g affords the character λ^g , it follows that $|\mathcal{D}| = \dim_F e_\lambda F G e_\lambda$. If $\eta \in e_\lambda F G e_\lambda$, then we may write $\eta = \sum_{g \in G} \mu_g e_\lambda g e_\lambda$ for suitable $\mu_g \in F$. By Remark 2.1 and Lemma 2.2 it is sufficient to sum over the set \mathcal{D} . Hence the set of condensed elements $e_\lambda \mathcal{D} e_\lambda$ is a generating set for the condensed algebra and we have $|\mathcal{D}| = |e_\lambda \mathcal{D} e_\lambda|$. \square

With Lemma 2.2 and Theorem 2.3 we can now characterize the elements of G which condense to zero.

Corollary 2.4. *Let $g \in G$. The element $e_\lambda g e_\lambda$ is zero if and only if the scalar product $(\lambda \downarrow_{K^g \cap K}, \lambda^g \downarrow_{K^g \cap K})$ is zero.*

We set $T := T(\lambda)$ to be the inertia subgroup of λ in N . By Remark 2.1 the condensation idempotent e_λ is central in the group algebra FT . This gives the following important observation.

Remark 2.5. For all $n, m \in T$ and $g \in G$ we have

$$e_\lambda n e_\lambda \cdot e_\lambda g e_\lambda \cdot e_\lambda m e_\lambda = e_\lambda n g m e_\lambda.$$

We can now give a generating set for the algebra $e_\lambda F T e_\lambda$.

Proposition 2.6. *Let $\mathcal{N} \subseteq T$ be a subset of T such that the images of its elements under the canonical projection generate the factor group T/K . Then the condensed algebra $e_\lambda F T e_\lambda$ is generated by the set $e_\lambda \mathcal{N} e_\lambda$.*

Proof. By Corollary 2.4 a condensed element of N is nonzero if and only if it is in T . Remark 2.5 gives $e_\lambda n e_\lambda \cdot e_\lambda m e_\lambda = e_\lambda n m e_\lambda$ for $n, m \in T$ and by Remark 2.1 we see that every condensed element of T is a product of elements of $e_\lambda \mathcal{N} e_\lambda$, as e_λ is central in FT . \square

We can now give the main result of this section which gives a set of generators for the condensed algebra.

Theorem 2.7. Let $\mathcal{N} \subseteq T$ be a subset of T such that the images of its elements under the canonical projection generate the factor group T/K , and \mathcal{T} a full set of representatives of the T - T -double cosets in G , which do not condense to zero. Then the condensed algebra $e_\lambda FGe_\lambda$ is generated by the set $e_\lambda \mathcal{N} e_\lambda \cup e_\lambda \mathcal{T} e_\lambda$.

Proof. Every element $g \in G$ can be written as $g = ntm$, where $n, m \in T$ and $t \in T \backslash G / T$. By Remark 2.5 the condensed element $e_\lambda g e_\lambda$ is the product $e_\lambda n e_\lambda \cdot e_\lambda t e_\lambda \cdot e_\lambda m e_\lambda$. The condensed elements of n and m lie in the algebra $e_\lambda FTe_\lambda$ and t is an element of \mathcal{T} , if $e_\lambda g e_\lambda \neq 0$. Thus Proposition 2.6 yields the claim \square

To stress that the generating sets of Theorem 2.7 are for the largest part condensed representatives of the inertia subgroup double cosets $T \backslash G / T$ we shall refer to them as *inert generating sets*.

In order to obtain a computationally tractable inert generating set we have to choose our condensation subgroup K and one of its linear characters such that the number of double cosets of the associated inertia subgroup is sufficiently small. Ideally the inertia subgroup is a maximal subgroup of low index in G and therefore we strive to choose K to be a normal subgroup of such a maximal subgroup. Using the linear idempotent corresponding to the trivial character of K , i.e. applying the traditional condensation method, this approach has successfully led to the solution of determining the 2-modular characters of the sporadic simple Fischer group Fi_{23} in [3]. In this case K was chosen to be an extraspecial 3-subgroup of order 19 683 in the seventh maximal subgroup $3_+^{1+8} \cdot 2_-^{1+6} \cdot 3_+^{1+2} \cdot 2S_4$ of Fi_{23} giving us 38 generators for the condensed algebra.

3. Reducing inert generating sets

Unfortunately we cannot always hope for such a benign situation as in the case of Fi_{23} in characteristic 2. To be able to exploit a full equivalence between the module categories $\text{mod-}FG$ and $\text{mod-}e_\lambda FGe_\lambda$ we may have to choose λ to be nontrivial and its inertia subgroup may no longer be maximal. In these cases the resulting increase in the cardinality of the inert generating set may lead to an infeasibly large number of generators. To counter this problem we can attempt to sort out redundant generators with the following.

For brevity we will write e for e_λ from now on. Let \mathcal{T} be as in Theorem 2.7, \mathcal{E} a subset of T and \mathcal{C} the subalgebra of $eFGe$ which is generated by $e\mathcal{E}e$ and $e\mathcal{T}e$.

Remark 3.1. Let $\eta \in \mathcal{C}$ and let it have the representation $\eta = \sum_{d \in \mathcal{D}} c_d e d e$ with $c_d \in F$ for all $d \in \mathcal{D}$ with respect to the basis given in Theorem 2.3. If $t \in \mathcal{E}$, then $\eta - c_d e d e$ is an element of \mathcal{C} for all $d \in \mathcal{D}$ with $KdK \subseteq TtT$, too.

Proof. Since $KdK \subseteq TtT$ there exist $n, n' \in T$ with $d = ntn'$. Furthermore we have $ene \cdot ete \cdot en'e = entn'e = ede \in \mathcal{C}$ and thus $\eta - c_d e d e \in \mathcal{C}$. \square

In order to use this observation we need some knowledge about the \mathcal{D} -basis representation of the product of two condensed group elements. For a group H we write $O_H(\omega)$ for the H -orbit containing an element ω of an H -set Ω .

Lemma 3.2. Let $g, h \in G$ with $ege \neq 0$ and $ehe \neq 0$. Then we have

$$ege \cdot ehe = \frac{1}{|O_K(Kg)|} \sum_{d \in \mathcal{D}} \left(\sum_{\substack{k \in (K^g \cap K) \setminus K \\ Kgh \subseteq KdK}} \lambda(k^{-1} \hat{d}(gkh)) \right) ede,$$

where $\hat{d}(y)$ denotes for any $y \in G$, for which Ky lies in the K -orbit of Kd , the unique element in the transversal $(K^d \cap K) \setminus K$ such that $Ky = Kd\hat{d}(y)$.

Proof. From the definition of the linear idempotent e the product of ege and ehe is the sum $|K|^{-1} \sum_{k' \in K} \lambda(k')^{-1} egk'he$. Writing every $k' \in K$ as $k''k$ with $k'' \in K^g \cap K$ and $k \in (K^g \cap K) \setminus K$, we can rewrite with Remark 2.1 this sum as

$$\frac{1}{|K|} \sum_{k'' \in K^g \cap K} \lambda(k'')^{-1} \lambda(gk''g^{-1}) \sum_{k \in (K^g \cap K) \setminus K} \lambda(k)^{-1} egkhe.$$

Inserting the quotient $|K^g \cap K|/|K^g \cap K|$ allows us to obtain the scalar product $(\lambda \downarrow_{K^g \cap K}, \lambda^g \downarrow_{K^g \cap K})$ from the first sum, which is 1 by Corollary 2.4. Thus we are left with the sum

$$\frac{1}{|O_K(Kg)|} \sum_{k \in (K^g \cap K) \setminus K} \lambda(k)^{-1} egkhe.$$

Every $egkhe$ is associated to exactly one ede , since by Remark 2.1 we have $egkhe = ed \cdot \hat{d}(gkh)e = \lambda(\hat{d}(gkh))ede$ and $egkhe$ is associated to ede if and only if Kgh is a subset of KdK . Now the claim follows by sorting the summands accordingly. \square

The trouble with the representation of Lemma 3.2 is that the knowledge of the coefficients of the basis elements is too detailed, since their calculation amounts to condensation. In order to determine the redundancy of a generator without prior condensation we content ourselves with less information.

Definition 3.3. Let $g, h \in G$ be as in Lemma 3.2. If we assume λ to be the trivial character in the inner sum of Lemma 3.2 we obtain the p -modular reduction of the integer $|\{\omega \in O_K(Kg) \mid \omega h \in O_K(Kd)\}|$ as the coefficient of ede for some $d \in \mathcal{D}$. For an element $t \in \mathcal{T}$ we define

$$c_{g,h}^t := \sum_{\substack{d \in \mathcal{D} \\ KdK \subseteq TtT}} |\{\omega \in O_K(Kg) \mid \omega h \in O_K(Kd)\}|.$$

We call the latter an *upper bound for the multiplicity* of elements ede for which d lies in the coset TtT and we call t the *germ* of these elements.

The appeal of these upper bounds is that in contrast to the coefficients of Lemma 3.2 they can be calculated with the following counting argument and the help of the permutation representation on $N \setminus G$.

Proposition 3.4. *We have*

$$c_{g,h}^t = \frac{|O_K(Kg)|}{|O_K(Tg)|} |\{\omega \in O_K(Tg) \mid \omega h \in O_T(Tt)\}|.$$

Proof. By Definition 3.3 the integer $c_{g,h}^t$ is given by the sum of cardinalities $|\{\omega \in O_K(Kg) \mid \omega h \in O_K(Kd)\}|$ for some $d \in \mathcal{D}$. The stabilizer of an ω in $O_K(Kg)$ is $K^g \cap K$ and therefore we have $\omega = Kgk$ for a unique k in the transversal $(K^g \cap K) \backslash K$. Hence $\omega h \in O_K(Kd)$ if and only if we have $Kgkh \subseteq KdK$ for this k . We can now rewrite the sum from Definition 3.3 as the sum of the cardinalities of the sets $\{k \in (K^g \cap K) \backslash K \mid KgkhK = KdK\}$ with $d \in \mathcal{D} \cap TtT$. The disjoint union of these sets gives the set $\{k \in (K^g \cap K) \backslash K \mid KgkhK \subseteq TtT\}$. Since $KgkhK$ is a subset of TtT if and only if $TgkhT = TtT$, we obtain as the right-hand side the natural number $|\{k \in (K^g \cap K) \backslash K \mid TgkhT = TtT\}|$. Rewriting every k as a product $k''k'$, where k'' lies in $(K^g \cap K) \backslash (T^g \cap K)$ and $k' \in (T^g \cap K) \backslash K$, this results in the product of $|T^g \cap K|/|K^g \cap K|$ and $|\{k' \in (T^g \cap K) \backslash K \mid Tgk'hT = TtT\}|$ as the right-hand side. \square

The upper bounds for the number of occurring basis elements with a certain germ can be exploited in the following way.

Theorem 3.5. *Let g and h be two elements of G and let the germs of their condensates lie in \mathcal{E} . If for $t \in \mathcal{T}$ we have $c_{g,h}^t = 1$ and furthermore for every $t \neq t' \in \mathcal{T}$ with $c_{g,h}^{t'} > 0$ we have $t' \in \mathcal{E}$ then ete is in \mathcal{C} .*

Proof. As $c_{g,h}^t = 1$ there occurs at most one basis vector ede with germ t in the decomposition of $ege \cdot ehe$ with respect to the basis eDe of Theorem 2.3. Therefore in the context of Lemma 3.2 we have exactly one $k \in (K^g \cap K) \backslash K$ with $Kgkh \subseteq KdK$. Thus the coefficient of ede is a unit in F and in particular nonzero. Since the germs of all remaining occurring basis vectors are contained in \mathcal{E} an iterated application of Remark 3.1 gives the claim. \square

If we find an element $t \in \mathcal{T}$ as in Theorem 2.7 the corresponding condensed element is already included in \mathcal{C} and therefore it is superfluous in the inert generating set. In order to sort out elements of \mathcal{T} which are redundant, in practice we need only consider products of the form $ete \cdot ene \cdot et'e$ for $t, t' \in \mathcal{T}$ and $n \in K \backslash T$: Any element of G lies in some double coset of $T \backslash G / T$. If we assume $g = nlm$ and $h = n'rm'$ for some $n, n', m, m' \in T$ and $l, r \in \mathcal{T}$ then we may write $ege \cdot ehe = ene \cdot ele \cdot en''e \cdot ere \cdot eme$ where $n'' = mn'$. From Definition 3.3 it follows that the multiplication from the left and right by elements of $eFTe$ has no effect on the defined upper bounds. Thus we may conclude $c_{g,h}^t = c_{ln'',r}^t$ for all $t \in \mathcal{T}$ and can formulate the following corollary.

Corollary 3.6. *Let $l, r \in \mathcal{E}$. An element $t \in \mathcal{T} \backslash \mathcal{E}$ which fulfills $c_{ln,r}^t = 1$ for some $n \in K \backslash T$ while all other $t \neq t' \in \mathcal{T}$ with $c_{ln,r}^{t'} > 0$ are in \mathcal{E} is not needed to generate $eFGe$ together with $eFTe$.*

In order to check for the redundancy of a generator ete for some $t \in \mathcal{T}$ with Theorem 3.5 we necessarily need $c_{g,h}^t = 1$ for some $g, h \in G$. This in turn can only be achieved if the ratio of the orbit lengths $|O_K(Kg)|/|O_K(Tg)|$ in Proposition 3.4 is one, or equivalently that $K \cap T^g = K \cap K^g$. In a practical application of Corollary 3.6 we therefore only allow such ele as left factors

in the product $ele \cdot ene \cdot ere$ as above, whose germ $l \in \mathcal{E}$ fulfills $|O_K(Kl)|/|O_K(Tl)| = 1$, to check which elements of eTe already lie in the algebra generated by $e\mathcal{E}e$.

This method has been applied successfully in the determination of the 2- and 3-modular character tables of Fischer's sporadic group Fi_{22} in [9], where for example with its help we were able to reduce an inert generating set of 68 elements to one of 41 elements.

4. A criterion for generation

In this section we introduce a second method to address the generation problem. In contrast to the preceding section we give a criterion that allows us to verify that an arbitrary subset of the condensed algebra is a generating set.

We shall keep the notation introduced so far, but we shall give our main result in a more general context. To this end let A and B be two finite-dimensional F -algebras with $B \leq A$. By $J(B)$ we denote the Jacobson radical of B and by Y the semi-simple module $B/J(B)$.

Our starting point for this section is the following lemma which turns out to be the key.

Lemma 4.1. *Assume the induction functor $\text{ind}_B^A := - \otimes_B A$ to be exact. Then we have the following short exact sequence of B - A -bimodules:*

$$0 \longrightarrow J(B)A \longrightarrow A \xrightarrow{\hat{\pi}} Y \otimes_B A \longrightarrow 0,$$

where $\hat{\pi} := \text{ind}_B^A(\pi)$ and $\pi : B \rightarrow Y$ is the epimorphism resulting from the canonical projection.

Proof. Considering the regular module for the algebra B we obtain the short exact sequence

$$0 \longrightarrow J(B) \longrightarrow B \xrightarrow{\pi} Y \longrightarrow 0$$

of B -modules. As both B and $J(B)$ are natural B - B -bimodules, so is Y , where B acts from the left via the isomorphism $B \cong \text{End}_B(B)$. Thus this is in fact a sequence of B - B -bimodules and an application of the exact induction functor ind_B^A yields the claim. \square

In the following we will denote a module generated by a particular set by enclosing this set in angle brackets and specifying the action of the algebra by appropriate subscripts. Considering the algebra A as a B - B -bimodule we can deduce from Lemma 4.1 how to test a subset for generation.

Corollary 4.2. *Let \mathcal{E} be an arbitrary subset of A . If the B - B -bimodule morphism*

$$\hat{\pi} : {}_B\langle \mathcal{E} \rangle_B \longrightarrow Y \otimes_B A$$

is onto, then the B - B -bimodule generated by \mathcal{E} is A , i.e. we have

$${}_B\langle \mathcal{E} \rangle_B = A.$$

Proof. By the surjectivity of $\hat{\pi}$ Lemma 4.1 gives the equation ${}_B\langle \mathcal{E} \rangle_B + J(B)A = A$ as B - B -bimodules. By [1, (10.38)] the kernel of $\hat{\pi}$, which is $J(B)A$, lies in the radical of the algebra A considered as an B - B -bimodule. Hence with the help of Nakayama's Lemma we obtain ${}_B\langle \mathcal{E} \rangle_B = A$. \square

Let \mathcal{S} be a set of representatives for the isomorphism classes of simple B -modules and let $\mathcal{H}(S)$ be the S -homogeneous component of Y , i.e. the sum of all submodules of Y which are isomorphic to S . By virtue of the associativity of the tensor product we can rewrite the right-hand side of $\hat{\pi}$ in Corollary 4.1 and obtain

$$\hat{\pi} : A \longrightarrow \bigoplus_{S \in \mathcal{S}} \mathcal{H}(S) \otimes_B A.$$

Lemma 4.3. *We have*

$$\bigoplus_{S \in \mathcal{S}} \mathcal{H}(S) \otimes_B A = \bigoplus_{S \in \mathcal{S}} {}_B \langle S \otimes_B A \rangle.$$

Proof. Let $S \in \mathcal{S}$. By identifying B with $\text{End}_B(B)$ the algebra B acts from the left on the homogeneous component $\mathcal{H}(S)$. Since this is a direct sum of copies of S and for any two summands there exists an endomorphism mapping one onto the other, we have ${}_B \langle S \rangle = \mathcal{H}(S)$. Analogously B acts on $S \otimes_B A$, and the claim follows. \square

From Lemma 4.3 we deduce that in order to establish the surjectivity of $\hat{\pi}$ in Corollary 4.2 it is sufficient to show that for all $S \in \mathcal{S}$ the morphisms $\hat{\pi}_S := \text{ind}_B^A(\pi_S)$, where $\pi_S : B \rightarrow S \leq Y$ is the projection onto S , are surjective.

Lemma 4.4. *The morphism $\hat{\pi}$ of Corollary 4.2 is onto if and only if*

$$\hat{\pi}_S : {}_B \langle \mathcal{E} \rangle_B \longrightarrow S \otimes_B A$$

is onto for all $S \in \mathcal{S}$.

Proof. If π is surjective, then so is π_S . Since the induction functor ind_B^A is exact, the surjectivity of $\hat{\pi}$ gives the surjectivity of $\hat{\pi}_S$. If $\hat{\pi}_S$ is onto for some $S \in \mathcal{S}$ it follows from Lemma 4.3 that $\mathcal{H}(S) \otimes_B A$ lies in the B - B -bimodule generated by the image of $\hat{\pi}_S$. Hence if $\hat{\pi}_S$ is surjective for all $S \in \mathcal{S}$, then $\hat{\pi}$ is surjective, too. \square

Let u denote the one in A and B . For any $S \in \mathcal{S}$ we may embed S in $S \otimes_B A$ via the map $s \mapsto s \otimes u$ and identify S with the corresponding subspace of $S \otimes_B A$.

Lemma 4.5. *Let $\mathcal{C} := {}_B \langle \mathcal{E} \rangle_B$. Then we have $\hat{\pi}_S(\mathcal{C}) = S \otimes_B \mathcal{C}$. In particular the B -module morphism $\hat{\pi}_S$ is an epimorphism if and only if the full image under right multiplication of an arbitrary $0 \neq v \in S$ by elements of \mathcal{C} is the whole module $S \otimes_B A$.*

Proof. By the definition of $\hat{\pi}_S$ we have $\hat{\pi}_S(a) = \pi_S(u) \otimes a = \pi_S(u) \otimes u \cdot a$ for any $a \in A$. Hence for the image of \mathcal{C} under $\hat{\pi}_S$ we get $\hat{\pi}_S(\mathcal{C}) = \pi_S(u) \otimes u \cdot \mathcal{C} = S \otimes_B \mathcal{C}$, since S is simple and B is included in \mathcal{C} . Due to the simplicity of S we may furthermore choose any nonzero vector $0 \neq v \in S$ to obtain the same result, i.e. $v \otimes u \cdot \mathcal{C} = \pi_S(u) \otimes u \cdot \mathcal{C}$. \square

We can now give our main theorem by combining the results we have achieved so far.

Theorem 4.6 (Main theorem). *Let $B \leq C \leq A$ be finite-dimensional F -algebras. Furthermore let S be a set of representatives of the isotypes of simple B -modules and let $- \otimes_B A$ be exact. Then the following holds: For every $S \in S$ we have $\dim_F(S \otimes_B C) = \dim_F(S \otimes_B A)$ if and only if $C = A$.*

Proof. As C contains B it is a natural B - B -bimodule and by Lemma 4.5 the image of C under $\hat{\pi}_S$ is $S \otimes_B C$ for any $S \in S$. The equality of dimensions forces $S \otimes_B C = S \otimes_B A$. Hence Lemma 4.4 gives that the hypothesis of Corollary 4.2 is met and we have $C = A$ as claimed. The converse is clear. \square

In order to apply Theorem 4.6 for condensation purposes we will now disband our general hypothesis and focus on condensed algebras. Therefore let H be some subgroup of G and we choose A and B above as $A := eFGe$ and $B := eFHe$ for some idempotent $e \in FH$.

Fortunately in this special situation the induction functor ind_{eFHe}^{eFGe} is exact, as $eFGe$ is a projective $eFHe$ -module. This follows from the fact that condensation preserves projectivity as follows.

Lemma 4.7. *A projective FG -module Q condenses to a projective $eFGe$ -module if for every indecomposable direct summand $P = \varepsilon FG$ of Q the condensation idempotent e possesses a decomposition into primitive orthogonal idempotents of which the idempotent ε is a summand. In particular the projective cover P of a simple FG -module S condenses to a projective $eFGe$ -module if $Se \neq 0$.*

Proof. Let $P = \varepsilon FG$ as above. By decomposing e into a set of primitive orthogonal idempotents which contains ε we obtain $e\varepsilon = \varepsilon e = \varepsilon$. Thus $Pe = \varepsilon FG e = e\varepsilon e FG e$ is projective. For the second claim we observe that the condensed module Se is nonzero if and only if $\text{Hom}_{FG}(eFG, S) \neq 0$ which is equivalent to P being isomorphic to a direct summand of eFG . \square

From Lemma 4.7 we may readily deduce the following corollary.

Corollary 4.8. *The condensed algebra $eFGe$ is a projective (left) $eFHe$ -module.*

Proof. Consider the projective FG -module generated by e . Since restriction preserves projectivity it is also projective as an FH -module. Hence by Lemma 4.7 the corresponding condensed module is a projective $eFHe$ -module. \square

From Lemma 4.5, Theorem 4.6, and Corollary 4.8 we now obtain a criterion for generation.

Corollary 4.9 (A criterion for generation). *Let $H \leq G$ and $e \in FH$ an idempotent. Furthermore let $\mathcal{N} \subseteq eFHe$ be a set of generators for $eFHe$ and S a set of representatives of the isotypes of simple $eFHe$ modules. Then the algebra C generated by a subset $\mathcal{E} \subseteq eFGe$ together with \mathcal{N} is equal to $eFGe$, if and only if for every $S \in S$ the C -module generated by an arbitrary $0 \neq v \in S$ has the same dimension as the induced module $S \otimes_{eFHe} eFGe$.*

For a practical application of Corollary 4.9 we need to know how to apply the induction functor ind_{eFHe}^{eFGe} to condensed modules.

Remark 4.10. Let V be an FH -module and $e \in FH$ an idempotent. Then we have

$$Ve \otimes_{eFHe} eFG e \cong \text{uncond}_{eFHe}^{FH}(Ve) \otimes_{FH} FGe,$$

where $\text{uncond}_{eFHe}^{FH} := - \otimes_{eFHe} eFH$ is the *uncondensation functor*.

Proof. Using the isomorphism $eFG e \cong eFH \otimes_{FH} FGe$ we may write the left-hand side as $(Ve \otimes_{eFHe} eFH) \otimes_{FH} FGe$. \square

Therefore we may induce a condensed module by uncondensing it first, then inducing it and finally condensing it again.

Unfortunately the uncondensation functor is in general not the left inverse of the condensation functor (see [2, Section 6.2] for details) and therefore we cannot hope to reclaim a module from its condensed module. In fact in practice we must avoid to apply $\text{uncond}_{eFHe}^{FH}$, as this would force us to work with modules whose dimensions generally exceed the computationally possible. However, if the idempotent e lies in the center of FH or it is faithful on FH , i.e. no simple FH -module condenses to the zero module, we have the following lemma.

Lemma 4.11. *If e is faithful on FH or e is central in FH then condensation commutes with induction, i.e. we have $\text{cond}_{eFG e}^{FG} \circ \text{ind}_{FH}^{FG} = \text{ind}_{eFHe}^{eFG e} \circ \text{cond}_{eFHe}^{FH}$.*

Proof. If e is faithful on FH then condensation induces a Morita-equivalence between FH and $eFHe$ and the composition $\text{uncond}_{eFHe}^{FH} \circ \text{cond}_{eFHe}^{FH}$ is isomorphic to the identity functor id_{FH} . Thus we have $\text{uncond}_{eFHe}^{FH}(Ve) \cong V$ for any FH -module V and Remark 4.10 gives $Ve \otimes_{eFHe} eFG e \cong V \otimes_{FH} FGe$. If e lies in the center of FH then every $eFHe$ -module is an FH -module and we obtain $Ve \otimes_{eFHe} eFG e \cong Ve \otimes_{FH} FGe$. Now since in both cases $- \otimes_{FH} FGe = \text{cond}_{eFG e}^{FG} \circ \text{ind}_{FH}^{FG}$ the commutation of induction and condensation is evident. \square

A practical implementation of Corollary 4.9 is now straightforward. In order to determine the dimension of the \mathcal{C} -module that is generated by some nonzero vector v of S embedded into $S \otimes_{eFHe} eFG e$, we need the matrices giving the elements of $\mathcal{N} \cup \mathcal{E}$ in their representation on this module. Under the hypothesis of Lemma 4.11 we may equivalently determine their images under a representation on the condensed induced module $\hat{S} \uparrow^G e$, where \hat{S} is a simple FH -module which condenses to S . Thus to determine the dimension of $\langle v \rangle_{\mathcal{C}}$ we need only compute the action of the assumed generators on $\hat{S} \uparrow^G e$ and apply the MEATAXE's spinning algorithm.

In practical applications we choose a condensation subgroup $K \leq H$ and a linear idempotent $e := e_{\lambda}$. In these cases the condensation of induced modules can be done efficiently, i.e. we may obtain representing matrices for the prospective generators of $eFG e$ without computing the induced module first. Confer [7] on how to condense induced representations in the special case $e = e_1$, i.e. λ is the trivial character of K . A description of the necessary algorithms to condense with an arbitrary linear idempotent may be found in [8] and they will also be published in a forthcoming GAP-package.

If the characters and representations of FH are known, the dimension of the induced module $\hat{S} \uparrow^G e$ can be determined by character theoretic means with relative ease: Let V be any FG -module. Then as $Ve_{\lambda} \cong \text{Hom}_{FG}(e_{\lambda} FG, V)$ we get $\dim_F Ve_{\lambda} = \dim_F \text{Hom}_{FK}(\Lambda, V \downarrow_K)$ by Frobenius reciprocity. Since FK is a semi-simple algebra the dimension of the condensed module

Ve_λ can therefore be computed by the ordinary character theoretic scalar product of λ and the restriction of the Brauer character afforded by V to K .

We would like to end this section with two concluding remarks. For one the condition that the idempotent e is faithful on FH is rather strong as this implies it is also faithful on FG .

Remark 4.12. If e is faithful on FH then it is also faithful on FG .

Proof. By [5, Theorem 3.2.3] for any finite-dimensional F -algebra A an idempotent $\varepsilon \in A$ is faithful if and only if the twosided ideal generated by ε is the algebra A itself. Therefore we have $FHeFH = FH$ and multiplying both sides by FG gives $FGeFG = FG$. \square

Secondly, the criterion presented here generalizes a criterion for generation which was previously discovered by Markus Wiegmann.

Remark 4.13. Our criterion in Corollary 4.9 contains as a special case an older criterion for generation that has been known for some time (see [13, Theorem 2.4.5]). In this case $e = e_1$, i.e. λ is the trivial character of K , the condensation subgroup K is normal in H and H/K is p -group. Therefore to verify a set as a generating set it is only necessary to determine the dimension of the \mathcal{C} -module which is generated by the trivial $eFHe$ -module.

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