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Character tables of parabolic subgroups of Steinberg's triality groups ${}^3D_4(2^n)$

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Abstract

We compute the conjugacy classes of elements and the character tables of the parabolic subgroups of Steinberg's triality groups ${}^3D_4(2^n)$.

Keywords: Character tables; Parabolic subgroups; Steinberg's triality groups

1. Introduction

Let ${}^3D_4(q)$ be Steinberg's simple triality group defined over a finite field with $q = p^n$ elements, where p is a prime number and n is a positive integer. The character table of ${}^3D_4(q)$ was computed by N. Spaltenstein, D.I. Deriziotis and G.O. Michler in [10] and [2]. In [6], the character tables of the proper parabolic subgroups of ${}^3D_4(q)$ were calculated for odd q.

In this paper we extend the results in [6] to even q, i.e., we construct the irreducible characters of the proper parabolic subgroups of ${}^3D_4(2^n)$ using methods similar to those in [6]. This completes the task of computing the character tables of the parabolic subgroups of Steinberg's triality groups ${}^3D_4(q)$ for all prime powers q. For the calculations we use computer programs written by C. Köhler and the author in the language of the GAP [3] and Maple [1] part of CHEVIE [5].

The results of this paper are used in the verification of Dade's conjecture and the Isaacs–Malle–Navarro version of McKay's conjecture for Steinberg's triality groups (see [7]). The

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character tables might also be helpful in getting new information on the decomposition numbers of ${}^3D_4(2^n)$ in non-defining characteristic along the same line as in [11].

2. Notation and group theoretical properties of ${}^3D_4(2^n)$

We use exactly the same notation as in [6, Section 2] with the only difference that $q = 2^n$ is now always a power of 2. Since we are working in characteristic 2 we do not have to deal with the signs of the structure constants N_{rs}. The relations in Tables 2.2–2.4 of [4] and in Table 2.1 of [6] still hold for even q, when one ignores the signs. So we can use these relations for computations in ${}^3D_4(2^n)$.

3. The conjugacy classes of the parabolic subgroups

We define parabolic subgroups B, P and Q of $G := {}^3D_4(2^n)$ in the same way as in Sections 4–6 of [6]. So B is a Borel subgroup of order $q^{12}(q^3-1)(q-1)$, and P and Q are maximal parabolic subgroups containing B. The order of P is $|P| = q^{12}(q^6-1)(q-1)$, the order of Q is $|Q| = q^{12}(q^3-1)(q^2-1)$. Up to conjugacy, B, P, Q and G are the only parabolic subgroups of G.

The conjugacy classes of G have been determined by Deriziotis and Michler in [2]. The conjugacy classes of B, P and Q can be computed in the same way as for odd q. Representatives for the conjugacy classes of B, P and Q are given in Tables A.1–A.4, A.7, A.8, A.11 and A.12 in Appendix A.

The parameter sets I_1, I_2, I_3 and the field elements ζ, η occurring in these tables are defined as follows. Let $\mathbb{F}_{q^3}$ be the field with q^3 elements and $\mathbb{F}_q$ its subfield with q elements. We write $\mathbb{F}_{q^3}^\times$ and $\mathbb{F}_q^\times$ for the multiplicative groups of these fields. The set I_2 is the set of all $t \in \mathbb{F}_q^\times$ such that $x_\beta(1)x_{2\alpha+\beta}(1)x_{3\alpha+\beta}(t)$ is G-conjugate to $x_\beta(1)x_{3\alpha+\beta}(1)$. The set I_1 is defined as $I_1 := \mathbb{F}_q^\times - I_2$. We call two elements $a', a'' \in I' := \mathbb{F}_{q^3} - \mathbb{F}_q$ equivalent if and only if there exist $t \in \mathbb{F}_q^\times$, $y \in \mathbb{F}_q$ such that $a'' = t(a' + y)$. Then, I_3 is defined as a set of representatives for the equivalence classes of this equivalence relation on I'. We choose an element $\eta \in \mathbb{F}_q - \{u^2 + u \mid u \in \mathbb{F}_q\}$ and define $\zeta := 1 + \eta$.

The class fusions of B in G are not given explicitly in the tables in Appendix A since they can be read off from Tables A.4 and A.8.

4. The character table of the Borel subgroup B

The Borel subgroup B is the semidirect product of the maximal torus $T = \mathbf{T}^F$ and the unipotent normal subgroup $U = X_\alpha X_\beta X_{\alpha+\beta} X_{2\alpha+\beta} X_{3\alpha+\beta} X_{3\alpha+2\beta}$. In this section we compute the irreducible characters of B.

Fix a linear character $\phi' : \mathbb{F}_{q^3} \to \mathbb{C}^\times$ of the additive group of $\mathbb{F}_{q^3}$, such that ϕ' restricts nontrivially on $\mathbb{F}_q$ and such that $\{r \in \mathbb{F}_{q^3} \mid r^{q^2} + r^q + r = 0\} \subseteq \ker(\phi')$. Let ϕ be the restriction of ϕ' to $\mathbb{F}_q$. So ϕ and ϕ' satisfy Eqs. (1) and (2) in [6, p. 779] and we have $\phi'(x^q) = \phi'(x)$ for all $x \in \mathbb{F}_{q^3}$.

Theorem 4.1. *The character table of the Borel subgroup B is given by Tables* A.5 *and* A.6 *in Appendix* A.

Proof. All irreducible characters of B can be obtained by inducing linear characters from suitable subgroups. The constructions are as follows:

${}_B\chi_1(k,l)$, ${}_B\chi_2(k), \ldots, {}_B\chi_7(k)$:
The constructions of ${}_B\chi_1(k,l), \ldots, {}_B\chi_7(k)$ for odd q carry over to even q without any changes.

${}_B\chi_{12}(k)$, ${}_B\chi_{13}(k)$, ${}_B\chi_{15}(k)$:
These characters correspond to the characters ${}_B\chi_{12}(k)$, ${}_B\chi_{13}(k)$, ${}_B\chi_{17}(k)$, respectively, for odd q. The constructions of these characters carry over to even q with some slight and obvious modifications.

${}_B\chi_8$, ${}_B\chi_9$, ${}_B\chi_{10}$, ${}_B\chi_{11}$:
For $x \in \{0, \eta\}$, we consider the linear characters

$$\varphi_x\colon\ x_\beta(d_2)x_{\alpha+\beta}(d_3)x_{2\alpha+\beta}(d_4)x_{3\alpha+\beta}(d_5)x_{3\alpha+2\beta}(d_6) \mapsto \phi'(x \cdot d_2 + d_3 + d_4)$$

of $U_P := X_\beta X_{\alpha+\beta} X_{2\alpha+\beta} X_{3\alpha+\beta} X_{3\alpha+2\beta}$. The two linear characters

$$x_\alpha(\delta)x_\beta(d_2)x_{\alpha+\beta}(d_3)x_{2\alpha+\beta}(d_4)\cdots \mapsto (-1)^{k\delta}\phi'(d_3 + d_4)$$

of $\langle x_\alpha(1)\rangle U_P$ $(k = 0, 1)$ are the extensions of φ_0 to $\langle x_\alpha(1)\rangle U_P$ and the two linear characters $x_\alpha(\delta)x_\beta(d_2)x_{\alpha+\beta}(d_3)x_{2\alpha+\beta}(d_4)\cdots \mapsto (-1)^{k\delta}\phi'(\eta \cdot d_2 + d_3 + d_4)$ of $\langle x_\alpha(1)\rangle U_P$ $(k = 0, 1)$ are the extensions of φ_η to $\langle x_\alpha(1)\rangle U_P$. So, by [8, (6.17)], $(\varphi_0 + \varphi_\eta)^B = \varphi_0^B + \varphi_\eta^B$ is the sum of four characters of degree $\frac{1}{2}q^3(q^3-1)(q-1)$. Let $\varphi := \varphi_0 + \varphi_\eta$. We can compute the values of φ^B and verify $(\varphi^B, \varphi^B)_B = 4$. Hence φ^B is the sum of four different irreducible characters of degree $\frac{1}{2}q^3(q^3-1)(q-1)$. The characters ${}_B\chi_8$, ${}_B\chi_9$, ${}_B\chi_{10}$, ${}_B\chi_{11}$ are, by definition, the constituents of φ^B and their values can be determined using orthogonality relations in a way similar to that for odd q (see [6, p. 783]).

${}_B\chi_{14}(k)$, ${}_B\chi_{16}$:
Number the elements of $\mathbb{F}_q$ in some way, say $\mathbb{F}_q = \{x_1, x_2, x_3, \ldots, x_q\}$ with $x_1 = 0$. Let ${}_B\chi_{14}(k)$, $k = 1, \ldots, q$, be the character of B induced from the following linear character of U_P: $x_\beta(d_2)x_{\alpha+\beta}(d_3)x_{2\alpha+\beta}(d_4)x_{3\alpha+\beta}(d_5)x_{3\alpha+2\beta}(d_6) \mapsto \phi'(x_k \cdot d_2 + d_4 + d_5)$ and let ${}_B\chi_{16}$ be the character of B induced from the following linear character of $X_\alpha X_{2\alpha+\beta} X_{3\alpha+\beta} X_{3\alpha+2\beta}$:

$$x_\alpha(d_1)x_{2\alpha+\beta}(d_4)x_{3\alpha+\beta}(d_5)x_{3\alpha+2\beta}(d_6) \mapsto \phi'(d_1 + d_6).$$

Computing scalar products with CHEVIE, we see that we have constructed $q^4 + 2q^3 + q^2 + 5q + 3$ different irreducible characters of B. Since this number equals the number of conjugacy classes of B, the character table is complete. □

We point out that we are not able to describe all values of the characters ${}_B\chi_6(k)$ and ${}_B\chi_{14}(k)$ generically because the values of these characters on the unipotent conjugacy classes $c_{1,9}(t)$, $c_{1,11}(t)$ or $c_{1,15}(a')$ of B depend on t or a' and we do not have a generic description of these classes. For fixed q (not too large), there is no difficulty in computing the values of these characters on all conjugacy classes only using the above definition of these characters.

5. The character table of the maximal parabolic subgroup P

The maximal subgroup P is the semidirect product of the Levi complement $L_P = \langle T, X_\alpha, X_{-\alpha}\rangle$ and the unipotent radical $U_P = X_\beta X_{\alpha+\beta} X_{2\alpha+\beta} X_{3\alpha+\beta} X_{3\alpha+2\beta}$. We choose $\psi_0, \psi_1, \psi_2, \psi_3, \psi_5 \in \mathrm{Irr}(U_P)$ in the same way as in [6, p. 786]. We use the same notation for inertia subgroups as in [6]. Inspecting the proof of Proposition 5.3 in [6] we see that this proposition carries over to even q without any changes.

Theorem 5.1. *The character table of the parabolic subgroup P is given by Tables* A.9 *and* A.10 *in Appendix* A.

Proof. Following [6], we use Clifford theory to determine the irreducible characters of P:

${}_P\chi_1(k), {}_P\chi_2(k), {}_P\chi_3(k), {}_P\chi_4(k)$:
These are the inflations of the irreducible characters of L_P.

${}_P\chi_5(k), {}_P\chi_6, {}_P\chi_7(k), {}_P\chi_8(k)$:
The construction of these characters is analogous to ${}_P\chi_5(k), {}_P\chi_6, {}_P\chi_7(k), {}_P\chi_8(k)$ for odd q.

${}_P\chi_9, {}_P\chi_{10}, \ldots, {}_P\chi_{18}(k)$:
We use the restrictions of the unipotent characters $\mathbf{1}$, $[\varepsilon_1]$, $[\rho_1]$, $[\rho_2]$, ${}^3D_4[-1]$, ${}^3D_4[1]$, $[\varepsilon_2]$, $\mathbf{St}$ of G to P (see [10]). The characters ${}_P\chi_{11}(k)$ and ${}_P\chi_{14}(k)$ for $k = 0, \ldots, q^2 + q$ and $k = 0, \ldots, q^2 - q$, respectively, are constructed in the same way as for odd q. The remaining irreducible characters of P are constructed as follows. We define ${}_P\chi_{17}(k) := {}_B\chi_{15}(k)^P$, $k = 0, \ldots, q^3 - 2$, ${}_P\chi_{15} := [\varepsilon_1]_P - \mathbf{1}_P - {}_P\chi_5(0)$,

$$ {}_P\chi_9 := {}^3D_4[1]_P - {}_P\chi_{15} - \sum{}' {}_P\chi_{17}(k) $$

where $\sum{}'$ is the sum over all *different* ${}_P\chi_{17}(k)$ with $q^2+q+1 \mid k, k \neq 0$ and ${}_P\chi_{10} := {}_P\chi_{11}(0) - {}_P\chi_9$. Furthermore, we set ${}_P\chi_{18}(k) := {}_P\chi_4(k) \cdot {}_P\chi_{15}$, $k = 0, \ldots, q^3$,

$$ {}_P\chi_{12} := {}^3D_4[-1]_P - \sum{}'' {}_P\chi_{18}(k) $$

where $\sum{}''$ is the sum over all *different* ${}_P\chi_{18}(k)$ with $q^2 - q + 1 \mid k, k \neq 0$, ${}_P\chi_{13} := {}_P\chi_{14}(0) - {}_P\chi_{12}$ and ${}_P\chi_{16} := {}_P\chi_{17}(0) - {}_P\chi_{15}$. Computing scalar products with CHEVIE, we see that ${}_P\chi_1(k), \ldots, {}_P\chi_{18}(k)$ are $q^4 + q^3 + q^2 + 3q + 4$ different irreducible characters. Since this number equals the number of conjugacy classes of P, the character table is complete. □

So $\chi_1(k,l), \ldots, \chi_4(k)$ cover ψ_0; $\chi_5(k), \chi_6$ cover ψ_1; $\chi_7(k), \chi_8(k)$ cover ψ_2; $\chi_9, \chi_{10}, \chi_{11}(k)$ cover ψ_3; $\chi_{12}, \chi_{13}, \chi_{14}(k)$ cover ψ_4 and $\chi_{15}, \ldots, \chi_{18}(k)$ cover ψ_5.

Corollary 5.2. *For $j = 0, \ldots, 5$, ψ_j extends to its inertia subgroup in P.*

Proof. The proof is analogous to Corollary 5.8 in [6]. □

6. The character table of the maximal parabolic subgroup Q

The maximal subgroup Q is the semidirect product of the Levi complement $L_Q = \langle T, X_\beta, X_{-\beta}\rangle$ and the unipotent radical $U_Q = X_\alpha X_{\alpha+\beta} X_{2\alpha+\beta} X_{3\alpha+\beta} X_{3\alpha+2\beta}$.

Theorem 6.1. *The character table of the parabolic subgroup Q is given by Tables* A.13 *and* A.14 *in Appendix* A.

Proof. The irreducible character of Q can be constructed as follows:

${}_Q\chi_1(k), {}_Q\chi_2(k), {}_Q\chi_3(k), {}_Q\chi_4(k)$:
These are the inflations of the irreducible characters of L_Q.

${}_Q\chi_5(k), {}_Q\chi_6, {}_Q\chi_7$:
The construction of these characters is analogous to ${}_Q\chi_5(k), {}_Q\chi_6, {}_Q\chi_7$ for odd q.

${}_Q\chi_8, \ldots, {}_Q\chi_{16}(k)$:
The definition and construction of these characters is different from odd q. Let ${}_Q\chi_{10}(k) := {}_B\chi_7(k)^Q$ for $k = 0, \ldots, q-2$ and ${}_Q\chi_8 := {}_B\chi_8{}^Q - \sum''' {}_Q\chi_{10}(k)$, where $\sum'''$ is the sum over all *different* ${}_Q\chi_{10}(k)$ with $k \neq 0$. Furthermore, we define ${}_Q\chi_9 := {}_Q\chi_{10}(0) - {}_Q\chi_8$. We define ${}_Q\chi_{14}(k) := {}_B\chi_{12}(k)^Q$ for $k = 0, \ldots, q^3-2$, ${}_Q\chi_{15}(k) := {}_B\chi_{13}(k)^Q$ for $k = 0, \ldots, q^2+q$ and ${}_Q\chi_{16}(k) := {}_B\chi_{14}(k)^Q$ for $k = 1, \ldots, q$. Since we do not know the values of ${}_B\chi_{14}(k)$ we are not able to compute the values of ${}_Q\chi_{16}(k)$ generically, but we can compute all values of $\sum_{k=1}^q {}_Q\chi_{16}(k)$ since we know $\sum_{k=1}^q {}_B\chi_{14}(k)$. Let

$$\begin{aligned}{}_Q\chi_{11} &:= [\rho_1]_Q + {}^3D_4[-1]_Q - \sum_{k=1}^q {}_Q\chi_{16}(k) - {}_Q\chi_1(0) - {}_Q\chi_2(0) - {}_Q\chi_5(0)\\ &\quad - {}_Q\chi_9 - {}_Q\chi_{14}(0) - {}_Q\chi_{15}(0)\end{aligned}$$

and

$$\begin{aligned}{}_Q\chi_{12} &:= [\rho_2]_Q + {}^3D_4[1]_Q - \sum_{k=1}^q {}_Q\chi_{16}(k) - {}_Q\chi_1(0) - {}_Q\chi_2(0) - {}_Q\chi_5(0)\\ &\quad - {}_Q\chi_8 - {}_Q\chi_{14}(0) - {}_Q\chi_{15}(0).\end{aligned}$$

To construct the remaining irreducible characters of Q, we consider the subgroup $\langle X_\beta, n_\beta\rangle$. The conjugacy classes of $\langle X_\beta, n_\beta\rangle$ can be computed analogously to those of L_P and L_Q and are shown in Tables 6.2 and 6.3.

Since $\langle X_\beta, n_\beta\rangle \cong SL_2(q)$ the character table of $\langle X_\beta, n_\beta\rangle$ is well known and can be taken from the CHEVIE-library or [9, p. 134]. In particular, for $k = 1, \ldots, q$, the group $\langle X_\beta, n_\beta\rangle$ has an irreducible character $\psi(k)$ with the values given in Table 6.4.

The subgroup $C_T(X_\beta)X_{2\alpha+\beta}$ has an irreducible character μ of degree q^3-1 whose values at $x_{2\alpha+\beta}(1)$ and $h(\tilde{\zeta}_3^{2i}, \tilde{\zeta}_1^{i}, \tilde{\zeta}_3^{2qi}, \tilde{\zeta}_3^{2q^2i}) \neq 1$ are respectively -1 and 0. For $k = 1, \ldots, q$, let $\chi(k)$ be the character of $\langle TX_\beta X_{2\alpha+\beta}, n_\beta\rangle = C_T(X_\beta)X_{2\alpha+\beta} \times \langle X_\beta, n_\beta\rangle$ defined by $\chi(k)(x) := \mu(x_1)\psi(k)(x_2)$ for $x = x_1x_2$ with $x_1 \in C_T(X_\beta)X_{2\alpha+\beta}$ and $x_2 \in \langle X_\beta, n_\beta\rangle$. Extend $\chi(k)$ to the

Table 6.2
Parameterization of the semisimple conjugacy classes of $\langle X_\beta, n_\beta\rangle$

Representative	Parameters	Number of classes
$h_1 := h(1,1,1,1)$		1
$h_2(i) := h(1, \tilde{\zeta}_1^i, 1, 1)$	$i = 0, \ldots, q-2$ $i \neq 0$	$(q-2)/2$
$h_3(i) := h(1, \tilde{\xi}_1^i, 1, 1)$	$i = 0, \ldots, q$ $i \neq 0$	$q/2$

Table 6.3
The conjugacy classes of $\langle X_\beta, n_\beta\rangle$

Notation	Representative	Order of centralizer(s)
$c_{1,0}$	1	$q(q^2-1)$
$c_{1,1}$	$x_\beta(1)$	q
$c_{2,0}(i)$	$h_2(i)$	$q-1$
$c_{3,0}(i)$	$h_3(i)$	$q+1$

Table 6.4
Values of the irreducible characters $\psi(k)$ of $\langle X_\beta, n_\beta\rangle$

	$c_{1,0}$	$c_{1,1}$	$c_{2,0}(i)$	$c_{3,0}(i)$
$\psi(k)$	$q-1$	-1	0	$-\xi_1^{ik} - \xi_1^{-ik}$

subgroup $\langle TX_\beta X_{2\alpha+\beta}, n_\beta\rangle X_{3\alpha+\beta}X_{3\alpha+2\beta}$ and induce it to Q. In this way we get the characters $\tilde{\chi}(k)$. For $k = 1, \ldots, q$ let

$$
\begin{aligned}
{}_Q\chi_{13}(k) := {} & q^2 \cdot {}_B\chi_8^Q + (q^2-2q+2) \cdot {}_B\chi_9^Q - \frac{q}{2}(q-1) \cdot {}_Q\chi_8 + \frac{q}{2}(q-1) \cdot {}_Q\chi_9 \\
& - \frac{(q-1)(q-2)}{2} \cdot {}_Q\chi_{11} + \frac{(q-1)(q-2)}{2} \cdot {}_Q\chi_{12} - \tilde{\chi}(k).
\end{aligned}
$$

Computing scalar products with CHEVIE, we see that ${}_Q\chi_1(k), \ldots, {}_Q\chi_{16}(k)$ are $q^4 + 2q^3 + q^2 + 3q + 3$ different irreducible characters. Since this number equals the number of conjugacy classes of Q, the character table is complete. $\square$

Remark. It is also possible to determine the orbits of Q on $\mathrm{Irr}(U_Q)$ and the corresponding inertia subgroups. In particular, it turns out that every irreducible character of U_Q extends to its inertia subgroup. But the determination and description of these subgroups is much more difficult and complicated than for odd q, so that this would fill an indisproportional part of this paper. Since these subgroups are not needed for the construction of the character table of U_Q, we omit the description of the inertia subgroups.

Appendix A

Table A.1
(cf. Deriziotis and Michler [2, Table 2.1]) Parameterization of the semisimple conjugacy classes of ${}^3D_4(q)$, q even

Representative	Parameters	Number of classes
$h_1 := h(1,1,1,1)$		1
$h_3(i) := h(\tilde{\zeta}_1^i, \tilde{\zeta}_1^{2i}, \tilde{\zeta}_1^i, \tilde{\zeta}_1^i)$	$i = 0, \dots, q-2$ $i \neq 0$	$\frac{q-2}{2}$
$h_4(i) := h(\tilde{\varphi}_3^i, 1, \tilde{\varphi}_3^{qi}, \tilde{\varphi}_3^{q^2 i})$	$i = 0, \dots, q^2+q$ $i \neq 0$	$\frac{q^2+q}{2}$
$h_5(i) := h(\tilde{\zeta}_3^i, 1, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0, \dots, q^3-2$ $i \neq (q-1)l,\ l = 0, \dots, q^2+q+1$	$\frac{q^3-q^2-q-2}{2}$
$h_6(i,j) := h(\tilde{\zeta}_3^i, \tilde{\zeta}_1^j, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0, \dots, q^3-2;\ j = 0, \dots, q-2$ $i, j \neq 0$ $i \neq (q^2+q+1)l$ or $j \neq l,\ l = 0, \dots, q-2$ $i \neq (q^2+q+1)l$ or $j \neq 2l,\ l = 0, \dots, q-2$ $i \neq j + (q-1)l,\ l = 0, \dots, q^2+q$ $i \neq 2j + (q-1)l,\ l = 0, \dots, q^2+q$	$\frac{1}{12}(q^4 - 4q^3$ $+ 2q^2 - 2q + 12)$
$h_7(i) := h(1, \tilde{\xi}_1^{ci}, 1, 1)$	$i = 0, \dots, q$ $i \neq 0$	$\frac{q}{2}$
$h_8(i) := h(\tilde{\mu}_3^{(q^2+q)ci}, \tilde{\mu}_3^{(q^2+q+1)ci},$ $\tilde{\mu}_3^{(q^3+q^2)ci}, \tilde{\mu}_3^{(q+1)ci})$	$i = 0, \dots, q^4+q^3-q-2$ $i \neq (q+1)l,\ l = 0, \dots, q^3-2$ $i \neq (q^3-1)l,\ l = 0, \dots, q$	$\frac{q^4-2q}{4}$
$h_9(i) := h(\tilde{\varphi}_6^i, 1, \tilde{\varphi}_6^{-qi}, \tilde{\varphi}_6^{(q-1)i})$	$i = 0, \dots, q^2-q$ $i \neq 0$	$\frac{q^2-q}{2}$
$h_{10}(i) := h(\tilde{\xi}_3^i, 1, \tilde{\xi}_3^{-qi}, \tilde{\xi}_3^{q^2 i})$	$i = 0, \dots, q^3$ $i \neq (q+1)l,\ l = 0, \dots, q^2-q$	$\frac{q^3-q^2+q}{2}$
$h_{11}(i) := h(\tilde{\eta}_3^i, \tilde{\eta}_3^{(q^3+1)i}, \tilde{\eta}_3^{q^4 i}, \tilde{\eta}_3^{q^2 i})$	$i = 0, \dots, q^4-q^3+q-2$ $i \neq (q-1)l,\ l = 0, \dots, q^3$ $i \neq (q^3+1)l,\ l = 0, \dots, q-2$	$\frac{q^4-2q^3}{4}$
$h_{12}(i,j) := h(\tilde{\varphi}_3^i, \tilde{\varphi}_3^j, \tilde{\varphi}_3^{qi+j}, \tilde{\varphi}_3^{(q+1)(j-i)})$	$i = 0, \dots, q^2+q$ $j = 0, \dots, q^2+q$ $j \neq 0, -2qi$ $2i \neq j, (1-q^2)j$	$\frac{q^4+2q^3-q^2-2q}{24}$
$h_{13}(i,j) := h(\tilde{\varphi}_6^i, \tilde{\varphi}_6^j, \tilde{\varphi}_6^{-qi+j}, \tilde{\varphi}_6^{(q-1)(i-j)})$	$i = 0, \dots, q^2-q$ $j = 0, \dots, q^2-q$ $j \neq 0, 2qi$ $2i \neq j, (1-q^2)j$	$\frac{q^4-2q^3-q^2+2q}{24}$
$h_{14}(i) := h(\tilde{\varphi}_{12}^i, \tilde{\varphi}_{12}^{(q^3+1)i}, \tilde{\varphi}_{12}^{qi}, \tilde{\varphi}_{12}^{q^2 i})$	$i = 0, \dots, q^4-q^2$ $i \neq 0$	$\frac{q^4-q^2}{4}$
$h_{15}(i,j) := h(\tilde{\xi}_3^i, \tilde{\xi}_1^j, \tilde{\xi}_3^{-qi}, \tilde{\xi}_3^{q^2 i})$	$i = 0, \dots, q^3$ $j = 0, \dots, q$ $i, j \neq 0$ $i \neq (q^2-q+1)l$ or $j \neq l,\ l = 0, \dots, q$ $i \neq (q^2-q+1)l$ or $j \neq 2l,\ l = 0, \dots, q$ $i \neq j + (q+1)l,\ l = 0, \dots, q^2-q$ $i \neq 2j + (q+1)l,\ l = 0, \dots, q^2-q$	$\frac{1}{12}(q^4 - 2q^3 + 2q^2 - 4q)$

(c is the multiplicative inverse of q^2+q-1 modulo $(q^3-1)(q+1)$.)

Table A.2
(cf. Deriziotis and Michler [2, Table 2.4]) The conjugacy classes of ${}^3D_4(q)$, q even

Notation	Representative	$\lvert C_{{}^3D_4(q)}\rvert$
$c_{1,0}$	1	$q^{12}(q^6-1)^2(q^4-q^2+1)$
$c_{1,1}$	$x_{3\alpha+2\beta}(1)$	$q^{12}(q^6-1)$
$c_{1,2}$	$x_{2\alpha+\beta}(1)$	$q^{10}(q^2-1)$
$c_{1,3}$	$x_\beta(1)x_{3\alpha+\beta}(1)$	$2q^8(q^2+q+1)$
$c_{1,4}$	$x_{\alpha+\beta}(1)x_{2\alpha+\beta}(1)x_{3\alpha+\beta}(\zeta)$	$2q^8(q^2-q+1)$
$c_{1,5}$	$x_\alpha(1)x_{\alpha+\beta}(a)$	q^6
$c_{1,6}$	$x_\alpha(1)x_\beta(1)$	$2q^4$
$c_{1,7}$	$x_\alpha(1)x_\beta(1)x_{2\alpha+\beta}(\eta)$	$2q^4$
$c_{3,0}(i)$	$h_3(i)$	$q^3(q^6-1)(q-1)$
$c_{3,1}(i)$	$h_3(i)x_\alpha(1)$	$q^3(q-1)$
$c_{4,0}(i)$	$h_4(i)$	$q^3(q^3-1)^2(q+1)$
$c_{4,1}(i)$	$h_4(i)x_{3\alpha+2\beta}(1)$	$q^3(q^3-1)$
$c_{4,2}(i)$	$h_4(i)x_\beta(1)x_{3\alpha+\beta}(1)$	$q^2(q^2+q+1)$
$c_{5,0}(i)$	$h_5(i)$	$q(q^3-1)(q^2-1)$
$c_{5,1}(i)$	$h_5(i)x_{3\alpha+2\beta}(1)$	$q(q^3-1)$
$c_{6,0}(i,j)$	$h_6(i,j)$	$(q^3-1)(q-1)$
$c_{7,0}(i)$	$h_7(i)$	$q^3(q^6-1)(q+1)$
$c_{7,1}(i)$	$h_7(i)x_{2\alpha+\beta}(1)$	$q^3(q+1)$
$c_{8,0}(i)$	$h_8(i)$	$(q^3-1)(q+1)$
$c_{9,0}(i)$	$h_9(i)$	$q^3(q^3+1)^2(q-1)$
$c_{9,1}(i)$	$h_9(i)x_{3\alpha+2\beta}(1)$	$q^3(q^3+1)$
$c_{9,2}(i)$	$h_9(i)x_\beta(s)x_{3\alpha+\beta}(s^q)x_{3\alpha+2\beta}(r)$	$q^2(q^2-q+1)$
$c_{10,0}(i)$	$h_{10}(i)$	$q(q^3+1)(q^2-1)$
$c_{10,1}(i)$	$h_{10}(i)x_{3\alpha+2\beta}(1)$	$q(q^3+1)$
$c_{11,0}(i)$	$h_{11}(i)$	$(q^3+1)(q-1)$
$c_{12,0}(i,j)$	$h_{12}(i,j)$	$(q^2+q+1)^2$
$c_{13,0}(i,j)$	$h_{13}(i,j)$	$(q^2-q+1)^2$
$c_{14,0}(i)$	$h_{14}(i)$	q^4-q^2+1
$c_{15,0}(i,j)$	$h_{15}(i,j)$	$(q^3+1)(q+1)$

(The field elements $\eta, \zeta \in \mathbb{F}_q$ are defined in Section 3, furthermore $s \in \mathbb{F}$ is a primitive (q^2-1)th root of unity, $r \in \mathbb{F}$ is a root of the polynomial $X^q + X + s^{q+1}$ and $a \in \mathbb{F}$ with $a^{q^3} = a$ and $a^q \neq a$.)

Table A.3
Parameterization of the semisimple conjugacy classes of B

Representative	Parameters	Number of classes
$h_1 := h(1,1,1,1)$		1
$h_5(i) := h(\tilde{\varphi}_3^i, 1, \tilde{\varphi}_3^{qi}, \tilde{\varphi}_3^{q^2 i})$	$i = 0,\dots,q^2+q$ $i \neq 0$	q^2+q
$h_6(i) := h(\tilde{\zeta}_1^i, \tilde{\zeta}_1^{2i}, \tilde{\zeta}_1^i, \tilde{\zeta}_1^i)$	$i = 0,\dots,q-2$ $i \neq 0$	$q-2$
$h_7(i) := h(\tilde{\zeta}_1^i, \tilde{\zeta}_1^i, \tilde{\zeta}_1^i, \tilde{\zeta}_1^i)$	$i = 0,\dots,q-2$ $i \neq 0$	$q-2$
$h_8(i) := h(1, \tilde{\zeta}_1^i, 1, 1)$	$i = 0,\dots,q-2$ $i \neq 0$	$q-2$
$h_9(i) := h(\tilde{\zeta}_3^{2i}, \tilde{\zeta}_1^i, \tilde{\zeta}_3^{2qi}, \tilde{\zeta}_3^{2q^2 i})$	$i = 0,\dots,q^3-2$ $i \neq (q-1)l$, $l = 0,\dots,q^2+q+1$	q^3-q^2-q-2
$h_{10}(i) := h(\tilde{\zeta}_3^i, \tilde{\zeta}_1^i, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0,\dots,q^3-2$ $i \neq (q-1)l$, $l = 0,\dots,q^2+q+1$	q^3-q^2-q-2
$h_{11}(i) := h(\tilde{\zeta}_3^i, 1, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0,\dots,q^3-2$ $i \neq (q-1)l$, $l = 0,\dots,q^2+q+1$	q^3-q^2-q-2
$h_{12}(i,j) := h(\tilde{\zeta}_3^i, \tilde{\zeta}_1^j, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0,\dots,q^3-2$ $j = 0,\dots,q-2$ $i \neq 0$ $j \neq 0$ $i \neq (q^2+q+1)l$ or $j \neq l$, $l = 0,\dots,q-2$ $i \neq (q^2+q+1)l$ or $j \neq 2l$, $l = 0,\dots,q-2$ $i \neq j+(q-1)l$, $l = 0,\dots,q^2+q$ $i \neq 2j+(q-1)l$, $l = 0,\dots,q^2+q$	$q^4-4q^3+2q^2-2q+12$

Table A.4
The conjugacy classes of B

Notation	Representative	$\lvert C_B\rvert$	Fusion in P	Fusion in Q
$c_{1,0}$	1	$q^{12}(q^3-1)(q-1)$	$c_{1,0}$	$c_{1,0}$
$c_{1,1}$	$x_{3\alpha+2\beta}(1)$	$q^{12}(q^3-1)$	$c_{1,1}$	$c_{1,1}$
$c_{1,2}$	$x_{3\alpha+\beta}(1)$	$q^{11}(q^3-1)$	$c_{1,2}$	$c_{1,1}$
$c_{1,3}$	$x_{2\alpha+\beta}(1)$	$q^{10}(q-1)$	$c_{1,3}$	$c_{1,2}$
$c_{1,4}$	$x_{\alpha+\beta}(1)$	$q^8(q-1)$	$c_{1,3}$	$c_{1,3}$
$c_{1,5}$	$x_{\alpha+\beta}(1)x_{2\alpha+\beta}(1)x_{3\alpha+\beta}(1)$	$2q^8$	$c_{1,4}$	$c_{1,4}$
$c_{1,6}$	$x_{\alpha+\beta}(1)x_{2\alpha+\beta}(1)x_{3\alpha+\beta}(\zeta)$	$2q^8$	$c_{1,5}$	$c_{1,5}$
$c_{1,7}$	$x_\beta(1)$	$q^8(q^3-1)$	$c_{1,2}$	$c_{1,7}$
$c_{1,8}$	$x_\beta(1)x_{2\alpha+\beta}(1)$	q^8	$c_{1,3}$	$c_{1,8}$
$c_{1,9}(t)$	$x_\beta(1)x_{2\alpha+\beta}(1)x_{3\alpha+\beta}(t)$	q^8	$c_{1,5}$	$c_{1,9}(t)$
$c_{1,10}$	$x_\beta(1)x_{3\alpha+\beta}(1)$	$q^8(q^2+q+1)$	$c_{1,4}$	$c_{1,10}$
$c_{1,11}(t)$	$x_\beta(1)x_{2\alpha+\beta}(1)x_{3\alpha+\beta}(t)$	q^8	$c_{1,4}$	$c_{1,11}(t)$
$c_{1,12}$	$x_\alpha(1)$	$q^7(q-1)$	$c_{1,6}$	$c_{1,3}$
$c_{1,13}$	$x_\alpha(1)x_{2\alpha+\beta}(1)x_{3\alpha+2\beta}(1)$	$2q^7$	$c_{1,7}$	$c_{1,4}$
$c_{1,14}$	$x_\alpha(1)x_{2\alpha+\beta}(1)x_{3\alpha+2\beta}(\zeta)$	$2q^7$	$c_{1,8}$	$c_{1,5}$
$c_{1,15}(a')$	$x_\alpha(1)x_{\alpha+\beta}(a')$	q^6	$c_{1,9}(a')$	$c_{1,6}$
$c_{1,16}$	$x_\alpha(1)x_\beta(1)$	$2q^4$	$c_{1,10}$	$c_{1,12}$
$c_{1,17}$	$x_\alpha(1)x_\beta(1)x_{2\alpha+\beta}(\eta)$	$2q^4$	$c_{1,11}$	$c_{1,13}$
$c_{5,0}(i)$	$h_5(i)$	$q^3(q^3-1)(q-1)$	$c_{6,0}(i)$	$c_{6,0}(i)$
$c_{5,1}(i)$	$h_5(i)x_{3\alpha+2\beta}(1)$	$q^3(q^3-1)$	$c_{6,1}(i)$	$c_{6,1}(i)$
$c_{5,2}(i)$	$h_5(i)x_{3\alpha+\beta}(1)$	$q^2(q^3-1)$	$c_{6,2}(i)$	$c_{6,1}(i)$
$c_{5,3}(i)$	$h_5(i)x_\beta(1)$	$q^2(q^3-1)$	$c_{6,3}(i)$	$c_{6,2}(i)$
$c_{5,4}(i)$	$h_5(i)x_\beta(1)x_{3\alpha+\beta}(1)$	$q^2(q^2+q+1)$	$c_{6,4}(i)$	$c_{6,3}(i)$
$c_{6,0}(i)$	$h_6(i)$	$q^3(q^3-1)(q-1)$	$c_{4,0}(i)$	$c_{4,0}(i)$
$c_{6,1}(i)$	$h_6(i)x_\alpha(1)$	$q^3(q-1)$	$c_{4,1}(i)$	$c_{4,1}(i)$
$c_{7,0}(i)$	$h_7(i)$	$q^3(q^3-1)(q-1)$	$c_{5,0}(i)$	$c_{4,0}(i)$
$c_{7,1}(i)$	$h_7(i)x_{\alpha+\beta}(1)$	$q^3(q-1)$	$c_{5,1}(i)$	$c_{4,1}(i)$
$c_{8,0}(i)$	$h_8(i)$	$q^3(q^3-1)(q-1)$	$c_{5,0}(i)$	$c_{5,0}(i)$
$c_{8,1}(i)$	$h_8(i)x_{2\alpha+\beta}(1)$	$q^3(q-1)$	$c_{5,1}(i)$	$c_{5,1}(i)$
$c_{9,0}(i)$	$h_9(i)$	$q(q^3-1)(q-1)$	$c_{7,0}(d\cdot i)$	$c_{7,0}(i)$
$c_{9,1}(i)$	$h_9(i)x_\beta(1)$	$q(q^3-1)$	$c_{7,1}(d\cdot i)$	$c_{7,1}(i)$
$c_{10,0}(i)$	$h_{10}(i)$	$q(q^3-1)(q-1)$	$c_{7,0}(i)$	$c_{8,0}(i)$
$c_{10,1}(i)$	$h_{10}(i)x_{3\alpha+\beta}(1)$	$q(q^3-1)$	$c_{7,1}(i)$	$c_{8,1}(i)$
$c_{11,0}(i)$	$h_{11}(i)$	$q(q^3-1)(q-1)$	$c_{8,0}(i)$	$c_{8,0}(i)$
$c_{11,1}(i)$	$h_{11}(i)x_{3\alpha+2\beta}(1)$	$q(q^3-1)$	$c_{8,1}(i)$	$c_{8,1}(i)$
$c_{12,0}(i,j)$	$h_{12}(i,j)$	$(q^3-1)(q-1)$	$c_{9,0}(i,j)$	$c_{9,0}(i,j)$

(The parameter t in the representatives for the conjugacy classes of type $c_{1,9}$, $c_{1,11}$ runs through the sets I_1, I_2 with $|I_1| = q/2$ and $|I_2| = q/2-1$, respectively. The parameter a' in the representatives for the conjugacy classes of type $c_{1,15}$ runs through the set I_3 with $|I_3| = q+1$. The sets I_1, I_2, I_3 are defined in Section 3. For abbreviation, $d := q^2+q-1$. The elements ζ, η are the same as in Table A.2.)

Table A.5
Parameterization of the irreducible characters of B

Character	Parameters	Number of characters
${}_B\chi_1(k,l)$	$k=0,\ldots,q^3-2; l=0,\ldots,q-2$	$(q^3-1)(q-1)$
${}_B\chi_2(k)$	$k=0,\ldots,q-2$	$q-1$
${}_B\chi_3(k)$	$k=0,\ldots,q^3-2$	q^3-1
${}_B\chi_4$		1
${}_B\chi_5(k)$	$k=0,\ldots,q-2$	$q-1$
${}_B\chi_6(k)$	$k=1,\ldots,q+1$	$q+1$
${}_B\chi_7(k)$	$k=0,\ldots,q-2$	$q-1$
${}_B\chi_8$		1
${}_B\chi_9$		1
${}_B\chi_{10}$		1
${}_B\chi_{11}$		1
${}_B\chi_{12}(k)$	$k=0,\ldots,q^3-2$	q^3-1
${}_B\chi_{13}(k)$	$k=0,\ldots,q^2+q$	q^2+q+1
${}_B\chi_{14}(k)$	$k=1,\ldots,q$	q
${}_B\chi_{15}(k)$	$k=0,\ldots,q^3-2$	q^3-1
${}_B\chi_{16}$		1

Table A.6
The character table of B

	$c_{1,0}$	$c_{1,1}$	$c_{1,2}$	$c_{1,3}$	$c_{1,4}$
${}_B\chi_1(k,l)$	1	1	1	1	1
${}_B\chi_2(k)$	q^3-1	q^3-1	q^3-1	q^3-1	q^3-1
${}_B\chi_3(k)$	$q-1$	$q-1$	$q-1$	$q-1$	$q-1$
${}_B\chi_4$	$(q-1)(q^3-1)$	$(q-1)(q^3-1)$	$(q-1)(q^3-1)$	$(q-1)(q^3-1)$	$(q-1)(q^3-1)$
${}_B\chi_5(k)$	$q(q^3-1)$	$q(q^3-1)$	$q(q^3-1)$	$q(q^3-1)$	$-q$
$\sum_{k=1}^{q+1} {}_B\chi_6(k)$	$q(q-1)(q+1)(q^3-1)$	$q(q-1)(q+1)(q^3-1)$	$q(q-1)(q+1)(q^3-1)$	$q(q-1)(q+1)(q^3-1)$	$-q(q-1)(q+1)$
${}_B\chi_7(k)$	$q^3(q^3-1)$	$q^3(q^3-1)$	$q^3(q^3-1)$	$-q^3$	$q^3(q-1)$
${}_B\chi_8$	$\frac{1}{2}q^3(q-1)(q^3-1)$	$\frac{1}{2}q^3(q-1)(q^3-1)$	$\frac{1}{2}q^3(q-1)(q^3-1)$	$-\frac{1}{2}q^3(q-1)$	$-\frac{1}{2}q^3(q-1)$
${}_B\chi_9$	$\frac{1}{2}q^3(q-1)(q^3-1)$	$\frac{1}{2}q^3(q-1)(q^3-1)$	$\frac{1}{2}q^3(q-1)(q^3-1)$	$-\frac{1}{2}q^3(q-1)$	$-\frac{1}{2}q^3(q-1)$
${}_B\chi_{10}$	$\frac{1}{2}q^3(q-1)(q^3-1)$	$\frac{1}{2}q^3(q-1)(q^3-1)$	$\frac{1}{2}q^3(q-1)(q^3-1)$	$-\frac{1}{2}q^3(q-1)$	$-\frac{1}{2}q^3(q-1)$
${}_B\chi_{11}$	$\frac{1}{2}q^3(q-1)(q^3-1)$	$\frac{1}{2}q^3(q-1)(q^3-1)$	$\frac{1}{2}q^3(q-1)(q^3-1)$	$-\frac{1}{2}q^3(q-1)$	$-\frac{1}{2}q^3(q-1)$
${}_B\chi_{12}(k)$	$q^3(q-1)$	$q^3(q-1)$	$-q^3$	$\cdot$	$\cdot$
${}_B\chi_{13}(k)$	$q^3(q-1)^2$	$q^3(q-1)^2$	$-q^3(q-1)$	$\cdot$	$\cdot$
$\sum_{k=1}^{q} {}_B\chi_{14}(k)$	$q^4(q-1)(q^3-1)$	$q^4(q-1)(q^3-1)$	$-q^4(q^3-1)$	$\cdot$	$\cdot$
${}_B\chi_{15}(k)$	$q^4(q-1)$	$-q^4$	$\cdot$	$\cdot$	$\cdot$
${}_B\chi_{16}$	$q^4(q-1)(q^3-1)$	$-q^4(q^3-1)$	$\cdot$	$\cdot$	$\cdot$

(*continued on next page*)

Table A.6 (*continued*)

	$c_{1,5}$	$c_{1,6}$	$c_{1,7}$	$c_{1,8}$	$c_{1,9}(t)$	$c_{1,10}$	$c_{1,11}(t)$	$c_{1,12}$
${}_B\chi_1(k,l)$	1	1	1	1	1	1	1	1
${}_B\chi_2(k)$	q^3-1	q^3-1	q^3-1	q^3-1	q^3-1	q^3-1	q^3-1	-1
${}_B\chi_3(k)$	$q-1$	$q-1$	-1	-1	-1	-1	-1	$q-1$
${}_B\chi_4$	$(q-1)(q^3-1)$	$(q-1)(q^3-1)$	$-(q^3-1)$	$-(q^3-1)$	$-(q^3-1)$	$-(q^3-1)$	$-(q^3-1)$	$-(q-1)$
${}_B\chi_5(k)$	$-q$	$-q$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$q(q^2-1)$
$\sum_{k=1}^{q+1} {}_B\chi_6(k)$	$-q(q-1)(q+1)$	$-q(q-1)(q+1)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$-q(q-1)(q+1)$
${}_B\chi_7(k)$	$-q^3$	$-q^3$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_B\chi_8$	$\frac{1}{2}q^3$	$\frac{1}{2}q^3$	$-\frac{1}{2}q^2(q^3-1)$	$\frac{1}{2}q^2$	$\frac{1}{2}q^2$	$-\frac{1}{2}q^2(q^3-1)$	$\frac{1}{2}q^2$	$\frac{1}{2}q^3(q-1)$
${}_B\chi_9$	$\frac{1}{2}q^3$	$\frac{1}{2}q^3$	$-\frac{1}{2}q^2(q^3-1)$	$\frac{1}{2}q^2$	$\frac{1}{2}q^2$	$-\frac{1}{2}q^2(q^3-1)$	$\frac{1}{2}q^2$	$-\frac{1}{2}q^3(q-1)$
${}_B\chi_{10}$	$\frac{1}{2}q^3$	$\frac{1}{2}q^3$	$\frac{1}{2}q^2(q^3-1)$	$-\frac{1}{2}q^2$	$-\frac{1}{2}q^2$	$\frac{1}{2}q^2(q^3-1)$	$-\frac{1}{2}q^2$	$\frac{1}{2}q^3(q-1)$
${}_B\chi_{11}$	$\frac{1}{2}q^3$	$\frac{1}{2}q^3$	$\frac{1}{2}q^2(q^3-1)$	$-\frac{1}{2}q^2$	$-\frac{1}{2}q^2$	$\frac{1}{2}q^2(q^3-1)$	$-\frac{1}{2}q^2$	$-\frac{1}{2}q^3(q-1)$
${}_B\chi_{12}(k)$	q^2	$-q^2$	$-q(q^2-1)$	q	$-q(q-1)$	$q(q+1)$	$q(q+1)$	$\cdot$
${}_B\chi_{13}(k)$	$q^2(q-1)$	$-q^2(q-1)$	$q(q^2-1)$	$-q$	$q(q-1)$	$-q(q+1)$	$-q(q+1)$	$\cdot$
$\sum_{k=1}^{q} {}_B\chi_{14}(k)$	$-q^3$	q^3	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_B\chi_{15}(k)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_B\chi_{16}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

Table A.6 (*continued*)

	$c_{1,13}$	$c_{1,14}$	$c_{1,15}(a')$	$c_{1,16}$	$c_{1,17}$	$c_{5,0}(i)$	$c_{5,1}(i)$	$c_{5,2}(i)$	$c_{5,3}(i)$	$c_{5,4}(i)$	$c_{6,0}(i)$
${}_B\chi_1(k,l)$	1	1	1	1	1	φ_3^{ik}	φ_3^{ik}	φ_3^{ik}	φ_3^{ik}	φ_3^{ik}	ζ_1^{ik+2il}
${}_B\chi_2(k)$	-1	-1	-1	-1	-1	.	.	.	.	.	$(q^3-1)\zeta_1^{ik}$
${}_B\chi_3(k)$	$q-1$	$q-1$	$q-1$	-1	-1	$(q-1)\varphi_3^{ik}$	$(q-1)\varphi_3^{ik}$	$(q-1)\varphi_3^{ik}$	$-\varphi_3^{ik}$	$-\varphi_3^{ik}$	.
${}_B\chi_4$	$-(q-1)$	$-(q-1)$	$-(q-1)$	1	1	.	.	.	.	.	.
${}_B\chi_5(k)$	$q(q^2-1)$	$q(q^2-1)$	$-q$	.	.	.	.	.	.	.	.
$\sum_{k=1}^{q+1} {}_B\chi_6(k)$	$-q(q-1)(q+1)$	$-q(q-1)(q+1)$	q	.	.	.	.	.	.	.	.
${}_B\chi_7(k)$	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_8$	$-\frac{1}{2}q^3$	$-\frac{1}{2}q^3$	.	$-\frac{1}{2}q^2$	$\frac{1}{2}q^2$	.	.	.	.	.	.
${}_B\chi_9$	$\frac{1}{2}q^3$	$\frac{1}{2}q^3$	.	$\frac{1}{2}q^2$	$-\frac{1}{2}q^2$	.	.	.	.	.	.
${}_B\chi_{10}$	$-\frac{1}{2}q^3$	$-\frac{1}{2}q^3$	.	$\frac{1}{2}q^2$	$-\frac{1}{2}q^2$	.	.	.	.	.	.
${}_B\chi_{11}$	$\frac{1}{2}q^3$	$\frac{1}{2}q^3$	.	$-\frac{1}{2}q^2$	$\frac{1}{2}q^2$	.	.	.	.	.	.
${}_B\chi_{12}(k)$	.	.	.	.	.	$(q-1)\varphi_3^{ik}$	$(q-1)\varphi_3^{ik}$	$-\varphi_3^{ik}$	$(q-1)\varphi_3^{ik}$	$-\varphi_3^{ik}$	.
${}_B\chi_{13}(k)$	.	.	.	.	.	$(q-1)^2\varphi_3^{ik}$	$(q-1)^2\varphi_3^{ik}$	$-(q-1)\varphi_3^{ik}$	$-(q-1)\varphi_3^{ik}$	φ_3^{ik}	.
$\sum_{k=1}^{q} {}_B\chi_{14}(k)$	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_{15}(k)$	q^2	$-q^2$	.	.	.	$q(q-1)\varphi_3^{ik}$	$-q\varphi_3^{ik}$	.	.	.	.
${}_B\chi_{16}$	$-q^2$	q^2	.	.	.	.	.	.	.	.	.

(*continued on next page*)

Table A.6 (*continued*)

	$c_{6,1}(i)$	$c_{7,0}(i)$	$c_{7,1}(i)$	$c_{8,0}(i)$	$c_{8,1}(i)$	$c_{9,0}(i)$	$c_{9,1}(i)$	$c_{10,0}(i)$	$c_{10,1}(i)$	$c_{11,0}(i)$	$c_{11,1}(i)$	$c_{12,0}(i,j)$
${}_B\chi_1(k,l)$	ζ_1^{ik+2il}	ζ_1^{ik+il}	ζ_1^{ik+il}	ζ_1^{il}	ζ_1^{il}	$\zeta_3^{2ik}\zeta_1^{il}$	$\zeta_3^{2ik}\zeta_1^{il}$	$\zeta_3^{ik}\zeta_1^{il}$	$\zeta_3^{ik}\zeta_1^{il}$	ζ_3^{ik}	ζ_3^{ik}	$\zeta_3^{ik}\zeta_1^{jl}$
${}_B\chi_2(k)$	$-\zeta_1^{ik}$	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_3(k)$	.	.	.	.	.	$(q-1)\zeta_3^{2ik}$	$-\zeta_3^{2ik}$	.	.	.	.	.
${}_B\chi_4$	.	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_5(k)$	.	$(q^3-1)\zeta_1^{ik}$	$-\zeta_1^{ik}$	.	.	.	.	.	.	.	.	.
$\sum_{k=1}^{q+1} {}_B\chi_6(k)$	.	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_7(k)$	.	.	.	$(q^3-1)\zeta_1^{ik}$	$-\zeta_1^{ik}$	.	.	.	.	.	.	.
${}_B\chi_8$	.	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_9$	.	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_{10}$	.	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_{11}$	.	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_{12}(k)$	.	.	.	.	.	.	.	$(q-1)\zeta_3^{ik}$	$-\zeta_3^{ik}$	.	.	.
${}_B\chi_{13}(k)$	.	.	.	.	.	.	.	.	.	.	.	.
$\sum_{k=1}^{q} {}_B\chi_{14}(k)$	.	.	.	.	.	.	.	.	.	.	.	.
${}_B\chi_{15}(k)$	.	.	.	.	.	.	.	.	.	$(q-1)\zeta_3^{ik}$	$-\zeta_3^{ik}$	.
${}_B\chi_{16}$	.	.	.	.	.	.	.	.	.	.	.	.

(In this table, zeros are replaced by dots. See Table 2.2 in [6] for notation for the irrational character values.)

Table A.7
Parameterization of the semisimple conjugacy classes of P

Representative	Parameters	Number of classes
$h_1 := h(1,1,1,1)$		1
$h_4(i) := h(\tilde{\zeta}_1^i, \tilde{\zeta}_1^{2i}, \tilde{\zeta}_1^i, \tilde{\zeta}_1^i)$	$i = 0, \ldots, q-2$; $i \neq 0$	$q-2$
$h_5(i) := h(1, \tilde{\zeta}_1^i, 1, 1)$	$i = 0, \ldots, q-2$; $i \neq 0$	$q-2$
$h_6(i) := h(\tilde{\varphi}_3^i, 1, \tilde{\varphi}_3^{qi}, \tilde{\varphi}_3^{q^2 i})$	$i = 0, \ldots, q^2+q$; $i \neq 0$	$\frac{q^2+q}{2}$
$h_7(i) := h(\tilde{\zeta}_3^i, \tilde{\zeta}_1^i, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0, \ldots, q^3-2$ $i \neq (q-1)l,\ l = 0, \ldots, q^2+q+1$	q^3-q^2-q-2
$h_8(i) := h(\tilde{\zeta}_3^i, 1, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0, \ldots, q^3-2$ $i \neq (q-1)l,\ l = 0, \ldots, q^2+q+1$	$(q^3-q^2-q-2)/2$
$h_9(i,j) := h(\tilde{\zeta}_3^i, \tilde{\zeta}_1^j, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0, \ldots, q^3-2$ $j = 0, \ldots, q-2$ $i, j \neq 0$ $i \neq (q^2+q+1)l$ or $j \neq l$, $i \neq (q^2+q+1)l$ or $j \neq 2l$, $l = 0, \ldots, q-2$ $i \neq j+(q-1)l,\ l = 0, \ldots, q^2+q$ $i \neq 2j+(q-1)l,\ l = 0, \ldots, q^2+q$	$\frac{q^4-4q^3+2q^2-2q+12}{2}$
$h_{10}(i) := h(\tilde{\varphi}_6^i, 1, \tilde{\varphi}_6^{-qi}, \tilde{\varphi}_6^{(q-1)i})$	$i = 0, \ldots, q^2-q$; $i \neq 0$	$\frac{q^2-q}{2}$
$h_{11}(i) := h(\tilde{\xi}_3^i, 1, \tilde{\xi}_3^{-qi}, \tilde{\xi}_3^{q^2 i})$	$i = 0, \ldots, q^3$ $i \neq (q+1)l,\ l = 0, \ldots, q^2-q$	$\frac{q^3-q^2+q}{2}$
$h_{12}(i) := h(\tilde{\eta}_3^i, \tilde{\eta}_3^{(q^3+1)i}, \tilde{\eta}_3^{q^4 i}, \tilde{\eta}_3^{q^2 i})$	$i = 0, \ldots, q^4-q^3+q-2$ $i \neq (q-1)l,\ l = 0, \ldots, q^3$ $i \neq (q^3+1)l,\ l = 0, \ldots, q-2$	$\frac{q^4-2q^3}{2}$

Table A.8
The conjugacy classes of P

Notation	Representative	$\|C_P\|$	Fusion in ${}^3D_4(q)$
$c_{1,0}$	1	$q^{12}(q^6-1)(q-1)$	$c_{1,0}$
$c_{1,1}$	$x_{3\alpha+2\beta}(1)$	$q^{12}(q^6-1)$	$c_{1,1}$
$c_{1,2}$	$x_{3\alpha+\beta}(1)$	$q^{11}(q^3-1)$	$c_{1,1}$
$c_{1,3}$	$x_{2\alpha+\beta}(1)$	$q^{10}(q-1)$	$c_{1,2}$
$c_{1,4}$	$x_{\beta}(1)x_{3\alpha+\beta}(1)$	$2q^8(q^2+q+1)$	$c_{1,3}$
$c_{1,5}$	$x_{\alpha+\beta}(1)x_{2\alpha+\beta}(1)x_{3\alpha+\beta}(\zeta)$	$2q^8(q^2-q+1)$	$c_{1,4}$
$c_{1,6}$	$x_{\alpha}(1)$	$q^7(q-1)$	$c_{1,2}$
$c_{1,7}$	$x_{\alpha}(1)x_{2\alpha+\beta}(1)x_{3\alpha+2\beta}(1)$	$2q^7$	$c_{1,3}$
$c_{1,8}$	$x_{\alpha}(1)x_{2\alpha+\beta}(1)x_{3\alpha+2\beta}(\zeta)$	$2q^7$	$c_{1,4}$
$c_{1,9}(a')$	$x_{\alpha}(1)x_{\alpha+\beta}(a')$	q^6	$c_{1,5}$
$c_{1,10}$	$x_{\alpha}(1)x_{\beta}(1)$	$2q^4$	$c_{1,6}$
$c_{1,11}$	$x_{\alpha}(1)x_{\beta}(1)x_{2\alpha+\beta}(\eta)$	$2q^4$	$c_{1,7}$
$c_{4,0}(i)$	$h_4(i)$	$q^3(q^6-1)(q-1)$	$c_{3,0}(i)$
$c_{4,1}(i)$	$h_4(i)x_{\alpha}(1)$	$q^3(q-1)$	$c_{3,1}(i)$
$c_{5,0}(i)$	$h_5(i)$	$q^3(q^3-1)(q-1)$	$c_{3,0}(i)$
$c_{5,1}(i)$	$h_5(i)x_{2\alpha+\beta}(1)$	$q^3(q-1)$	$c_{3,1}(i)$
$c_{6,0}(i)$	$h_6(i)$	$q^3(q^3-1)(q-1)$	$c_{4,0}(i)$
$c_{6,1}(i)$	$h_6(i)x_{3\alpha+2\beta}(1)$	$q^3(q^3-1)$	$c_{4,1}(i)$
$c_{6,2}(i)$	$h_6(i)x_{3\alpha+\beta}(1)$	$q^2(q^3-1)$	$c_{4,1}(i)$
$c_{6,3}(i)$	$h_6(i)x_{\beta}(1)$	$q^2(q^3-1)$	$c_{4,1}(i)$
$c_{6,4}(i)$	$h_6(i)x_{\beta}(1)x_{3\alpha+\beta}(1)$	$q^2(q^2+q+1)$	$c_{4,2}(i)$
$c_{7,0}(i)$	$h_7(i)$	$q(q^3-1)(q-1)$	$c_{5,0}(i)$
$c_{7,1}(i)$	$h_7(i)x_{3\alpha+\beta}(1)$	$q(q^3-1)$	$c_{5,1}(i)$
$c_{8,0}(i)$	$h_8(i)$	$q(q^3-1)(q-1)$	$c_{5,0}(i)$
$c_{8,1}(i)$	$h_8(i)x_{3\alpha+2\beta}(1)$	$q(q^3-1)$	$c_{5,1}(i)$
$c_{9,0}(i,j)$	$h_9(i,j)$	$(q^3-1)(q-1)$	$c_{6,0}(i,j)$
$c_{10,0}(i)$	$h_{10}(i)$	$q^3(q^3+1)(q-1)$	$c_{9,0}(i)$
$c_{10,1}(i)$	$h_{10}(i)x_{3\alpha+2\beta}(1)$	$q^3(q^3+1)$	$c_{9,1}(i)$
$c_{10,2}(i)$	$h_{10}(i)x_{\beta}(s)x_{3\alpha+\beta}(s^q)x_{3\alpha+2\beta}(r)$	$q^2(q^2-q+1)$	$c_{9,2}(i)$
$c_{11,0}(i)$	$h_{11}(i)$	$q(q^3+1)(q-1)$	$c_{10,0}(i)$
$c_{11,1}(i)$	$h_{11}(i)x_{3\alpha+2\beta}(1)$	$q(q^3+1)$	$c_{10,1}(i)$
$c_{12,0}(i)$	$h_{12}(i)$	$(q^3+1)(q-1)$	$c_{11,0}(i)$

(The parameter a' in the representatives for the conjugacy classes of type $c_{1,9}$ runs through the set I_3 with $|I_3| = q+1$ which is defined in Section 3. The elements $\zeta, \eta, r, s \in \mathbb{F}$ are the same as in Table A.2.)

Table A.9
Parameterization of the irreducible characters of P

Character	Parameters	Number of characters
${}_P\chi_1(k)$	$k = 0, \dots, q-2$	$q-1$
${}_P\chi_2(k)$	$k = 0, \dots, q-2$	$q-1$
${}_P\chi_3(k,l)$	$k = 0, \dots, q^3-2; l = 0, \dots, q-2; k \neq 0$	$(q^3-2)(q-1)/2$
${}_P\chi_4(k)$	$k = 0, \dots, q^4-q^3+q-2$ $k \neq (q^3+1)m,\ m = 0, \dots, q-2$	$\frac{1}{2}q^3(q-1)$
${}_P\chi_5(k)$	$k = 0, \dots, q^3-2$	q^3-1
${}_P\chi_6$		1
${}_P\chi_7(k)$	$k = 0, \dots, q-2$	$q-1$
${}_P\chi_8(k)$	$k = 1, \dots, q+1$	$q+1$
${}_P\chi_9$		1
${}_P\chi_{10}$		1
${}_P\chi_{11}(k)$	$k = 0, \dots, q^2+q; k \neq 0$	$(q^2+q)/2$
${}_P\chi_{12}$		1
${}_P\chi_{13}$		1
${}_P\chi_{14}(k)$	$k = 0, \dots, q^2-q; k \neq 0$	$\frac{1}{2}(q^2-q)$
${}_P\chi_{15}$		1
${}_P\chi_{16}$		1
${}_P\chi_{17}(k)$	$k = 0, \dots, q^3-2; k \neq 0$	$(q^3-2)/2$
${}_P\chi_{18}(k)$	$k = 0, \dots, q^3; k \neq 0$	$\frac{1}{2}q^3$

Table A.10
The character table of P

	$c_{1,0}$	$c_{1,1}$	$c_{1,2}$	$c_{1,3}$	$c_{1,4}$	$c_{1,5}$
${}_P\chi_1(k)$	1	1	1	1	1	1
${}_P\chi_2(k)$	q^3	q^3	q^3	q^3	q^3	q^3
${}_P\chi_3(k,l)$	q^3+1	q^3+1	q^3+1	q^3+1	q^3+1	q^3+1
${}_P\chi_4(k)$	q^3-1	q^3-1	q^3-1	q^3-1	q^3-1	q^3-1
${}_P\chi_5(k)$	$(q-1)(q^3+1)$	$(q-1)(q^3+1)$	$-(q^3-q+1)$	$q-1$	q^2+q-1	$-(q^2-q+1)$
${}_P\chi_6$	$(q-1)(q^6-1)$	$(q-1)(q^6-1)$	$-(q^3-1)(q^3-q+1)$	$(q-1)(q^3-1)$	$(q^3-1)(q^2+q-1)$	$-(q^2-q+1)(q^3-1)$
${}_P\chi_7(k)$	$q(q^6-1)$	$q(q^6-1)$	$q(q^3-1)$	$q(q^3-q^2-1)$	$-q(q^2+q+1)$	$-q(q^2-q+1)$
$\sum_{k=1}^{q+1} {}_P\chi_8(k)$	$q(q-1)(q+1)(q^6-1)$	$q(q-1)(q+1)(q^6-1)$	$q(q-1)(q+1)(q^3-1)$	$q(q^2-1)(q^3-q^2-1)$	$-q(q+1)(q^3-1)$	$-q(q-1)(q^3+1)$
${}_P\chi_9$	$\frac{1}{2}q^3(q-1)^2(q^3+1)$	$\frac{1}{2}q^3(q-1)^2(q^3+1)$	$\frac{1}{2}q^3(q-1)(q^2+q-1)$	$-\frac{1}{2}q^3(q-1)$	q^3	$\cdot$
${}_P\chi_{10}$	$\frac{1}{2}q^3(q-1)^2(q^3+1)$	$\frac{1}{2}q^3(q-1)^2(q^3+1)$	$\frac{1}{2}q^3(q-1)(q^2+q-1)$	$-\frac{1}{2}q^3(q-1)$	q^3	$\cdot$
${}_P\chi_{11}(k)$	$q^3(q-1)^2(q^3+1)$	$q^3(q-1)^2(q^3+1)$	$q^3(q-1)(q^2+q-1)$	$-q^3(q-1)$	$2q^3$	$\cdot$
${}_P\chi_{12}$	$\frac{1}{2}q^3(q^2-1)(q^3-1)$	$\frac{1}{2}q^3(q^2-1)(q^3-1)$	$-\frac{1}{2}q^3(q^3-1)$	$-\frac{1}{2}q^3(q-1)$	$\cdot$	q^3
${}_P\chi_{13}$	$\frac{1}{2}q^3(q^2-1)(q^3-1)$	$\frac{1}{2}q^3(q^2-1)(q^3-1)$	$-\frac{1}{2}q^3(q^3-1)$	$-\frac{1}{2}q^3(q-1)$	$\cdot$	q^3
${}_P\chi_{14}(k)$	$q^3(q^2-1)(q^3-1)$	$q^3(q^2-1)(q^3-1)$	$-q^3(q^3-1)$	$-q^3(q-1)$	$\cdot$	$2q^3$
${}_P\chi_{15}$	$q^4(q-1)$	$-q^4$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_P\chi_{16}$	$q^7(q-1)$	$-q^7$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_P\chi_{17}(k)$	$q^4(q-1)(q^3+1)$	$-q^4(q^3+1)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_P\chi_{18}(k)$	$q^4(q-1)(q^3-1)$	$-q^4(q^3-1)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

Table A.10 (*continued*)

	$c_{1,6}$	$c_{1,7}$	$c_{1,8}$	$c_{1,9}(a')$	$c_{1,10}$	$c_{1,11}$	$c_{4,0}(i)$	$c_{4,1}(i)$	$c_{5,0}(i)$	$c_{5,1}(i)$	$c_{6,0}(i)$
$_P\chi_1(k)$	1	1	1	1	1	1	ζ_1^{2ik}	ζ_1^{2ik}	ζ_1^{ik}	ζ_1^{ik}	1
$_P\chi_2(k)$	.	.	.	.	.	.	$q^3\zeta_1^{2ik}$	.	ζ_1^{ik}	ζ_1^{ik}	1
$_P\chi_3(k,l)$	1	1	1	1	1	1	$(q^3+1)\zeta_1^{ik+2il}$	ζ_1^{ik+2il}	$\zeta_1^{ik+il}+\zeta_1^{il}$	$\zeta_1^{ik+il}+\zeta_1^{il}$	$\varphi_3^{ik}+\varphi_3^{-ik}$
$_P\chi_4(k)$	-1	-1	-1	-1	-1	-1	$(q^3-1)\zeta_1^{ik}$	$-\zeta_1^{ik}$	.	.	.
$_P\chi_5(k)$	$q-1$	$q-1$	$q-1$	$q-1$	-1	-1	.	.	.	.	$(q-1)(\varphi_3^{ik}+\varphi_3^{-ik})$
$_P\chi_6$	$-(q-1)$	$-(q-1)$	$-(q-1)$	$-(q-1)$	1	1	.	.	.	.	.
$_P\chi_7(k)$	$q(q^2-1)$	$q(q^2-1)$	$q(q^2-1)$	$-q$	.	.	.	.	$(q^3-1)\zeta_1^{ik}$	$-\zeta_1^{ik}$	.
$\sum_{k=1}^{q+1} {}_P\chi_8(k)$	$-q(q^2-1)$	$-q(q^2-1)$	$-q(q^2-1)$	q	.	.	.	.	.	.	.
$_P\chi_9$	$-\frac{1}{2}q^3(q-1)$	$\frac{1}{2}q^3$	$\frac{1}{2}q^3$	.	$-\frac{1}{2}q^2$	$\frac{1}{2}q^2$	.	.	.	.	$(q-1)^2$
$_P\chi_{10}$	$\frac{1}{2}q^3(q-1)$	$-\frac{1}{2}q^3$	$-\frac{1}{2}q^3$	.	$\frac{1}{2}q^2$	$-\frac{1}{2}q^2$	.	.	.	.	$(q-1)^2$
$_P\chi_{11}(k)$	$(q-1)^2(\varphi_3^{ik}+\varphi_3^{-ik})$	.	.	.	.	.	.	.	.	.	.
$_P\chi_{12}$	$-\frac{1}{2}q^3(q-1)$	$\frac{1}{2}q^3$	$\frac{1}{2}q^3$	.	$\frac{1}{2}q^2$	$-\frac{1}{2}q^2$	.	.	.	.	.
$_P\chi_{13}$	$\frac{1}{2}q^3(q-1)$	$-\frac{1}{2}q^3$	$-\frac{1}{2}q^3$	.	$-\frac{1}{2}q^2$	$\frac{1}{2}q^2$	.	.	.	.	.
$_P\chi_{14}(k)$	.	.	.	.	.	.	.	.	.	.	.
$_P\chi_{15}$	.	q^2	$-q^2$	.	.	.	.	.	.	.	$q(q-1)$
$_P\chi_{16}$	.	.	.	.	.	.	.	.	.	.	$q(q-1)$
$_P\chi_{17}(k)$	.	q^2	$-q^2$	.	.	.	.	.	.	.	$q(q-1)(\varphi_3^{ik}+\varphi_3^{-ik})$
$_P\chi_{18}(k)$	.	$-q^2$	q^2	.	.	.	.	.	.	.	.

(*continued on next page*)

Table A.10 (*continued*)

	$c_{6,1}(i)$	$c_{6,2}(i)$	$c_{6,3}(i)$	$c_{6,4}(i)$	$c_{7,0}(i)$	$c_{7,1}(i)$	$c_{8,0}(i)$
${}_P\chi_1(k)$	1	1	1	1	ζ_1^{ik}	ζ_1^{ik}	1
${}_P\chi_2(k)$	1	1	1	1	ζ_1^{ik}	ζ_1^{ik}	1
${}_P\chi_3(k,l)$	$\varphi_3^{ik}+\varphi_3^{-ik}$	$\varphi_3^{ik}+\varphi_3^{-ik}$	$\varphi_3^{ik}+\varphi_3^{-ik}$	$\varphi_3^{ik}+\varphi_3^{-ik}$	$\zeta_3^{ik}\zeta_1^{il}+\zeta_3^{(q+1)qik}\zeta_1^{il}$	$\zeta_3^{ik}\zeta_1^{il}+\zeta_3^{(q+1)qik}\zeta_1^{il}$	$\zeta_3^{ik}+\zeta_3^{-ik}$
${}_P\chi_4(k)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_P\chi_5(k)$	$(q-1)(\varphi_3^{ik}+\varphi_3^{-ik})$	$q\varphi_3^{ik}-\varphi_3^{-ik}-\varphi_3^{ik}$	$q\varphi_3^{-ik}-\varphi_3^{-ik}-\varphi_3^{ik}$	$-\varphi_3^{-ik}-\varphi_3^{ik}$	$(q-1)\zeta_3^{(q+1)qik}$	$-\zeta_3^{(q+1)qik}$	$\cdot$
${}_P\chi_6$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_P\chi_7(k)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\sum_{k=1}^{q+1} {}_P\chi_8(k)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_P\chi_9$	$(q-1)^2$	$-(q-1)$	$-(q-1)$	1	$\cdot$	$\cdot$	$\cdot$
${}_P\chi_{10}$	$(q-1)^2$	$-(q-1)$	$-(q-1)$	1	$\cdot$	$\cdot$	$\cdot$
${}_P\chi_{11}(k)$	$(q-1)^2(\varphi_3^{ik}+\varphi_3^{-ik})$	$-(q-1)(\varphi_3^{ik}+\varphi_3^{-ik})$	$-(q-1)(\varphi_3^{ik}+\varphi_3^{-ik})$	$\varphi_3^{ik}+\varphi_3^{-ik}$	$\cdot$	$\cdot$	$\cdot$
${}_P\chi_{12}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_P\chi_{13}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_P\chi_{14}(k)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_P\chi_{15}$	$-q$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$q-1$
${}_P\chi_{16}$	$-q$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$q-1$
${}_P\chi_{17}(k)$	$-q(\varphi_3^{ik}+\varphi_3^{-ik})$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$(q-1)(\zeta_3^{ik}+\zeta_3^{-ik})$
${}_P\chi_{18}(k)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

Table A.10 (*continued*)

	$c_{8,1}(i)$	$c_{9,0}(i,j)$	$c_{10,0}(i)$	$c_{10,1}(i)$	$c_{10,2}(i)$	$c_{11,0}(i)$	$c_{11,1}(i)$	$c_{12,0}(i)$
${}_P\chi_1(k)$	1	ζ_1^{jk}	1	1	1	1	1	ζ_1^{ik}
${}_P\chi_2(k)$	1	ζ_1^{jk}	-1	-1	-1	-1	-1	$-\zeta_1^{ik}$
${}_P\chi_3(k,l)$	$\zeta_3^{ik}+\zeta_3^{-ik}$	$\zeta_3^{ik}\zeta_1^{jl}+\zeta_3^{-ik}\zeta_1^{(k+l)j}$	.	.	.	.	.	.
${}_P\chi_4(k)$	.	.	$-\varphi_6^{ik}-\varphi_6^{-ik}$	$-\varphi_6^{ik}-\varphi_6^{-ik}$	$-\varphi_6^{ik}-\varphi_6^{-ik}$	$-\xi_3^{ik}-\xi_3^{-ik}$	$-\xi_3^{ik}-\xi_3^{-ik}$	$-\eta_3^{ik}-\eta_3^{q^3ik}$
${}_P\chi_5(k)$	.	.	.	.	.	.	.	.
${}_P\chi_6$	.	.	.	.	.	.	.	.
${}_P\chi_7(k)$	.	.	.	.	.	.	.	.
$\sum_{k=1}^{q+1} {}_P\chi_8(k)$	.	.	.	.	.	.	.	.
${}_P\chi_9$	.	.	.	.	.	.	.	.
${}_P\chi_{10}$	.	.	.	.	.	.	.	.
${}_P\chi_{11}(k)$	.	.	.	.	.	.	.	.
${}_P\chi_{12}$	.	.	q^2-1	q^2-1	-1	.	.	.
${}_P\chi_{13}$	.	.	q^2-1	q^2-1	-1	.	.	.
${}_P\chi_{14}(k)$	.	.	$(q^2-1)(\varphi_6^{ik}+\varphi_6^{-ik})$	$(q^2-1)(\varphi_6^{ik}+\varphi_6^{-ik})$	$-\varphi_6^{ik}-\varphi_6^{-ik}$	.	.	.
${}_P\chi_{15}$	-1	.	$-q(q-1)$	q	.	$q-1$	-1	.
${}_P\chi_{16}$	-1	.	$q(q-1)$	$-q$	.	$-(q-1)$	1	.
${}_P\chi_{17}(k)$	$-\zeta_3^{-ik}-\zeta_3^{ik}$	.	.	.	.	.	.	.
${}_P\chi_{18}(k)$	.	.	$q(q-1)(\varphi_6^{ik}+\varphi_6^{-ik})$	$-q(\varphi_6^{ik}+\varphi_6^{-ik})$	.	$-(q-1)(\xi_3^{-ik}+\xi_3^{ik})$	$\xi_3^{-ik}+\xi_3^{ik}$	.

(In this table, zeros are replaced by dots. See Table 2.2 in [6] for notation for the irrational character values.)

Table A.11
Parameterization of the semisimple conjugacy classes of Q

Representative	Parameters	Number of classes
$h_1 := h(1,1,1,1)$		1
$h_4(i) := h(\tilde{\zeta}_1^i, \tilde{\zeta}_1^{2i}, \tilde{\zeta}_1^i, \tilde{\zeta}_1^i)$	$i = 0,\dots,q-2;\ i \neq 0$	$q-2$
$h_5(i) := h(1, \tilde{\zeta}_1^i, 1, 1)$	$i = 0,\dots,q-2;\ i \neq 0$	$\frac{q-2}{2}$
$h_6(i) := h(\tilde{\varphi}_3^i, 1, \tilde{\varphi}_3^{qi}, \tilde{\varphi}_3^{q^2 i})$	$i = 0,\dots,q^2+q;\ i \neq 0$	q^2+q
$h_7(i) := h(\tilde{\zeta}_3^{2i}, \tilde{\zeta}_1^i, \tilde{\zeta}_3^{2qi}, \tilde{\zeta}_3^{2q^2 i})$	$i = 0,\dots,q^3-2$ $i \neq (q-1)l,\ l = 0,\dots,q^2+q+1$	q^3-q^2-q-2
$h_8(i) := h(\tilde{\zeta}_3^i, \tilde{\zeta}_1^i, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0,\dots,q^3-2$ $i \neq (q-1)l,\ l = 0,\dots,q^2+q+1$	q^3-q^2-q-2
$h_9(i,j) := h(\tilde{\zeta}_3^i, \tilde{\zeta}_1^j, \tilde{\zeta}_3^{qi}, \tilde{\zeta}_3^{q^2 i})$	$i = 0,\dots,q^3-2;\ j = 0,\dots,q-2$ $i, j \neq 0$ $i \neq (q^2+q+1)l$ or $j \neq l$, $l = 0,\dots,q-2$ $i \neq (q^2+q+1)l$ or $j \neq 2l$, $l = 0,\dots,q-2$ $i \neq j+(q-1)l,\ l = 0,\dots,q^2+q$ $i \neq 2j+(q-1)l,\ l = 0,\dots,q^2+q$	$\frac{q^4-4q^3+2q^2-2q+12}{2}$
$h_{10}(i) := h(1, \tilde{\xi}_1^{ci}, 1, 1)$	$i = 0,\dots,q;\ i \neq 0$	$\frac{q}{2}$
$h_{11}(i) := h(\tilde{\mu}_3^{(q^2+q)ci}, \tilde{\mu}_3^{(q^2+q+1)ci}, \tilde{\mu}_3^{(q^3+q^2)ci}, \tilde{\mu}_3^{(q+1)ci})$	$i = 0,\dots,q^4+q^3-q-2$ $i \neq (q+1)l,\ l = 0,\dots,q^3-2$ $i \neq (q^3-1)l,\ l = 0,\dots,q$	$\frac{q(q^3-2)}{2}$

(c is the multiplicative inverse of q^2+q-1 modulo $(q^3-1)(q+1)$.)

Table A.12
The conjugacy classes of Q

Notation	Representative	$\lvert C_Q\rvert$	Fusion in ${}^3D_4(q)$
$c_{1,0}$	1	$q^{12}(q^3-1)(q^2-1)$	$c_{1,0}$
$c_{1,1}$	$x_{3\alpha+2\beta}(1)$	$q^{12}(q^3-1)$	$c_{1,1}$
$c_{1,2}$	$x_{2\alpha+\beta}(1)$	$q^{10}(q^2-1)$	$c_{1,2}$
$c_{1,3}$	$x_{\alpha+\beta}(1)$	$q^8(q-1)$	$c_{1,2}$
$c_{1,4}$	$x_{\alpha+\beta}(1)x_{2\alpha+\beta}(1)x_{3\alpha+\beta}(1)$	$2q^8$	$c_{1,3}$
$c_{1,5}$	$x_{\alpha+\beta}(1)x_{2\alpha+\beta}(1)x_{3\alpha+\beta}(\zeta)$	$2q^8$	$c_{1,4}$
$c_{1,6}$	$x_\alpha(1)x_{\alpha+\beta}(a)$	q^6	$c_{1,5}$
$c_{1,7}$	$x_\beta(1)$	$q^8(q^3-1)$	$c_{1,1}$
$c_{1,8}$	$x_\beta(1)x_{2\alpha+\beta}(1)$	q^8	$c_{1,2}$
$c_{1,9}(t)$	$x_\beta(1)x_{2\alpha+\beta}(1)x_{3\alpha+\beta}(t)$	q^8	$c_{1,4}$
$c_{1,10}$	$x_\beta(1)x_{3\alpha+\beta}(1)$	$q^8(q^2+q+1)$	$c_{1,3}$
$c_{1,11}(t)$	$x_\beta(1)x_{2\alpha+\beta}(1)x_{3\alpha+\beta}(t)$	q^8	$c_{1,3}$
$c_{1,12}$	$x_\alpha(1)x_\beta(1)$	$2q^4$	$c_{1,6}$
$c_{1,13}$	$x_\alpha(1)x_\beta(1)x_{2\alpha+\beta}(\eta)$	$2q^4$	$c_{1,7}$
$c_{4,0}(i)$	$h_4(i)$	$q^3(q^3-1)(q-1)$	$c_{3,0}(i)$
$c_{4,1}(i)$	$h_4(i)x_\alpha(1)$	$q^3(q-1)$	$c_{3,1}(i)$
$c_{5,0}(i)$	$h_5(i)$	$q^3(q^3-1)(q-1)$	$c_{3,0}(i)$
$c_{5,1}(i)$	$h_5(i)x_{2\alpha+\beta}(1)$	$q^3(q-1)$	$c_{3,1}(i)$
$c_{6,0}(i)$	$h_6(i)$	$q^3(q^3-1)(q^2-1)$	$c_{4,0}(i)$
$c_{6,1}(i)$	$h_6(i)x_{3\alpha+2\beta}(1)$	$q^3(q^3-1)$	$c_{4,1}(i)$
$c_{6,2}(i)$	$h_6(i)x_\beta(1)$	$q^2(q^3-1)$	$c_{4,1}(i)$
$c_{6,3}(i)$	$h_6(i)x_\beta(1)x_{3\alpha+\beta}(1)$	$q^2(q^2+q+1)$	$c_{4,2}(i)$
$c_{7,0}(i)$	$h_7(i)$	$q(q^3-1)(q^2-1)$	$c_{5,0}((q^2+q-1)i)$
$c_{7,1}(i)$	$h_7(i)x_\beta(1)$	$q(q^3-1)$	$c_{5,1}((q^2+q-1)i)$
$c_{8,0}(i)$	$h_8(i)$	$q(q^3-1)(q-1)$	$c_{5,0}(i)$
$c_{8,1}(i)$	$h_8(i)x_{3\alpha+\beta}(1)$	$q(q^3-1)$	$c_{5,1}(i)$
$c_{9,0}(i,j)$	$h_9(i,j)$	$(q^3-1)(q-1)$	$c_{6,0}(i,j)$
$c_{10,0}(i)$	$h_{10}(i)$	$q^3(q^3-1)(q+1)$	$c_{7,0}(i)$
$c_{10,1}(i)$	$h_{10}(i)x_{2\alpha+\beta}(1)$	$q^3(q+1)$	$c_{7,1}(i)$
$c_{11,0}(i)$	$h_{11}(i)$	$(q^3-1)(q+1)$	$c_{8,0}(i)$

(The parameter t in the representatives for the conjugacy classes of type $c_{1,9}$ and $c_{1,11}$ runs through the sets I_1, I_2 with $|I_1| = q/2$ and $|I_2| = q/2-1$, respectively, which are defined in Section 3. The elements $\zeta, \eta, a \in \mathbb{F}$ are the same as in Table A.2.)

Table A.13
Parameterization of the irreducible characters of Q

Character	Parameters	Number of characters
${}_Q\chi_1(k)$	$k = 0, \ldots, q^3 - 2$	$q^3 - 1$
${}_Q\chi_2(k)$	$k = 0, \ldots, q^3 - 2$	$q^3 - 1$
${}_Q\chi_3(k, l)$	$k = 0, \ldots, q^3 - 2$ $l = 0, \ldots, q - 2;\ l \neq 0$	$\frac{1}{2}(q^3 - 1)(q - 2)$
${}_Q\chi_4(k)$	$k = 0, \ldots, q^4 + q^3 - q - 2$ $k \neq (q + 1)m, m = 0, \ldots, q^3 - 2$	$\frac{1}{2}q(q^3 - 1)$
${}_Q\chi_5(k)$	$k = 0, \ldots, q - 2$	$q - 1$
${}_Q\chi_6$		1
${}_Q\chi_7$		1
${}_Q\chi_8$		1
${}_Q\chi_9$		1
${}_Q\chi_{10}(k)$	$k = 0, \ldots, q - 2; k \neq 0$	$(q - 2)/2$
${}_Q\chi_{11}$		1
${}_Q\chi_{12}$		1
${}_Q\chi_{13}(k)$	$k = 0, \ldots, q; k \neq 0$	$q/2$
${}_Q\chi_{14}(k)$	$k = 0, \ldots, q^3 - 2$	$q^3 - 1$
${}_Q\chi_{15}(k)$	$k = 0, \ldots, q^2 + q$	$q^2 + q + 1$
${}_Q\chi_{16}(k)$	$k = 1, \ldots, q$	q

Table A.14
The character table of Q

	$c_{1,0}$	$c_{1,1}$	$c_{1,2}$	$c_{1,3}$	$c_{1,4}$	$c_{1,5}$
${}_Q\chi_1(k)$	1	1	1	1	1	1
${}_Q\chi_2(k)$	q	q	q	q	q	q
${}_Q\chi_3(k,l)$	$q+1$	$q+1$	$q+1$	$q+1$	$q+1$	$q+1$
${}_Q\chi_4(k)$	$q-1$	$q-1$	$q-1$	$q-1$	$q-1$	$q-1$
${}_Q\chi_5(k)$	$(q+1)(q^3-1)$	$(q-1)(q+1)(q^2+q+1)$	$(q+1)(q^3-1)$	q^3-q-1	q^3-q-1	q^3-q-1
${}_Q\chi_6$	$(q-1)(q+1)(q^3-1)$	$(q-1)^2(q+1)(q^2+q+1)$	$(q-1)(q+1)(q^3-1)$	$(q-1)(q^3-q-1)$	$(q-1)(q^3-q-1)$	$(q-1)(q^3-q-1)$
${}_Q\chi_7$	$q(q^2-1)(q^3-1)$	$q(q^2-1)(q^3-1)$	$q(q^2-1)(q^3-1)$	$-q(q^2-1)$	$-q(q^2-1)$	$-q(q^2-1)$
${}_Q\chi_8$	$\frac{1}{2}q^3(q+1)(q^3-1)\vert$	$\frac{1}{2}q^3(q+1)(q^3-1)$	$-\frac{1}{2}q^3(q+1)$	$\frac{1}{2}q^3(q-1)$	$-\frac{1}{2}q^3$	$-\frac{1}{2}q^3$
${}_Q\chi_9$	$\frac{1}{2}q^3(q+1)(q^3-1)$	$\frac{1}{2}q^3(q+1)(q^3-1)$	$-\frac{1}{2}q^3(q+1)$	$\frac{1}{2}q^3(q-1)$	$-\frac{1}{2}q^3$	$-\frac{1}{2}q^3$
${}_Q\chi_{10}(k)$	$q^3(q+1)(q^3-1)$	$q^3(q+1)(q^3-1)$	$-q^3(q+1)$	$q^3(q-1)$	$-q^3$	$-q^3$
${}_Q\chi_{11}$	$\frac{1}{2}q^3(q-1)(q^3-1)$	$\frac{1}{2}q^3(q-1)(q^3-1)$	$-\frac{1}{2}(q^4-q^3)$	$-\frac{1}{2}(q^4-q^3)$	$\frac{1}{2}q^3$	$\frac{1}{2}q^3$
${}_Q\chi_{12}$	$\frac{1}{2}q^3(q-1)(q^3-1)$	$\frac{1}{2}q^3(q-1)(q^3-1)$	$-\frac{1}{2}(q^4-q^3)$	$-\frac{1}{2}(q^4-q^3)$	$\frac{1}{2}q^3$	$\frac{1}{2}q^3$
${}_Q\chi_{13}(k)$	$q^3(q-1)(q^3-1)$	$q^3(q-1)(q^3-1)$	$-(q^4-q^3)$	$-(q^4-q^3)$	q^3	q^3
${}_Q\chi_{14}(k)$	$q^3(q-1)(q+1)$	$-q^3$	$\cdot$	$\cdot$	q^2	$-q^2$
${}_Q\chi_{15}(k)$	$q^3(q-1)^2(q+1)$	$-q^3(q-1)$	$\cdot$	$\cdot$	$q^2(q-1)$	$-q^2(q-1)$
$\sum_{k=1}^{q} {}_Q\chi_{16}(k)$	$q^4(q-1)(q+1)(q^3-1)$	$-q^4(q-1)(q^2+q+1)$	$\cdot$	$\cdot$	$-q^3$	q^3

(*continued on next page*)

Table A.14 (*continued*)

	$c_{1,6}$	$c_{1,7}$	$c_{1,8}$	$c_{1,9}(t)$	$c_{1,10}$	$C_{1,11}(t)$	$c_{1,12}$
${}_Q\chi_1(k)$	1	1	1	1	1	1	1
${}_Q\chi_2(k)$	q	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_Q\chi_3(k,l)$	$q+1$	1	1	1	1	1	1
${}_Q\chi_4(k)$	$q-1$	-1	-1	-1	-1	-1	-1
${}_Q\chi_5(k)$	$-(q+1)$	$(q-1)(q^2+q+1)$	$(q-1)(q^2+q+1)$	$(q-1)(q^2+q+1)$	$(q-1)(q^2+q+1)$	$(q-1)(q^2+q+1)$	-1
${}_Q\chi_6$	$-(q-1)(q+1)$	$-(q-1)(q^2+q+1)$	$-(q-1)(q^2+q+1)$	$-(q-1)(q^2+q+1)$	$-(q-1)(q^2+q+1)$	$-(q-1)(q^2+q+1)$	1
${}_Q\chi_7$	q	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_Q\chi_8$	$\cdot$	$-\frac{1}{2}q^2(q^3-1)$	$\frac{1}{2}q^2$	$\frac{1}{2}q^2$	$-\frac{1}{2}q^2(q^3-1)$	$\frac{1}{2}q^2$	$-\frac{1}{2}q^2$
${}_Q\chi_9$	$\cdot$	$\frac{1}{2}q^2(q^3-1)$	$-\frac{1}{2}q^2$	$-\frac{1}{2}q^2$	$\frac{1}{2}q^2(q^3-1)$	$-\frac{1}{2}q^2$	$\frac{1}{2}q^2$
${}_Q\chi_{10}(k)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_Q\chi_{11}$	$\cdot$	$-\frac{1}{2}(q^5-q^2)$	$\frac{1}{2}q^2$	$\frac{1}{2}q^2$	$-\frac{1}{2}(q^5-q^2)$	$\frac{1}{2}q^2$	$\frac{1}{2}q^2$
${}_Q\chi_{12}$	$\cdot$	$\frac{1}{2}(q^5-q^2)$	$-\frac{1}{2}q^2$	$-\frac{1}{2}q^2$	$\frac{1}{2}(q^5-q^2)$	$-\frac{1}{2}q^2$	$-\frac{1}{2}q^2$
${}_Q\chi_{13}(k)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_Q\chi_{14}(k)$	$\cdot$	$-q(q-1)(q+1)$	q	$-q(q-1)$	$q(q+1)$	$q(q+1)$	$\cdot$
${}_Q\chi_{15}(k)$	$\cdot$	$q(q-1)(q+1)$	$-q$	$q(q-1)$	$-q(q+1)$	$-q(q+1)$	$\cdot$
$\sum_{k=1}^{q} {}_Q\chi_{16}(k)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

Table A.14 (*continued*)

	$c_{1,13}$	$c_{4,0}(i)$	$c_{4,1}(i)$	$c_{5,0}(i)$	$c_{5,1}(i)$	$c_{6,0}(i)$	$c_{6,1}(i)$	$c_{6,2}(i)$
${}_Q\chi_1(k)$	1	ζ_1^{ik}	ζ_1^{ik}	1	1	φ_3^{ik}	φ_3^{ik}	φ_3^{ik}
${}_Q\chi_2(k)$	$\cdot$	ζ_1^{ik}	ζ_1^{ik}	1	1	$q\varphi_3^{ik}$	$q\varphi_3^{ik}$	$\cdot$
${}_Q\chi_3(k,l)$	1	$\zeta_1^{ik+2il}+\zeta_1^{ik+il}$	$\zeta_1^{ik+2il}+\zeta_1^{ik+il}$	$\zeta_1^{-il}+\zeta_1^{il}$	$\zeta_1^{-il}+\zeta_1^{il}$	$(q+1)\varphi_3^{ik}$	$(q+1)\varphi_3^{ik}$	φ_3^{ik}
${}_Q\chi_4(k)$	-1	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$(q-1)\varphi_3^{ik}$	$(q-1)\varphi_3^{ik}$	$-\varphi_3^{ik}$
${}_Q\chi_5(k)$	-1	$(q-1)(q^2+q+1)\zeta_1^{ik}$	$-\zeta_1^{ik}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_Q\chi_6$	1	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_Q\chi_7$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_Q\chi_8$	$\frac{1}{2}q^2$	$\cdot$	$\cdot$	q^3-1	-1	$\cdot$	$\cdot$	$\cdot$
${}_Q\chi_9$	$-\frac{1}{2}q^2$	$\cdot$	$\cdot$	q^3-1	-1	$\cdot$	$\cdot$	$\cdot$
${}_Q\chi_{10}(k)$	$\cdot$	$\cdot$	$\cdot$	$(q^3-1)(\zeta_1^{ik}+\zeta_1^{-ik})$	$-\zeta_1^{ik}-\zeta_1^{-ik}$	$\cdot$	$\cdot$	$\cdot$
${}_Q\chi_{11}$	$-\frac{1}{2}q^2$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_Q\chi_{12}$	$\frac{1}{2}q^2$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_Q\chi_{13}(k)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
${}_Q\chi_{14}(k)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$(q-1)\varphi_3^{ik}(q+1)$	$-\varphi_3^{ik}$	$(q-1)\varphi_3^{ik}$
${}_Q\chi_{15}(k)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$(q-1)^2\varphi_3^{ik}(q+1)$	$-(q-1)\varphi_3^{ik}$	$-(q-1)\varphi_3^{ik}$
$\sum_{k=1}^{q}{}_Q\chi_{16}(k)$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

(*continued on next page*)

Table A.14 (*continued*)

	$c_{6,3}(i)$	$c_{7,0}(i)$	$c_{7,1}(i)$	$c_{8,0}(i)$	$c_{8,1}(i)$	$c_{9,0}(i,j)$	$c_{10,0}(i)$	$c_{10,1}(i)$	$c_{11,0}(i)$
${}_Q\chi_1(k)$	φ_3^{ik}	ζ_3^{2ik}	ζ_3^{2ik}	ζ_3^{ik}	ζ_3^{ik}	ζ_3^{ik}	1	1	$\zeta_3^{q^2ik-qik+ik}$
${}_Q\chi_2(k)$	.	$q\zeta_3^{2ik}$	.	ζ_3^{ik}	ζ_3^{ik}	ζ_3^{ik}	-1	-1	$-\zeta_3^{q^2ik-qik+ik}$
${}_Q\chi_3(k,l)$	φ_3^{ik}	$(q+1)\zeta_3^{2ik}\zeta_1^{il}$	$\zeta_3^{2ik}\zeta_1^{il}$	$\zeta_3^{ik}\zeta_1^{il}+\zeta_3^{ik}$	$\zeta_3^{ik}\zeta_1^{il}+\zeta_3^{ik}$	$\zeta_3^{ik}\zeta_1^{jl}+\zeta_3^{ik}\zeta_1^{(i-j)l}$	.	.	.
${}_Q\chi_4(k)$	$-\varphi_3^{ik}$	$(q-1)\zeta_3^{2ik}$	$-\zeta_3^{2ik}$	.	.	.	$-\xi_1^{-2ik}-\xi_1^{2ik}$	$-\xi_1^{-2ik}-\xi_1^{2ik}$	$-\mu_3^{2ik}-\mu_3^{2q^3ik}$
${}_Q\chi_5(k)$	.	.	.	.	.	.	.	.	.
${}_Q\chi_6$	.	.	.	.	.	.	.	.	.
${}_Q\chi_7$	.	.	.	.	.	.	.	.	.
${}_Q\chi_8$	.	.	.	.	.	.	.	.	.
${}_Q\chi_9$	.	.	.	.	.	.	.	.	.
${}_Q\chi_{10}(k)$	.	.	.	.	.	.	.	.	.
${}_Q\chi_{11}$	.	.	.	.	.	.	q^3-1	-1	.
${}_Q\chi_{12}$	.	.	.	.	.	.	q^3-1	-1	.
${}_Q\chi_{13}(k)$	.	.	.	.	.	.	$(q^3-1)(\xi_1^{ik}+\xi_1^{-ik})$	$-\xi_1^{ik}-\xi_1^{-ik}$	.
${}_Q\chi_{14}(k)$	$-\varphi_3^{ik}$	.	.	$(q-1)\zeta_3^{ik}$	$-\zeta_3^{ik}$	.	.	.	.
${}_Q\chi_{15}(k)$	φ_3^{ik}	.	.	.	.	.	.	.	.
$\sum_{k=1}^{q} {}_Q\chi_{16}(k)$	.	.	.	.	.	.	.	.	.

(In this table, zeros are replaced by dots. See Table 2.2 in [6] for notation for the irrational character values.)

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