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Krull dimension of bimodules

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Abstract

Let R and S be rings and let M be a left S-, right R-bimodule such that M has Krull dimension both as a left S-module and as a right R-module and M/N has the same Krull dimension as a left S- and as a right R-module for every sub-bimodule N of M. Suppose that M is a finitely generated left S-module. Then it is proved that the Krull dimension of the right R-module M is the Krull dimension of a k-critical right R-module M/K for some prime submodule K of the right R-module M.

Keywords: Artinian rings and modules; Noetherian rings and modules; Krull dimension; Bimodules

1. Introduction

In this note all rings are associative with identity and, unless stated otherwise, all modules are unital right modules. Let R be a ring and let M be an R-module. We shall denote the *annihilator* of M in R by $\mathrm{ann}_R(M)$, i.e. $\mathrm{ann}_R(M) = \{r \in R \mid Mr = 0\}$. The module M is called *prime* if $M \neq 0$ and $\mathrm{ann}_R(N) = \mathrm{ann}_R(M)$ for every non-zero submodule N of M. A submodule K of an arbitrary R-module M will be called a *prime submodule* of M provided the R-module M/K is prime. By a *prime right ideal* of R we mean a prime submodule of the right R-module R. For example, an ideal P of R is a prime right ideal if and only if P is a prime ideal of R. A submodule L of a module M is called *irreducible* if M/L is a uniform module, i.e. $L \neq M$

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and whenever L_1 and L_2 are submodules of M such that $L = L_1 \cap L_2$ then either $L = L_1$ or $L = L_2$.

For the definition and properties of Krull dimension see [6]. Let M be an R-module. If M has Krull dimension then $\mathrm{k}(M_R)$, or simply $\mathrm{k}(M)$, will denote the Krull dimension of M. If S is also a ring and M is a left S-, right R-bimodule we shall say that M *has Krull dimension* in case the left S-module M has Krull dimension and the right R-module M has Krull dimension, and in this case $\mathrm{k}({}_SM)$ and $\mathrm{k}(M_R)$ will denote these Krull dimensions.

It is well known that a commutative ring R is Artinian if and only if R is Noetherian and every prime ideal of R is maximal (see, for example, [1, Theorem 4.15 and Proposition 4.20]). However, the first Weyl algebra $A_1(\mathbb{C})$ is a simple right and left Noetherian ring which is not Artinian (see [1, Corollary 2.2]). It is also well known that a ring R is right Artinian if and only if R is right Noetherian and R/P is a right Artinian ring for every prime ideal P of R. In [4, Theorem 3.6] Lambek and Michler prove that a ring is right Artinian if and only if it is right Noetherian and every irreducible prime right ideal is maximal. The purpose of this note is to investigate this theorem of Lambek and Michler. We begin with a simple observation about the Krull dimension of prime modules.

Lemma 1. *Let R be any ring and let M be a prime R-module with Krull dimension. Then* $\mathrm{k}(M) = \mathrm{k}(M/L)$ *for some irreducible prime submodule L of M.*

Proof. By [6, Lemma 6.2.6], M has finite uniform dimension and by [7, Corollary 2.4], $0 = L_1 \cap \cdots \cap L_n$ for some positive integer n and irreducible prime submodules L_i $(1 \leqslant i \leqslant n)$ of M. Clearly M embeds in the module $(M/L_1) \oplus \cdots \oplus (M/L_n)$. By [6, Lemma 6.2.4], $\mathrm{k}(M) = \mathrm{k}(M/L_i)$ for some $1 \leqslant i \leqslant n$. □

Corollary 2. *Let R be a ring with right Krull dimension. Then* $\mathrm{k}(R) = \mathrm{k}(R/A)$ *for some irreducible prime right ideal A of R.*

Proof. By [6, Corollary 6.3.8], there exists a prime ideal P of R such that $\mathrm{k}(R_R) = \mathrm{k}((R/P)_R)$. The result follows by Lemma 1. □

In particular, Corollary 2 shows that if every irreducible prime right ideal is maximal then $\mathrm{k}(R_R) = 0$, i.e. R is right Artinian (cf. [4, Theorem 3.6]). Corollary 2 and [4, Theorem 3.6] fail spectacularly for right modules, as the following example shows.

Example 3. For each ordinal $\alpha \geqslant 1$ there exists a right Noetherian PI algebra R and a cyclic projective uniform R-module M such that every prime submodule of M is maximal but $\mathrm{k}(M) = \alpha$.

Proof. Let $\alpha \geqslant 1$ be an ordinal. By [3, Theorem 9.8], for any field F, there exists a commutative Noetherian F-algebra domain S such that $\mathrm{k}(S) = \alpha$. Let R be the matrix ring

$$R = \begin{pmatrix} F & S \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \,\middle|\, a \in F,\ b, c \in S \right\}.$$

Then R is a right (but not left) Noetherian ring.

Let M be the R-module

$$M = \begin{pmatrix} F & S \\ 0 & 0 \end{pmatrix}$$

and note that M is cyclic, projective and uniform. If K is a prime submodule of M then

$$K = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$$

so $M/K \cong F$ is simple. Thus every prime submodule of the R-module M is maximal. Clearly $\mathrm{k}(M) = \mathrm{k}(S) = \alpha$. $\square$

We shall prove that for certain bimodules we can recover and in fact improve on Corollary 2. Specifically, one consequence of our result is that, if S and R are rings and ${}_SM_R$ is a left S-, right R-bimodule having Krull dimension such that $\mathrm{k}({}_S(M/N)) = \mathrm{k}((M/N)_R)$ for all sub-bimodules N of ${}_SM_R$ and if, moreover, M is a finitely generated left S-module then $\mathrm{k}(M_R) = \mathrm{k}((M/K)_R)$ for some irreducible prime submodule K of M_R.

2. Krull dimension of bimodules

Let R and S be rings. A left S-, right R-bimodule M will be called *Noetherian* if M is Noetherian both as a left S-module and as a right R-module. We shall call a (not necessarily Noetherian) bimodule ${}_SM_R$ *Krull symmetric* if M has Krull dimension (i.e. ${}_SM$ and M_R both have Krull dimension) and $\mathrm{k}({}_S(M/N)) = \mathrm{k}((M/N)_R)$ for all sub-bimodules N of ${}_SM_R$. No example is known of a Noetherian bimodule which is not Krull symmetric (see [6, 6.4.11]). Note that a Noetherian bimodule ${}_SM_R$ is Krull symmetric in case M is Artinian either as a left S-module or as a right R-module [6, 4.1.6]. Moreover, if S is a left FBN ring and R is a right FBN ring then any Noetherian bimodule ${}_SM_R$ is Krull symmetric by [6, 6.4.13]. For more information on Krull symmetric bimodules see [1, Appendix 9, p. 323].

Following [2], an R-module M will be called *cocritical* provided M is non-zero and there exists an hereditary torsion theory τ on Mod-R such that M is τ-torsion-free but M/N is τ-torsion for every non-zero submodule N of M. Cocritical modules are discussed in [2, Section 18]. In [4] a right ideal A of R is called *critical* provided the cyclic R-module R/A is cocritical. Let M be a module with Krull dimension. Then M will be called *k-critical* provided M is non-zero and $\mathrm{k}(M/N) < \mathrm{k}(M)$ for every non-zero submodule N of M. Note that every k-critical module M is cocritical with respect to the hereditary torsion theory cogenerated by the injective hull $\mathrm{E}(M)$ of M.

In this section we shall prove that, for any rings R and S and for any Krull symmetric left S-, right R-bimodule M such that the left S-module M is finitely generated and the right R-module M is finitely generated or has finite Krull dimension, $\mathrm{k}(M_R) = \mathrm{k}((M/K)_R)$ for some prime submodule K of M_R such that $(M/K)_R$ is k-critical. We require a number of lemmas.

Lemma 4. *Let R be a ring, let M be an R-module and let K be a submodule of M. Then K is a prime submodule of M if and only if $P = \mathrm{ann}_R(M/K)$ is a prime ideal of R and L/K is a faithful (R/P)-module for every submodule L of M properly containing K.*

Proof. See [5, Proposition 1.1]. $\square$

Corollary 5. *Let R be a ring and let M be an R-module with Krull dimension such that* $\mathrm{k}(M) = \mathrm{k}(M/K)$ *for some prime submodule K of M. Then* $\mathrm{k}(M) = \mathrm{k}(M/MP)$ *for some prime ideal P of R.*

Proof. Let $P = \mathrm{ann}_R(M/K)$. Then P is a prime ideal of R such that $MP \subseteq K$, by Lemma 4. Thus

$$\mathrm{k}(M) = \mathrm{k}(M/K) \leqslant \mathrm{k}(M/MP) \leqslant \mathrm{k}(M),$$

by [6, Lemma 6.2.4]. □

We mention Corollary 5 because our strategy in proving that $\mathrm{k}(M) = \mathrm{k}(M/K)$ for a certain R-module M and prime submodule K of M is to first prove that $\mathrm{k}(M) = \mathrm{k}(M/MP)$ for some prime ideal P of R.

Lemma 6. *Let S and R be rings and let ${}_SM_R$ be a Krull symmetric bimodule. Let A and B be ideals of R. Then*

$$\mathrm{k}\big((M/MAB)_R\big) = \sup\big\{\mathrm{k}\big((M/MA)_R\big), \mathrm{k}\big((M/MB)_R\big)\big\}.$$

Proof. Let $b \in B$. Define a mapping $\varphi : M/MA \to (Mb + MAB)/MAB$ by $\varphi(m + MA) = mb + MAB$ for all $m \in M$. Then φ is well defined and is a left S-epimorphism. It follows, by [6, Lemma 6.2.4], that $\mathrm{k}({}_S((Mb + MAB)/MAB)) \leqslant \mathrm{k}({}_S(M/MA))$. Now

$$MB/MAB = \sum_{b \in B} \big((Mb + MAB)/MAB\big),$$

so that, by [6, Lemma 6.2.17], $\mathrm{k}({}_S(MB/MAB)) \leqslant \mathrm{k}({}_S(M/MA))$. Again using [6, Lemma 6.2.4] we have

$$\mathrm{k}\big({}_S(M/MAB)\big) \leqslant \sup\big\{\mathrm{k}\big({}_S(M/MA)\big), \mathrm{k}\big({}_S(M/MB)\big)\big\}.$$

But ${}_SM_R$ is Krull symmetric, so that we have

$$\mathrm{k}\big((M/MAB)_R\big) \leqslant \sup\big\{\mathrm{k}\big((M/MB)_R\big), \mathrm{k}\big((M/MA)_R\big)\big\}.$$

The result follows since M/MA and M/MB are both isomorphic to factor modules of the right R-module M/MAB. □

Given a ring R and an R-module M, the *singular submodule* of M, denoted by $\mathrm{Z}(M_R)$, is the set of elements of M which are annihilated by essential right ideals of R, i.e. $\mathrm{Z}(M_R) = \{m \in M \mid mE = 0$ for some essential right ideal E of $R\}$. A module M is called *singular* if $\mathrm{Z}(M_R) = M$ and *non-singular* if $\mathrm{Z}(M_R) = 0$.

Lemma 7. *Let S be a ring, let R be a prime right Goldie ring and let ${}_SM_R$ be a bimodule such that ${}_SM$ is finitely generated and M_R is faithful. Let $\mathrm{Z} = \mathrm{Z}(M_R)$. Then Z is a prime submodule of M_R and M/Z is a faithful right R-module.*

Proof. Suppose that $M = Z$. There exist a positive integer n and elements $m_i \in M$ $(1 \leqslant i \leqslant n)$ such that $M = Sm_1 + \cdots + Sm_n$. For each $1 \leqslant i \leqslant n$ there exists an essential right ideal E_i of R such that $m_i E_i = 0$. Then $M(E_1 \cap \cdots \cap E_n) = 0$ and hence $E_1 \cap \cdots \cap E_n = 0$, a contradiction. Thus $M \neq Z$. The right R-module M/Z is non-singular and hence, by [1, Lemma 7.22] and Lemma 4, Z is a prime submodule of M_R and, moreover, $(M/Z)_R$ is faithful. □

Lemma 8. *Let S and R be rings and let ${}_SM_R$ be a bimodule such that ${}_SM$ is finitely generated and M_R is faithful and has Krull dimension. Then R has right Krull dimension and* $\mathrm{k}(R_R) = \mathrm{k}(M_R)$.

Proof. See [6, Lemma 6.3.12]. □

Lemma 9.

(i) *Every cocritical module is uniform.*
(ii) *Every non-singular uniform module is cocritical.*

Proof. Clear. □

Lemma 10. *Let M be a non-zero (prime) non-singular module with finite uniform dimension. Then there exist a positive integer n and submodules L_i $(1 \leqslant i \leqslant n)$ of M such that $0 = L_1 \cap \cdots \cap L_n$ and M/L_i is a (prime) non-singular uniform module for each $1 \leqslant i \leqslant n$.*

Proof. There exist a positive integer n and uniform submodules U_i $(1 \leqslant i \leqslant n)$ of M such that $U_1 \oplus \cdots \oplus U_n$ is an essential submodule of M. For each $1 \leqslant i \leqslant n$, Zorn's Lemma gives a submodule L_i of M maximal with the properties $\bigoplus_{j \neq i} U_j \subseteq L_i$ and $L_i \cap U_i = 0$. It is easy to check that $(L_1 \cap \cdots \cap L_n) \cap (U_1 \oplus \cdots \oplus U_n) = 0$ and hence $L_1 \cap \cdots \cap L_n = 0$. Next, the choice of L_i implies that M/L_i is uniform for all $1 \leqslant i \leqslant n$. Moreover, for each $1 \leqslant i \leqslant n$, $(U_i \oplus L_i)/L_i \cong U_i \leqslant M$ so that $(U_i \oplus L_i)/L_i$ is non-singular and hence M/L_i is non-singular.

Now suppose that, in addition, M is a prime module. By [7, Lemmas 2.1 and 2.2], M/L_i is a prime module for all $1 \leqslant i \leqslant n$. □

Lemma 11. *Let R be a prime ring with right Krull dimension and let M be a non-singular uniform right R-module with Krull dimension. Suppose further that M is finitely generated or has finite Krull dimension. Then M is k-critical.*

Proof. By [6, Proposition 6.3.5], R is a right Goldie ring and by [1, Corollary 7.25], $\mathrm{k}(M) = \mathrm{k}(R_R)$. Suppose that M is a finitely generated module. For any non-zero submodule N of M, M/N is a finitely generated singular module and hence $\mathrm{k}(M/N) < \mathrm{k}(R_R)$ by [6, Proposition 6.3.11]. Thus M is k-critical.

Now suppose that $\mathrm{k}(M)$ is finite. Let N be a non-zero submodule of M. For any $m \in M$, $(mR + N)/N$ is a cyclic singular module. Again by [6, Proposition 6.3.11], $\mathrm{k}((mR + N)/N) < \mathrm{k}(R_R)$. By [6, Lemma 6.2.17], $\mathrm{k}(M/N) < \mathrm{k}(R_R)$. It follows again that M is k-critical. □

Lemmas 10 and 11 have the following interesting consequence.

Corollary 12. *Let R be a ring with right Krull dimension. Then* $\mathrm{k}(R) = \mathrm{k}((R/A)_R)$ *for some right ideal A of R such that R/A is a prime k-critical right R-module.*

Proof. By [3, Corollary 7.5], $\mathrm{k}(R)=\mathrm{k}(R/P)$ for some prime ideal P of R. Thus we may assume that R is a prime ring. By Lemma 10, it follows that $\mathrm{k}(R)=\mathrm{k}((R/A)_R)$ for some right ideal of A of R such that R/A is a prime uniform non-singular right R-module. But then, by Lemma 11, R/A is a k-critical right R-module. $\square$

Theorem 13. *Let S and R be rings and let ${}_SM_R$ be a Krull symmetric bimodule such that the left S-module M is finitely generated. Then $\mathrm{k}(M_R)=\mathrm{k}(M'_R)$ for some prime cocritical homomorphic image M' of the right R-module M.*

If, in addition, the right R-module M is finitely generated or has finite Krull dimension then $\mathrm{k}(M_R)=\mathrm{k}(M'_R)$ for some prime k-critical homomorphic image M' of the right R-module M.

Proof. Let $A=\mathrm{ann}_R(M)$. By Lemma 8, the ring R/A has right Krull dimension. By [6, Theorem 6.3.7, Corollary 6.3.8(ii)], there exist an integer $t\geqslant 1$ and prime ideals P_i $(1\leqslant i\leqslant t)$ of R such that $P_1\cdots P_t\subseteq A\subseteq P_1\cap\cdots\cap P_t$ and hence $MP_1\cdots P_t=0$. By Lemma 6, $\mathrm{k}(M_R)=\mathrm{k}((M/MP_i)_R)$ for some $1\leqslant i\leqslant t$.

Suppose that $\mathrm{k}((M/MP_i)_R)\neq\mathrm{k}((M/K)_R)$ for all prime submodules K of M_R such that $(M/K)_R$ is cocritical. By [3, Theorem 7.1], the ring R/A has the ascending chain condition on prime ideals. Thus, there exists a prime ideal P of R maximal such that $\mathrm{k}((M/MP)_R)\neq\mathrm{k}((M/K)_R)$ for all prime submodules K of M_R such that $(M/K)_R$ is cocritical. Let $B=\mathrm{ann}_R(M/MP)$. Then B is an ideal of R and $P\subseteq B$. Suppose that $P\neq B$. By the above argument, $\mathrm{k}((M/MP)_R)=\mathrm{k}((M/MQ)_R)$ for some prime ideal Q of R containing B. The maximal choice of P implies that $\mathrm{k}((M/MP)_R)=\mathrm{k}((M/MQ)_R)=\mathrm{k}((M/K)_R)$ for some prime submodule K of M_R such that $(M/K)_R$ is cocritical, a contradiction. Thus $P=B$ and hence M/MP is a faithful right (R/P)-module. Note that R/P is a prime right Goldie ring. Let Z denote the submodule of M containing MP such that Z/MP is the singular submodule of the right (R/P)-module M/MP. By Lemma 7, it follows that Z is a prime submodule of M_R and M/Z is a faithful right (R/P)-module. Now Lemma 8 gives

$$\mathrm{k}\big((M/MP)_{R/P}\big)=\mathrm{k}\big((R/P)_{R/P}\big)=\mathrm{k}\big((M/Z)_{R/P}\big)$$

and hence $\mathrm{k}((M/MP)_R)=\mathrm{k}((M/Z)_R)$. By Lemmas 10 and 9(ii), there exist a positive integer n and submodules L_i $(1\leqslant i\leqslant n)$ of M such that $Z=L_1\cap\cdots\cap L_n$ and M/L_i is a prime cocritical module for each $1\leqslant i\leqslant n$. Then there is an embedding of right R-modules $M/Z\hookrightarrow(M/L_1)\oplus\cdots\oplus(M/L_n)$, and so $\mathrm{k}((M/MP)_R)=\mathrm{k}((M/Z)_R)=\mathrm{k}((M/L_i)_R)$ for some $1\leqslant i\leqslant n$, a contradiction. It follows that $\mathrm{k}(M_R)=\mathrm{k}((M/MP_i)_R)=\mathrm{k}((M/K)_R)$ for some prime submodule K of M_R such that $(M/K)_R$ is cocritical.

If, in addition, the right R-module M is finitely generated or has finite Krull dimension then, using Lemma 11, the result follows similarly, with the supposition for contradiction being that $\mathrm{k}((M/MP_i)_R)\neq\mathrm{k}((M/K)_R)$ for all prime submodules K of M_R such that $(M/K)_R$ is k-critical. $\square$

Theorem 13 has the following immediate consequence.

Corollary 14. *Let R be a commutative ring and let M be a finitely generated R-module with Krull dimension. Then $\mathrm{k}(M)=\mathrm{k}(M')$ for some prime k-critical homomorphic image M' of M.*

Another consequence of Theorem 13 is the following result.

Corollary 15. *Let R be a right FBN ring, let S be a left FBN ring and let ${}_SM_R$ be a bimodule such that ${}_SM$ and M_R are both finitely generated. Then $\mathrm{k}(M_R) = \mathrm{k}(M'_R)$ for some prime k-critical homomorphic image M' of the right R-module M.*

Proof. By Theorem 13 and [6, Proposition 6.4.13]. □

3. Artinian bimodules

We now present some results concerning Artinian bimodules. Our first result gives a bimodule analogue of the Lambek–Michler result [4, Theorem 3.6]. Compare with Theorem 13. Recall that an element c of a ring R is called *regular* if $rc \neq 0$ and $cr \neq 0$ for every non-zero element r of R.

Theorem 16. *Let R and S be rings and let ${}_SM_R$ be a Noetherian bimodule. Then the right R-module M is Artinian if and only if every irreducible prime submodule of M_R is maximal.*

Proof. ($\Rightarrow$) Let $A = \mathrm{ann}_R(M)$. By Lemma 8, R/A is a right Artinian ring. Let L be an irreducible prime submodule of M_R. If $P = \mathrm{ann}_R(M/L)$ then P is a prime ideal of R and $A \subseteq P$, so that R/P is a simple Artinian ring. It follows that M/L is a simple R-module.

($\Leftarrow$) Suppose that every irreducible prime submodule of the right R-module M is maximal. By Lemma 1, M/K is an Artinian right R-module for every prime submodule K of M_R. Suppose that M is not an Artinian right R-module. Because M_R (or ${}_SM$) is Noetherian, we can suppose without loss of generality that M/N is an Artinian right R-module for every non-zero sub-bimodule N of ${}_SM_R$. In addition, we can suppose without loss of generality that M is a faithful right R-module. Let A and B be non-zero ideals of R. Because M_R is faithful, we have $MA \neq 0$ and $MB \neq 0$. By hypothesis, M/MA and M/MB are Artinian right R-modules. Using the method of the proof of Lemma 6 and Lenagan's Theorem [6, 4.1.6], it follows that M/MAB is an Artinian right R-module. This implies that $AB \neq 0$. It follows that R is a prime ring. Moreover, because ${}_SM$ is finitely generated and M_R is faithful Noetherian, R is a right Noetherian ring.

Let $T = \mathrm{Z}(M_R)$. By Lemma 7, T is a prime submodule of M_R and M/T is a faithful right R-module. By Lemma 8, R is a right Artinian ring. But M is a finitely generated right R-module and hence is Artinian, a contradiction. □

We now aim to extend Theorem 16. We first require a lemma.

Lemma 17. *Let R be any ring. Let $X = X_1 \oplus \cdots \oplus X_n$ be a finite direct sum of R-modules such that X_i/K is simple for all irreducible prime submodules K of X_i for all $1 \leqslant i \leqslant n$. Then X/L is simple for all irreducible prime submodules L of X.*

Proof. Let L be any irreducible prime submodule of X. Let $1 \leqslant i \leqslant n$. Then $X_i/(L \cap X_i) \cong (X_i + L)/L \leqslant X/L$ so that $X_i/(L \cap X_i)$ is either zero or a uniform prime module, and hence is simple. It follows that $X/(\bigoplus_{i=1}^n (L \cap X_i))$ is semisimple and hence so too is X/L. Clearly X/L is, in fact, simple. □

Theorem 18. *Let R and S be rings and let ${}_SM_R$ be a Noetherian bimodule. Let N be a submodule of M_R. Then N is Artinian if and only if every irreducible prime submodule of N is maximal.*

Proof. First consider the sub-bimodule SN of ${}_SM_R$. Now $(SN)_R$ is finitely generated, so $SN = s_1N + \cdots + s_kN$ for some positive integer k and elements s_i $(1 \leqslant i \leqslant k)$ of S. Define a map $\varphi : N^{(k)} \to SN$ by $\varphi(n_1, \dots, n_k) = s_1n_1 + \cdots + s_kn_k$ for all $n_i \in N$ $(1 \leqslant i \leqslant k)$. Then φ is an epimorphism from the right R-module $N^{(k)}$ to the right R-module SN.

Now suppose that N is an Artinian R-module. Then SN is an Artinian R-module. Let $A = \mathrm{ann}_R(N)$ and note that $A = \mathrm{ann}_R(SN)$. By Lemma 8, R/A is a right Artinian ring. Let L be an irreducible prime submodule of N. If $P = \mathrm{ann}_R(N/L)$ then P is a prime ideal of R and $A \subseteq P$ so that R/P is a simple Artinian ring. It follows that N/L is a simple R-module.

Conversely, suppose that every irreducible prime submodule of N is maximal. By Lemma 17, every irreducible prime submodule of $(SN)_R$ is maximal. But ${}_S(SN)_R$ is a Noetherian bimodule. By Theorem 16, $(SN)_R$ is Artinian. Thus N_R is Artinian. □

Corollary 19. *Let R be a (right and left) Noetherian ring and let M be an R-module which embeds in a finitely generated free R-module. Then M is Artinian if and only if every irreducible prime submodule of M is maximal.*

Proof. Suppose M embeds in a finitely generated free R-module F. Suppose $F \cong R_R^{(n)}$ for some positive integer n. Let $S = \mathrm{End}(F_R) \cong \mathrm{End}(R_R^{(n)}) \cong M_n(R)$ (see [6, 1.1.1]). Then S is a right and left Noetherian ring [6, Proposition 1.1.2] and F is a finitely generated left S-module. Now $M \hookrightarrow F_R$, that is M is isomorphic to a right R-submodule M'_R of the left S-, right R-bimodule ${}_SF_R$. Note that both ${}_SF$ and F_R are Noetherian. The result follows by Theorem 18. □

Note in particular that Corollary 19 applies in case M is a finitely generated projective R-module or a right ideal of R (compare Example 3 and note that the ring R in Example 3 is not left Noetherian). The next result gives an application of Corollary 19.

Corollary 20. *Let R be a semiprime Noetherian ring and let M be a finitely generated non-singular R-module. Then M is Artinian if and only if every irreducible prime submodule of M is maximal.*

Proof. A result of Gentile and Levy [1, Proposition 7.19] says that such a module can be embedded in a finitely generated free R-module. The result follows by Corollary 19. □

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