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Mickelsson algebras and Zhelobenko operators

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Abstract

We construct a family of automorphisms of Mickelsson algebra, satisfying braid group relations. The construction uses 'Zhelobenko cocycle' and includes the dynamical Weyl group action as a particular case.

Keywords: Contragredient Lie algebras of finite growth; Quantized universal enveloping algebras; Extremal projector; Mickelsson algebras; Zhelobenko cocycles; Braid group actions; Dynamical Weyl group

Contents

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1. Introduction

Mickelsson algebras were introduced in [M] for the study of Harish-Chandra modules of reductive groups. The Mickelsson algebra, related to a real reductive group, acts on the space of highest weight vectors of its maximal compact subgroup, and each irreducible Harish-Chandra module of the initial reductive group is uniquely characterized by this action.

A similar construction can be given for any associative algebra $\mathcal{A}$, which contains a universal enveloping algebra $U(\mathfrak{g})$ (or its q-analog) of a contragredient Lie algebra $\mathfrak{g}$ with a fixed Gauss decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}$. Namely, we define the Mickelsson algebra $S^{\mathfrak{n}}(\mathcal{A})$ as the quotient of the normalizer $N(\mathcal{A}\mathfrak{n})$ of the left ideal $\mathcal{A}\mathfrak{n}$ by this ideal. For any representation V of $\mathcal{A}$ the Mickelsson algebra $S^{\mathfrak{n}}(\mathcal{A})$ acts on the space $V^{\mathfrak{n}}$ of $\mathfrak{n}$-invariant vectors. This construction provides a reduction of a representation of $\mathcal{A}$ with respect to the action of $U(\mathfrak{g})$ and can be viewed as a counterpart of Hamiltonian reduction. It has been applied for various problems of representation theory, see the survey [T2] and references therein.

The structure of Mickelsson algebra simplifies after localizing it over a certain multiplicative subset of $U(\mathfrak{h})$, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$. The corresponding algebra $Z^{\mathfrak{n}}(\mathcal{A})$ is generated by a finite-dimensional space of generators, which obey quadratic-linear relations. These generators can be defined with a help of an extremal projector of Asherova–Smirnov–Tolstoy [AST]. An application of the extremal projector to the study of the Mickelsson algebras $Z^{\mathfrak{n}}(\mathcal{A})$ was proposed by Zhelobenko [Z2]. Besides, Zhelobenko developed the so called 'dual methods,' and constructed another set of generators of the Mickelsson algebra by means of a family of special operators, which form a cocycle on the Weyl group [Z1].

Later Mickelsson algebras appeared in the theory of dynamical quantum groups. Their basic ingredients, the intertwining operators between Verma modules and the tensor products of Verma modules with finite-dimensional representations actually form special Mickelsson algebras. Matrix coefficients of these intertwining operators are very useful in the study of quantum integrable models [ES]. Tarasov and Varchenko [TV] found the symmetries of the algebra of intertwining operators, which originate from the morphisms of Verma modules. They satisfy the braid group relations and transform the weights of the Cartan subalgebra by means of a shifted Weyl group action. These symmetries got the name of a 'dynamical Weyl group.' The theory of dynamical Weyl groups was generalized to the quantum groups setup in [EV]. The form of operators of the dynamical Weyl group is very close to the factorized expressions for the extremal projector and for the Zhelobenko cocycles [Z1]. However, the precise statements and the origin of such a relation were not clear. One of our goals is to clarify this relation.

In this paper we describe a family of symmetries for a wide class of Mickelsson algebras. They form a representation of the related braid group by automorphisms of the Mickelsson algebra $Z^{\mathfrak{n}}(\mathcal{A})$ and transform the Cartan elements by means of the shifted Weyl group action. Each generating automorphism is a product of the Zhelobenko 'cocycle' map q_{α_i} and of an automorphism T_i of the algebra $\mathcal{A}$, extending the action of the Weyl group on the Cartan subalgebra. The new feature of our approach is the homomorphism property of Zhelobenko maps, that was not noticed before. However, the proof of this property is not short and requires calculations with the extremal projector.

The construction of automorphisms of Mickelsson algebra $Z^{\mathfrak{n}}(\mathcal{A})$ is quite general. In particular, it covers the examples of Mickelsson algebras, related to reductions $\mathfrak{g}' \supset \mathfrak{g}$ from one reductive Lie algebra to another, and of the smash product $U(\mathfrak{g}) \ltimes S(V)$ of $U(\mathfrak{g})$ with the symmetric algebra of a $U(\mathfrak{g})$-module V. Here it becomes the dynamical Weyl group action after certain specialization of the Cartan elements of $\mathfrak{g}$. It can be applied to the construction of finite-dimensional representations of Yangians and of quantum affine algebras, see [KN1,KN2].

The paper is organized as follows. In Section 2 we collect all necessary information about the extremal projector and the required extensions of $U(\mathfrak{g})$.

In Section 3 for a fixed contragredient Lie algebra $\mathfrak{g}$ of finite growth we introduce a class of associative algebras $\mathcal{A}$, which we call $\mathfrak{g}$-admissible. They contain $U(\mathfrak{g})$ as a subalgebra, with a requirement that the adjoint action of $\mathfrak{g}$ in $\mathcal{A}$ has some special properties. In particular, as a $\mathfrak{g}$-module with respect to the adjoint action $\mathcal{A}$ is isomorphic to a tensor product of $U(\mathfrak{g})$ and of some subspace $\mathcal{V} \subset \mathcal{A}$. Mickelsson algebras, related to admissible algebras, have specific properties. The crucial one is the existence of two distinguished subspaces of generators z_v and z'_v, $v \in \mathcal{V}$.

Section 4 is an exposition of the 'Zhelobenko cocycle' construction [Z1]. We present it with complete proofs in order to eliminate the unnecessary restrictions, made in [Z1]. The story begins from the map q_α, which relates the universal Verma modules, attached to different maximal nilpotent subalgebras of $\mathfrak{g}$. The product of such operators over the system of positive roots maps the vectors $v \in \mathcal{V}$ to the generators z'_v of Mickelsson algebras. This invariant description proves the cocycle properties of the maps q_α.

Section 5 describes the homomorphism properties of the Zhelobenko maps. We prove first that for a simple root α_i the Zhelobenko map q_{α_i} establishes an isomorphism of a double coset algebra and of the Mickelsson algebra. This implies that the compositions $\check{\mathrm{q}}_i$ of the Zhelobenko maps q_{α_i} with the extensions of Weyl group automorphisms are automorphisms of the Mickelsson algebras, satisfying the braid group relations.

In Sections 6 and 8 we calculate the images of the generators of Mickelsson algebras and of the standard modules with respect to $\check{\mathrm{q}}_i$ and show that the dynamical Weyl group is a particular case of our construction. Section 7 is devoted to the Mickelsson algebra $Z_{\mathfrak{n}_-}(\mathcal{A})$, defined as the localizations of the quotient of the normalizer $N(\mathfrak{n}_-\mathcal{A})$ of the right ideal $\mathfrak{n}_-\mathcal{A}$ by this ideal. These algebras are used for the study of $\mathfrak{n}_-$-coinvariants of $\mathcal{A}$-modules.

Section 9 is a sketch of extensions of our constructions to the quantized universal enveloping algebras $U_q(\mathfrak{g})$. The new important detail here is that the compositions of the Zhelobenko maps q_{α_i} with the Lusztig automorphisms coincide with the compositions of q_{α_i} with the adjoint action of the Lusztig automorphisms, see Propositions 9.2 and 9.4. This allows us to prove both the homomorphisms properties and the braid group relations. We conclude with remarks about the range of assumptions on $\mathfrak{g}$-admissible algebras, used in the paper.

2. Extremal projector

In this section we review Zhelobenko's approach to the extremal projector of Asherova–Smirnov–Tolstoy [AST]. Our exposition follows [Z2] in the main details.

2.1. *Taylor extension of $U(\mathfrak{g})$*

Let $\mathfrak{g}$ be a contragredient Lie algebra of finite growth with symmetrizable Cartan matrix $a_{i,j}$, $i, j = 1, \ldots, r$. Let

$$\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n} \tag{2.1}$$

be its Gauss decomposition, where $\mathfrak{h}$ is a Cartan subalgebra, $\mathfrak{n} = \mathfrak{n}_+ \subset \mathfrak{b}$ and $\mathfrak{n}_- \subset \mathfrak{b}_-$ are nilradicals of two opposite Borel subalgebras $\mathfrak{b}_+$ (which is also denoted as $\mathfrak{b}$) and $\mathfrak{b}_-$. We use the notation Π for the system of simple positive roots; Δ_+, Δ_- and $\Delta = \Delta_+ \sqcup \Delta_-$ for the system of positive, negative and all roots; Δ_+^{re}, Δ_-^{re} and $\Delta^{\mathrm{re}} = \Delta_+^{\mathrm{re}} \sqcup \Delta_-^{\mathrm{re}}$ for the system of positive, negative and all real roots. Let $(\,,)$ be the scalar product in $\mathfrak{h}^*$, such that $(\alpha_i, \alpha_j) = d_i a_{i,j} = d_j a_{j,i}$, for $\alpha_i, \alpha_j \in \Pi$ and $d_i \in \mathbb{N}$.

Denote by $Q \subset \mathfrak{h}^*$ the root lattice, $Q = \mathbb{Z} \cdot \Delta$, and put $Q_\pm = \mathbb{Z}_{\geqslant 0} \cdot \Delta_\pm$. For any $\mathfrak{g}$-module M and $\mu \in \mathfrak{h}^*$ we denote by M_μ the subspace of elements $m \in M$, such that $h(m) = \langle \mu, h\rangle m$, in particular the space $U(\mathfrak{n})_\mu$ consists of elements $x \in U(\mathfrak{n})_\mu$, such that $[h, x] = \langle \mu, h\rangle x$. We adopt the normalization of the Chevalley generators $e_{\alpha_i} \in \mathfrak{n}$, $e_{-\alpha_i} \in \mathfrak{n}_-$, and of the coroots $h_{\alpha_i} = \alpha_i{}^\vee \in \mathfrak{h}$, where $\alpha_i \in \Pi$, such that

$$[e_{\alpha_i}, e_{-\alpha_j}] = \delta_{i,j} h_{\alpha_i}, \qquad [h_{\alpha_i}, e_{\pm\alpha_j}] = \pm a_{i,j} e_{\pm\alpha_j} = \pm\langle h_{\alpha_i}, \alpha_j\rangle e_{\pm\alpha_j},$$
$$\mathrm{ad}_{e_{\pm\alpha_i}}^{1-a_{i,j}} e_{\pm\alpha_j} = 0 \quad \text{if } i \neq j. \tag{2.2}$$

For any $\gamma \in \Delta_+$ we define a coroot $h_\gamma \in \mathfrak{h}$ by the rule

$$(\alpha, \alpha) h_\alpha + (\beta, \beta) h_\beta = (\alpha + \beta, \alpha + \beta) h_{\alpha+\beta}, \quad \text{if } \alpha, \beta, \alpha + \beta \in \Delta_+.$$

Due to (2.2), we have

$$[h_\gamma, e_{\pm\alpha_j}] = \pm\langle h_\gamma, \alpha_j\rangle e_{\pm\alpha_j}, \quad \langle h_\gamma, \alpha_j\rangle \in \mathbb{Z}. \tag{2.3}$$

Let W be the Weyl group of $\mathfrak{g}$. For any $w \in W$ we denote by $T_w : U(\mathfrak{g}) \to U(\mathfrak{g})$ a lift of the map $w : \mathfrak{h} \to \mathfrak{h}$ to the automorphism of the algebra $U(\mathfrak{g})$, satisfying the braid group relations $T_{ww'} = T_w T_{w'}$, if $l(ww') = l(w) + l(w')$, where $l(w)$ is the length of w. For instance, we may choose T_w, as in [L] (see (9.2) for the quantum group version). We adopt a shortened notation T_i for automorphisms $T_{s_{\alpha_i}}$, where $\alpha_i \in \Pi$.

Denote by D the localization of the free commutative algebra $U(\mathfrak{h})$ with respect to the multiplicative set of denominators, generated by

$$\{h_\alpha + k \mid \alpha \in \Delta,\ k \in \mathbb{Z}\}. \tag{2.4}$$

To any $\mu \in \mathfrak{h}^*$ we associate an automorphism τ_μ of the algebra D. It is uniquely defined by the conditions

$$\tau_\mu(h) = h + \langle h, \mu\rangle \quad \text{for any } h \in \mathfrak{h}. \tag{2.5}$$

Due to (2.3) the algebra $U(\mathfrak{g})$ satisfies the Ore condition with respect to the above set of denominators. Denote by $U'(\mathfrak{g})$ the extension of $U(\mathfrak{g})$ by means of D:

$$U'(\mathfrak{g}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} D \approx D \otimes_{U(\mathfrak{h})} U(\mathfrak{g}).$$

Note that the algebra $U'(\mathfrak{g})$ is a D-bimodule and any automorphism T_w admits a canonical extension to an automorphism of $U'(\mathfrak{g})$, which we denote by the same symbol.

Choose a normal ordering (see [T] for the definition) $\gamma_1 \prec \gamma_2 \prec \cdots \prec \gamma_n$ of the system Δ_+ of positive roots of $\mathfrak{g}$ ($n = |\Delta_+|$ may be infinite). Let $e_{\pm\alpha}$, where $\alpha \in \Delta$, be the Cartan–Weyl generators, constructed by the recursive procedure, attached to this order, see [T]. We assume that they are normalized in such a way that

$$[e_\alpha, e_{-\alpha}] = h_\alpha, \qquad [h_\alpha, e_{\pm\beta}] = \pm\langle h_\alpha, \beta\rangle e_{\pm\beta}. \tag{2.6}$$

For any $\bar{k} \in \mathbb{Z}^n_{\geqslant 0}$, $\bar{k} = (k_1, \dots, k_n)$, with $\sum k_i < \infty$ denote by $e_{\bar{k}}$ the monomial $e_{\bar{k}} = e_{\gamma_1}^{k_1} \cdots e_{\gamma_n}^{k_n} \in U(\mathfrak{b}_+)$ and by $\tilde{e}_{\bar{k}}$ the monomial $e_{-\gamma_1}^{k_1} \cdots e_{-\gamma_n}^{k_n} \in U(\mathfrak{b}_-)$. For every $\mu \in Q$ denote by $(F_{\mathfrak{g},\mathfrak{n}})_\mu$ the vector space of series

$$x_\mu = \sum_{\bar{k},\bar{r} \in \mathbb{Z}^n_{\geqslant 0}} \tilde{e}_{\bar{k}} x_{\bar{k},\bar{r}} e_{\bar{r}}, \quad x_{\bar{k},\bar{r}} \in D, \tag{2.7}$$

of the total weight μ. Set

$$F_{\mathfrak{g},\mathfrak{n}} = \bigoplus_{\mu \in Q} (F_{\mathfrak{g},\mathfrak{n}})_\mu.$$

Proposition 2.1. *(See [Z2, Section 3.2.3].) The space $F_{\mathfrak{g},\mathfrak{n}}$ is an associative algebra with respect to the multiplication of formal series. Its definition does not depend on a choice of the normal ordering $\prec$.*

Clearly, the algebra $F_{\mathfrak{g},\mathfrak{n}}$ contains $U'(\mathfrak{g})$ as a subalgebra. We call $F_{\mathfrak{g},\mathfrak{n}}$ *Taylor extension* of $U'(\mathfrak{g})$, related to the decomposition (2.1).

A choice of normal ordering is a technical tool for a description of the algebra $F_{\mathfrak{g},\mathfrak{n}}$. It is used for constructing particular bases in the weight components of the algebras $U(\mathfrak{n}_\pm)$. Instead, one can fix for any $\nu \in Q_+$ a basis $\{e_{\nu,j}\}$ of the finite-dimensional space $U(\mathfrak{n})_\nu$ and, for any $\nu \in Q_-$, a basis $\{\tilde{e}_{\nu,j}\}$ of the finite-dimensional space $U(\mathfrak{n}_-)_\nu$. Then the space $F_{\mu,\mathfrak{g}}$ consists of formal series

$$x_\mu = \sum_{\nu \in Q_+,\, \nu' \in Q_-,\, j, j'} \tilde{e}_{\nu',j'} x_{\nu',j',\nu,j} e_{\nu,j}, \quad x_{\nu',j',\nu,j} \in D,$$

of the total weight $\mu \in Q$.

2.2. Universal Verma module and the extremal projector

Set

$$M_{\mathfrak{n}}(\mathfrak{g}) = U'(\mathfrak{g})/U'(\mathfrak{g})\mathfrak{n}.$$

The space $M_{\mathfrak{n}}(\mathfrak{g})$ is a left $U'(g)$-module and a D-bimodule. It is called the *universal Verma module*. Since $M_{\mathfrak{n}}(\mathfrak{g})$ is a $U(\mathfrak{h})$-bimodule, we have an adjoint action of $U(\mathfrak{h})$ in $M_{\mathfrak{n}}(\mathfrak{g})$, defined as $\mathrm{ad}_h(m) = [h, m]$ for any $h \in \mathfrak{h}$ and $m \in M_{\mathfrak{n}}(\mathfrak{g})$. We have the weight decomposition of $M_{\mathfrak{n}}(\mathfrak{g})$ with respect to the adjoint action of $U(\mathfrak{h})$:

$$M_{\mathfrak{n}}(\mathfrak{g}) = \bigoplus_{\mu \in Q_-} \big(M_{\mathfrak{n}}(\mathfrak{g})\big)_\mu.$$

Denote by $E'(\mathfrak{g})$ the algebra of those endomorphisms of $M_{\mathfrak{n}}(\mathfrak{g})$, which commute with the right action of $U(\mathfrak{h})$. We have a linear map $\xi : U'(\mathfrak{g}) \to E'(\mathfrak{g})$, induced by the multiplication in $U'(\mathfrak{g})$ and establishing on $M_{\mathfrak{n}}(\mathfrak{g})$ the structure of the left $U'(\mathfrak{g})$-module. For any $\mu \in Q$, define

$$E_\mu(\mathfrak{g}) = \bigl\{a \in E'(\mathfrak{g}) \bigm| \bigl[\xi(h), a\bigr] = \langle \mu, h\rangle a \text{ for any } h \in \mathfrak{h}\bigr\},$$

and set

$$E(\mathfrak{g}) = \bigoplus_{\mu \in Q} E_\mu(\mathfrak{g}).$$

Proposition 2.2. *(See [Z2, Section 3.2.4–3.2.5].) The map ξ induces an isomorphism of algebras*:

$$\xi : F_{\mathfrak{g},\mathfrak{n}} \to E(\mathfrak{g}). \tag{2.8}$$

The proof of Proposition 2.2 is based on the nondegeneracy of the Shapovalov form. Recall [S] that the Shapovalov form $A(x, y) : U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to U(\mathfrak{h})$ is defined by the relation $A(x, y) = \beta(x^t \cdot y)$, where $x \mapsto x^t$ is the Chevalley antiinvolution ($e^t_{\pm\alpha} = e_{\mp\alpha},\ h^t = h,\ (xy)^t = y^t x^t$) in $U(\mathfrak{g})$ and $\beta : U(\mathfrak{g}) \to U(\mathfrak{h})$ is the projection with respect to the decomposition

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus \bigl(\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}\bigr).$$

The Shapovalov form vanishes on the left ideal $U(\mathfrak{g})\mathfrak{n}$ and is nondegenerate on $U(\mathfrak{n}_-)$. It admits an extension to a nondegenerate form on $M_{\mathfrak{n}}(\mathfrak{g})$ with values in D.

For the proof of the isomorphism (2.8) we choose for any $\nu \in Q^-$ a basis $\{\tilde{e}_{\nu,j}\}$ of $U(\mathfrak{n}_-)_\nu$, which is orthogonal with respect to the Shapovalov form, and take $e_{-\nu,j} = (\tilde{e}^-_{-\nu,j})^t$.

Proposition 2.3. *(See [Z2, Section 3.2.8].) There exists a unique element $\mathrm{P}_{\mathfrak{n}} \in F_{\mathfrak{g},\mathfrak{n}}$, satisfying equations*

$$e_\alpha \mathrm{P}_{\mathfrak{n}} = \mathrm{P}_{\mathfrak{n}} e_{-\alpha} = 0 \quad \textit{for all } \alpha \in \Delta_+ \tag{2.9}$$

with the zero term $\beta(\mathrm{P}_{\mathfrak{n}}) = 1$. It is a self-adjoint projector of zero weight, $\mathrm{P}^2_{\mathfrak{n}} = \mathrm{P}_{\mathfrak{n}}$, $\mathrm{P}^t_{\mathfrak{n}} = \mathrm{P}_{\mathfrak{n}}$.

The definition and the construction of the projector $\mathrm{P}_{\mathfrak{n}}$ depend on a choice of the nilpotent subalgebra $\mathfrak{n} \subset \mathfrak{g}$. When $\mathfrak{n}$ coincides with a fixed nilpotent subalgebra $\mathfrak{n}$, appearing in the decomposition (2.1), we omit the label $\mathfrak{n}$ and denote the projector simply by P.

Note an important property of P, which follows from Proposition 2.3:

$$\mathrm{P} - 1 \in F_{\mathfrak{g},\mathfrak{n}}\mathfrak{n} \cap \mathfrak{n}_- F_{\mathfrak{g},\mathfrak{n}}. \tag{2.10}$$

By means of Proposition 2.2, the element P is described as an element of $E(\mathfrak{g})$, which projects the universal Verma module $M_{\mathfrak{n}}(\mathfrak{g})$ to the subspace $(M_{\mathfrak{n}}(\mathfrak{g}))^{\mathfrak{n}} = M_{\mathfrak{n}}(\mathfrak{g})_0$ of $\mathfrak{n}$-invariants along $\mathfrak{n}_- M_{\mathfrak{n}}(\mathfrak{g}) = \bigoplus_{\gamma<0}(M_{\mathfrak{n}}(\mathfrak{g}))_\gamma$. The element P is called the *extremal projector*.

There are three distinguished cases of the use of the extremal projector.

(1) Let V be a $F_{\mathfrak{g},\mathfrak{n}}$-module. Then P projects V on the subspace $V^{\mathfrak{n}}$ of $\mathfrak{n}$-invariants along the subspace $\mathfrak{n}_- V$.

(2) Let V be a module over $U'(\mathfrak{g})$, locally finite with respect to $\mathfrak{n}$. Then P projects V on the subspace $V^{\mathfrak{n}}$ of $\mathfrak{n}$-invariants along the subspace $\mathfrak{n}_- V$.
(3) Let V be a module over $U(\mathfrak{g})$, locally finite with respect to $\mathfrak{b}$. Assume that $\mu \in \mathfrak{h}^*$ satisfies the conditions:

$$\langle \mu + \rho, h_\alpha \rangle \neq -1, -2, \ldots \quad \text{for any } \alpha \in \Delta_+, \tag{2.11}$$

where ρ is a weight such that $(\rho, \alpha_i) = (\alpha_i, \alpha_i)/2$ for all i. Denote by V_μ the generalized weight subspace of V of the weight μ. Then P projects V_μ on $V_\mu \cap V^{\mathfrak{n}}$ along $V_\mu \cap \mathfrak{n}_- V$.

In the following, for any left $U(\mathfrak{g})$-module M, on which the action of the projector P is defined, we denote the corresponding element of $\operatorname{End} M$ by p, and for any right $U(\mathfrak{g})$-module N, on which the action of the projector P is defined, we denote the corresponding element of $\operatorname{End} N$ by $\bar{\mathrm{p}}$.

The operator p satisfies the relations

$$\mathrm{p}(e_{-\gamma} m) = e_\gamma \mathrm{p}(m) = 0 \quad \text{for any } \gamma \in \Delta_+,\ m \in M,\ \mathrm{p}^2 = \mathrm{p}. \tag{2.12}$$

Similarly, the operator $\bar{\mathrm{p}}$ satisfies the relations

$$\bar{\mathrm{p}}(n e_\gamma) = \bar{\mathrm{p}}(n) e_{-\gamma} = 0 \quad \text{for any } \gamma \in \Delta_+,\ n \in N,\ \bar{\mathrm{p}}^2 = \bar{\mathrm{p}}. \tag{2.13}$$

2.3. Multiplicative formula for extremal projector

The extremal projector P for any simple Lie algebra was discovered and investigated by Asherova, Smirnov and Tolstoy [AST]. They gave a multiplicative expression for P, which was later generalized to affine Lie superalgebras and their q-analogs. We reproduce here the multiplicative formula from [AST].

For any $\alpha \in \Delta_+$ and any $\lambda \in \mathfrak{h}^*$ let $f_{\alpha,n}[\lambda]$ and $g_{\alpha,n}[\lambda]$ be the following elements of D:

$$f_{\alpha,n}[\lambda] = \prod_{j=1}^{n} \bigl(h_\alpha + \langle h_\alpha, \lambda \rangle + j\bigr)^{-1}, \qquad g_{\alpha,n}[\lambda] = \prod_{j=1}^{n} \bigl(-h_\alpha + \langle h_\alpha, \lambda \rangle + j\bigr)^{-1}. \tag{2.14}$$

Define $\mathrm{P}_\alpha[\lambda] \in F_{\mathfrak{g},\mathfrak{n}}$ and $\mathrm{P}_{-\alpha}[\lambda] \in F_{\mathfrak{g},\mathfrak{n}_-}$ by the relations

$$\mathrm{P}_\alpha[\lambda] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} f_{\alpha,n}[\lambda] e_{-\alpha}^n e_\alpha^n, \qquad \mathrm{P}_{-\alpha}[\lambda] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} g_{\alpha,n}[\lambda] e_\alpha^n e_{-\alpha}^n. \tag{2.15}$$

Set

$$\mathrm{P}_\alpha = \mathrm{P}_\alpha[\rho], \qquad \mathrm{P}_{-\alpha} = \mathrm{P}_{-\alpha}[\rho]. \tag{2.16}$$

Proposition 2.4. *(See [AST].) Let $\gamma_1 \prec \cdots \prec \gamma_n$ be a normal ordering of Δ_+. Then the extremal projector* P *is equal to the product*

$$\mathrm{P} = \prod_{\gamma \in \Delta_+}^{\prec} \mathrm{P}_\gamma, \tag{2.17}$$

where the order in the product coincides with the chosen normal order $\prec$.

Analogously, the product $\prod^{\prec}_{\gamma\in\Delta_+} \mathrm{P}_{-\gamma}$ is equal in $F_{\mathfrak{g},\mathfrak{n}-}$ to the projector $\mathrm{P}_{\mathfrak{n}_-}$.

For a generalization of (2.17) to arbitrary contragredient Kac–Moody Lie algebras of finite growth and their q-analogs, see [T,KT].

We can also define the elements $\mathrm{P}[\lambda]$ and $\mathrm{P}_-[\lambda]$ for any $\lambda \in \mathfrak{h}^*$ by the relations

$$\mathrm{P}[\lambda] = \prod^{\prec}_{\gamma\in\Delta_+} \mathrm{P}_{\gamma}[\lambda], \qquad \mathrm{P}_-[\lambda] = \prod^{\prec}_{\gamma\in\Delta_+} \mathrm{P}_{-\gamma}[\lambda]. \tag{2.18}$$

It is known (see [Z2], and also Section 3.5 of the present paper), that $\mathrm{P}[\lambda]$ and $\mathrm{P}_-[\lambda]$ do not depend on a choice of the normal order. In this notation, $\mathrm{P} = \mathrm{P}[\rho]$ and $\mathrm{P}_{\mathfrak{n}_-} = \mathrm{P}_-[\rho]$.

3. Mickelsson algebras

3.1. $\mathfrak{g}$-Admissible algebras

Let $\mathcal{A}$ be an associative algebra, which contains $U(\mathfrak{g})$ as a subalgebra. Then $\mathcal{A}$ has a natural structure of a $U(\mathfrak{g})$-bimodule. Since $U(\mathfrak{g})$ is a Hopf algebra, the bimodule structure produces an adjoint action of $U(\mathfrak{g})$ in $\mathcal{A}$. We have

$$\mathrm{ad}_g(x) = \sum_i g_i' x S\big(g_i''\big), \tag{3.1}$$

where the coproduct $\Delta(\mathfrak{g})$ of an element $g \in U(\mathfrak{g})$ has a form $\Delta(g) = \sum_i g_i' \otimes g_i''$ and $S(g)$ is the antipode of g. The adjoint action ad_g of $g \in \mathfrak{g}$ on $a \in \mathcal{A}$ is the commutator, $\mathrm{ad}_g(a) = ga - ag$. In the sequel we use the following notation for the adjoint action:

$$\hat{g}(a) \equiv \mathrm{ad}_g(a), \quad g \in U(\mathfrak{g}),\ a \in \mathcal{A}.$$

We call $\mathcal{A}$ a $\mathfrak{g}$-*admissible* algebra if:

(a) there is a subspace $\mathcal{V} \subset \mathcal{A}$, invariant with respect to the adjoint action of $U(\mathfrak{g})$, such that the multiplication m in $\mathcal{A}$ induces the following isomorphisms of vector spaces

$$\text{(a1)}\quad m: U(\mathfrak{g}) \otimes \mathcal{V} \to \mathcal{A}, \qquad \text{(a2)}\quad m: \mathcal{V} \otimes U(\mathfrak{g}) \to \mathcal{A};$$

(b) the adjoint action of real root vectors $e_\gamma \in U(\mathfrak{g})$ on $\mathcal{V}$ is locally nilpotent, the adjoint action of the Cartan subalgebra $\mathfrak{h}$ in $\mathcal{V}$ is semisimple.

Sometimes we call $\mathcal{V}$ an *ad-invariant generating subspace* of the $\mathfrak{g}$-admissible algebra $\mathcal{A}$. The condition (a) says, in particular, that $\mathcal{A}$ is a free left $U(\mathfrak{h})$-module and a free right $U(\mathfrak{h})$-module.

Since the adjoint action of real root vectors in $U(\mathfrak{g})$ is locally nilpotent, the conditions (a) and (b) imply that the adjoint action of real root vectors e_γ is locally nilpotent in $\mathcal{A}$, that is, for any $a \in \mathcal{A}$ and $\gamma \in \Delta^{\mathrm{re}}$ vectors $\mathrm{ad}^n_{e_\gamma}(a)$ are zero for sufficiently large n. Thus the restriction of the adjoint action of $U(\mathfrak{g})$ to any $\mathfrak{sl}_2$-subalgebra, generated by real root vectors $e_{\pm\gamma}$, where $\gamma \in \Delta^{\mathrm{re}}_+$, is locally finite in $\mathcal{A}$. Since the adjoint action of the Cartan subalgebra $U(\mathfrak{h})$ is semisimple, $\mathcal{A}$ admits a weight decomposition with respect to the adjoint action of $U(\mathfrak{h})$.

There are two main classes of $\mathfrak{g}$-admissible algebras.

(1) Let $\mathfrak{g}$ be a reductive finite-dimensional Lie algebra and $\mathfrak{g}_1$ a contragredient Lie algebra of finite growth, which contains $\mathfrak{g}$. Then the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}_1$ is locally finite and $\mathcal{A} = U(\mathfrak{g}_1)$ is a $\mathfrak{g}$-admissible algebra.

(2) Let $\mathcal{V}$ be a $U(\mathfrak{g})$-module algebra with a locally nilpotent action of real root vectors. This means that $\mathcal{V}$ is an associative algebra, equipped with a structure of a $U(\mathfrak{g})$-module, such that the action of real root vectors is locally finite. These two structures are related: the action of Lie generators $g \in \mathfrak{g}$ satisfies the Leibniz rule:

$$g(v_1v_2) = g(v_1)v_2 + v_1g(v_2). \tag{3.2}$$

Denote by $U(\mathfrak{g}) \ltimes \mathcal{V}$ the smash product of $U(\mathfrak{g})$ and $\mathcal{V}$. It is an associative algebra, generated by elements $g \in U(\mathfrak{g})$ and $v \in \mathcal{V}$, satisfying the relation

$$gv - vg = g(v), \quad g \in \mathfrak{g},\ v \in \mathcal{V}, \tag{3.3}$$

and, more generally,

$$\sum_i g_i' v S\big(g_i''\big) = g(v), \quad g \in U(\mathfrak{g}),\ v \in \mathcal{V}, \tag{3.4}$$

where S is the antipode in $U(\mathfrak{g})$, $\Delta(g) = \sum_i g_i' \otimes g_i''$ is the comultiplication in $U(\mathfrak{g})$. The smash product $U(\mathfrak{g}) \ltimes \mathcal{V}$ is a $\mathfrak{g}$-admissible algebra.

(2a) Let V be a $U(\mathfrak{g})$-module and $\mathrm{End}^0 V$ be the algebra of the endomorphisms of V, finite with respect to the adjoint action of real root vectors of $U(\mathfrak{g})$. Then the tensor product $U(\mathfrak{g}) \otimes \mathrm{End}^0 V$ is a $\mathfrak{g}$-admissible algebra. This construction is a particular case of the previous one: the tensor product $U(\mathfrak{g}) \otimes \mathrm{End}^0 V$ is a smash product of $\mathbb{C} \otimes \mathrm{End}^0 V$ and of diagonally embedded $U(g)$, generated by the elements $g \otimes 1 + 1 \otimes g$, $g \in \mathfrak{g}$.

3.2. *Definitions of Mickelsson algebras*

Let $\mathcal{A}$ be a $\mathfrak{g}$-admissible algebra. Let $\mathrm{Nr}(\mathcal{A}\mathfrak{n})$ be the normalizer of the left ideal $\mathcal{A}\mathfrak{n}$:

$$x \in \mathrm{Nr}(\mathcal{A}\mathfrak{n}) \quad \Leftrightarrow \quad \mathfrak{n}x \subset \mathcal{A}\mathfrak{n}.$$

Denote by $S^{\mathfrak{n}}(\mathcal{A})$ the quotient space

$$S^{\mathfrak{n}}(\mathcal{A}) = \mathrm{Nr}(\mathcal{A}\mathfrak{n})/\mathcal{A}\mathfrak{n}.$$

Proposition 3.1.

(i) *The space $S^{\mathfrak{n}}(\mathcal{A})$ is an algebra with respect to the multiplication in $\mathcal{A}$*;
(ii) *Let M be an $\mathcal{A}$-module. Then the space $M^{\mathfrak{n}}$ of $\mathfrak{n}$-invariant vectors in M is a $S^{\mathfrak{n}}(\mathcal{A})$-module.*

The algebra $S^{\mathfrak{n}}(\mathcal{A})$ is called the *Mickelsson algebra* [M]. Since $\mathfrak{h}$ normalizes $\mathfrak{n}_\pm$, we have the inclusion $U(\mathfrak{h}) \subset S^{\mathfrak{n}}(\mathcal{A})$. Denote by $\mathcal{A}'$ the localization

$$\mathcal{A}' = D \otimes_{U(\mathfrak{h})} \mathcal{A}.$$

The condition (a), (b) of a $\mathfrak{g}$-admissible algebra imply that the eigenvalues of operators $\hat{h}_\gamma : \mathcal{A} \to \mathcal{A}$ are integers for any $\gamma \in \Delta$. Therefore the algebra $\mathcal{A}$ satisfies the Ore conditions with respect to the set of denominators (2.4). Besides, we have a canonical embedding of $\mathcal{A}$ into $\mathcal{A}'$ and thus an adjoint action of $U(\mathfrak{g})$ in $\mathcal{A}'$, compatible with the adjoint action of $U(\mathfrak{g})$ in $\mathcal{A}$.

Define the *Mickelsson algebra* $Z^{\mathfrak{n}}(\mathcal{A})$ as the quotient

$$Z^{\mathfrak{n}}(\mathcal{A}) = \mathrm{Nr}(\mathcal{A}'\mathfrak{n})/\mathcal{A}'\mathfrak{n},$$

where $\mathrm{Nr}(\mathcal{A}'\mathfrak{n})$ is the normalizer of the left ideal $\mathcal{A}'\mathfrak{n}$ of $\mathcal{A}'$. The algebra $Z^{\mathfrak{n}}(\mathcal{A})$ is a localization of the algebra $S^{\mathfrak{n}}(\mathcal{A})$:

$$Z^{\mathfrak{n}}(\mathcal{A}) = D \otimes_{U(\mathfrak{h})} S^{\mathfrak{n}}(\mathcal{A}).$$

We can change the order of taking quotients and subspaces in the definition of Mickelsson algebra. Then the Mickelsson algebra $Z^{\mathfrak{n}}(\mathcal{A})$ is defined as a subspace of $\mathfrak{n}$-invariants in a left $U'(\mathfrak{g})$-module $M_{\mathfrak{n}}(\mathcal{A}') = \mathcal{A}'/\mathcal{A}'\mathfrak{n}$:

$$Z^{\mathfrak{n}}(\mathcal{A}) = \big(M_{\mathfrak{n}}(\mathcal{A}')\big)^{\mathfrak{n}} = \big\{m \in M_{\mathfrak{n}}(\mathcal{A}') \bigm| \mathfrak{n}m = 0\big\}.$$

The algebra $Z^{\mathfrak{n}}$ acts in the space $M^{\mathfrak{n}}$ of $\mathfrak{n}$-invariants of any $\mathcal{A}'$-module M.

3.3. *Double coset algebra*

Suppose that a g-admissible algebra $\mathcal{A}$ satisfies the additional *local highest weight* condition:

(HW) For any $v \in \mathcal{V}$, the adjoint action of elements $x \in U(\mathfrak{n})_\mu$ on v is nontrivial, $\hat{x}(v) \neq 0$, only for a finite number of $\mu \in \mathfrak{h}^*$.

With this assumption the quotient $M_{\mathfrak{n}}(\mathcal{A}') = \mathcal{A}'/\mathcal{A}'n$ has a structure of a left $F_{\mathfrak{g},\mathfrak{n}}$-module, extending the action of $\mathcal{A}'$ by the left multiplication. In particular, the extremal projector P acts on the left $F_{\mathfrak{g},\mathfrak{n}}$-module $M_{\mathfrak{n}}(\mathcal{A}')$.

The properties of the extremal projectors imply the relation

$$Z^{\mathfrak{n}}(\mathcal{A}) = \mathrm{Im}\,\mathrm{p} \subset M_{\mathfrak{n}}(\mathcal{A}'), \tag{3.5}$$

where $\mathrm{p} \in \mathrm{End}\, M_{\mathfrak{n}}(\mathcal{A}')$ is the action of P in $M_{\mathfrak{n}}(\mathcal{A}')$, see Section 2.2.

Denote by ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$ the double coset space

$${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}} = \mathfrak{n}_-\mathcal{A}' \setminus \mathcal{A}'/\mathcal{A}'\mathfrak{n} \equiv \mathcal{A}'/(\mathfrak{n}_-\mathcal{A}' + \mathcal{A}'\mathfrak{n}). \tag{3.6}$$

Equip ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$ with a binary operation $\circ : {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}} \otimes {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}} \to {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$:

$$a \circ b = a\mathrm{P}b \stackrel{\mathrm{def}}{=} a \cdot \mathrm{p}(b). \tag{3.7}$$

The rule (3.7) means the following. For a class $\bar{x}$ in ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$, we take its representative $x \in \mathcal{A}'$. For a class $\bar{y}$ in ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$, we take its representative $y \in M_{\mathfrak{n}}(\mathcal{A}')$. Consider an element $x\mathrm{p}(y)$ in $M_{\mathfrak{n}}(\mathcal{A}') = \mathcal{A}'/\mathcal{A}'\mathfrak{n}$. Then its class modulo $\mathfrak{n}_- M_{\mathfrak{n}}(\mathcal{A}')$ defines an element $\bar{x} \circ \bar{y}$ of ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$. It does not depend on a choice of the representatives.

We call the double coset space ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$, equipped with the operation (3.7) the *double coset algebra* ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$.

Define the linear maps $\phi^+ : Z^{\mathfrak{n}}(\mathcal{A}) \to {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$ and $\psi^+ : {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}} \to Z^{\mathfrak{n}}(\mathcal{A})$ by the rules

$$\phi^+(x) = x \mod \mathfrak{n}_- M_{\mathfrak{n}}(\mathcal{A}'), \qquad \psi^+(y) = \mathrm{p}(y), \quad x \in Z^{\mathfrak{n}}(\mathcal{A}),\ y \in {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}. \tag{3.8}$$

Let us explain the formula $\psi^+(y) = \mathrm{p}(y)$. For a class y in ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}} = \mathcal{A}'/\mathfrak{n}_-\mathcal{A}' + \mathcal{A}'\mathfrak{n}$, we choose its representative $\bar{y} \in M_{\mathfrak{n}}(\mathcal{A}') = \mathcal{A}'/\mathcal{A}'\mathfrak{n}$ and take $\mathrm{p}(\bar{y})$. The result does not depend on a choice of a representative and is denoted by $\psi^+(y)$.

Proposition 3.2. *Assume that a $\mathfrak{g}$-admissible algebra $\mathcal{A}$ satisfies the condition* (HW). *Then*

(i) *The operation* (3.7) *equips ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$ with the structure of an associative algebra.*
(ii) *The linear maps ϕ^+ and ψ^+ are inverse to each other and establish an isomorphism of algebras $Z^{\mathfrak{n}}(\mathcal{A})$ and ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$.*

Proof. Let $x \in Z^{\mathfrak{n}}(\mathcal{A}')$. Then $\mathrm{p}(x) = x \mod \mathcal{A}'\mathfrak{n}$ due to (2.10). Thus $\psi^+ \cdot \phi^+ = \mathrm{Id}_{Z^{\mathfrak{n}}(\mathcal{A})}$. On the other hand, due to the same property (2.10) of P, for any $y \in \mathcal{A}'/\mathcal{A}'\mathfrak{n}$,

$$\mathrm{p}(y) = y \mod \mathfrak{n}_-\mathcal{A}' + \mathcal{A}'\mathfrak{n}.$$

Thus $\phi^+ \cdot \psi^+ = \mathrm{Id}_{{}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}}$. So the maps ϕ^+ and ψ^+ are inverse to each other.

Let now $x \in \mathcal{A}'$ and $y \in \mathcal{A}'$ be representatives of classes $\tilde{x}$ and $\tilde{y}$ in ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$. Let $\bar{x} = x \mod \mathcal{A}'\mathfrak{n}$ and $\bar{y} = y \mod \mathcal{A}'\mathfrak{n}$ be their images in $M_{\mathfrak{n}}(\mathcal{A}')$. We have $\psi^+(\tilde{x}) = \mathrm{p}(\bar{x})$, $\psi^+(\tilde{y}) = \mathrm{p}(\bar{y})$ and $\psi^+(\tilde{x} \circ \tilde{y}) = \mathrm{p}(x \cdot \mathrm{p}(\bar{y}))$.

On the other hand, the multiplication rule m in $Z^{\mathfrak{n}}(\mathcal{A})$ can be written as follows. Let $z, u \in Z^{\mathfrak{n}}(\mathcal{A}')$. Let $z' \in \mathrm{Nr}(\mathcal{A}'\mathfrak{n})$ be a representative of a class $z \in \mathcal{A}'/\mathcal{A}'\mathfrak{n}$. Then $m(z \otimes u) = z' \cdot u$ as an element of $M_{\mathfrak{n}}(\mathcal{A}')$. By (3.5), $m(z, u) = \mathrm{p}(z' \cdot u)$. Thus we have in $Z^{\mathfrak{n}}(\mathcal{A}')$

$$m\big(\mathrm{p}(x) \otimes \mathrm{p}(y)\big) = \mathrm{p}\big(\mathrm{p}(x)' \cdot \mathrm{p}(y)\big),$$

where $\mathrm{p}(x)'$ is a representative of a class $\mathrm{p}(x)$ in $\mathcal{A}'$. By the property (2.10),

$$\mathrm{p}(x)' = x + x' + x'',$$

where $x' \in \mathfrak{n}_-\mathcal{A}'$ and $x'' \in \mathcal{A}'\mathfrak{n}$. Thus

$$\mathrm{p}\big(\mathrm{p}(x)' \cdot \mathrm{p}(y)\big) = \mathrm{p}\big(x \cdot \mathrm{p}(y)\big)$$

due to (2.10) and (2.12). Thus ψ^+ is a homomorphism, which proves simultaneously (i) and (ii). □

3.4. Generators of Mickelsson algebras

Let $\mathcal{A}$ be a $\mathfrak{g}$-admissible algebra with an ad-invariant generating subspace $\mathcal{V}$, satisfying the highest weight condition (HW). By the condition (a) of a $\mathfrak{g}$-admissible algebra (see Section 3.1) and the PBW theorem for the algebra $U(\mathfrak{g})$, any element of $\mathcal{A}'$ can be uniquely presented in the following form

$$x = \sum_i f_i d_i e_i v_i, \quad \text{where } f_i \in U(\mathfrak{n}_-),\ d_i \in D,\ e_i \in U(\mathfrak{n}),\ v_i \in \mathcal{V}.$$

Due to the highest weight condition (HW), we can move all e_i to the right and get a presentation

$$x = \sum_i f_i' d_i' v_i' e_i', \quad \text{where } f_i' \in U(\mathfrak{n}_-),\ d_i' \in D,\ e_i' \in U(\mathfrak{n}),\ v_i' \in \mathcal{V}.$$

In the double coset space this presentation gives

$$x = \sum_i d_i' v_i', \quad \operatorname{mod} \mathfrak{n}_-\mathcal{A}' + \mathcal{A}'\mathfrak{n}, \quad \text{where } d_i' \in D,\ v_i' \in \mathcal{V}.$$

Proposition 3.3. *Let $\mathcal{A}$ be a $\mathfrak{g}$-admissible algebra satisfying the highest weight condition* (HW). *Then*

(i) *Each element of the double coset algebra ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$ can be uniquely presented in a form $x = \sum_i d_i v_i$, where $d_i \in D$, $v_i \in \mathcal{V}$, so that ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$ is a free left* (*and right*) *D module, isomorphic to $D \otimes \mathcal{V}$ ($\mathcal{V} \otimes D$).*
(ii) *For each $v \in \mathcal{V}$ there exists a unique element $z_v \in Z^{\mathfrak{n}}(\mathcal{A})$ of the form*

$$z_v = v + \sum_{i=1} d_i f_i v_i, \quad f_i \in \mathfrak{n}_- U(\mathfrak{n}_-),\ d_i \in D,\ v_i \in \mathcal{V}, \tag{3.9}$$

such that the algebra $Z^{\mathfrak{n}}(\mathcal{A})$ is a free left (*and right*) *D-module, generated by the elements z_v. The element z_v is equal to* $\mathrm{p}(v)$.

Proof. The part (i) is already proved. Applying Proposition 3.2, we see that any element of $Z^{\mathfrak{n}}(\mathcal{A})$ can be presented in a form $\sum_i d_i \mathrm{p}(w_i)$, where $d_i \in D$, $v_i \in \mathcal{V}$. The element $z_v = \mathrm{p}(v)$ has a form (3.9) due to the definition of the operator p and is uniquely characterized by this presentation. □

Mickelsson algebras have distinguished generators of another type. Their existence is implied by the following proposition. Let $\mathcal{A}$ be an arbitrary $\mathfrak{g}$-admissible algebra with an ad-invariant generating subspace $\mathcal{V}$.

Proposition 3.4. *For each $v \in \mathcal{V}$ there exists at most one element $z_v' \in Z^{\mathfrak{n}}(\mathcal{A})$ of the form*

$$z_v' = v + \sum_i d_i v_i f_i, \quad f_i \in \mathfrak{n}_- U(\mathfrak{n}_-),\ d_i \in D,\ v_i \in \mathcal{V}. \tag{3.10}$$

Proof. If $m \in M_{\mathfrak{n}}(\mathcal{A}') = \mathcal{A}'/\mathcal{A}'\mathfrak{n}$ is a highest weight vector, that is $e_\alpha m = 0$ for any $\alpha \in \Delta_+$, then dm is also a highest weight vector for any $d \in D$. Thus (i) is equivalent to the statement that for any $\gamma \in \mathfrak{h}^*$ there is no highest weight vector of the form

$$x = \sum_i d_i v_i f_i, \quad f_i \in \mathfrak{n}_- U(\mathfrak{n}_-),\ d_i \in D,\ v_i \in \mathcal{V}, \tag{3.11}$$

where all the terms have the weight γ with respect to the adjoint representation of $\mathfrak{h}$. In other words, we should prove that the conditions $[e_{\alpha_i}, x] = 0$ in $M_\mathfrak{n}(\mathcal{A})$ imply $x = 0$ in $M_\mathfrak{n}(\mathcal{A})$ if all $f_j \in \mathfrak{n}_- U(\mathfrak{n}_-)$.

By the condition (a2) of a $\mathfrak{g}$-admissible algebra and the PBW theorem for $U(\mathfrak{g})$ the elements $v_i f_i$ form a basis of $M_\mathfrak{n}(\mathcal{A}')$ over D if v_i form a basis of $\mathcal{V}$ and f_i form a basis of $U(\mathfrak{n}_-)$. Consider those terms at the right-hand side of (3.11), where v_j have minimal weights comparing to the other weights which occur in (3.11). Then the expression $[e_{\alpha_i}, x]$ contains terms $v_j[e_{\alpha_i}, f_j]$ which are nonzero for some α_i if $f_j \in \mathfrak{n}_- U(\mathfrak{n}_-)$. This is because all the highest weight vectors of $M_\mathfrak{n}(\mathfrak{g})$ have zero weight. Thus x cannot be a highest weight vector. □

Theorem 3.5. *Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra and $\mathcal{A}$ a $\mathfrak{g}$-admissible algebra with generating subspace $\mathcal{V}$. Then for any $v \in \mathcal{V}$ there exists a unique element $z'_v \in \mathrm{Z}^\mathfrak{n}(\mathcal{A})$ (3.10). The algebra $\mathrm{Z}^\mathfrak{n}(\mathcal{A})$ is generated by elements z'_v as a free left (and right) D-module.*

The proof of Theorem 3.5 will be given in the next section.

Remark. When the choice of a nilpotent subalgebra $\mathfrak{n}$ is not unique, we use the notation $z_{\mathfrak{n},v}$ and $z'_{\mathfrak{n},v}$ for the elements (3.9) and (3.10).

3.5. Relations between two sets of generators

Extend the notation of canonical generators of Mickelsson algebras to the elements of $D \otimes_\mathbb{C} \mathcal{V}$ and $\mathcal{V} \otimes_\mathbb{C} D$. We set for any $d \in D$ and $v \in \mathcal{V}$

$$z_{d\otimes v} = d \cdot z_v, \qquad z'_{d\otimes v} = d \cdot z'_v, \qquad z_{v\otimes d} = z_v \cdot d, \qquad z'_{v\otimes d} = z'_v \cdot d \quad \text{in } \mathrm{Z}^\mathfrak{n}(\mathcal{A}). \tag{3.12}$$

Fix a positive real root α. We define now certain operators on a vector space $\mathcal{V} \otimes D$. Adopt the notation $A^{(1)}$ for the operator $A \otimes 1$ on a vector space $\mathcal{V} \otimes D$, and $A^{(2)}$ for the operator $1 \otimes A$.

Let α be a real root. For any $\mu \in \mathfrak{h}^*$ and $n \geqslant 0$ define the operators $\overline{f}^{(1)}_{n,\alpha}[\mu]$, $\overline{g}^{(1)}_{n,\alpha}[\mu]$, $\mathrm{B}^{(1)}_\alpha[\mu]$ and $\mathrm{C}^{(1)}_{-\alpha}[\lambda] \in \mathrm{End}(\mathcal{V} \otimes D)$ by the relations

$$\begin{aligned}
\overline{f}^{(1)}_{n,\alpha}[\mu] &= \prod_{k=1}^{n} \bigl(\hat{h}^{(1)}_\alpha + h^{(2)}_\alpha + \langle h_\alpha, \mu\rangle + k\bigr)^{-1}, \\
\overline{g}^{(1)}_{n,\alpha}[\mu] &= \prod_{k=1}^{n} \bigl(-\hat{h}^{(1)}_\alpha - h^{(2)}_\alpha + \langle h_\alpha, \mu\rangle + k\bigr)^{-1},
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
\mathrm{B}^{(1)}_\alpha[\mu] &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \overline{f}^{(1)}_{n,\alpha}[\mu] \bigl(\hat{e}^{(1)}_{-\alpha}\bigr)^n \bigl(\hat{e}^{(1)}_\alpha\bigr)^n, \\
\mathrm{C}^{(1)}_{-\alpha}[\mu] &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \overline{g}^{(1)}_{n,\alpha}[\mu] \bigl(\hat{e}^{(1)}_\alpha\bigr)^n \bigl(\hat{e}^{(1)}_{-\alpha}\bigr)^n.
\end{aligned} \tag{3.14}$$

Here $\hat{h}_\alpha^{(1)} = \mathrm{ad}^{(1)}_{h_\alpha}$ is the adjoint action of h_α on $\mathcal{V}$, $h_\alpha^{(2)}$ is the operator of multiplication by h_α on D.

Define also the operators $\overline{f}^{(2)}_{n,\alpha}[\mu]$, $\overline{g}^{(2)}_{n,\alpha}[\mu]$, $\mathrm{C}^{(2)}_\alpha[\mu]$ and $\mathrm{B}^{(2)}_{-\alpha}[\lambda] \in \mathrm{End}(D \otimes \mathcal{V})$ by the following relations:

$$\overline{f}^{(2)}_{n,\alpha}[\mu] = \prod_{k=1}^{n} \bigl(\hat{h}_\alpha^{(2)} - h_\alpha^{(1)} + \langle h_\alpha, \mu\rangle + k\bigr)^{-1},$$

$$\overline{g}^{(2)}_{n,\alpha}[\mu] = \prod_{k=1}^{n} \bigl(-\hat{h}_\alpha^{(2)} + h_\alpha^{(1)} + \langle h_\alpha, \mu\rangle + k\bigr)^{-1}, \tag{3.15}$$

$$\mathrm{C}^{(2)}_\alpha[\mu] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \overline{f}^{(2)}_{n,\alpha}[\mu] \bigl(\hat{e}^{(2)}_{-\alpha}\bigr)^n \bigl(\hat{e}^{(2)}_{\alpha}\bigr)^n,$$

$$\mathrm{B}^{(2)}_{-\alpha}[\mu] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \overline{g}^{(2)}_{n,\alpha}[\mu] \bigl(\hat{e}^{(2)}_{\alpha}\bigr)^n \bigl(\hat{e}^{(2)}_{-\alpha}\bigr)^n. \tag{3.16}$$

Here $\hat{h}_\alpha^{(2)} = \mathrm{ad}^{(2)}_{h_\alpha}$ is the adjoint action of h_α in $\mathcal{V}$, $h_\alpha^{(1)}$ is the operator of multiplication by h_α in D.

Let $\mathfrak{g}$ be a reductive finite-dimensional Lie algebra. Let $\prec$ be a normal ordering of the system Δ_+ of positive roots. Set

$$\mathrm{B}^{(1)}[\lambda] = \prod_{\gamma\in\Delta_+}^{\prec} \mathrm{B}^{(1)}_\gamma[\lambda], \qquad \mathrm{C}^{(1)}_-[\lambda] = \prod_{\gamma\in\Delta_+}^{\prec} \mathrm{C}^{(1)}_{-\gamma}[\lambda],$$

$$\mathrm{C}^{(2)}[\lambda] = \prod_{\gamma\in\Delta_+}^{\prec} \mathrm{C}^{(2)}_\gamma[\lambda], \qquad \mathrm{B}^{(2)}_-[\lambda] = \prod_{\gamma\in\Delta_+}^{\prec} \mathrm{B}^{(2)}_{-\gamma}[\lambda].$$

Theorem 3.6. *Let $\mathcal{A}$ be an admissible algebra with a generating subspace $\mathcal{V}$ over a finite-dimensional reductive Lie algebra $\mathfrak{g}$. Then for any $v \in \mathcal{V}$ we have the following equality in $Z^{\mathfrak{n}}(\mathcal{A})$*

$$z_v = z'_{\mathrm{B}^{(1)}[\rho](v\otimes 1)}. \tag{3.17}$$

In particular, operators $\mathrm{B}^{(1)}[\rho](v) : \mathcal{V} \otimes D \to \mathcal{V} \otimes D$ do not depend on a choice of the normal ordering $\prec$.

Proof. Consider the left $F_{\mathfrak{g},\mathfrak{n}}$-module $M_{\mathfrak{n}}(\mathcal{A}) = \mathcal{A}'/\mathcal{A}'\mathfrak{n}$. The multiplication in $\mathcal{A}'$ induces an isomorphism of vector spaces $M_{\mathfrak{n}}(\mathcal{A})$ and $\mathcal{V} \otimes M_{\mathfrak{n}}(\mathfrak{g})$, where $M_{\mathfrak{n}}(\mathfrak{g}) = U'(\mathfrak{g})/U'(\mathfrak{g})\mathfrak{n}$:

$$m : \mathcal{V} \otimes M_{\mathfrak{n}}(\mathfrak{g}) \to M_{\mathfrak{n}}(\mathcal{A}). \tag{3.18}$$

With this identification the tensor product $\mathcal{V} \otimes M_{\mathfrak{n}}(\mathfrak{g})$ becomes a $F_{\mathfrak{g},\mathfrak{n}}$-module. As a $U(\mathfrak{g})$-module it coincides with the tensor product of $\mathcal{V}$, equipped with a structure of the adjoint representation of $U(\mathfrak{g})$, and of the left $U(\mathfrak{g})$-module $M_{\mathfrak{n}}(\mathfrak{g})=U'(\mathfrak{g})/U'(\mathfrak{g})\mathfrak{n}$. This follows from the Leibniz rule

$$g(v\cdot x)=(gv-vg)\cdot x+v\cdot(gx)=\hat{g}(v)\cdot x+v\cdot gx,$$

for any $g\in\mathfrak{g}$, $v\in\mathcal{V}$ and $x\in U'(\mathfrak{g})/U'(\mathfrak{g})\mathfrak{n}$.

The elements of D act by the following rule: for any $d\in D$, $v\in\mathcal{V}$ of the weight μ_v and $x\in M_{\mathfrak{n}}(\mathcal{A})$ we have $d\cdot(v\otimes x)=v\otimes\tau_{\mu_v}(d)x$. Due to local finiteness of the adjoint action on $\mathcal{V}$ these rules define correctly an action of $F_{\mathfrak{g},\mathfrak{n}}$ in $\mathcal{V}\otimes M_{\mathfrak{n}}(\mathfrak{g})$.

Under the identification (3.18) we have

$$z_v=\mathrm{p}(v\otimes 1).$$

In order to express z_v via z'_v, we should write it in a form $v'_i\otimes d_i+{}$lower order terms, where 'lower order terms' contain vectors, whose second tensor component lies in $D\cdot\mathfrak{n}_-U(\mathfrak{n}_-)$. Write P as a series over ordered monomials of e_{γ_i}, and $e_{-\gamma_i}$, where $\gamma_i\in\Delta_+$, with coefficients being rational functions of h_{γ_i} such that in any monomial all $e_{-\gamma_i}$ stand before all e_{γ_j}, in accordance with the rules of $F_{\mathfrak{g},\mathfrak{n}}$,

$$\mathrm{P}=\mathrm{P}(h_{\gamma_i},e_{-\gamma_i},e_{\gamma_i}).$$

By the coproduct rule, in the action of P in $\mathcal{V}\otimes M_{\mathfrak{n}}(\mathfrak{g})$ we substitute instead of $e_{\pm\gamma_i}$, the sum $\hat{e}^{(1)}_{\pm\gamma_i}+e^{(2)}_{\pm\gamma_i}$, and instead of h_{γ_i}, the sum $\hat{h}^{(1)}_{\gamma_i}+h^{(2)}_{\gamma_i}$ with each term acting on the corresponding tensor component. The action of $e^{(2)}_{\gamma_i}$ on $v\otimes 1$ vanishes, the action of $e^{(2)}_{-\gamma_i}$ gives 'lower order terms.' So the term we are looking for is equal to

$$\mathrm{p}\big(\big(\hat{h}^{(1)}_{\gamma_i}+h^{(2)}_{\gamma_i}\big),\hat{e}^{(1)}_{-\gamma_i},\hat{e}^{(1)}_{\gamma_i}\big)(v\otimes 1).$$

This is precisely $\mathrm{B}^{(1)}[\rho](v\otimes 1)$. $\square$

The operators $\mathrm{B}^{(1)}[\rho]$, $\mathrm{B}^{(2)}_-[\rho]$, $\mathrm{C}^{(2)}[-\rho]$ and $\mathrm{C}^{(1)}_-[-\rho]$, are closely related to the operators $\mathrm{P}[\lambda]$ and $\mathrm{P}_-[\lambda]$, see (2.18). Namely, denote by $\rho^{(1)}$ and $\rho^{(2)}$ the expressions

$$\rho^{(1)}=\frac{1}{2}\sum_{\gamma\in\Delta_+}h_\gamma\otimes\gamma,\qquad \rho^{(2)}=\frac{1}{2}\sum_{\gamma\in\Delta_+}\gamma\otimes h_\gamma.$$

Then

$$\begin{aligned}\mathrm{B}^{(1)}[\rho]&=\hat{\mathrm{p}}^{(1)}\big[\big(\rho+\rho^{(2)}\big)\big],&\qquad \mathrm{C}^{(1)}_-[-\rho]&=\hat{\mathrm{p}}^{(1)}_-\big[-\big(\rho+\rho^{(2)}\big)\big],\\ \mathrm{B}^{(2)}_-[\rho]&=\hat{\mathrm{p}}^{(2)}_-\big[\big(\rho+\rho^{(1)}\big)\big],&\qquad \mathrm{C}^{(2)}[-\rho]&=\hat{\mathrm{p}}^{(2)}\big[-\big(\rho+\rho^{(1)}\big)\big].\end{aligned}$$

The operator $\mathrm{B}^{(1)}[\rho]$ of the 'change of coordinates' in Theorem 3.6 (see also Theorem 7.3 below) can be reversed by means of the relation

$$\mathrm{P}_\alpha[\lambda]P_{-\alpha}[-\lambda]=\frac{h_\alpha+\langle h_\alpha,\lambda\rangle}{\langle h_\alpha,\lambda\rangle}.\tag{3.19}$$

This relation makes sense for a generic λ in any finite-dimensional representation of $\mathbf{sl}_2$, see [TV, Theorem 10].

We have so

$$\mathrm{B}^{(1)}[\rho]^{-1} = \mathrm{C}_-^{(1)}[-\rho] \prod_{\alpha\in\Delta_+} \frac{h_\alpha^{(2)} + \langle h_\alpha, \rho\rangle}{\hat{h}_\alpha^{(1)} + h_\alpha^{(2)} + \langle h_\alpha, \rho\rangle},$$

$$\mathrm{B}_-^{(2)}[\rho]^{-1} = \mathrm{C}^{(2)}[-\rho] \prod_{\alpha\in\Delta_+} \frac{h_\alpha^{(1)} + \langle h_\alpha, \rho\rangle}{h_\alpha^{(1)} - \hat{h}_\alpha^{(2)} + \langle h_\alpha, \rho\rangle}. \tag{3.20}$$

Proof of Theorem 3.5. The inversion relations (3.20) imply that

$$z'_v = z_{\gamma_1 \mathrm{C}_-^{(1)}[-\rho](1\otimes v)}, \tag{3.21}$$

where $\gamma_1 = \prod_{\alpha\in\Delta_+} \frac{h_\alpha^{(2)}+\langle h_\alpha,\rho\rangle}{\hat{h}_\alpha^{(1)}+h_\alpha^{(2)}+\langle h_\alpha,\rho\rangle}$. Thus we have correctly defined elements z'_v, which proves the first statement of the theorem. Other statements follow from the corresponding statements of Proposition 3.3. □

Remark. If we replace the ring D by the field of fractions $\tilde{D} = \mathrm{Frac}(U(\mathfrak{h}))$, the statement of Theorem 3.5 does not require the precise inversion relation (3.21). We just note that all the elements v_i, appearing at the right-hand side of (3.9), belong to a finite-dimensional ad-invariant subspace $V \subset \mathcal{V}$, generated by v. We move then all f_i to the right and get a sum of elements z'_{v_k} with coefficients in D. If we allow the coefficients to be in $\tilde{D}$, such a transformation defines an injective operator in $\tilde{D} \otimes V$, which is a finite-dimensional vector space over $\tilde{D}$. Thus this operator is invertible. A use of the precise formula (3.21) is to show that the inverse matrix has coefficients in D.

4. Zhelobenko maps

4.1. Maps $\mathrm{q}_\alpha^{(k)}$ and q_α

Let α be a real root of $\mathfrak{g}$. For any $x \in \mathcal{A}$ and $k \geqslant 0$ denote by $\mathrm{q}_\alpha^{(k)}(x)$ the following element of $\mathcal{A}'/\mathcal{A}'e_\alpha$:

$$\mathrm{q}_\alpha^{(k)}(x) = \sum_{n=k}^{\infty} \frac{(-1)^n}{(n-k)!} \hat{e}_\alpha^{n-k}(x) \cdot e_{-\alpha}^n \cdot g_{n,\alpha} \mod \mathcal{A}'e_\alpha, \tag{4.1}$$

where

$$g_{n,\alpha} = \big(h_\alpha(h_\alpha - 1)\cdots(h_\alpha - n + 1)\big)^{-1}. \tag{4.2}$$

The assignment (4.1) has the following properties [Z2]:

$$
\begin{aligned}
&\text{(i)} && \mathrm{q}_\alpha^{(k)}(xe_{-\alpha}) = 0, && \\
&\text{(ii)} && \bigl[h, \mathrm{q}_\alpha^{(k)}(x)\bigr] = \mathrm{q}_\alpha^{(k)}\bigl([h,x]\bigr), && h \in \mathfrak{h}, \\
&\text{(iii)} && \mathrm{q}_\alpha^{(k)}(xh) = \mathrm{q}_\alpha^{(k)}(x)\bigl(h + \langle h,\alpha\rangle\bigr), && h \in \mathfrak{h}, \\
&\text{(iv)} && e_\alpha \mathrm{q}_\alpha^{(k)}(x) = \mathrm{q}_\alpha^{(k)}(e_\alpha x) = -k\mathrm{q}_\alpha^{(k-1)}(x). &&
\end{aligned}
\tag{4.3}
$$

Note that the quotient $\mathcal{A}'/\mathcal{A}'e_\alpha$ admits the left and right actions of D, so the commutator $[h, \mathrm{q}_\alpha^{(k)}(x)]$ is well defined in (ii). The property (iii) in the notation (2.5) can be written as

$$
\mathrm{q}_\alpha^{(k)}(xh) = \mathrm{q}_\alpha^{(k)}(x)\tau_\alpha(h), \quad h \in U(\mathfrak{h}). \tag{4.4}
$$

We extend the assignment (4.1) to the linear map $\mathrm{q}_\alpha^{(k)} : \mathcal{A}' \to \mathcal{A}'/\mathcal{A}'e_\alpha$ by means of the relation (iii) and denote by q_α the linear map $\mathrm{q}_\alpha^{(0)} : \mathcal{A}' \to \mathcal{A}'/\mathcal{A}'e_\alpha$,

$$
\mathrm{q}_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \hat{e}_\alpha^n(x) \cdot e_{-\alpha}^n \cdot g_{n,\alpha} \mod \mathcal{A}'e_\alpha. \tag{4.5}
$$

The relations (4.3)(i), (iv) show that the image of q_α belongs to the normalizer of $\mathcal{A}'e_\alpha$ in $\mathcal{A}'$ and that the ideals $e_\alpha\mathcal{A}'$ and $\mathcal{A}'e_{-\alpha}$ are in the kernel of q_α.

Consider the one-dimensional vector space $\mathbb{C}e_\alpha$ as an abelian Lie algebra $\mathfrak{n}_\alpha = \mathbb{C}e_\alpha$, and the one-dimensional vector space $\mathbb{C}e_{-\alpha}$ as an abelian Lie algebra $\mathfrak{n}_{-\alpha} = \mathbb{C}e_{-\alpha}$. Extending the notation of Section 3.2, denote by $Z^{\mathfrak{n}_\alpha}(\mathcal{A}) = \mathrm{Nr}(\mathcal{A}'e_\alpha)/\mathcal{A}'e_\alpha$ the Mickelsson algebra, corresponding to the reduction over the subalgebra, generated by e_α.

Proposition 4.1. *The map q_α defines an isomorphism of the vector spaces ${}_{\mathfrak{n}_\alpha}\mathcal{A}_{\mathfrak{n}_{-\alpha}} \equiv e_\alpha\mathcal{A}' \backslash \mathcal{A}'/\mathcal{A}'e_{-\alpha}$ and $Z^{\mathfrak{n}_\alpha}(\mathcal{A})$, such that for any $x \in {}_{\mathfrak{n}_\alpha}\mathcal{A}_{\mathfrak{n}_{-\alpha}}$ and $d \in D$*

$$
\bigl[d, \mathrm{q}_\alpha(x)\bigr] = \mathrm{q}_\alpha\bigl([d,x]\bigr), \qquad \mathrm{q}_\alpha(xd) = \mathrm{q}_\alpha(x)\tau_\alpha(d). \tag{4.6}
$$

Proof. First of all note that the properties (4.3)(i) and (iv) say that the map q_α vanishes on $\mathfrak{n}_\alpha\mathcal{A}' + \mathcal{A}'\mathfrak{n}_{-\alpha}$ and thus defines a map of ${}_{\mathfrak{n}_\alpha}\mathcal{A}_{\mathfrak{n}_{-\alpha}}$ to $\mathcal{A}'/\mathcal{A}'\mathfrak{n}_\alpha$. Its image belongs to $Z^{\mathfrak{n}}(\mathcal{A})$ by (4.3)(iv).

Let $\mathfrak{g}_\alpha$ be the subalgebra of $\mathfrak{g}$, generated by e_α, $e_{-\alpha}$ and $\mathfrak{h}$.

Since α is a real root, the adjoint action of $\mathfrak{g}_\alpha$ in $\mathfrak{g}$ is locally finite and semisimple. So there is a decomposition $\mathfrak{g} = \mathfrak{g}_\alpha + \mathfrak{p}$, invariant with respect to adjoint action of $\mathfrak{g}_\alpha$. Poincaré–Birkhoff–Witt theorem implies that the multiplication in $U(\mathfrak{g})$ defines an isomorphism of tensor products $U(\mathfrak{g}_\alpha) \otimes S(\mathfrak{p})$ and $S(\mathfrak{p}) \otimes U(\mathfrak{g}_\alpha)$ with $U(\mathfrak{g})$. Here $S(\mathfrak{p})$ is regarded as a subspace of $U(\mathfrak{g})$, which consists of symmetric noncommutative polynomials on $\mathfrak{p}$. The space $S(\mathfrak{p})$ is invariant with respect to adjoint action of $\mathfrak{g}_\alpha$. Thus $U(\mathfrak{g})$ is $\mathfrak{g}_\alpha$-admissible. Since the adjoint representation of $U(\mathfrak{g}_\alpha)$ in $U(\mathfrak{g})$ is locally finite, and $\mathcal{A}$ is a $\mathfrak{g}$-admissible algebra, it is $\mathfrak{g}_\alpha$ admissible as well. Let $\mathcal{V}_\alpha \subset \mathcal{A}$ be a subspace of $\mathcal{A}$, invariant with respect to the adjoint action of $\mathfrak{g}_a$, such that the multiplication in $\mathcal{A}$ induces an isomorphism of vector spaces $U(\mathfrak{g}_\alpha) \otimes \mathcal{V}_\alpha$ and $\mathcal{A}$.

The double coset space ${}_{\mathfrak{n}_\alpha}\mathcal{A}_{\mathfrak{n}_{-\alpha}}$ is a free D-module, generated by the vector space $\mathcal{V}_\alpha$ (see Section 3.4 where $\mathfrak{n}$ has been replaced by $\mathfrak{n}_{-\alpha}$). On the other hand, the Mickelsson algebra

$Z^{\mathfrak{n}_\alpha}(\mathcal{A})$ is also a free D-module generated, in the notation of Remark in Section 3.4, by the vectors $z'_{\mathfrak{n}_\alpha,v}$, $v \in \mathcal{V}_\alpha$, see Theorem 3.5. Using the structure of the map q_α, see (4.1), we have

$$\mathrm{q}_\alpha(v) = v + \sum v_i f_i d_i,$$

where $v_i \in \mathcal{V}_\alpha$, f_i is a polynomial on $e_{-\alpha}$ without a constant term, and $d_i \in D$. In other words,

$$\mathrm{q}_\alpha(v) = z'_{\mathfrak{n}_\alpha,v}. \tag{4.7}$$

The relations (4.7) and (4.4) prove the proposition. $\square$

Let us restrict the map q_α to the normalizer $\mathrm{Nr}(\mathcal{A}'\mathfrak{n}_{-\alpha})$. Due to (4.3)(i), this restriction defines a map $\mathrm{q}_\alpha|_{\mathrm{Nr}(\mathcal{A}'\mathfrak{n}_{-\alpha})} : Z^{\mathfrak{n}_{-\alpha}}(\mathcal{A}) \to Z^{\mathfrak{n}_\alpha}(\mathcal{A})$. We have also a map in the opposite direction, $\mathrm{q}_{-\alpha}|_{\mathrm{Nr}(\mathcal{A}'\mathfrak{n}_\alpha)} : Z^{\mathfrak{n}_\alpha}(\mathcal{A}) \to Z^{-\mathfrak{n}_\alpha}(\mathcal{A})$.

Proposition 4.2. *We have equalities*

$$\begin{aligned} \mathrm{q}_{-\alpha}\mathrm{q}_\alpha(x) &= (h_\alpha+1)x(h_\alpha+1)^{-1} \quad \textit{for any } x \in Z^{\mathfrak{n}_{-\alpha}}(\mathcal{A}), \\ \mathrm{q}_\alpha\mathrm{q}_{-\alpha}(y) &= (h_\alpha+1)^{-1}y(h_\alpha+1) \quad \textit{for any } y \in Z^{\mathfrak{n}_\alpha}(\mathcal{A}). \end{aligned} \tag{4.8}$$

In particular, the restriction of the map q_α to the normalizer $\mathrm{Nr}(\mathcal{A}'e_{-\alpha})$ defines an isomorphism of the vector spaces $Z^{\mathfrak{n}_{-\alpha}}(\mathcal{A})$ and $Z^{\mathfrak{n}_\alpha}(\mathcal{A})$. The inverse map is given by the formula

$$y \mapsto (h_\alpha+1)^{-1}\mathrm{q}_{-\alpha}(y)(h_\alpha+1), \quad y \in Z^{\mathfrak{n}_\alpha}(\mathcal{A}).$$

Proof. Take $y \in Z^{\mathfrak{n}_\alpha}(\mathcal{A})$. We have

$$\begin{aligned} \mathrm{q}_\alpha\mathrm{q}_{-\alpha}(y) &= \mathrm{q}_\alpha\left(\sum_{n\geqslant 0}\frac{(-1)^n}{n!}\hat{e}^n_{-\alpha}(y)e^n_\alpha \cdot g_{n,-\alpha}\right) \\ &= \mathrm{q}_\alpha\left(\sum_{n\geqslant 0}\frac{1}{n!}\hat{e}^n_{-\alpha}(y)e^n_\alpha\right)\cdot\big((h_\alpha+2)(h_\alpha+3)\cdots(h_\alpha+n+1)\big)^{-1} \\ &= \mathrm{q}_\alpha\left(\sum_{n\geqslant 0}\frac{(-1)^n}{n!}\hat{e}^n_\alpha\hat{e}^n_{-\alpha}(y)\right)\cdot\big((h_\alpha+2)(h_\alpha+3)\cdots(h_\alpha+n+1)\big)^{-1} \\ &= \sum_{m,n\geqslant 0}\frac{(-1)^{n+m}}{n!m!}\hat{e}^m_\alpha\hat{e}^n_\alpha\hat{e}^n_{-\alpha}(y)\cdot e^m_{-\alpha}\big((h_\alpha+2)\cdots(h_\alpha+n+1)\big)^{-1}\cdot g_{m,\alpha}. \end{aligned}$$

Since $\mathrm{q}_\alpha\mathrm{q}_{-\alpha}(y) \in Z^{\mathfrak{n}_\alpha}(\mathcal{A})$, we can replace it by $\mathrm{p}_\alpha(\mathrm{q}_\alpha\mathrm{q}_{-\alpha}(y))$, where P_α is the projection operator P, related to the algebra $\mathfrak{g}_\alpha$, and p_α is the action of P_α in $\mathcal{A}'/\mathcal{A}'\mathfrak{n}_\alpha$. Since $\mathrm{p}_\alpha(e_{-\alpha}z) = 0$ for any $z \in \mathcal{A}'/\mathcal{A}'\mathfrak{n}_\alpha$, we have

$$
\begin{aligned}
&\mathrm{q}_\alpha \mathrm{q}_{-\alpha}(y)\\
&\quad= \mathrm{p}_\alpha\left(\sum_{m,n\geqslant 0}\frac{(-1)^n}{n!m!}\hat{e}_{-\alpha}^m\hat{e}_\alpha^m\hat{e}_\alpha^n\hat{e}_{-\alpha}^n(y)\big((h_\alpha+2)\cdots(h_\alpha+n+1)\big)^{-1}\cdot g_{m,\alpha}\right)\\
&\quad= \mathrm{p}_\alpha\left(\sum_{m,n\geqslant 0}\frac{(-1)^n}{n!m!}\prod_{k=1}^m(\hat{h}_\alpha-h_\alpha+k-1)^{-1}\prod_{k=1}^n(-\hat{h}_\alpha+h_\alpha+k+1)^{-1}\hat{e}_{-\alpha}^m\hat{e}_\alpha^m\hat{e}_\alpha^n\hat{e}_{-\alpha}^n(y)\right).
\end{aligned}
$$

The expression inside brackets can be interpreted as

$$\hat{\mathrm{p}}_\alpha^{(2)}\big[-h_\alpha^{(1)}\otimes\rho-\rho\big]\cdot\hat{\mathrm{p}}_{-\alpha}^{(2)}\big[h_\alpha^{(1)}\otimes\rho+\rho\big](1\otimes y)$$

in $D\otimes Z^{\mathfrak{n}_\alpha}(\mathcal{A})$. Due to (3.19), it is equal to $\frac{\hat{h}_\alpha-h_\alpha^{(1)}-1}{-h_\alpha^{(1)}-1}(1\otimes y)=\frac{-\hat{h}_\alpha+h_\alpha^{(1)}+1}{h_\alpha^{(1)}+1}(1\otimes y)$. It means that

$$
\begin{aligned}
\mathrm{q}_\alpha\mathrm{q}_{-\alpha}(y) &= \mathrm{p}_\alpha\left(\frac{-\hat{h}_\alpha+h_\alpha+1}{h_\alpha+1}y\right)=\mathrm{p}_\alpha\big((h_\alpha+1)^{-1}y(h_\alpha+1)\big)\\
&=(h_\alpha+1)^{-1}\mathrm{p}_\alpha(y)(h_\alpha+1)=(h_\alpha+1)^{-1}y(h_\alpha+1).
\end{aligned}
$$

In the last line we used again the relation $y=\mathrm{p}_\alpha(y)$ mod $\mathcal{A}'\mathfrak{n}_\alpha$ for any $y\in Z^{\mathfrak{n}_\alpha}(\mathcal{A})$. The second relation is proved in an analogous manner. $\square$

4.2. *Maps* $\mathrm{q}_{\alpha,\mathfrak{m}}^{(k)}$ *and* $\mathrm{q}_{\alpha,\mathfrak{m}}$

Let α be a real root of $\mathfrak{g}$, e_α the corresponding root vector with respect to the decomposition (2.1). Let $\mathfrak{m}$ be a maximal nilpotent subalgebra of $\mathfrak{g}$, conjugated to $\mathfrak{n}$ by means of an element of the Weyl group W, such that e_α is a simple positive root vector of $\mathfrak{m}$. Set $\mathfrak{m}_-=\mathfrak{m}^t$, where $x\mapsto x^t$ is the Chevalley antiinvolution in $U(\mathfrak{g})$, see Section 2.2. For any $x\in\mathcal{A}$ and $k\geqslant 0$ denote by $\mathrm{q}_{\alpha,\mathfrak{m}}^{(k)}(x)$ the following element of $\mathcal{A}'/\mathcal{A}'\mathfrak{m}$:

$$\mathrm{q}_{\alpha,\mathfrak{m}}^{(k)}(x)=\sum_{n=k}^\infty\frac{(-1)^n}{(n-k)!}\hat{e}_\alpha^{n-k}(x)\cdot e_{-\alpha}^n\cdot g_{n,\alpha}\quad \mathrm{mod}\ \mathcal{A}'\mathfrak{m}, \tag{4.9}$$

where $g_{n,\alpha}$ is given by the relation (4.2). In other words,

$$\mathrm{q}_{\alpha,\mathfrak{m}}^{(k)}(x)=\pi_{m,\alpha}\cdot\mathrm{q}_\alpha^{(k)}(x), \tag{4.10}$$

where $\pi_{m,\alpha}:\mathcal{A}/\mathcal{A}e_\alpha\to\mathcal{A}/\mathcal{A}\mathfrak{m}$ is the natural factorization map. Due to (4.10), the assignment (4.9) enjoys the property (4.3) and admits an extension to a map $\mathrm{q}_{\alpha,\mathfrak{m}}^{(k)}:\mathcal{A}'\to\mathcal{A}'/\mathcal{A}'\mathfrak{m}$, satisfying the relation

$$\big[d,\mathrm{q}_{\alpha,\mathfrak{m}}^{(k)}(x)\big]=\mathrm{q}_{\alpha,\mathfrak{m}}^{(k)}\big([d,x]\big),\qquad \mathrm{q}_{\alpha,\mathfrak{m}}^{(k)}(xd)=\mathrm{q}_{\alpha,\mathfrak{m}}^{(k)}(x)\tau_\alpha(d), \tag{4.11}$$

for any $x\in\mathcal{A}'$ and $d\in D$. We denote by $\mathrm{q}_{\alpha,\mathfrak{m}}$ the map $\mathrm{q}_{\alpha,\mathfrak{m}}^{(0)}:\mathcal{A}'\to\mathcal{A}'/\mathcal{A}'\mathfrak{m}$.

For any element $w\in W$ of the Weyl group of $\mathfrak{g}$, and for any maximal nilpotent subalgebra $\mathfrak{m}\subset\mathfrak{g}$, we denote by $\mathfrak{m}^w\subset\mathfrak{g}$ the nilpotent subalgebra $\mathfrak{m}^w=T_w(\mathfrak{m})$, see Section 2.1.

Lemma 4.3. *For any $k \geqslant 0$,*

$$\mathrm{q}^{(k)}_{\alpha,\mathfrak{m}}\big(\mathcal{A}'\mathfrak{m}^{s_\alpha}\big)=0.$$

Proof. Denote by $\mathfrak{m}(\alpha)\subset\mathfrak{m}$ the nilpotent subalgebra of $\mathfrak{m}$, generated as a vector space by all root vectors of $\mathfrak{m}$ except for e_α. Due to (4.3)(i), it is sufficient to prove that $\mathrm{q}_{\alpha,k}(\mathcal{A}\mathfrak{m}(\alpha))=0$. The basic theory of root systems for simple Lie algebras says that $\hat{e}_{\pm\alpha}(\mathfrak{m}(\alpha))\subset\mathfrak{m}(\alpha)$. Thus for any $n\geqslant 0$ we have $\hat{e}^n_\alpha(\mathcal{A}\mathfrak{m}(\alpha))\subset\mathcal{A}\mathfrak{m}(\alpha)$ and $\mathfrak{m}(\alpha)e^n_{-\alpha}g^{-1}_{n,\alpha}\subset\mathcal{A}'\mathfrak{m}(\alpha)$, which imply the statement of the lemma. $\square$

Due to Lemma 4.3, the maps $\mathrm{q}^{(k)}_{\alpha,\mathfrak{m}}$ induce linear maps of $\mathcal{A}'/\mathcal{A}'\mathfrak{m}^{s_\alpha}$ to $\mathcal{A}'/\mathcal{A}'\mathfrak{m}$. We denote them by the same symbol:

$$\mathrm{q}^{(k)}_{\alpha,\mathfrak{m}}:\mathcal{A}'/\mathcal{A}'\mathfrak{m}^{s_\alpha}\to\mathcal{A}'/\mathcal{A}'\mathfrak{m}.$$

Proposition 4.4.

(i) *The map* $\mathrm{q}_{\alpha,\mathfrak{m}}$ *transforms the normalizer* $\mathrm{Nr}(\mathcal{A}'\mathfrak{m}^{s_\alpha})$ *to the Mickelsson algebra* $Z^m(\mathcal{A})=\mathrm{Nr}(\mathcal{A}'\mathfrak{m})/\mathcal{A}'\mathfrak{m}$.

(ii) *The restriction of the map* $\mathrm{q}_{\alpha,\mathfrak{m}}$ *to the normalizer* $\mathrm{Nr}(\mathcal{A}'\mathfrak{m}^{s_\alpha})$ *defines an isomorphism of vector spaces* $Z^{m^{s_\alpha}}(\mathcal{A})$ *and* $Z^m(\mathcal{A})$, *satisfying* (4.11) *with* $k=0$.

Proof. We prove the statement (i) first. Let $\Delta_+(\mathfrak{m})$ be the system of positive roots, related to the decomposition $\mathfrak{g}=\mathfrak{m}_-+\mathfrak{h}+\mathfrak{m}$. By assumption, α is a simple root of $\Delta_+(\mathfrak{m})$.

Let x be an element of the normalizer of the ideal $\mathcal{A}'\mathfrak{m}^{s_\alpha}$, $x\in\mathrm{Nr}(\mathcal{A}'\mathfrak{m}^{s_\alpha})$. It means that $e_{-\alpha}x\in\mathcal{A}'\mathfrak{m}^{s_\alpha}$ and $e_\gamma x\in\mathcal{A}'\mathfrak{m}^{s_\alpha}$ for any root $\gamma\in\Delta_+(\mathfrak{m})$, $\gamma\neq\alpha$.

Since $e_\alpha\mathrm{q}_{\alpha,\mathfrak{m}}=0$ by (4.3)(iv), we have to prove that

$$e_\gamma\mathrm{q}_{\alpha,\mathfrak{m}}(x)=0\quad\text{for any }\gamma\in\Delta_+(\mathfrak{m})\setminus\{\alpha\}.\tag{4.12}$$

Fix a positive root $\gamma\in\Delta_+(\mathfrak{m})\setminus\{\alpha\}$. Let $\gamma_0,\dots,\gamma_m$ be an 'α-string' of roots, starting with γ, that is $\gamma_0=\gamma$, and $\gamma_{k+1}=\gamma_k+\alpha$. Since α is simple, all roots γ_k are positive, $\gamma_k\in\Delta_+(\mathfrak{m})\setminus\{\alpha\}$, and for any $k=0,\dots,m$ we have

$$\hat{e}^k_\alpha(e_\gamma)=a_ke_{\gamma_k},\quad k=0,\dots,m,\ a_k\in\mathbb{C},\qquad \hat{e}^k_\alpha(e_\gamma)=0,\quad k>m.\tag{4.13}$$

For any $y\in\mathcal{A}$ we have by (4.13)

$$\mathrm{q}_\alpha(e_\gamma y)=\sum_{k=0}^m a_k\sum_{n=k}^\infty\frac{(-1)^n}{(n-k)!}e_{\gamma_k}\hat{e}_\alpha(y)e^n_{-\alpha}\cdot g_{n,\alpha},$$

therefore

$$e_\gamma\mathrm{q}_\alpha(y)=\mathrm{q}_\alpha(e_\gamma y)-\sum_{k=1}^m a_ke_{\gamma_k}\mathrm{q}^{(k)}_\alpha(y).\tag{4.14}$$

Iterations of (4.14) and the factorization by $\mathcal{A}'\mathfrak{m}$ give the relation

$$e_\gamma \mathrm{q}_{\alpha,\mathfrak{m}}(x) = \mathrm{q}_{\alpha,\mathfrak{m}}(e_\gamma x) + \sum_{k=1}^{m} b_k \mathrm{q}^{(k)}_{\alpha,\mathfrak{m}}(e_{\gamma_k} x), \quad b_k \in \mathbb{C}. \tag{4.15}$$

The right-hand side of (4.15) is zero by our assumption. This proves (4.12) and the statement (i) of the proposition.

Let us prove (ii). The root $-\alpha$ is a simple positive root for the algebra $\mathfrak{m}^{s_\alpha}$. Thus by (i) the map $\mathrm{q}_{-\alpha,\mathfrak{m}^{s_\alpha}}$ maps the normalizer $\mathrm{Nr}(\mathcal{A}'\mathfrak{m})$ and the Mickelsson algebra $Z^{\mathfrak{m}}(\mathcal{A})$ to the Mickelsson algebra $Z^{\mathfrak{m}^{s_\alpha}}(\mathcal{A})$. This implies that the map

$$\mathrm{q}'_{-\alpha,\mathfrak{m}^{s_\alpha}} = \mathrm{Ad}^{-1}_{h_\alpha+1} \cdot \mathrm{q}_{-\alpha,\mathfrak{m}^{s_\alpha}},$$

which sends $x \in \mathrm{Nr}(\mathcal{A}'\mathfrak{m})$ to $(h_\alpha+1)^{-1} \cdot \mathrm{q}_{-\alpha,\mathfrak{m}^{s_\alpha}}(x) \cdot (h_\alpha+1)$, also maps $Z^{\mathfrak{m}}(\mathcal{A})$ to the Mickelsson algebra $Z^{\mathfrak{m}^{s_\alpha}}(\mathcal{A})$. Proposition 4.2 says that $\mathrm{q}'_{-\alpha,\mathfrak{m}^{s_\alpha}}$ is inverse to the map of $Z^{\mathfrak{m}^{s_\alpha}}(\mathcal{A})$ to $Z^{\mathfrak{m}}(\mathcal{A})$, induced by the restriction of $\mathrm{q}_{\alpha,\mathfrak{m}}$ to $\mathrm{Nr}(\mathcal{A}'\mathfrak{m}^{s_\alpha})$. This proves the statement (ii). □

4.3. Maps $\mathrm{q}_{\overline{w},\mathfrak{m}}$

Let $\mathfrak{m}$ be a maximal nilpotent subalgebra of $\mathfrak{g}$, conjugated to $\mathfrak{n}$ by an automorphism $T_{w'}$, where $w' \in W$: $\mathfrak{m} = T_{w'}(\mathfrak{n})$, and $\mathfrak{m}_- = T_{w'}(\mathfrak{n}_-)$ the opposite maximal nilpotent subalgebra. Assume that $w \in W$ satisfies the condition

$$\dim T_w(\mathfrak{m}) \cap \mathfrak{m}_- = l(w), \tag{4.16}$$

where $l(w)$ is the length of w in W. Since $\mathfrak{m} = T_{w'}(\mathfrak{n})$, the condition (4.16) is equivalent to the relation $l(w'w) = l(w') + l(w)$. Let $\overline{w}$ denote a pair, consisting of an element $w \in W$ and of a reduced decomposition of w:

$$\overline{w} := \{w, w = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_n}}\}. \tag{4.17}$$

Let $n = l(w)$. Define a sequence $w_0, w_1, \ldots, w_n, w_n = w$ of elements of the Weyl group W, a sequence $\gamma_1, \gamma_2, \ldots, \gamma_n$ of positive roots, and a sequence $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ of maximal nilpotent subalgebras of $\mathfrak{g}$ by the following inductive rules:

$$w_0 = 1, \qquad w_{k+1} = w_k \cdot s_{\alpha_{i_{k+1}}}, \tag{4.18}$$

$$\gamma_k = w' w_{k-1}(\alpha_{i_k}), \qquad \mathfrak{m}_k = T_{w'w_{k-1}}(\mathfrak{n}). \tag{4.19}$$

The relations (4.16), (4.18)–(4.19) imply that for any $k = 1, \ldots, n$ the root vector e_{γ_k} is a simple positive root vector for the algebra $\mathfrak{m}_k$, and the composition

$$\mathrm{q}_{\overline{w},\mathfrak{m}} = \mathrm{q}_{\gamma_1,\mathfrak{m}_1} \cdot \mathrm{q}_{\gamma_2,\mathfrak{m}_2} \cdot \cdots \cdot \mathrm{q}_{\gamma_n,\mathfrak{m}_n}, \tag{4.20}$$

is a well-defined map $\mathrm{q}_{\overline{w},\mathfrak{m}} : \mathcal{A}'/\mathcal{A}'\mathfrak{m}^w \to \mathcal{A}'/\mathcal{A}'\mathfrak{m}$. The index $\overline{w}$ indicates that this map is related by the construction to the reduced decomposition (4.17).

Denote by $\Delta_w(\mathfrak{m})$ the set of roots $\gamma_1, \ldots, \gamma_n$, defined in (4.19). Alternatively, $\Delta_w(\mathfrak{m})$ consists of all roots $\gamma \in \Delta(\mathfrak{m}) = w'(\Delta_+)$, such that $w^{-1}(\gamma) \in \Delta(\mathfrak{m}_-) = w'(\Delta_-)$.

Lemma 4.5. *(See [Z2, Section 5.2.4].) Let $x \in \mathcal{A}'$. Then in the notation* (4.18), (4.20)

(i)	$\mathrm{q}_{\overline{w},\mathfrak{m}}(xd) = \mathrm{q}_{\overline{w},\mathfrak{m}}(x) \cdot \tau_{w'(\rho)-w'w(\rho)}(d)$	*for any* $d \in D$,
(ii)	$e_\alpha \cdot \mathrm{q}_{\overline{w},\mathfrak{m}}(x) = 0$	*for any* $\alpha \in \Delta_w(\mathfrak{m})$,
(iii)	$\mathrm{q}_{\overline{w},\mathfrak{m}}(e_\alpha x) = 0$	*for any* $\alpha \in \Delta_w(\mathfrak{m})$,
(iv)	$e_{w'w(\alpha)}\mathrm{q}_{\overline{w},\mathfrak{m}}(x) = \mathrm{q}_{\overline{w},\mathfrak{m}}(e_{w'w(\alpha)}x)$	*if* $l(w'ws_\alpha) > l(w'w)$,
(v)	$e_{w'(\alpha)}\mathrm{q}_{\overline{w},\mathfrak{m}^{s_\alpha}}(x) = \mathrm{q}_{\overline{w},\mathfrak{m}^{s_\alpha}}(e_{w'(\alpha)}x)$	*if* $l(w's_\alpha w) > l(w'w)$.

Proof. The property (i) is a direct consequence of (4.11)(ii). Indeed, the relation (4.11)(ii), for $k = 0$, implies that for any $x \in \mathcal{A}'$ and $d \in D$ we have

$$\mathrm{q}_{\overline{w},\mathfrak{m}}(xd) = \mathrm{q}_{\overline{w},\mathfrak{m}}(x)\tau_{\gamma_1+\cdots+\gamma_n}(d) = \mathrm{q}_{\overline{w},\mathfrak{m}}(x) \cdot \tau_{w'(\rho)-w'w(\rho)}(d).$$

Suppose that the relations (ii)–(iv) take place for all $\mathfrak{m} = w'(\mathfrak{n})$, and for all reduced decompositions of any element w of the Weyl group of length less than n, such that $l(w'w) = l(w') + l(w)$. Let (4.17) be a reduced decomposition of the element w of length n.

In the notation of (4.18)–(4.20) we have the equality $\mathrm{q}_{\overline{w},\mathfrak{m}} = \mathrm{q}_{\overline{w}_{n-1},\mathfrak{m}}\mathrm{q}_{\gamma_n,\mathfrak{m}_n}$, with $l(w_{n-1}) = n-1$. Thus, by the induction assumption, for any $x \in \mathcal{A}'$, $e_{\gamma_i}\mathrm{q}_{\overline{w}_{n-1},\mathfrak{m}}(x) = 0$ for $i = 1, \ldots, n-1$ and $e_{\gamma_n}\mathrm{q}_{\overline{w}_{n-1},\mathfrak{m}}(x) = \mathrm{q}_{\overline{w}_{n-1},\mathfrak{m}}e_{\gamma_n}(x)$. This implies the equalities $e_{\gamma_i}\mathrm{q}_{\overline{w},\mathfrak{m}}(x) = 0$ for $i = 1, \ldots, n-1$ and

$$e_{\gamma_n}\mathrm{q}_{\overline{w},\mathfrak{m}}(x) = \mathrm{q}_{\overline{w}_{n-1},\mathfrak{m}}(x)e_{\gamma_n}\mathrm{q}_{\gamma_n,\mathfrak{m}_n}(x).$$

The expression in the last line is zero due to (4.11)(iii). This makes the induction step for the statement (ii).

On the other hand, let us present w as a product $w = s_{\gamma_1}w''$, where w'' is an element of the length $n-1$ with a given reduced decomposition $w'' = s_{\alpha_{i_2}} \cdot \ldots \cdot s_{\alpha_{i_n}}$. Again, we denote by $\overline{w}''$ the pair $\{w'', w'' = s_{\alpha_{i_2}} \cdot \ldots \cdot s_{\alpha_{i_n}}\}$. We have a decomposition $\mathrm{q}_{\overline{w},\mathfrak{m}} = \mathrm{q}_{\gamma_1,\mathfrak{m}} \cdot \mathrm{q}_{\overline{w}'',\mathfrak{m}_2}$, where

$$\mathrm{q}_{\overline{w}'',\mathfrak{m}_2} = \mathrm{q}_{\gamma_2,\mathfrak{m}_2}\mathrm{q}_{\gamma_3,\mathfrak{m}_3} \cdot \ldots \cdot \mathrm{q}_{\gamma_n,\mathfrak{m}_n}. \tag{4.21}$$

The induction assumptions say that $\mathrm{q}_{\overline{w}'',\mathfrak{m}_2}(e_{\gamma_i}x) = 0$ for $i = 2, \ldots, n$ by (ii) and that

$$\mathrm{q}_{\overline{w}'',\mathfrak{m}_2}(e_{\gamma_1}x) = e_{\gamma_1}\mathrm{q}_{\overline{w}'',\mathfrak{m}_2}(x)$$

by (v). Thus $\mathrm{q}_{\overline{w},\mathfrak{m}}(e_{\gamma_i}x) = 0$ for $i = 2, \ldots, n$ and

$$\mathrm{q}_{\overline{w},\mathfrak{m}}(e_{\gamma_1}x) = \mathrm{q}_{\gamma_1,\mathfrak{m}}\mathrm{q}_{\overline{w}'',\mathfrak{m}_2}(e_{\gamma_1}x) = \mathrm{q}_{\gamma_1,\mathfrak{m}}\big(e_{\gamma_1}\mathrm{q}_{\overline{w}'',\mathfrak{m}_2}(x)\big) = 0$$

by (4.3)(iv). This makes the induction step for (iii).

Let us make the induction step for (iv). Take a simple root α, such that $l(w'ws_\alpha) > l(w'w)$. Let $\gamma = w'w(\alpha)$. Then γ is a positive root, and the sequence

$$\gamma_1, \ldots, \gamma_n, \gamma \tag{4.22}$$

is convex, that is, the sum of any two elements of the sequence lies between them, if the sum is a root. Let $\mu_0, \ldots, \mu_m$ be the finite 'γ_1-sequence' of positive roots, starting with γ, that is, $\mu_0 = \gamma$,

$\mu_{k+1} = \mu_k + \gamma_1$. Then each μ_k belongs to the set (4.22), $\mu_k = \gamma_{i_k}$, such that $i_k \in \{2, \dots, n\}$ if $k > 0$, and $\hat{e}^k_\alpha(e_\gamma) = a_k e_{\mu_k}$ with $a_k \in \mathbb{C}$, and $\hat{e}^k_\alpha(e_\gamma) = 0$ for $k > m$. This implies, see the proof of Proposition 4.4, that

$$e_\gamma \mathrm{q}_{\gamma_1}(y) = \mathrm{q}_{\gamma_1}(e_\gamma y) + \sum_{k=1}^{m} b_k \mathrm{q}^{(k)}_{\gamma_1}(e_{\mu_k} y), \quad b_k \in \mathbb{C}. \tag{4.23}$$

The relation (4.23) implies the equality

$$e_\gamma \mathrm{q}_{\overline{w},\mathfrak{m}}(x) = \mathrm{q}_{\gamma_1,\mathfrak{m}}\bigl(e_\gamma \mathrm{q}_{\overline{w}'',\mathfrak{m}_2}(x)\bigr) + \sum_{k=1}^{m} b_k \mathrm{q}^{(k)}_{\gamma_1,\mathfrak{m}}\bigl(e_{\mu_k} \mathrm{q}_{\overline{w}'',\mathfrak{m}_2}(x)\bigr),$$

where $\mathrm{q}_{\overline{w}'',\mathfrak{m}_2}$ is defined in (4.21). The induction assumption says that $e_\gamma \mathrm{q}_{\overline{w}'',\mathfrak{m}_2}(x) = \mathrm{q}_{\overline{w}'',\mathfrak{m}_2}(e_\gamma x)$ and $e_{\mu_k} \mathrm{q}_{\overline{w}'',\mathfrak{m}_2} = 0$ for any $k \geqslant 1$. This makes the induction step for (iv). The statement (v) is proved in an analogous manner.

4.4. Map q_{w_0}

In this section we assume that $\mathfrak{g}$ is a finite-dimensional reductive Lie algebra and $\mathcal{A}$ is a $\mathfrak{g}$-admissible algebra. In this case, the Weyl group W is finite and the adjoint action of $\mathfrak{g}$ on $\mathcal{A}$ is locally finite.

Set $\mathfrak{m} = \mathfrak{n}$ and for any $w \in W$ and any reduced decomposition $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_k}}$ denote the map $\mathrm{q}_{\overline{w},\mathfrak{n}}$ as $\mathrm{q}_{\overline{w}}$:

$$\mathrm{q}_{\overline{w}} \equiv \mathrm{q}_{\overline{w},\mathfrak{n}}.$$

Let $w_0 = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_n}}$ be a reduced decomposition of the longest element $w_0 \in W$ and $\overline{w}_0$ the corresponding pair $\overline{w}_0 = \{w_0, w_0 = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_n}}\}$. The map $\mathrm{q}_{\overline{w}_0}$ has the following properties: the right ideal $\mathfrak{n}\mathcal{A}'$ is in the kernel of $\mathrm{q}_{\overline{w}_0}$ by Lemma 4.5(ii), and the image of $\mathrm{q}_{\overline{w}_0}$ is in the normalizer of the left ideal $\mathcal{A}'\mathfrak{n}$ by Lemma 4.5(iii). It means that it induces the map of the double coset space ${}_\mathfrak{n}\mathcal{A}_{\mathfrak{n}_-} = \mathfrak{n}\mathcal{A}' \setminus \mathcal{A}'/\mathcal{A}'\mathfrak{n}_-$ to the Mickelsson algebra $Z^\mathfrak{n}(\mathcal{A})$:

$$\mathrm{q}_{\overline{w}_0} : {}_\mathfrak{n}\mathcal{A}_{\mathfrak{n}_-} \to Z^\mathfrak{n}(\mathcal{A}).$$

Let $\mathcal{V} \subset \mathcal{A}$ be an ad-invariant generating subspace of $\mathcal{A}$. Let z'_w, where $w \in \mathcal{V}$, be the canonical generators of the D-module $Z^\mathfrak{n}(\mathcal{A})$, see Theorem 3.5.

Proposition 4.6. *The mapping $\mathrm{q}_{\overline{w}_0}$ of vector spaces defines an isomorphism of the double coset space ${}_\mathfrak{n}\mathcal{A}_{\mathfrak{n}_-}$ and $Z^\mathfrak{n}(\mathcal{A})$, such that for any $x \in {}_{\mathfrak{n}_-}\mathcal{A}_\mathfrak{n}$, $d \in D$ and $v \in \mathcal{V}$*

$$\bigl[d, \mathrm{q}_{\overline{w}_0}(x)\bigr] = \mathrm{q}_{\overline{w}_0}\bigl([d, x]\bigr), \qquad \mathrm{q}_{\overline{w}_0}(xd) = \mathrm{q}_{\overline{w}_0}(x)\tau_{\rho - w_0(\rho)}(d), \tag{4.24}$$

$$\mathrm{q}_{\overline{w}_0}(v) = z'_v. \tag{4.25}$$

Proof. The arguments are the same as in the proof of Proposition 4.1.

The double coset space ${}_\mathfrak{n}\mathcal{A}_{\mathfrak{n}_-}$ is a free right (and left) D-module, generated by the vector space $\mathcal{V}$. On the other hand, the Mickelsson algebra $Z^\mathfrak{n}(\mathcal{A})$ is also a free right (and left)

D-module, generated by the vectors z'_v, $v \in \mathcal{V}_\alpha$, see Theorem 3.5. Using the structure of the map $\mathrm{q}_{\overline{w}_0}$,

$$\mathrm{q}_{\overline{w}_0}(v) = w + \sum v_i g_i,$$

where $g_i \in U'(\mathfrak{g})$, and $v_i \in \mathcal{V}$ have the weight strictly bigger then the weight of v, that is, $\mu(v - v_i) \in Q_+$, $\mu(v - v_i) \neq 0$, where $\mu(x) \in \mathfrak{h}^*$ denotes the weight of x. We can present further any g_i as a sum $g_i = \sum_j f_{i,j} d_{i,j} e_{i,j}$, where $f_{i,j} \in U(\mathfrak{n}_-)$, $d_{i,j} \in D$ and $e_{i,j} \in U(\mathfrak{n})$. The terms where $e_{i,j} \neq 1$, vanish by definition in $Z^{\mathfrak{n}}(\mathcal{A})$, so we have the equality (4.25). $\square$

Proposition 4.6 implies that the map $\mathrm{q}_{\overline{w}_0}$ does not depend on a reduced decomposition of w_0. We will denote it by q_{w_0},

$$\mathrm{q}_{w_0} \equiv \mathrm{q}_{\overline{w}_0}.$$

Corollary 4.7. *The restriction of the map q_{w_0} to the normalizer $\mathrm{Nr}(\mathcal{A}'\mathfrak{n}_-)$ defines an isomorphism of the vector spaces $Z^{\mathfrak{n}_-}(\mathcal{A})$ and $Z^{\mathfrak{n}}(\mathcal{A})$, such that*

$$\begin{aligned} &\bigl[d, \mathrm{q}_{\overline{w}_0}(x)\bigr] = \mathrm{q}_{\overline{w}_0}\bigl([d,x]\bigr), \qquad \mathrm{q}_{w_0}(xd) = \mathrm{q}_{w_0}(x)\tau_{\rho - w_0(\rho)}(d), \quad d \in D, \\ &\mathrm{q}_{w_0}(z_{\mathfrak{n}_-,v}) = z'_{\mathfrak{n},v}, \quad v \in \mathcal{V}. \end{aligned} \tag{4.26}$$

Here $z_{\mathfrak{n}_-,v}$ are generators of the Mickelsson algebra $Z^{\mathfrak{n}_-}(\mathcal{A})$ of the 'first type,' see Proposition 3.3, $z'_{\mathfrak{n},v} \equiv z'_v$ are generators of the Mickelsson algebra $Z^{\mathfrak{n}}(\mathcal{A})$ of the 'second type,' see Proposition 3.4.

Proof. We have $\mathfrak{n}_- = \mathfrak{n}^{w_0}$ and all the statements of the corollary follow by induction from Proposition 4.4. We should prove only the equality (4.26). By the definition (3.9), the element $z_{\mathfrak{n}_-,v} \in Z^{\mathfrak{n}_-}(\mathcal{A})$ has a form

$$z_{\mathfrak{n}_-,v} = v + \sum_{i=1} d_i e_i v_i, \quad e_i \in \mathfrak{n}U(\mathfrak{n}),\ d_i \in D,\ v_i \in \mathcal{V},$$

that is, $z_{\mathfrak{n}_-,v} = v \bmod \mathfrak{n}\mathcal{A}'$. Using the properties of the map q_{w_0}, given in Proposition 4.6, we have $\mathrm{q}_{w_0}(z_{\mathfrak{n}_-,v}) = \mathrm{q}_{w_0}(v) = z'_v$. $\square$

4.5. Cocycle properties

In this section $\mathfrak{g}$ is an arbitrary contragredient Lie algebra of finite growth, $\mathcal{A}$ is a g-admissible algebra, $\mathfrak{m}$ a maximal nilpotent subalgebra of $\mathfrak{g}$, conjugated to $\mathfrak{n}$ by an element of the Weyl group W, and w an element of W, satisfying the condition (4.16).

Proposition 4.8. *Maps $\mathrm{q}_{\overline{w},\mathfrak{m}}$ do not depend on reduced decompositions of $w \in W$.*

Proof. First we take an element $w \in W$, equal to the longest element of a reductive subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ of rank two. Then the algebra $\mathfrak{g}' = \mathfrak{n}'_- + \mathfrak{h} + \mathfrak{n}'$ is generated by the elements $e_{\pm\gamma_i}$ and $h \in \mathfrak{h}$, where $\gamma_1, \ldots, \gamma_n$ are the terms of the sequence (4.19), such that all $e_{\gamma_i} \in \mathfrak{m}$ by the condition (4.16). By Proposition 4.6, the map

$$\mathrm{q}_{\overline{w},\mathfrak{n}'} : \mathcal{A}' \to \mathcal{A}'/\mathcal{A}'\mathfrak{n}'$$

does not depend on a reduced decomposition of w. The map

$$\mathrm{q}_{\overline{w},\mathfrak{m}} : \mathcal{A}' \to \mathcal{A}'/\mathcal{A}'\mathfrak{m}$$

is the composition of $\mathrm{q}_{\overline{w},\mathfrak{n}'}$ with the natural projection of $\mathcal{A}'/\mathcal{A}'\mathfrak{n}'$ to $\mathcal{A}'/\mathcal{A}'\mathfrak{m}$, and thus also does not depend on a choice of the reduced decomposition of w.

This result implies the statement of the proposition for a general w, since any two reduced decompositions are related by a sequence of flips of reduced decompositions of longest elements of rank two subalgebras. □

With the use of Proposition 4.8, we further simplify the notation for the maps $\mathrm{q}_{\overline{w},\mathfrak{m}}$ and $\mathrm{q}_{\overline{w}}$ and write them as $\mathrm{q}_{w,\mathfrak{m}}$ and q_w:

$$\mathrm{q}_{w,\mathfrak{m}} \equiv \mathrm{q}_{\overline{w},\mathfrak{m}}, \qquad \mathrm{q}_w \equiv \mathrm{q}_{\overline{w}} \equiv \mathrm{q}_{\overline{w},\mathfrak{n}}.$$

Proposition 4.8 can be formulated as the condition

$$\mathrm{q}_{w'w,\mathfrak{m}} = \mathrm{q}_{w',\mathfrak{m}}\mathrm{q}_{w,\mathfrak{m}^{w'}}, \quad \text{if } l(w'w) = l(w') + l(w). \tag{4.27}$$

This statement is known as 'Zhelobenko cocycle condition' [Z1].

5. Homomorphism properties of Zhelobenko maps

5.1. Homomorphism property of maps q_α

Let $\mathcal{A}$ be an admissible algebra over a contragredient Lie algebra $\mathfrak{g}$ of finite growth. Let α be a real root of $\mathfrak{g}$, and $\mathfrak{g}_\alpha$ the subalgebra of $\mathfrak{g}$, generated by $e_{\pm\alpha}$ and $\mathfrak{h}$, $\mathfrak{n}_{\pm\alpha} = \mathbb{C}e_{\pm\alpha}$. Since the algebra $\mathcal{A}$ is $\mathfrak{g}$-admissible and the adjoint action of $\mathfrak{g}_\alpha$ on $\mathfrak{g}$ is locally finite, $\mathcal{A}$ is $\mathfrak{g}_\alpha$-admissible as well, see Section 4.1. In this setting we proved in Section 3.3 that the double coset space ${}_{\mathfrak{n}_\alpha}\mathcal{A}_{\mathfrak{n}_{-\alpha}} = \mathfrak{n}_\alpha\mathcal{A}' \setminus \mathcal{A}'/\mathcal{A}'\mathfrak{n}_{-\alpha}$ can be equipped with a structure of an associative algebra with the multiplication rule

$$a \circ b = a\mathrm{p}_{-\alpha}(b) = \bar{\mathrm{p}}_{-\alpha}(a)b,$$

in the notation of Sections 2.2 and 3.3.

The following theorem is the basic source for applications of the Zhelobenko operators to the representation theory of Mickelsson algebras.

Theorem 5.1. *The map* $\mathrm{q}_\alpha : {}_{\mathfrak{n}_\alpha}\mathcal{A}_{\mathfrak{n}_{-\alpha}} \to Z^{\mathfrak{n}_\alpha}(\mathcal{A})$ *is a homomorphism of algebras.*

Theorem 5.1 and Proposition 4.1 imply that q_α establishes an *isomorphism* of the double coset algebra ${}_{\mathfrak{n}_\alpha}\mathcal{A}_{\mathfrak{n}_{-\alpha}}$ and Mickelsson algebra $Z^{\mathfrak{n}_\alpha}(\mathcal{A})$. On the other hand, Theorem 5.1 implies the equality

$$\mathrm{q}_\alpha(xy) = \mathrm{q}_\alpha(x)\mathrm{q}_\alpha(y), \quad \text{for any } x \in \mathcal{A}',\ y \in \mathrm{Nr}(\mathcal{A}'\mathfrak{n}_{-\alpha}). \tag{5.1}$$

Indeed, if $y \in \mathrm{Nr}(\mathcal{A}'\mathfrak{n}_{-\alpha})$, and $\bar{y}$ is the class of y in $\mathcal{A}'/\mathcal{A}'\mathfrak{n}_{-\alpha}$ then $\bar{y} = \mathrm{p}_{-\alpha}(\bar{y})$ and the first equality in (4.12) is a corollary of the first statement of Theorem 5.1.

Proof. Let $\mathrm{P}_{-\alpha}$ be the extremal projector, related to the decomposition $\mathfrak{g}_\alpha = \mathfrak{n}_\alpha + \mathbb{C}h_\alpha + \mathfrak{n}_{-\alpha}$ of the algebra $\mathfrak{g}_\alpha$. It is given by the relations (2.15) and (2.16). The corresponding operators $\mathrm{p}_{-\alpha} : \mathcal{A}'/\mathfrak{n}_{-\alpha}\mathcal{A}' \to \mathcal{A}'/\mathfrak{n}_{-\alpha}\mathcal{A}'$ and $\bar{\mathrm{p}}_{-\alpha} : \mathcal{A}'\mathfrak{n}_\alpha \setminus \mathcal{A}' \to \mathcal{A}'\mathfrak{n}_\alpha \setminus \mathcal{A}'$ can be written as

$$\mathrm{p}_{-\alpha}(x) = \sum_{m\geqslant 0} \frac{1}{m!}\frac{1}{(h_\alpha - 2)\cdots(h_\alpha - m - 1)} e_\alpha \hat{e}^m_{-\alpha}(x) \mod \mathcal{A}'e_{-\alpha}, \tag{5.2}$$

$$\bar{\mathrm{p}}_{-\alpha}(x) = \sum_{m\geqslant 0} \frac{(-1)^m}{m!}\hat{e}^m_\alpha(x)e^m_{-\alpha}\frac{1}{(h_\alpha - 2)\cdots(h_\alpha - m - 1)} \mod e_\alpha\mathcal{A}'. \tag{5.3}$$

We should establish an equality

$$\mathrm{q}_\alpha\big(\bar{\mathrm{p}}_{-\alpha}(x)y\big) = \mathrm{q}_\alpha(x)\mathrm{q}_\alpha(y) \quad \text{for any } x \in \mathfrak{n}_\alpha\mathcal{A}' \setminus \mathcal{A}',\ y \in \mathcal{A}'. \tag{5.4}$$

Suppose $y \in \mathcal{A}$ is a weight vector such that $[h_\alpha, y] = \mu_y y$. We have

$$\begin{aligned}
\mathrm{q}_\alpha\big(\bar{\mathrm{p}}_{-\alpha}(x)y\big) &= \mathrm{q}_\alpha\left(\sum_{m\geqslant 0}\frac{(-1)^m}{m!}\hat{e}^m_\alpha(x)e^m_{-\alpha}\frac{1}{(h_\alpha - 2)\cdots(h_\alpha - m - 1)}y\right)\\
&= \mathrm{q}_\alpha\left(\sum_{m\geqslant 0}\frac{(-1)^m}{m!}\hat{e}^m_\alpha(x)e^m_{-\alpha}y\frac{1}{(h_\alpha - 2 + \mu_y)\cdots(h_\alpha - m - 1 + \mu_y)}\right)\\
&= \mathrm{q}_\alpha\left(\sum_{m\geqslant 0}\frac{(-1)^m}{m!}\hat{e}^m_\alpha(x)e^m_{-\alpha}y\right)\frac{1}{(h_\alpha + \mu_y)\cdots(h_\alpha - m + 1 + \mu_y)}\\
&= \sum_{n,m\geqslant 0}\frac{(-1)^{n+m}}{n!m!}\hat{e}^n_\alpha\big(\hat{e}^m_\alpha(x)e^m_{-\alpha}y\big)e^n_{-\alpha}\frac{1}{g_{n,\alpha}}\frac{1}{(h_\alpha + \mu_y)\cdots(h_\alpha - m + 1 + \mu_y)}\\
&= \sum_{n,m\geqslant 0}\frac{(-1)^{n+m}}{n!m!}\\
&\quad\cdot\sum_{k=0}^{n}\binom{n}{k}\hat{e}^k_\alpha\big(\hat{e}^m_\alpha(x)e^m_{-\alpha}\big)\hat{e}^{n-k}_\alpha(y)e^n_{-\alpha}\frac{1}{g_{n,\alpha}}\frac{1}{(h_\alpha + \mu_y)\cdots(h_\alpha - m + 1 + \mu_y)}\\
&= \sum_{k,m=0}^{\infty}\frac{(-1)^m}{m!k!}\hat{e}^k_\alpha\big(\hat{e}^m_\alpha(x)e^m_{-\alpha}\big)\frac{1}{(h_\alpha + 2k)\cdots(h_\alpha + 2k - m + 1)}\mathrm{q}^{(k)}_\alpha(y).
\end{aligned} \tag{5.5}$$

The sum of the terms with $k=0$ in (5.5) is $\mathrm{q}_\alpha(x)\cdot\mathrm{q}_\alpha(y)$. So it is sufficient to prove the following identity in $\mathcal{A}'/\mathcal{A}'e_\alpha$:

$$\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!k!}\hat{e}_\alpha^k\big(\hat{e}_\alpha^m(x)e_{-\alpha}^m\big)\frac{1}{(h_\alpha+2k)\cdots(h_\alpha+2k-m+1)}\mathrm{q}_\alpha^{(k)}(y)=0 \tag{5.6}$$

for any $x,y\in\mathcal{A}$.

We have for any $k\geqslant 1$:

$$\begin{aligned}\hat{e}_\alpha^k\big(\hat{e}_\alpha^m(x)e_{-\alpha}^m\big)&=\hat{e}_\alpha^{k-1}\big(\hat{e}_\alpha^{m+1}(x)e_{-\alpha}^m+m\hat{e}_\alpha^m(x)e_{-\alpha}^{m-1}(h_\alpha-m+1)\big)\\&=\hat{e}_\alpha^{k-1}\big(\hat{e}_\alpha^{m+1}(x)e_{-\alpha}^m\big)+m\hat{e}_\alpha^{k-1}\big(\hat{e}_\alpha^m(x)e_{-\alpha}^{m-1}\big)(h_\alpha-m+1)\\&\quad-2m(k-1)\hat{e}_\alpha^{k-2}\big(\hat{e}_\alpha^m(x)e_{-\alpha}^{m-1}\big)e_\alpha.\end{aligned} \tag{5.7}$$

Denote the left-hand side of (5.6) by S. Substitute (5.7) into S. We get

$$\begin{aligned}S=\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!k!}\bigg(&\hat{e}_\alpha^{k-1}\big(\hat{e}_\alpha^{m+1}(x)e_{-\alpha}^m\big)\frac{1}{(h_\alpha+2k)\cdots(h_\alpha+2k-m+1)}\mathrm{q}_\alpha^{(k)}(y)\\&+m\hat{e}_\alpha^{k-1}\big(\hat{e}_\alpha^m(x)e_{-\alpha}^{m-1}\big)\frac{(h_\alpha-m+1)}{(h_\alpha+2k)\cdots(h_\alpha+2k-m+1)}\mathrm{q}_\alpha^{(k)}(y)\\&-2m(k-1)\hat{e}_\alpha^{k-2}\big(\hat{e}_\alpha^m(x)e_{-\alpha}^{m-1}\big)e_\alpha\frac{1}{(h_\alpha+2k)\cdots(h_\alpha+2k-m+1)}\mathrm{q}_\alpha^{(k)}(y)\bigg).\end{aligned}$$

Substitute $(h_\alpha-m+1)=(h_\alpha+2k-m+1)-2k$ into the second sum and use the relation (4.3)(iv) in the third sum. We get

$$\begin{aligned}S=\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!k!}\bigg(&\hat{e}_\alpha^{k-1}\big(\hat{e}_\alpha^{m+1}(x)e_{-\alpha}^m\big)\frac{1}{(h_\alpha+2k)\cdots(h_\alpha+2k-m+1)}\mathrm{q}_\alpha^{(k)}(y)\\&+m\hat{e}_\alpha^{k-1}\big(\hat{e}_\alpha^m(x)e_{-\alpha}^{m-1}\big)\frac{1}{(h_\alpha+2k)\cdots(h_\alpha+2k-m+2)}\mathrm{q}_\alpha^{(k)}(y)\\&-2km\hat{e}_\alpha^{k-1}\big(\hat{e}_\alpha^m(x)e_{-\alpha}^{m-1}\big)\frac{1}{(h_\alpha+2k)\cdots(h_\alpha+2k-m+1)}\mathrm{q}_\alpha^{(k)}(y)\\&+2mk(k-1)\hat{e}_\alpha^{k-2}\big(\hat{e}_\alpha^m(x)e_{-\alpha}^{m-1}\big)\frac{1}{(h_\alpha+2k-2)\cdots(h_\alpha+2k-m-1)}\mathrm{q}_\alpha^{(k-1)}(y)\bigg).\end{aligned}$$

Change the indices of summation: m to $m+1$ in the second and in the third sum; m to $m+1$ and k to $k+1$ in the last sum. Then

$$\begin{aligned}S=\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!k!}\bigg(&\hat{e}_\alpha^{k-1}\big(\hat{e}_\alpha^{m+1}(x)e_{-\alpha}^m\big)\frac{1}{(h_\alpha+2k)\cdots(h_\alpha+2k-m+1)}\mathrm{q}_\alpha^{(k)}(y)\\&-\hat{e}_\alpha^{k-1}\big(\hat{e}_\alpha^{m+1}(x)e_{-\alpha}^m\big)\frac{1}{(h_\alpha+2k)\cdots(h_\alpha+2k-m+1)}\mathrm{q}_\alpha^{(k)}(y)\end{aligned}$$

$$
\begin{aligned}
&+2k\hat{e}_\alpha^{k-1}\big(\hat{e}_\alpha^m(x)e_{-\alpha}^{m-1}\big)\frac{1}{(h_\alpha+2k)\cdots(h_\alpha+2k-m)}\mathrm{q}_\alpha^{(k)}(y)\\
&-2k\hat{e}_\alpha^{k-2}\big(\hat{e}_\alpha^m(x)e_{-\alpha}^{m-1}\big)\frac{1}{(h_\alpha+2k)\cdots(h_\alpha+2k-m)}\mathrm{q}_\alpha^{(k)}(y)\bigg)=0.
\end{aligned}
$$

The theorem is proved. □

5.2. Properties of maps $\mathrm{q}_{\alpha,\mathfrak{m}}$ *and* $\mathrm{q}_{w,\mathfrak{m}}$

Let α be a real root of the Lie algebra $\mathfrak{g}$, e_α the corresponding root vector with respect to the decomposition (2.1). Put $\mathfrak{m}=\mathfrak{n}^w$, where $w\in W$, a maximal nilpotent subalgebra of $\mathfrak{g}$, such that e_α is a simple positive root vector of $\mathfrak{m}$.

Theorem 5.2. *For any* $x\in\mathcal{A}'$, $y\in\mathrm{Nr}(\mathcal{A}'\mathfrak{m}^{s_\alpha})$

$$\mathrm{q}_{\alpha,\mathfrak{m}}(xy)=\mathrm{q}_{\alpha,\mathfrak{m}}(x)\mathrm{q}_{\alpha,\mathfrak{m}}(y).$$

Proof. Since $y\in\mathrm{Nr}(\mathcal{A}'\mathfrak{m}^{s_\alpha})$, its image $\bar{y}$ in $M_{\mathfrak{m}^{s_\alpha}}(\mathcal{A}')=\mathcal{A}'/\mathcal{A}'\mathfrak{m}^{s_\alpha}$ satisfies the relations $e_\gamma\bar{y}=0$ for all $\gamma\in\Delta_+(\mathfrak{m}^{s_\alpha})$. In particular, we have the equality

$$e_{-\alpha}\bar{y}=0\quad\text{in }\mathcal{A}'/\mathcal{A}'\mathfrak{m}^{s_\alpha}. \tag{5.8}$$

Let $\mathfrak{g}_\alpha\subset\mathfrak{g}$ be the subalgebra, generated by $e_{\pm\alpha}$ and $\mathfrak{h}$. Since $\mathcal{A}$ is $\mathfrak{g}$-admissible, it is $\mathfrak{g}_\alpha$-admissible as well and $M_{\mathfrak{m}^{s_\alpha}}(\mathcal{A}')$ is locally nilpotent with respect to $e_{-\alpha}$. Thus the extremal projector $\mathrm{P}_{-\alpha}$, related to $\mathfrak{n}_{-\alpha}=\mathbb{C}e_{-\alpha}$, acts on $M_{\mathfrak{m}^{s_\alpha}}(\mathcal{A}')$. Denote, following the notation of Section 2.2, its image in $\mathrm{End}\,M_{\mathfrak{m}^{s_\alpha}}(\mathcal{A}')$ by $\mathrm{p}_{-\alpha}$. By (5.8) and the properties of the extremal projector, we have

$$\bar{y}=\mathrm{p}_{-\alpha}(\bar{y})\quad\text{in }\mathcal{A}'/\mathcal{A}'\mathfrak{m}^{s_\alpha}. \tag{5.9}$$

The equality (5.9) can be read as

$$y\in\mathrm{Nr}(\mathcal{A}'\mathfrak{n}_{-\alpha})\quad\mathrm{mod}\ \mathcal{A}'\mathfrak{m}^{s_\alpha},$$

that is $y=y'+z$, where $y'\in\mathrm{Nr}(\mathcal{A}'\mathfrak{m}_{-\alpha})$ and $z\in\mathcal{A}'\mathfrak{m}^{s_\alpha}$.

Indeed, $\mathrm{p}_{-\alpha}(\bar{y})\in\mathrm{Nr}(\mathcal{A}'\mathfrak{m}_{-\alpha})$ mod $\mathcal{A}'\mathfrak{m}_{-\alpha}$ and $\mathcal{A}'\mathfrak{m}_{-\alpha}\subset\mathcal{A}'\mathfrak{m}^{s_\alpha}$. Due to Theorem 5.1, see (5.1) and the properties of the maps $\mathrm{q}_{\alpha,\mathfrak{m}}$, we have

$$\mathrm{q}_{\alpha,\mathfrak{m}}(xy)=\mathrm{q}_{\alpha,\mathfrak{m}}(xy')=\mathrm{q}_{\alpha,\mathfrak{m}}(x)\mathrm{q}_{\alpha,\mathfrak{m}}(y')=\mathrm{q}_{\alpha,\mathfrak{m}}(x)\mathrm{q}_{\alpha,\mathfrak{m}}(y). \qquad \square$$

Combining Theorem 5.1 with the statements of Propositions 4.4 and 7.5, we conclude that the restriction of the map $\mathrm{q}_{\alpha,\mathfrak{m}}$ to the normalizer $\mathrm{Nr}(\mathcal{A}'\mathfrak{m}^{s_\alpha})$ defines an isomorphism of the algebras $Z^{\mathfrak{m}^{s_\alpha}}(\mathcal{A})$ and $Z^{\mathfrak{m}}(\mathcal{A})$.

Iterations of these conclusions yield the following statement.

Proposition 5.3. *For any w', w, such that $l(w'w) = l(w') + l(w)$ and $\mathfrak{m} = \mathfrak{n}^{w'}$, the restriction of the map $\mathrm{q}_{w,\mathfrak{m}}$ to the normalizer $\mathrm{Nr}(\mathcal{A}'\mathfrak{m}^w)$ defines an isomorphism of the algebras $Z^{\mathfrak{m}^w}(\mathcal{A})$ and $Z^{\mathfrak{m}}(\mathcal{A})$ such that for any $x \in Z^{\mathfrak{m}^w}(\mathcal{A})$ and $d \in D$*

$$\big[d, \mathrm{q}_{w,\mathfrak{m}}(x)\big] = \mathrm{q}_{w,\mathfrak{m}}\big([d,x]\big), \qquad \mathrm{q}_{w,\mathfrak{m}}(xd) = \mathrm{q}_{w,\mathfrak{m}}(x)\cdot \tau_{w'(\rho)-w'w(\rho)}(d).$$

The case of a finite-dimensional reductive Lie algebra $\mathfrak{g}$ and the element $w_0 \in W$ of the maximal length is special. In this case we have the following statements:

Proposition 5.4.

(i) *The map q_{w_0} defines an isomorphism of the double coset algebra ${}_{\mathfrak{n}}\mathcal{A}_{\mathfrak{n}_-}$ and Mickelsson algebra $Z^{\mathfrak{n}}(\mathcal{A})$ such that for any $d \in D$, $v \in \mathcal{V}$ and $x \in {}_{\mathfrak{n}}\mathcal{A}_{\mathfrak{n}_-}$*

$$\mathrm{q}_{w_0}(v) = z'_v, \qquad \big[d, \mathrm{q}_{w_0}(x)\big] = \mathrm{q}_{w_0}\big([d,x]\big), \qquad \mathrm{q}_{w_0}(xd) = \mathrm{q}_{w_0}(x)\tau_{\rho-w_0(\rho)}(d);$$

(ii) *the restriction of the map q_{w_0} to the normalizer $\mathrm{Nr}(\mathcal{A}'\mathfrak{n}_-)$ defines an isomorphism of the Mickelsson algebras $Z^{\mathfrak{n}_-}(\mathcal{A})$ and $Z^{\mathfrak{n}}(\mathcal{A})$ such that for any $d \in D$, $v \in \mathcal{V}$ and $x \in Z^{\mathfrak{n}_-}(\mathcal{A})$*

$$\mathrm{q}_{w_0}(z_{\mathfrak{n}_-,v}) = z'_v, \qquad \big[d, \mathrm{q}_{w_0}(x)\big] = \mathrm{q}_{w_0}\big([d,x]\big), \qquad \mathrm{q}_{w_0}(xd) = \mathrm{q}_{w_0}(x)\tau_{\rho-w_0(\rho)}(d).$$

6. Braid group action

6.1. *Operators* $\check{\mathrm{q}}_i$

Suppose that the automorphisms $T_w : U(\mathfrak{g}) \to U(\mathfrak{g})$, $w \in W$, admit extensions to automorphisms $T_w : \mathcal{A} \to \mathcal{A}$ of a $\mathfrak{g}$-admissible algebra $\mathcal{A}$. Such an extension is uniquely determined by the automorphisms $T_i : \mathcal{A} \to \mathcal{A}$, defined for all simple positive roots α_i, which extend the automorphisms $T_i : U(\mathfrak{g}) \to U(\mathfrak{g})$, see Section 2.1, and satisfy braid group relations, related to $\mathfrak{g}$:

$$\underbrace{T_iT_jT_i\cdots}_{m_{i,j}} = \underbrace{T_jT_iT_j\cdots}_{m_{i,j}}, \quad i \neq j, \tag{6.1}$$

where $m_{i,j} = 2$, if $a_{i,j} = 0$; $m_{i,j} = 3$, if $a_{i,j}a_{j,i} = 1$; $m_{i,j} = 4$, if $a_{i,j}a_{j,i} = 2$; $m_{i,j} = 6$, if $a_{i,j}a_{j,i} = 3$. There is no relation, if $a_{i,j}a_{j,i} > 3$. Having (6.1), for any $w \in W$ we define the automorphism $T_w : \mathcal{A} \to \mathcal{A}$ by the relation $T_w = T_{i_1}\cdots T_{i_k}$, where $w = s_{\alpha_{i_1}}\cdots s_{\alpha_{i_k}}$ is a reduced decomposition of w. Then the elements T_w do not depend on a choice of the reduced decomposition and satisfy the relations

$$T_{w'w} = T_{w'}\cdot T_w, \quad \text{if } l(w'w) = l(w') + l(w). \tag{6.2}$$

The automorphisms T_i (and T_w) admit unique extensions to automorphisms of the algebra $\mathcal{A}'$, satisfying (6.1). We denote them by the same symbols.

For any maximal nilpotent subalgebra $\mathfrak{m} = \mathfrak{n}^w$, where $w \in W$, and any root α, such that the root vector e_α is a simple positive root vector of $\mathfrak{m}$, we have the following relation:

$$\mathrm{q}_{\alpha,\mathfrak{m}} = T_w\mathrm{q}_{w^{-1}(\alpha),\mathfrak{n}}T_w^{-1}. \tag{6.3}$$

For each $\alpha_i \in \Pi$ define the operators $\check{\mathrm{q}}_i : \mathcal{A}'/\mathcal{A}'\mathfrak{n} \to \mathcal{A}'/\mathcal{A}'\mathfrak{n}$ by the relations

$$\check{\mathrm{q}}_i = \mathrm{q}_{s_{\alpha_i}} \cdot T_i. \tag{6.4}$$

In (6.4), we understand $\mathrm{q}_{s_{\alpha_i}} \equiv \mathrm{q}_{\alpha_i,\mathfrak{n}}$ as the maps $\mathrm{q}_{\alpha_i,\mathfrak{n}} : \mathcal{A}'/\mathcal{A}'\mathfrak{n}^{s_{\alpha_i}} \to \mathcal{A}'/\mathcal{A}'\mathfrak{n}$, given by the relations (4.9)–(4.11). Using the same agreement for any $w \in W$ we define the operators $\check{\mathrm{q}}_w : \mathcal{A}'/\mathcal{A}'\mathfrak{n} \to \mathcal{A}'/\mathcal{A}'\mathfrak{n}$ as

$$\check{\mathrm{q}}_w = \mathrm{q}_w \cdot T_w. \tag{6.5}$$

The relation (6.3), Proposition 4.8 and its analog for the maps $\tilde{\mathrm{q}}_{\overline{w},\mathfrak{m}}$ imply

Proposition 6.1. *Operators* $\check{\mathrm{q}}_i$ *satisfy the braid group relations*

$$\underbrace{\check{\mathrm{q}}_i\check{\mathrm{q}}_j\cdots}_{m_{i,j}} = \underbrace{\check{\mathrm{q}}_j\check{\mathrm{q}}_i\cdots}_{m_{i,j}}, \quad i \neq j. \tag{6.6}$$

In other words, for any reduced decomposition $w = s_{\alpha_{i_1}}\cdots s_{\alpha_{i_m}}$ we have equalities $\check{\mathrm{q}}_w = \check{\mathrm{q}}_{i_1}\cdots\check{\mathrm{q}}_{i_m}$ such that

$$\check{\mathrm{q}}_{w'w} = \check{\mathrm{q}}_{w'}\check{\mathrm{q}}_w, \quad \text{if } l(w'w) = l(w') + l(w). \tag{6.7}$$

Proof. Let $w', w \in W$ and $l(w'w) = l(w') + l(w)$. We have by (4.27) and (6.3)

$$\begin{aligned}\check{\mathrm{q}}_{w'w} &= \mathrm{q}_{w'w,\mathfrak{n}}T_{w'w} = \mathrm{q}_{w',\mathfrak{n}}\mathrm{q}_{w,\mathfrak{n}^{w'}}T_{w'w}\\ &= \mathrm{q}_{w',\mathfrak{n}}\big(T_{w'}\mathrm{q}_{w,\mathfrak{n}}T_{w'}^{-1}\big)T_{w'w} = \mathrm{q}_{w',\mathfrak{n}}T_{w'}\mathrm{q}_{w,\mathfrak{n}}T_w = \check{\mathrm{q}}_{w'}\check{\mathrm{q}}_w.\end{aligned}$$

Thus we have (6.7), which are equivalent to the braid group relations. □

For any $w \in W$ denote by $w\circ$ the shifted action of w in $\mathfrak{h}^*$:

$$w \circ \mu = w(\mu + \rho) - \rho.$$

It induces the shifted action by automorphisms of W on D, characterized by the relations

$$w \circ h_\alpha = h_{w(\alpha)} + \big\langle h_\alpha, w(\rho) - \rho\big\rangle. \tag{6.8}$$

Theorem 5.2 and Proposition 5.3 imply

Proposition 6.2. *For any* $x \in \mathcal{A}'/\mathcal{A}'\mathfrak{n}$, $y \in Z^{\mathfrak{n}}(\mathcal{A})$ *and* $d \in D$ *we have*

$$\begin{aligned}&\check{\mathrm{q}}_i\big([d,x]\big) = \big[s_{\alpha_i}(d), \check{\mathrm{q}}_i(x)\big], \qquad \check{\mathrm{q}}_i(xd) = \check{\mathrm{q}}_i(x)\cdot(s_{\alpha_i}\circ d),\\ &\check{\mathrm{q}}_i(xy) = \check{\mathrm{q}}_i(x)\cdot\check{\mathrm{q}}_i(y).\end{aligned} \tag{6.9}$$

Corollary 6.3.

(i) *The restriction of operators* $\check{\mathfrak{q}}_i$ *to* $Z^{\mathfrak{n}}(\mathcal{A})$ *defines an automorphism of* $Z^{\mathfrak{n}}(\mathcal{A})$*, satisfying* (6.9).
(ii) *For any* $x \in Z^{\mathfrak{n}}(\mathcal{A})$ *we have*

$$\check{\mathfrak{q}}_i^2(x) = (h_{\alpha_i}+1)^{-1} T_i^2(x)(h_{\alpha_i}+1).$$

Proof. The statement (i) of the corollary is a direct consequence of Proposition 6.2. The statement (ii) follows from Proposition 4.2 and the relation (6.3). Namely, for any $x \in Z^{\mathfrak{n}}(\mathcal{A})$ we have by Proposition 4.2

$$\check{\mathfrak{q}}_i^2(x) = \mathfrak{q}_{\alpha_i,\mathfrak{n}} T_i \mathfrak{q}_{\alpha_i,\mathfrak{n}} T_i(x) = \mathfrak{q}_{\alpha_i,\mathfrak{n}} \mathfrak{q}_{-\alpha_i,\mathfrak{n}^{s_\alpha}}\big(T_i^2(x)\big) = (h_{\alpha_i}+1)^{-1} T_i^2(x)(h_{\alpha_i}+1). \qquad \square$$

Clearly, all the statements of Proposition 6.2 remain valid for all operators $\check{\mathfrak{q}}_w$, $w \in W$. The properties (6.9), (7.16) look as

$$\check{\mathfrak{q}}_w\big([h,x]\big) = \big[w(h), \check{\mathfrak{q}}_w(x)\big], \qquad \check{\mathfrak{q}}_w(xd) = \check{\mathfrak{q}}_w(x)\cdot(w \circ d),$$
$$\check{\mathfrak{q}}_w(xy) = \check{\mathfrak{q}}_w(x)\cdot\check{\mathfrak{q}}_w(y). \tag{6.10}$$

6.2. Calculation of $\check{\mathfrak{q}}_i(z_v)$

Denote by $I_{\mathfrak{n}_-}$ the image of the right ideal $\mathfrak{n}_-\mathcal{A}'$ in $\mathcal{A}'/\mathcal{A}'\mathfrak{n}$:

$$I_{\mathfrak{n}_-} = (\mathfrak{n}_-\mathcal{A}' + \mathcal{A}'\mathfrak{n})/\mathcal{A}'\mathfrak{n}.$$

Lemma 6.4. *For any* $\alpha_i \in \Pi$ *we have an inclusion*

$$\check{\mathfrak{q}}_i(I_{\mathfrak{n}_-}) \subset I_{\mathfrak{n}_-}. \tag{6.11}$$

Proof. We have to prove that for any $\gamma \in \Delta_+$ and any $x \in \mathcal{A}'$

$$\mathfrak{q}_{\alpha_i}\big(T_i(e_{-\gamma}x)\big) = \sum_j e_{-\mu_j} y_j \tag{6.12}$$

for some $\mu_j \in \Delta_+$ and $y_j \in \mathcal{A}'/\mathcal{A}'\mathfrak{n}$. If $\gamma \neq \alpha_i$ then

$$\mathfrak{q}_{\alpha_i}\big(T_i(e_{-\gamma}x)\big) = \mathfrak{q}_{\alpha_i}\big(e_{-\gamma'}\cdot T_i(x)\big),$$

where $\gamma' = s_{\alpha_i}(\gamma) \in \Delta_+ \setminus \{\alpha_i\}$, and the statement of the lemma follows from the invariance of the subalgebra $\mathfrak{n}_-(\alpha_i)$ with respect to the action of $\hat{e}_{\alpha_i}$. Here $\mathfrak{n}_-(\alpha_i)$ is generated by root vectors $e_{-\gamma}$, where $\gamma \in \Delta_+ \setminus \{\alpha_i\}$.

If $\gamma = \alpha_i$ then the right-hand side of (6.12) vanishes due to (4.3)(i). $\square$

Suppose that an ad-invariant generating subspace $\mathcal{V}$ of a $\mathfrak{g}$-admissible algebra $\mathcal{A}$ is invariant with respect to the action of the automorphisms T_i, $T_i(\mathcal{V}) = \mathcal{V}$. Suppose that $\mathcal{A}$ satisfies the highest weight (HW) condition, see Section 3.3. With this assumption we calculate the elements $\check{\mathfrak{q}}_i(z_v)$, where $v \in \mathcal{V}$ and z_v are the generators of $Z^n(\mathcal{A})$, defined in Section 3.4.

Keep the notation of Sections 3.4 and 3.5. The operators $\mathrm{C}^{(2)}_{\pm\alpha}[\lambda]$ were defined in Section 3.5.

Proposition 6.5. *Assume that $\mathcal{A}$ satisfies the* HW *condition. Then*

$$\check{\mathrm{q}}_i(z_v) = z_{\mathrm{C}^{(2)}_{\alpha_i}[-\rho](1\otimes T_i(v))}.$$

Proof. Assume that $\mathcal{A}$ satisfies the HW condition. By definition of the elements z_v, we have $z_v = v \bmod I_{\mathfrak{n}_-}$. Lemma 6.4 then implies the equality

$$\check{\mathrm{q}}_i(z_v) = \check{\mathrm{q}}_i(v) \mod I_{\mathfrak{n}_-}.$$

We have

$$\begin{aligned}
\check{\mathrm{q}}_i(z_v) &= \check{\mathrm{q}}_i(v) \mod I_{\mathfrak{n}_-} \\
&= \sum_{n\geqslant 0} \frac{(-1)^n}{n!}\hat{e}^n_\alpha(v) e^n_{-\alpha} g_{\alpha,n} \mod I_{\mathfrak{n}_-} \\
&= \sum_{n\geqslant 0} \frac{1}{n!}\hat{e}^n_{-\alpha}\hat{e}^n_\alpha(v) g_{\alpha,n} \mod I_{\mathfrak{n}_-} \\
&= \sum_{n\geqslant 0} \frac{(-1)^n}{n!}\frac{(-1)^n}{(\hat{h}_\alpha - h_\alpha)(\hat{h}_\alpha - h_\alpha + 1)\cdots(\hat{h}_\alpha - h_\alpha + n - 1)}\hat{e}^n_{-\alpha}\hat{e}^n_\alpha(v) \mod I_{\mathfrak{n}_-}.
\end{aligned}$$

This is precisely the statement of the proposition. $\square$

Remark. Let $w = s_{\alpha_1}\cdots s_{\alpha_n}$ be a reduced decomposition of an element $w \in W$. Let $\gamma_1, \dots, \gamma_n$ be the corresponding sequence of positive roots: $\gamma_1 = \alpha_1,\ \gamma_2 = s_{\alpha_1}(\alpha_2), \dots$. Then the properties of the maps $\check{\mathrm{q}}_i$, see Proposition 6.2, imply the following relation:

$$\check{\mathrm{q}}_w(z_v) = z_{\mathrm{C}^{(2)}_{\gamma_1}[-\rho]\cdots\mathrm{C}^{(2)}_{\gamma_n}[-\rho](1\otimes T_w(v))}.$$

6.3. *Calculation of* $\check{\mathrm{q}}_i(z'_v)$

In this section we assume that $\mathfrak{g}$ is an arbitrary contragredient Lie algebra of finite growth and that the generating subspace $\mathcal{V}$ of the $\mathfrak{g}$-admissible algebra $\mathcal{A}$ is invariant with respect to the action of the automorphisms T_i. With this assumption we calculate the elements $\check{\mathrm{q}}_i(z'_v)$, where $v \in \mathcal{V}$ and z'_v are the generators of $Z^n(\mathcal{A})$, defined in Section 3.4.

Proposition 6.6. *Assume that the element $z'_v \in Z^{\mathfrak{n}}(\mathcal{A})$ is defined. Then the element $\check{\mathrm{q}}_i(z'_v)$ is also defined and is given by the relation*

$$\check{\mathrm{q}}_i\left(z'_v\right) = z'_{\mathrm{B}^{(2)}_{-\alpha_i}[\rho](1\otimes T_i(v))}. \tag{6.13}$$

The proof of Proposition 6.6 is based on the following lemma.

Let α be a simple positive root, and $\mathfrak{n}_\pm(\alpha)$ the subalgebras of $\mathfrak{n}_\pm$, generated by all root vectors except $e_{\pm\alpha}$ respectively.

Lemma 6.7. *Suppose that the element z'_v exists. Then it admits a presentation*

$$z'_v = \mathrm{q}_\alpha(v) + \sum_{n\geqslant 0, j} e^n_{-\alpha} v_{j,n} d_{j,n} f_{j,n}, \tag{6.14}$$

where $v_{j,n} \in \mathcal{V}$, $d_{j,n} \in D$, $f_{j,n} \in \mathfrak{n}_-(\alpha)U(\mathfrak{n}_-(\alpha))$.

Proof. Using the PBW theorem, present z'_v as $z'_v = x + y$, where

$$x = v + \sum_k v_k d_k e^k_{-\alpha}, \quad v_k \in \mathcal{V},\ d_k \in D,$$

$$y = \sum_{n\geqslant 0, j} \tilde{v}_{j,n} e^n_{-\alpha} d_{j,n} f_{j,n}, \quad \tilde{v}_{j,n} \in \mathcal{V},\ d_{j,n} \in D,\ f_{j,n} \in \mathfrak{n}_-(\alpha)U\big(\mathfrak{n}_-(\alpha)\big).$$

By definition, z'_w satisfies the relation $[e_\alpha, z'_v] = 0 \bmod \mathcal{A}'\mathfrak{n}$. Since the algebra $\mathfrak{n}_-(\alpha)$ is invariant with respect to the adjoint action of e_α, the commutator $[e_\alpha, y]$ is an element z of the same kind, so we have two equations

$$[e_\alpha, x] = 0 \mod \mathcal{A}'\mathfrak{n} \quad \text{and} \quad [e_\alpha, y] = 0 \mod \mathcal{A}'\mathfrak{n}.$$

The first equation has a unique solution $x = \mathrm{q}_\alpha(v)$. To finish the proof of the lemma, we move all factors $e^n_{-\alpha}$ in the presentation of y to the left, using the commutation relations in $\mathcal{A}$. □

Proof of Proposition 6.6. The application of the automorphism T_i to the presentation (6.14) gives:

$$T_i \mathrm{q}_{\alpha_i,\mathfrak{n}}(u) = T_i \mathrm{q}_{\alpha_i,\mathfrak{n}}(v) + \sum_{n\geqslant 0, j} e^n_{\alpha_i} \bar{v}_{j,n} \bar{d}_{j,n} \bar{f}_{j,n}, \tag{6.15}$$

where $\bar{v}_{j,n} = T_i(\tilde{v}_{j,n}) \in \mathcal{V}$, $\bar{f}_{j,n} = T_i(\tilde{f}_{j,n}) \in \mathfrak{n}_-(\alpha_i)U(\mathfrak{n}_-(\alpha_i))$ and $\bar{d}_{j,n} = T_i(\tilde{d}_{j,n}) \in D$. Now we apply the map $\mathrm{q}_{\alpha_i,\mathfrak{n}}$ to both sides of (6.15). The images of the terms in the last sum with $n > 0$ vanish, since $\mathrm{q}_{\alpha_i,\mathfrak{n}}(e_{\alpha_i} x) = 0$ for any $x \in \mathcal{A}'$ by (4.3)(iv). The images of the terms in the last sum with $n = 0$ do not contribute to the 'leading term,' since the algebra $\mathfrak{n}_-(\alpha_i)$ is invariant with respect to the adjoint action of $\hat{e}_{\alpha_i}$. We obtain

Lemma 6.8. *The leading term of $\check{\mathrm{q}}_i(z'_v)$ is equal to the leading term of $\mathrm{q}_{\alpha_i,\mathfrak{n}} T_i \mathrm{q}_{\alpha_i}(v)$.*

Let us compute the leading term of $\mathrm{q}_{\alpha_i,\mathfrak{n}} T_i \mathrm{q}_{\alpha_i}(v)$. We have

$$\begin{aligned}
\mathrm{q}_{\alpha_i,\mathfrak{n}} T_i \mathrm{q}_{\alpha_i}(v) &= \mathrm{q}_{\alpha_i,\mathfrak{n}} T_i \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \hat{e}^n_{\alpha_i}(v) e^n_{-\alpha_i} g_{n,\alpha_i} \right) \\
&= \mathrm{q}_{\alpha_i,\mathfrak{n}} \left(\sum_{n=0}^{\infty} \frac{1}{n!} \hat{e}^n_{-\alpha_i}\big(T_i(v)\big) e^n_{\alpha_i} \big((h_{\alpha_i})(h_{\alpha_i}+1)\cdots(h_{\alpha_i}+n-1)\big)^{-1} \right) \\
&= \mathrm{q}_{\alpha_i,\mathfrak{n}} \left(\sum_{n=0}^{\infty} \frac{1}{n!} \hat{e}^n_{-\alpha_i}\big(T_i(v)\big) e^n_{\alpha_i} \right) \big((h_{\alpha_i}+2)(h_{\alpha_i}+3)\cdots(h_{\alpha_i}+n+1)\big)^{-1},
\end{aligned}$$

by the property (4.3)(iii). Since $\mathrm{q}_{\alpha_i,\mathfrak{n}}(e_{\alpha_i}x)=0$ for any $x\in\mathcal{A}'$, we further get

$$\mathrm{q}_{\alpha_i,\mathfrak{n}}T_i\mathrm{q}_{\alpha_i}(v)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\mathrm{q}_{\alpha_i,\mathfrak{n}}\big(\hat{e}^n_{\alpha_i}\hat{e}^n_{-\alpha_i}\big(T_i(v)\big)\big)\big((h_{\alpha_i}+2)\cdots(h_{\alpha_i}+n+1)\big)^{-1}.$$

Since $\hat{e}^n_{\alpha_i}\hat{e}^n_{-\alpha_i}(T_i(v))$ belongs to $\mathcal{V}$, the leading term of $\mathrm{q}_{\alpha_i,\mathfrak{n}}(\hat{e}^n_{\alpha_i}\hat{e}^n_{-\alpha_i}(T_i(v)))$ is equal to $\hat{e}^n_{\alpha_i}\hat{e}^n_{-\alpha_i}(T_i(v))$ and the leading term of $\mathrm{q}_{\alpha_i,\mathfrak{n}}T_i\mathrm{q}_{\alpha_i,\mathfrak{n}}(v)$ is equal to the sum

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\hat{e}^n_{\alpha_i}\hat{e}^n_{-\alpha_i}\big(T_i(v)\big)\big((h_{\alpha_i}+2)\cdots(h_{\alpha_i}+n+1)\big)^{-1},$$

which can be written as $\mathrm{B}^{(2)}_{-\alpha_i}[\rho](1\otimes T_i(v))$. This ends the proof of Proposition 6.6. □

As well as in the previous section, for any reduced decomposition $w=s_{\alpha_1}\cdots s_{\alpha_n}$ of an element $w\in W$ and the corresponding sequence of positive roots: $\gamma_1=\alpha_1$, $\gamma_2=s_{\alpha_1}(\alpha_2),\dots$ we have, assuming the existence of z'_v,

$$\check{\mathrm{q}}_w\big(z'_v\big)=z'_{\mathrm{B}^{(2)}_{-\gamma_1}[\rho]\cdots\mathrm{B}^{(2)}_{\gamma_n}[\rho](1\otimes T_w(v))}.$$

7. Mickelsson algebra $Z_{\mathfrak{n}_-}(\mathcal{A})$

In this section we collect the results of the previous section for the Mickelsson algebra $Z_{\mathfrak{n}_-}(\mathcal{A})$, see the definition below. This algebra deserves the special attention, since it acts on the space of $\mathfrak{n}_-$-coinvariants, which are sometimes more convenient than $\mathfrak{n}$-invariants. If a $\mathfrak{g}$-admissible algebra $\mathcal{A}$ admits an antiinvolution, which extends the Cartan antiinvolution ${}^t: U(\mathfrak{g})\to U(\mathfrak{g})$, then all the results of this section can be obtained by an application of this antiinvolution to corresponding results from the previous sections.

7.1. Algebra $Z_{\mathfrak{n}_-}(\mathcal{A})$ and related structures

For any $\mathfrak{g}$-admissible algebra $\mathcal{A}$, we define Mickelsson algebras $S_{\mathfrak{n}_-}(\mathcal{A})$ and $Z_{\mathfrak{n}_-}(\mathcal{A})$ as the quotients

$$S_{\mathfrak{n}_-}(\mathcal{A})=\mathfrak{n}_-\mathcal{A}\setminus\mathrm{Nr}(\mathfrak{n}_-\mathcal{A}),\qquad Z_{\mathfrak{n}_-}(\mathcal{A})=\mathfrak{n}_-\mathcal{A}'\setminus\mathrm{Nr}(\mathfrak{n}_-\mathcal{A}'),$$

where $\mathrm{Nr}(\mathfrak{n}_-\mathcal{A})$ (respectively $\mathrm{Nr}(\mathfrak{n}_-\mathcal{A}')$) is the normalizer of the right ideal $\mathfrak{n}_-\mathcal{A}$ (respectively $\mathfrak{n}_-\mathcal{A}'$). The algebra $Z_{\mathfrak{n}_-}(\mathcal{A})$ is a localization of the algebra $S_{\mathfrak{n}_-}(\mathcal{A})$: $Z_{\mathfrak{n}_-}(\mathcal{A})=D\otimes_{U(\mathfrak{h})}S_{\mathfrak{n}_-}(\mathcal{A})$.

Alternatively, the Mickelsson algebra $Z_{\mathfrak{n}_-}(\mathcal{A})$ can be defined as the subspace of $\mathfrak{n}_-$-invariants in a right $U(\mathfrak{g})$-module $\tilde{M}_{\mathfrak{n}_-}(\mathcal{A}')=\mathfrak{n}_-\mathcal{A}'\setminus\mathcal{A}'$:

$$Z_{\mathfrak{n}_-}(\mathcal{A})=\big(\tilde{M}_{\mathfrak{n}_-}(\mathcal{A}')\big)^{\mathfrak{n}_-}=\big\{m\in\tilde{M}_{\mathfrak{n}}(\mathcal{A}')\bigm| m\mathfrak{n}_-=0\big\}.$$

As well as $S^{\mathfrak{n}}(\mathcal{A})$, the space $S_{\mathfrak{n}_-}(\mathcal{A})$ is an associative algebra, containing $U(\mathfrak{h})$, and for any left $\mathcal{A}$-module M, the space $M_{\mathfrak{n}_-}=M/\mathfrak{n}_-M$ of $\mathfrak{n}_-$-coinvariants is a $S_{\mathfrak{n}_-}(\mathcal{A})$-module, see Proposition 3.1. The algebra $Z_{\mathfrak{n}_-}$ acts in the space $M_{\mathfrak{n}_-}$ of $\mathfrak{n}_-$-coinvariants of any left $\mathcal{A}'$-module M.

Suppose that a $\mathfrak{g}$-admissible algebra $\mathcal{A}$ satisfies the additional *local lowest weight* condition

(LW) For any $v \in \mathcal{V}$ the adjoint action of elements $x \in U(\mathfrak{n}_-)_\mu$ on v is nontrivial, $\hat{x}(v) \neq 0$, only for a finite number of $\mu \in \mathfrak{h}^*$.

Then the quotient $\tilde{M}_{\mathfrak{n}_-}(\mathcal{A}') = \mathfrak{n}_-\mathcal{A}' \setminus \mathcal{A}'\mathfrak{n}$ has a structure of a right $F_{\mathfrak{g},\mathfrak{n}}$-module, extending the action of $\mathcal{A}'$ by the right multiplication. In particular, the extremal projector P acts in the right $F_{\mathfrak{g},\mathfrak{n}}$-module $\tilde{M}_{\mathfrak{n}_-}(\mathcal{A}')$. Denote the corresponding operator by $\bar{\mathrm{p}} \in \operatorname{End} \tilde{M}_{\mathfrak{n}_-}(\mathcal{A}')$, see Section 2.2.

The properties of the extremal projectors imply the relation

$$Z_{\mathfrak{n}_-}(\mathcal{A}) = \operatorname{Im} \bar{\mathrm{p}} \subset \tilde{M}_{\mathfrak{n}_-}(\mathcal{A}'). \tag{7.1}$$

Equip the double coset space ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$, see (3.6), with a multiplication $\circ : {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}} \otimes_{\mathfrak{n}_-} \mathcal{A}_{\mathfrak{n}} \to {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$:

$$a \circ b = a\mathrm{P}b \stackrel{\text{def}}{=} \bar{\mathrm{p}}(a) \cdot b. \tag{7.2}$$

We also call the double coset space ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$, equipped with the operation (7.2), the *double coset algebra* ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$. In a case, when both conditions (HW) and (LW) are satisfied, the multiplication rules (3.7) and (7.2) coincide.

Define the linear maps $\phi_- : Z_{\mathfrak{n}_-}(\mathcal{A}) \to {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$ and $\psi_- : {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}} \to Z_{\mathfrak{n}_-}(\mathcal{A})$ by the rules

$$\phi_-(x) = x \mod \mathcal{A}'\mathfrak{n}, \qquad \psi^+(y) = \bar{\mathrm{p}}(y), \quad x \in Z_{\mathfrak{n}_-}(\mathcal{A}),\ y \in {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}. \tag{7.3}$$

Proposition 7.1. *Assume that a $\mathfrak{g}$-admissible algebra $\mathcal{A}$ satisfies the condition* (LW). *Then*

(i) *The operation* (3.7) *equips ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$ with a structure of an associative algebra.*
(ii) *The linear maps ϕ_- and ψ_- are inverse to each other and establish an isomorphism of the algebras $Z_{\mathfrak{n}_-}(\mathcal{A})$ and ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$.*

We have the 'lowest weight counterpart' of Propositions 3.3 and 3.4.

Proposition 7.2. *Let $\mathcal{A}$ be a $\mathfrak{g}$-admissible algebra satisfying the condition* (LW). *Then*

(i) *Each element of the double coset algebra ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$ can be uniquely presented in a form $x = \sum_i d_i v_i$, where $d_i \in D$, $w_i \in \mathcal{V}$, so that ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$ is a free left* (*and right*) *D-module, isomorphic to $D \otimes \mathcal{V}$.*
(ii) *For each $v \in \mathcal{V}$ there exists a unique element $\tilde{z}_v \in Z_{\mathfrak{n}_-}(\mathcal{A})$ of the form*

$$\tilde{z}_v = v + \sum_{i=1,\dots,k} v_i e_i d_i, \quad e_i \in \mathfrak{n}U(\mathfrak{n}),\ d_i \in D,\ v_i \in \mathcal{V}, \tag{7.4}$$

so that the algebra $Z_{\mathfrak{n}_-}(\mathcal{A})$ is a free left (*and right*) *D-module, generated by the elements $\tilde{z}_v$. The element $\tilde{z}_v$ is equal to $\bar{\mathrm{p}}(v)$.*
(iii) *For each $v \in \mathcal{V}$ there exists at most one element $\tilde{z}'_v \in Z_{\mathfrak{n}_-}(\mathcal{A})$ of the form*

$$\tilde{z}'_v = v + \sum_j e_j v_j d_j, \quad e_j \in \mathfrak{n}U(\mathfrak{n}),\ d_j \in D,\ v_j \in \mathcal{V}. \tag{7.5}$$

Next, we have an analog of Theorems 3.5 and 3.6:

Theorem 7.3. *Let $\mathfrak{g}$ be reductive and finite-dimensional, and let $\mathcal{A}$ be a $\mathfrak{g}$-admissible algebra with a generating subspace $\mathcal{V}$. Then for any $v \in \mathcal{V}$*

(i) *there exists a unique element $\tilde{z}'_v \in Z_{\mathfrak{n}_-}(\mathcal{A})$ of the form* (7.5). *The algebra $Z_{\mathfrak{n}_-}(\mathcal{A})$ is generated by the elements $\tilde{z}'_v$ as a free left* (*and right*) *D-module*;

(ii) *we have the following equality in $Z_{\mathfrak{n}_-}(\mathcal{A})$*

$$\tilde{z}_v = \tilde{z}'_{\mathrm{B}_-^{(2)}[\rho](1\otimes v)}. \tag{7.6}$$

Here $\tilde{z}_{d\otimes v} = d\cdot\tilde{z}_v$, $\tilde{z}'_{d\otimes v} = d\cdot\tilde{z}'_v$, $\tilde{z}_{v\otimes d} = \tilde{z}_v\cdot d$, $\tilde{z}'_{v\otimes d} = \tilde{z}'_v\cdot d$.

7.2. Zhelobenko maps

For any real root α, the relations

$$\tilde{\mathrm{q}}_\alpha(x) = \sum_{n=0}^{\infty}\frac{1}{(n)!}g_{n,\alpha}\cdot e_\alpha^n\cdot\hat{e}_{-\alpha}^n(x) \quad \mathrm{mod}\ \mathfrak{n}_{-\alpha}\mathcal{A}',\ x\in\mathcal{A},$$
$$\tilde{\mathrm{q}}_\alpha(dx) = \tau_\alpha(d)\tilde{\mathrm{q}}_\alpha(x), \quad d\in D,$$

define a map $\tilde{\mathrm{q}}_\alpha : \mathcal{A}' \to \mathfrak{n}_{-\alpha}\mathcal{A}' \setminus \mathcal{A}'$, such that for any $x \in \mathcal{A}'$

$$\tilde{\mathrm{q}}_\alpha(e_\alpha x) = 0, \qquad \tilde{\mathrm{q}}_\alpha(xe_{-\alpha}) = \tilde{\mathrm{q}}_\alpha(x)e_{-\alpha} = 0. \tag{7.7}$$

Here $g_{n,\alpha}$ is given by (4.2), $\mathfrak{n}_{\pm\alpha} = \mathbb{C}e_{\pm\alpha}$. We have the analogs of Propositions 4.1, 4.2 and Theorem 5.1.

Theorem 7.4.

(i) *The map $\tilde{\mathrm{q}}_\alpha$ defines an isomorphism of algebras $\tilde{\mathrm{q}}_\alpha : {}_{\mathfrak{n}_\alpha}\mathcal{A}_{\mathfrak{n}_{-\alpha}} \to Z_{\mathfrak{n}_{-\alpha}}(\mathcal{A})$, such that for any $d \in D$,*

$$\big[d, \tilde{\mathrm{q}}_\alpha(x)\big] = \tilde{\mathrm{q}}_\alpha\big([d,x]\big), \qquad \tilde{\mathrm{q}}_\alpha(dx) = \tau_\alpha(d)\tilde{\mathrm{q}}_\alpha(x).$$

(ii) *For any $x \in Z_{\mathfrak{n}_\alpha}(\mathcal{A})$ and $y \in Z_{\mathfrak{n}_{-\alpha}}(\mathcal{A})$ we have*

$$\tilde{\mathrm{q}}_{-\alpha}\tilde{\mathrm{q}}_\alpha(x) = (h_\alpha+1)x(h_\alpha+1)^{-1}, \qquad \tilde{\mathrm{q}}_\alpha\tilde{\mathrm{q}}_{-\alpha}(y) = (h_\alpha+1)^{-1}y(h_\alpha+1). \tag{7.8}$$

Theorem 7.4 implies the equality

$$\tilde{\mathrm{q}}_\alpha(xy) = \tilde{\mathrm{q}}(x)\tilde{\mathrm{q}}(y) \quad \text{for any } x\in\mathrm{Nr}(\mathfrak{n}_\alpha\mathcal{A}'),\ y\in\mathcal{A}'.$$

For a maximal nilpotent subalgebra $\mathfrak{m}$ of $\mathfrak{g}$, such that e_α is a simple positive root vector of $\mathfrak{m}$, define the linear map $\tilde{\mathrm{q}}^{(k)}_{\alpha,\mathfrak{m}} : \mathcal{A}' \to \mathfrak{m}_-\mathcal{A}' \setminus \mathcal{A}'$ by the prescriptions

$$\tilde{\mathrm{q}}_{\alpha,\mathfrak{m}}(x)=\sum_{n=0}^{\infty}\frac{1}{(n)!}g_{n,\alpha}\cdot e_{\alpha}^{n}\cdot\hat{e}_{-\alpha}^{n}(x)\quad \operatorname{mod}\mathfrak{m}_{-}\mathcal{A}',\qquad x\in\mathcal{A},\tag{7.9}$$

$$\tilde{\mathrm{q}}_{\alpha,\mathfrak{m}}(dx)=\tau_{\alpha}(d)\tilde{\mathrm{q}}_{\alpha}(x),\qquad d\in D.\tag{7.10}$$

The assignment $\tilde{\mathrm{q}}_{\alpha,\mathfrak{m}}$ satisfies the relation $\tilde{\mathrm{q}}_{\alpha,\mathfrak{m}}(\mathfrak{m}_{-}^{s_\alpha}\mathcal{A}')=0$ and determines the map

$$\tilde{\mathrm{q}}_{\alpha,\mathfrak{m}}:\mathfrak{m}_{-}^{s_\alpha}\mathcal{A}'\setminus\mathcal{A}'\to\mathfrak{m}_{-}\mathcal{A}'\setminus\mathcal{A}'.$$

We have (see Proposition 4.4 and Theorem 5.2):

Proposition 7.5.

(i) *The map* $\tilde{\mathrm{q}}_{\alpha,\mathfrak{m}}$ *transforms* $\mathrm{Nr}(\mathfrak{m}_{-}^{s_\alpha}\mathcal{A}')$ *to the Mickelsson algebra* $Z_{m_-}(\mathcal{A})$, *and*

$$\tilde{\mathrm{q}}_{\alpha,\mathfrak{m}}(zu)=\tilde{\mathrm{q}}_{\alpha,\mathfrak{m}}(z)\tilde{\mathrm{q}}_{\alpha,\mathfrak{m}}(u)\quad \textit{for any } z\in\mathrm{Nr}\big(\mathfrak{m}_{-}^{s_\alpha}\mathcal{A}'\big),\ u\in\mathcal{A}'.$$

(ii) *The restriction of the map* $\tilde{\mathrm{q}}_{\alpha,\mathfrak{m}}$ *to the normalizer* $\mathrm{Nr}(\mathfrak{m}_{-}^{s_\alpha}\mathcal{A}')$ *defines an isomorphism of the algebras* $Z_{m_-^{s_\alpha}}(\mathcal{A})$ *and* $Z_{m_-}(\mathcal{A})$, *satisfying the relations*

$$\big[d,\tilde{\mathrm{q}}_{\alpha,\mathfrak{m}}(x)\big]=\tilde{\mathrm{q}}_{\alpha,\mathfrak{m}}\big([\,,x]\big),\qquad \tilde{\mathrm{q}}_{\alpha,\mathfrak{m}}(xd)=\tilde{\mathrm{q}}_{\alpha,\mathfrak{m}}(x)\tau_{\alpha}(d),\quad d\in D.\tag{7.11}$$

For any $w\in W$, satisfying the condition (4.16), and its reduced decomposition (4.17), we define a map $\tilde{\mathrm{q}}_{w,\mathfrak{m}}:\ \mathfrak{m}_{-}^{w}\mathcal{A}'\setminus\mathcal{A}'\to\mathfrak{m}_{-}\mathcal{A}'\setminus\mathcal{A}'$ by the relation

$$\tilde{\mathrm{q}}_{w,\mathfrak{m}}=\tilde{\mathrm{q}}_{\gamma_1,\mathfrak{m}_1}\cdot\tilde{\mathrm{q}}_{\gamma_2,\mathfrak{m}_2}\cdot\dots\cdot\tilde{\mathrm{q}}_{\gamma_n,\mathfrak{m}_n},$$

where the positive roots γ_k and the maximal nilpotent subalgebras $\mathfrak{m}_k$ are defined by the prescriptions (4.18)–(4.19). This map does not depend on the choice of a reduced decomposition of w and satisfies the relations

$$\tilde{\mathrm{q}}_{w'w,\mathfrak{m}}=\tilde{\mathrm{q}}_{w',\mathfrak{m}}\tilde{\mathrm{q}}_{w,\mathfrak{m}^{w'}},\quad \text{if } l(w'w)=l(w')+l(w),\tag{7.12}$$

$$\big[h,\tilde{\mathrm{q}}_{w,\mathfrak{m}}(x)\big]=\tilde{\mathrm{q}}_{w,\mathfrak{m}}\big([h,x]\big),\qquad \tilde{\mathrm{q}}_{w,\mathfrak{m}}(dx)=\tau_{w'(\rho)-w'w(\rho)}(d)\cdot\tilde{\mathrm{q}}_{w,\mathfrak{m}}(x),\tag{7.13}$$

for any $x\in\mathcal{A}'$, $h\in\mathfrak{h}$ and $d\in D$.

The restriction of $\tilde{\mathrm{q}}_{w,\mathfrak{m}}$ to the normalizer $\mathrm{Nr}\,(\mathfrak{m}_{-}^{w}\mathcal{A}')$ defines an isomorphism of the algebras $Z_{\mathfrak{m}_{-}^{w}}(\mathcal{A})$ and $Z_{\mathfrak{m}_-}(\mathcal{A})$, satisfying (7.13). We denote $\tilde{\mathrm{q}}_w\equiv\tilde{\mathrm{q}}_{w,\mathfrak{n}}$.

The following counterpart of Proposition 4.6 is valid.

Proposition 7.6. *Let* $\mathfrak{g}$ *be a finite-dimensional reductive Lie algebra. Let* w_0 *be the longest element of the Weyl group* W. *Then*

(i) *The map* $\tilde{\mathrm{q}}_{w_0}$ *defines an isomorphism of the algebras* ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$ *and* $Z_{\mathfrak{n}_-}(\mathcal{A})$, *such that for any* $x\in{}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$, $d\in D$ *and* $v\in\mathcal{V}$

$$\begin{gathered}\big[d,\tilde{\mathrm{q}}_{w_0}(x)\big]=\tilde{\mathrm{q}}_{w_0}\big([d,x]\big),\qquad \tilde{\mathrm{q}}_{w_0}(dx)=\tau_{\rho-w_0(\rho)}(d)\tilde{\mathrm{q}}_{w_0}(x),\\ \tilde{\mathrm{q}}_{w_0}(v)=\tilde{z}'_{v}.\end{gathered}\tag{7.14}$$

(ii) *The restriction of* $\tilde{\mathrm{q}}_{w_0}$ *to the normalizer* $\mathrm{Nr}(\mathfrak{n}\mathcal{A}')$ *defines an isomorphism of the algebras* $Z_{\mathfrak{n}}(\mathcal{A})$ *and* $Z_{\mathfrak{n}_-}(\mathcal{A})$*, satisfying* (7.14)*, such that*

$$\tilde{\mathrm{q}}_{w_0}(\tilde{z}_{\mathfrak{n},v}) = \tilde{z}'_{\mathfrak{n}_-,v}, \quad v \in \mathcal{V}.$$

Here $\tilde{z}_{\mathfrak{n},v}$ are the generators of the Mickelsson algebra $Z_{\mathfrak{n}}(\mathcal{A})$ of the 'first type,' see Proposition 7.2(i), $\tilde{z}'_{\mathfrak{n}_-,v}$ are the generators of the Mickelsson algebra $Z_{\mathfrak{n}-}(\mathcal{A})$ of the 'second type,' see Proposition 7.2(ii).

7.3. Braid group action

Keep the notation of Section 6. We suppose again that the automorphisms $T_i : U(\mathfrak{g}) \to U(\mathfrak{g})$, $i = 1, \ldots, r$, see Section 2.1, admit extensions to automorphisms $T_i : \mathcal{A} \to \mathcal{A}$ of a $\mathfrak{g}$-admissible algebra $\mathcal{A}$, satisfying the braid group relations (6.1).

Then for each maximal nilpotent subalgebra $\mathfrak{m} = \mathfrak{n}^w$, where $w \in W$, and any root α, such that the root vector e_α is a simple positive root vector of $\mathfrak{m}$, we have the relations:

$$\tilde{\mathrm{q}}_{\alpha,\mathfrak{m}} = T_w \tilde{\mathrm{q}}_{w^{-1}(\alpha),\mathfrak{n}} T_w^{-1}. \tag{7.15}$$

For each $\alpha_i \in \Pi$ define the operators $\check{\mathrm{q}}_i : \mathfrak{n}_-\mathcal{A}' \backslash \mathcal{A}' \to \mathfrak{n}_-\mathcal{A}' \backslash \mathcal{A}'$ and $\check{\mathrm{q}}_w : \mathfrak{n}_-\mathcal{A}' \backslash \mathcal{A}' \to \mathfrak{n}_-\mathcal{A}' \backslash \mathcal{A}'$ as

$$\check{\mathrm{q}}_i = \tilde{\mathrm{q}}_{s_{\alpha_i}} \cdot T_i, \qquad \check{\mathrm{q}}_w = \tilde{\mathrm{q}}_w \cdot T_w.$$

The operators $\check{\mathrm{q}}_i$ satisfy the braid group relations,

$$\underbrace{\check{\mathrm{q}}_i \check{\mathrm{q}}_j \cdots}_{m_{i,j}} = \underbrace{\check{\mathrm{q}}_j \check{\mathrm{q}}_i \cdots}_{m_{i,j}}, \quad i \neq j,$$

that is,

$$\check{\mathrm{q}}_{w'w} = \check{\mathrm{q}}_{w'} \check{\mathrm{q}}_w, \quad \text{if } l(w'w) = l(w') + l(w).$$

For any $w \in W$, $x \in Z_{\mathfrak{n}_-}(\mathcal{A})$, $y \in \mathfrak{n}_-\mathcal{A}' \setminus \mathcal{A}'$ and $d \in D$ we have by Proposition 7.5:

$$\begin{aligned} &\check{\mathrm{q}}_w\big([h, x]\big) = \big[w(h), \check{\mathrm{q}}_w(x)\big], \qquad \check{\mathrm{q}}_w(dx) = (w \circ d) \cdot \check{\mathrm{q}}_w(x), \\ &\check{\mathrm{q}}_w(xy) = \check{\mathrm{q}}_w(x) \cdot \check{\mathrm{q}}_w(y). \end{aligned} \tag{7.16}$$

Corollary 7.7.

(i) *The restriction of* $\check{\mathrm{q}}_i$ *to* $Z_{\mathfrak{n}_-}(\mathcal{A})$ *defines an automorphism of the algebra* $Z_{\mathfrak{n}_-}(\mathcal{A})$*, satisfying* (7.16).

(ii) *For any* $y \in Z_{\mathfrak{n}_-}(\mathcal{A})$ *we have*

$$\check{\mathrm{q}}_i^2(y) = (h_{\alpha_i} + 1)^{-1} T_i^2(y)(h_{\alpha_i} + 1).$$

The following proposition describes the action of the automorphisms $\check{\mathrm{q}}_i$ on the canonical generators of the Mickelsson algebra $Z_{\mathfrak{n}_-}(\mathcal{A})$.

Proposition 7.8.

(i) *Assume that $\mathcal{A}$ satisfies the* LW *condition. Then*

$$\check{\mathrm{q}}_i(\tilde{z}_v) = \tilde{z}_{\mathrm{C}^{(1)}_{-\alpha_i}[-\rho](T_i(v)\otimes 1)}.$$

(ii) *Assume that the element $\tilde{z}'_v \in Z_{\mathfrak{n}_-}(\mathcal{A})$ is defined. Then $\check{\mathrm{q}}_i(\tilde{z}'_v)$ is also defined and is given by the relation*

$$\check{\mathrm{q}}_i(\tilde{z}'_v) = \tilde{z}'_{\mathrm{B}^{(1)}_{\alpha_i}[\rho](T_i(v)\otimes 1)}.$$

The operators $\mathrm{C}^{(1)}_{\pm\alpha}[\lambda]$ and $\mathrm{B}^{(1)}_{\pm\alpha}[\lambda]$ were defined in Section 3.5.

Remark. Let $w = s_{\alpha_1}\cdots s_{\alpha_n}$ be a reduced decomposition of an element $w \in W$. Let $\gamma_1, \dots, \gamma_n$ be the corresponding sequence of positive roots: $\gamma_1 = \alpha_1,\ \gamma_2 = s_{\alpha_1}(\alpha_2),\ \dots$. Then the properties of the maps $\check{\mathrm{q}}_i$ imply the following relations:

$$\check{\mathrm{q}}_w(\tilde{z}_v) = \tilde{z}_{\mathrm{C}^{(1)}_{-\gamma_1}[-\rho]\cdots\mathrm{C}^{(1)}_{-\gamma_n}[-\rho](T_w(v)\otimes 1)}, \qquad \check{\mathrm{q}}_w(\tilde{z}'_v) = \tilde{z}'_{\mathrm{B}^{(1)}_{\gamma_1}[\rho]\cdots\mathrm{B}^{(1)}_{-\gamma_n}[\rho](T_w(v)\otimes 1)}.$$

8. Standard modules and dynamical Weyl group

8.1. Double coset space

Recall the notation ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}} = \mathfrak{n}_-\mathcal{A}' \setminus \mathcal{A}'/\mathcal{A}'\mathfrak{n}$ of Section 3.3.

Lemma 8.1. *The multiplication in $\mathcal{A}'$ equips the double coset space ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$ with the structure of a left $Z_{\mathfrak{n}_-}(\mathcal{A})$-module and of a right $Z^{\mathfrak{n}}(\mathcal{A})$-module.*

Proof. It follows from the definition of normalizers. □

If $\mathcal{A}$ satisfies the HW condition, Proposition 3.2 says that the double coset space ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$ is a free right $Z^{\mathfrak{n}}(\mathcal{A})$-module of rank one, generated by the class of 1. If $\mathcal{A}$ satisfies the LW condition, Proposition 7.1 says that the double coset space ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$ is a free left $Z_{\mathfrak{n}_-}(\mathcal{A})$-module of rank one, generated by the class of 1.

Lemma 6.4 says that the operators $\check{\mathrm{q}}_i : \mathcal{A}'/\mathcal{A}'\mathfrak{n} \to \mathcal{A}'/\mathcal{A}'\mathfrak{n}$ and $\check{\mathrm{q}}_i : \mathfrak{n}_-\mathcal{A}' \setminus \mathcal{A}' \to \mathfrak{n}_-\mathcal{A}' \setminus \mathcal{A}'$ correctly define operators on the double coset space ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$. We denote them by the same symbol. According to definitions, they are given by the formulas

$$\check{\mathrm{q}}_i(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \hat{e}^n_{\alpha_i}\big(T_i(x)\big) \cdot e^n_{-\alpha_i} \cdot g_{n,\alpha_i} \mod \mathfrak{n}_-\mathcal{A}' + \mathcal{A}'\mathfrak{n},$$

$$\check{\mathrm{q}}_i(x) = \sum_{n=0}^{\infty} \frac{1}{n!} g_{n,\alpha_i} \cdot e^n_{\alpha_i} \cdot \hat{e}^n_{-\alpha_i}\big(T_i(x)\big) \mod \mathfrak{n}_-\mathcal{A}' + \mathcal{A}'\mathfrak{n},$$

where $g_{n,\alpha_i} = (h_{\alpha_i}(h_{\alpha_i} - 1)\cdots(h_{\alpha_i} - n + 1))^{-1}$. Equivalently,

$$\check{\mathrm{q}}_i(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{e}^n_{-\alpha_i} \hat{e}^n_{\alpha_i}\bigl(T_i(x)\bigr) \cdot g_{n,\alpha_i} \quad \mathrm{mod}\ \mathfrak{n}_-\mathcal{A}' + \mathcal{A}'\mathfrak{n}, \tag{8.1}$$

$$\check{\mathrm{q}}_i(x) = \sum_{n=0}^{\infty} \frac{1}{n!} g_{n,\alpha_i} \cdot \hat{e}^n_{\alpha_i} \hat{e}^n_{-\alpha_i}\bigl(T_i(x)\bigr) \quad \mathrm{mod}\ \mathfrak{n}_-\mathcal{A}' + \mathcal{A}'\mathfrak{n}. \tag{8.2}$$

They satisfy all the properties, mentioned in Propositions 6.1, 6.2, Corollary 6.3.

In the notation of Section 3.5, the formulas (8.1) and (8.2) mean that for any $v \in \mathcal{V}$ we have the following equalities in ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$:

$$\check{\mathrm{q}}_i(v) = m\bigl(\mathrm{C}^{(2)}_{\alpha_i}[-\rho]\bigl(1 \otimes T_i(v)\bigr)\bigr), \qquad \check{\mathrm{q}}_i(v) = m\bigl(\mathrm{C}^{(1)}_{-\alpha_i}[-\rho]\bigl(T_i(v) \otimes 1\bigr)\bigr), \tag{8.3}$$

where $m : D \otimes \mathcal{V} \to \mathcal{A}'$ and $m : \mathcal{V} \otimes D \to \mathcal{A}'$ are the multiplication maps.

Let M be a module over an associative algebra U. Denote by ξ_M the corresponding homomorphism $\xi_M : U \to \mathrm{End}(M)$. Let $T : U \to U$ be an automorphism of the algebra U. Denote by M^T the U-module M, conjugated by the automorphism T. It can be described as follows. M^T coincides with M as a vector space, while the map $\xi_{M^T} : U \to \mathrm{End}(M) \equiv \mathrm{End}(M^T)$ is

$$\xi_{M^T} = \xi_M \cdot T.$$

In this notation, Proposition 6.2 states the equivariance of the maps $\check{\mathrm{q}}_i$ and $\check{\mathrm{q}}_i$:

Proposition 8.2.

(i) *The map* $\check{\mathrm{q}}_i$, *given by* (8.1) *is a morphism of the right* $Z^{\mathfrak{n}}(\mathcal{A})$*-modules*:

$$\check{\mathrm{q}}_i : {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}} \to ({}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}})^{\check{\mathrm{q}}_i}.$$

(ii) *The map* $\check{\mathrm{q}}_i$, *given by* (8.2) *is a morphism of the left* $Z_{\mathfrak{n}_-}(\mathcal{A})$*-modules*:

$$\check{\mathrm{q}}_i : {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}} \to ({}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}})^{\check{\mathrm{q}}_i}.$$

The same statement holds for the operators $\check{\mathrm{q}}_w$ and $\check{\mathrm{q}}_w$, defined as products of (8.1) and (8.2), for any $w \in W$.

Let $\lambda \in \mathfrak{h}^*$ be a generic weight, that is, $\langle h_\alpha, \lambda\rangle \notin \mathbb{Z}$ for all $\alpha \in \Delta$. Then the following quotients of the double coset space are well defined:

$$\begin{aligned} {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n},\lambda} &= \mathfrak{n}_-\mathcal{A}' \setminus \mathcal{A}'/\mathcal{A}' \cdot \bigl(\mathfrak{n}, \bigl(h - \langle h, \lambda\rangle\bigr)\big|_{h \in \mathfrak{h}}\bigr), \\ {}_{\lambda,\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}} &= \bigl(\bigl(h - \langle h, \lambda\rangle\bigr)\big|_{h \in \mathfrak{h}}, \mathfrak{n}_-\bigr)\mathcal{A}' \setminus \mathcal{A}'/\mathcal{A}'\mathfrak{n}. \end{aligned} \tag{8.4}$$

The space ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n},\lambda}$ is a left $Z_{\mathfrak{n}_-}(\mathcal{A})$-module, the space ${}_{\lambda,\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$ is a right $Z^{\mathfrak{n}}(\mathcal{A})$-module.

Corollary 8.3. *For a generic* $\lambda \in \mathfrak{h}^*$,

(i) *the map* (8.1) *defines a morphism of the right* $Z^{\mathfrak{n}}(\mathcal{A})$*-modules*:

$$\check{\mathrm{q}}_{i,\lambda} : {}_{\lambda,\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}} \to ({}_{s_{\alpha_i}\circ\lambda,\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}})^{\check{\mathrm{q}}_i};$$

(ii) *the map* (8.2) *defines a morphism of the left* $Z_{\mathfrak{n}_-}(\mathcal{A})$*-modules*:

$$\check{\mathrm{q}}_{i,\lambda} : {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n},\lambda} \to ({}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n},s_{\alpha_i}\circ\lambda})^{\check{\mathrm{q}}_i}.$$

Due to (8.3), we have

$$\check{\mathrm{q}}_{i,\lambda}(v) = \hat{\mathrm{p}}_{\alpha_i}[\lambda-\rho]\big(T_i(v)\big), \qquad \check{\mathrm{q}}_{i,\lambda}(v) = \hat{\mathrm{p}}_{-\alpha_i}[\lambda-\rho]\big(T_i(v)\big),$$

that is,

$$\check{\mathrm{q}}_{i,\lambda}(v) = \sum_{n\geqslant 0}\frac{(-1)^n}{n!}\prod_{k=1}^{n}\big(\hat{h}_{\alpha_i} + \langle h_{\alpha_i}, \lambda-\rho\rangle + k\big)^{-1}\hat{e}^n_{-\alpha_i}\hat{e}^n_{\alpha_i}\big(T_i(v)\big),$$
$$\check{\mathrm{q}}_{i,\lambda}(v) = \sum_{n\geqslant 0}\frac{(-1)^n}{n!}\prod_{k=1}^{n}\big(-\hat{h}_{\alpha_i} + \langle h_{\alpha_i}, \lambda-\rho\rangle + k\big)^{-1}\hat{e}^n_{\alpha_i}\hat{e}^n_{-\alpha_i}\big(T_i(v)\big).$$

More generally, for any element $w \in W$, the maps $\check{\mathrm{q}}_w : \mathcal{A}'/\mathcal{A}'\mathfrak{n} \to \mathcal{A}'/\mathcal{A}'\mathfrak{n}$ and $\check{\mathrm{q}}_w : \mathfrak{n}_-\mathcal{A}' \setminus \mathcal{A}' \to \mathfrak{n}_-\mathcal{A}' \setminus \mathcal{A}'$ define morphisms of modules over Mickelsson algebras $\check{\mathrm{q}}_{w,\lambda} : {}_{\lambda,\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}} \to ({}_{w\circ\lambda,\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}})^{\check{\mathrm{q}}_w}$ and $\check{\mathrm{q}}_{w,\lambda} : {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n},\lambda} \to ({}_{\mathfrak{n}_-}\mathcal{A}_{w\circ\lambda,\mathfrak{n}})^{\check{\mathrm{q}}_w}$. For a reduced decomposition $w = s_{\alpha_1}\cdots s_{\alpha_n}$ of an element $w \in W$ and a corresponding sequence of positive roots $\gamma_1, \dots, \gamma_n$, we have

$$\check{\mathrm{q}}_{w,\lambda}(v) = \hat{\mathrm{p}}_{\gamma_1}[\lambda-\rho]\cdots\hat{\mathrm{p}}_{\gamma_n}[\lambda-\rho]\big(T_w(v)\big),$$
$$\check{\mathrm{q}}_{i,\lambda}(v) = \hat{\mathrm{p}}_{-\gamma_1}[\lambda-\rho]\cdots\hat{\mathrm{p}}_{-\gamma_n}[\lambda-\rho]\big(T_w(v)\big).$$

Remark. By the definition, the double coset space ${}_{\mathfrak{n}_-}\tilde{\mathcal{A}}_{\mathfrak{n},\lambda} = \mathfrak{n}_-\mathcal{A} \setminus \mathcal{A}/\mathcal{A}\mathfrak{n}$ is a left $S_{\mathfrak{n}_-}(\mathcal{A})$ and a right $S^{\mathfrak{n}}(\mathcal{A})$ module. Its quotients

$${}_{\mathfrak{n}_-}\tilde{\mathcal{A}}_{\mathfrak{n},\lambda} = \mathfrak{n}_-\mathcal{A} \setminus \mathcal{A}/\mathcal{A}\cdot\big(\mathfrak{n}, \big(h - \langle h,\lambda\rangle\big)\big|_{h\in\mathfrak{h}}\big),$$
$${}_{\lambda,\mathfrak{n}_-}\tilde{\mathcal{A}}_{\mathfrak{n}} = \big(\mathfrak{n}_-, \big(h - \langle h,\lambda\rangle\big)\big|_{h\in\mathfrak{h}}\big)\mathcal{A} \setminus \mathcal{A}/\mathcal{A}\mathfrak{n}$$

coincide with the spaces (8.4) for generic λ: ${}_{\mathfrak{n}_-}\tilde{\mathcal{A}}_{\mathfrak{n},\lambda} = {}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n},\lambda}$ and ${}_{\lambda,\mathfrak{n}_-}\tilde{\mathcal{A}}_{\mathfrak{n}} = {}_{\lambda,\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$. They have the structure of a left $S_{\mathfrak{n}_-}(\mathcal{A})$ and a right $S^{\mathfrak{n}}(\mathcal{A})$ module, correspondingly. The module ${}_{\mathfrak{n}_-}\tilde{\mathcal{A}}_{\mathfrak{n},\lambda}$ can be interpreted as a space of $\mathfrak{n}_-$-coinvariants in the left $\mathcal{A}$-module $M_{\mathfrak{n},\lambda}(\mathcal{A}) = \mathcal{A}/\mathcal{A}\cdot(\mathfrak{n}, h - \langle h,\lambda\rangle \mid h \in \mathfrak{h})$. The module ${}_{\lambda,\mathfrak{n}_-}\tilde{\mathcal{A}}_{\mathfrak{n}}$ can be interpreted as a space of $\mathfrak{n}$-coinvariants in the right $\mathcal{A}$-module $\tilde{M}_\lambda(\mathcal{A}) = (\mathfrak{n}_-, h - \langle h,\lambda\rangle \mid h \in \mathfrak{h})\mathcal{A} \setminus \mathcal{A}$.

For each i, the operators $\check{\mathrm{q}}_i$ and $\check{\mathrm{q}}_i$ define homomorphisms of Mickelsson algebras $S^{\mathfrak{n}}(\mathcal{A})$ and $S_{\mathfrak{n}_-}(\mathcal{A})$ to their localizations with respect to denominators, generated by monomials $(h_{\alpha_i} + k)$, $k \in \mathbb{Z}$. If $\lambda \in \mathfrak{h}^*$ satisfies the condition $\langle h_{\alpha_i}, \lambda\rangle \notin \mathbb{Z}$, these localizations act on $({}_{s_{\alpha_i}\circ\lambda,\mathfrak{n}_-}\tilde{\mathcal{A}}_{\mathfrak{n}})^{\check{\mathrm{q}}_i}$ and

$({}_{\mathfrak{n}_-}\tilde{\mathcal{A}}_{\mathfrak{n},s_{\alpha_i}\circ\lambda})^{\check{\mathrm{q}}_i}$ correspondingly. In this sense the operators $\check{\mathrm{q}}_i$ and $\check{\mathrm{q}}_i$ define morphisms of the right $S^{\mathfrak{n}}(\mathcal{A})$ modules and of the left $S_{\mathfrak{n}_-}(\mathcal{A})$ modules:

$$\check{\mathrm{q}}_{i,\lambda} : {}_{\lambda,\mathfrak{n}_-}\tilde{\mathcal{A}}_{\mathfrak{n}} \to ({}_{s_{\alpha_i}\circ\lambda,\mathfrak{n}_-}\tilde{\mathcal{A}}_{\mathfrak{n}})^{\check{\mathrm{q}}_i} \quad \text{and} \quad \check{\mathrm{q}}_{i,\lambda} : {}_{\mathfrak{n}_-}\tilde{\mathcal{A}}_{\mathfrak{n},\lambda} \to ({}_{\mathfrak{n}_-}\tilde{\mathcal{A}}_{\mathfrak{n},s_{\alpha_i}\circ\lambda})^{\check{\mathrm{q}}_i}. \tag{8.5}$$

One can regard (8.5) as a family of operators with the meromorphic dependence on a parameter λ, study their singularities, residues, etc.

8.2. Quotients of free modules

As any associative algebra with unit, the Mickelsson algebra is a free left and a free right module over itself of rank one. Let us restrict ourselves to the Mickelsson algebra $Z^{\mathfrak{n}}(A)$ and the corresponding free right module of rank one.

Let $\lambda \in \mathfrak{h}^*$ be a generic weight. Consider the following quotient of the free right $Z^{\mathfrak{n}}(A)$-module

$$\Phi_\lambda = \big(h - \langle h, \lambda\rangle\big)\big|_{h\in\mathfrak{h}} Z^{\mathfrak{n}}(\mathcal{A}) \setminus Z^{\mathfrak{n}}(\mathcal{A}). \tag{8.6}$$

It can be realized as follows. The multiplication m in $\mathcal{A}$ induces an isomorphism of the two left $U(\mathfrak{g})$-modules:

$$\mathcal{V} \otimes M_{\mathfrak{n}}(\mathfrak{g}) \cong M_{\mathfrak{n}}(\mathcal{A}'), \tag{8.7}$$

where $M_{\mathfrak{n}}(\mathfrak{g})$ is the 'universal Verma module' $U'(\mathfrak{g})/U'(\mathfrak{g})\mathfrak{n}$, $M_{\mathfrak{n}}(\mathcal{A}') = \mathcal{A}'/A'\mathfrak{n}$, and the module structure of $\mathcal{V}$ is the restriction of the adjoint representation of $\mathfrak{g}$ in $\mathcal{A}$. The map (8.7) is also an isomorphism of the right D-modules, where the structure of D-modules in the left-hand side of (8.7) is given by the rule $(v \otimes m) \cdot d = v \otimes (m \cdot d)$ for any $v \in \mathcal{V}$, $m \in M_{\mathfrak{n}}(\mathfrak{g})$, $d \in D$. The Mickelsson algebra is the space of highest weight vectors in $M_{\mathfrak{n}}(\mathcal{A}')$, so with the identification (8.7) we have the following isomorphism of D-bimodules:

$$Z^{\mathfrak{n}}(\mathcal{A}) \cong \big(\mathcal{V} \otimes M_{\mathfrak{n}}(\mathfrak{g})\big)^{\mathfrak{n}}. \tag{8.8}$$

Recall that $Z^{\mathfrak{n}}(\mathcal{A})$ is a $U(\mathfrak{h})$-bimodule and admits the weight decomposition with respect to the adjoint action of $\mathfrak{h}$. This implies that the right $Z^{\mathfrak{n}}(\mathcal{A})$-module $\Phi_\lambda(\mathcal{A})$ is a semisimple right $U(\mathfrak{h})$-module and admits a decomposition

$$\Phi_\lambda = \bigoplus_{\nu} \Phi_{\lambda,\lambda-\nu},$$

where the sum is taken over the weights ν of $\mathcal{A}$ such that for any $\varphi \in \Phi_{\lambda,\mu}$ we have

$$\varphi \cdot h = \langle h, \mu\rangle\varphi, \tag{8.9}$$

and $\Phi_{\lambda,\mu}$ coincides with the double coset

$$\Phi_{\lambda,\mu} = \big(h - \langle h, \lambda\rangle\big)\big|_{h\in\mathfrak{h}} \cdot Z^{\mathfrak{n}}(\mathcal{A}) \setminus Z^{\mathfrak{n}}(\mathcal{A})/Z^{\mathfrak{n}}(\mathcal{A}) \cdot \big(h - \langle h, \mu\rangle\big)\big|_{h\in\mathfrak{h}}.$$

Analogously, the left $Z^{\mathfrak{n}}(A)$-module $\tilde{\Phi}_\mu$

$$\tilde{\Phi}_\mu = Z^{\mathfrak{n}}(\mathcal{A})/Z^{\mathfrak{n}}(\mathcal{A})\cdot\big(h-\langle h,\mu\rangle\big)\big|_{h\in\mathfrak{h}}$$

admits a presentation

$$\tilde{\Phi}_\mu = \bigoplus_\nu \Phi_{\mu+\nu,\mu}.$$

With the identification (8.8), the relation (8.9) can be interpreted as follows: any element $\phi\in\Phi_{\lambda,\mu}$ is a highest weight vector in the tensor product $\mathcal{V}\otimes M_\mu$ of the weight λ. Here $M_\mu = U(\mathfrak{g})/U(\mathfrak{g})\cdot(\mathfrak{n},(h-\langle h,\mu\rangle)h\in\mathfrak{h})$ is the Verma module of g with the highest weight μ. We have proved

Lemma 8.4. *Let $\lambda\in\mathfrak{h}^*$ be generic and $\nu\in\mathfrak{h}^*$ a weight of $Z^{\mathfrak{n}}(\mathcal{A})$. Then the weight space $\Phi_{\lambda,\lambda-\nu}$ of the right $Z^{\mathfrak{n}}(\mathcal{A})$-module Φ_λ is isomorphic to the space of intertwining operators*

$$\Phi_{\lambda,\lambda-\nu}\cong \mathrm{Hom}_{U(\mathfrak{g})}(M_\lambda,\mathcal{V}\otimes M_{\lambda-\nu}). \tag{8.10}$$

Denote by $\mathbb{I}_\lambda$ the class of unit $1\in Z^{\mathfrak{n}}(\mathcal{A})$ in Φ_λ. The vector $\mathbb{I}_\lambda$ generates Φ_λ as $Z^{\mathfrak{n}}(\mathcal{A})$-module. For any $v\in\mathcal{V}$ denote by Φ_λ^v the element of the right $Z^{\mathfrak{n}}(\mathcal{A})$-module Φ_λ, obtained by applying of the element z'_v to $\mathbb{I}_\lambda$. It is equal to the class of z'_v in Φ_λ. Let ν be the weight of v. In the description (8.10) of $\Phi_{\lambda,\lambda-\nu}$, the element Φ_λ^v corresponds to a map from $\mathrm{Hom}_{U(\mathfrak{g})}(M_\lambda,\mathcal{V}\otimes M_{\lambda-\nu})$, such that

$$\Phi_\lambda^v(1_\lambda) = v\otimes 1_{\lambda-\nu} + \text{l.o.t.}$$

where 1_λ is the highest weight vector of Verma module M_λ and l.o.t. means the terms which have lower weight on the second tensor component.

The properties of the operators $\check{\mathrm{q}}_w$ imply that $\check{\mathrm{q}}_w(\Phi_{\lambda,\mu}) = \Phi_{w\circ\lambda,w\circ\mu}$, so that each operator $\check{\mathrm{q}}_w$ defines the morphisms of the right and left $Z^{\mathfrak{n}}(\mathcal{A})$-modules:

$$\check{\mathrm{q}}_{w,\lambda}:\Phi_\lambda\to(\Phi_{w\circ\lambda})^{\check{\mathrm{q}}_w},\qquad \check{\mathrm{q}}_{w,\cdot\,\mu}:\tilde{\Phi}_\mu\to(\tilde{\Phi}_{w\circ\mu})^{\check{\mathrm{q}}_w}(\mathcal{A}). \tag{8.11}$$

Proposition 6.6 gives a formula for transformations of vectors Φ_λ^v:

$$\check{\mathrm{q}}_{w,\lambda}\big(\Phi_\lambda^v\big) = \Phi_{w\circ\lambda}^{\hat{\mathrm{p}}_{-\gamma_1}[\lambda+\rho]\cdots\hat{\mathrm{p}}_{-\gamma_n}[\lambda+\rho](T_w(v))}, \tag{8.12}$$

where $\gamma_1,\dots,\gamma_n$ is the sequence of positive roots, attached to a reduced decomposition $w = s_{\alpha_{i_1}}\cdots s_{\alpha_{i_n}}$ by the standard rule $\gamma_1=\alpha_{i_1}$, $\gamma_2 = s_{\alpha_{i_1}}(\alpha_{i_2})$, ...

8.3. Connections to dynamical Weyl group

Let $\mathcal{V}$ be a $U(\mathfrak{g})$-module algebra with a locally nilpotent action of real root vectors and $\mathcal{A}_{\mathcal{V}} = U(\mathfrak{g})\ltimes\mathcal{V}$ be a smash product of $U(\mathfrak{g})$ and $\mathcal{V}$, see Example 2 in Section 3.1.

The typical examples are: the tensor algebra of an integrable highest weight representation of $\mathfrak{g}$ and the symmetric algebra of an integrable highest weight representation of $\mathfrak{g}$.

Due to assumptions above, the $\mathfrak{g}$-module structure in $\mathcal{V}$ lifts to an action of the Weyl group of $\mathfrak{g}$ in $\mathcal{V}$ by standard formulas

$$\hat{T}_i = \exp \hat{e}_{\alpha_i} \cdot \exp -\hat{e}_{-\alpha_i} \cdot \exp \hat{e}_{\alpha_i}. \tag{8.13}$$

Due to (8.13), the operators $\hat{T}_i$ are automorphisms of the algebra $\mathcal{V}$.

The operators $\hat{T}_i$ admit lifts to automorphisms of the algebra $\mathcal{A}$ by the relation $T_i(gv) = T_i(g)\hat{T}_i(v)$, where $g \in U(\mathfrak{g})$, $v \in \mathcal{V}$, and $T_i(g)$ is the automorphism of $U(\mathfrak{g})$, as in Section 2.1. Actually, they are given by the same relation (8.13) with respect to the adjoint action of $\mathfrak{g}$ on $\mathcal{A}$.

The elements of the right $Z^{\mathfrak{n}}(\mathcal{A})$-modules $\Phi_\lambda(\mathcal{A})$ are known in this case as the *intertwining operators*. The morphisms $\check{\mathrm{q}}_{w,\lambda}$ form the *dynamical Weyl group* action, see [EV,TV].

The space of intertwining operators $\Phi_\lambda(\mathcal{A})$ form an algebroid with respect to the composition operation. The composition of intertwining operators can be described as follows.

For a generic $\lambda \in \mathfrak{h}^*$, any morphism $\varphi_\lambda : M_\lambda \to \mathcal{V} \otimes M_{\lambda-\nu}$ of $\mathfrak{g}$-modules admits a lift to a morphism $\overline{\varphi}_\lambda : \mathcal{V} \otimes M_\lambda \to \mathcal{V} \otimes M_{\lambda-\nu}$ of $\mathcal{A}'$-modules by the rule $\overline{\varphi}_\lambda(v \otimes m) = v \cdot \varphi_\lambda(m)$ for any $m \in M_\lambda$. Then the composition $\varphi'_{\lambda-\nu} \circ \varphi_\lambda$ of the intertwining operators $\varphi_\lambda \in \Phi_{\lambda,\lambda-\nu}$ and $\varphi'_{\lambda-\nu} \in \Phi_{\lambda-\nu,\lambda-\nu-\nu'}$ is an element $\varphi''_\lambda \in \Phi_{\lambda,\lambda-\nu-\nu'}$, such that

$$\overline{\varphi}''_\lambda = \overline{\varphi}'_{\lambda-\nu} \circ \overline{\varphi}_\lambda.$$

The composition of intertwining operators coincides with the structure multiplication map of the right $Z^{\mathfrak{n}}(\mathcal{A})$-module in Φ_λ. Namely, in the notation of the previous section, for any $x \in Z^{\mathfrak{n}}(\mathcal{A})$ denote by Φ_λ^x its class in Φ_λ, considered as intertwining operator. Then we have

Proposition 8.5. *(See [K].) Let $z', z'' \in Z^{\mathfrak{n}}(\mathcal{A})$ and the weight of z' with respect to the adjoint action of $\mathfrak{h}$ is ν. Then*

$$\Phi_\lambda^{z'z''} = \Phi_{\lambda-\nu}^{z''} \circ \Phi_\lambda^{z'}.$$

In this context the statement that the maps $\check{\mathrm{q}}_{w,\lambda}$ are morphisms of $Z^{\mathfrak{n}}(\mathcal{A})$-modules, see (8.11), is equivalent to the compatibility of the dynamical Weyl group action with the composition of intertwining operators.

9. Quantum group settings

9.1. Notation and assumptions

In this section we announce basic results of this paper for Mickelsson algebras, related to reductions relative to quantized universal enveloping algebras. We restrict our attention to the Mickelsson algebras $Z^{\mathfrak{n}}(\mathcal{A})$.

Keep the notation of Section 2.1. Let q be an indeterminate; $d_i \in \mathbb{N}$ are defined by the condition that the matrix $(\alpha_i, \alpha_j) = d_i a_{i,j} = d_j a_{j,i}$ is symmetric. Here $a_{i,j}$ is the Cartan matrix of $\mathfrak{g}$. For any root $\gamma \in \Delta$ put $q_\gamma = q^{(\gamma,\gamma)/2}$, $[a]_p = \frac{p^a - p^{-a}}{p - p^{-1}}$, and $(a)_p = \frac{p^a - 1}{p - 1}$ for any symbols a and p. We also use the notation $q_i = q_{\alpha_i} = q^{d_i}$ for simple roots α_i.

Denote by $U_q(\mathfrak{g})$ the Hopf algebra, generated by the Chevalley generators $e_{\alpha_i} \in U_q(\mathfrak{n})$, $e_{-\alpha_i} \in U_q(\mathfrak{n}_-)$, $k_{\alpha_i}^{\pm 1} = q_i^{\pm h_{\alpha_i}} \in U_q(\mathfrak{h})$, where $\alpha_i \in \Pi$, so that

$$[e_{\alpha_i}, e_{-\alpha_j}] = \delta_{i,j}[h_{\alpha_i}]_{q_i}, \quad \text{where } [h_{\alpha_i}]_{q_i} := \frac{k_{\alpha_i} - k_{\alpha_i}^{-1}}{q_i - q_i^{-1}},$$

$$k_{\alpha_i} e_{\pm\alpha_j} k_{\alpha_i}^{-1} = q_i^{\pm a_{i,j}} e_{\pm\alpha_j} = q^{\pm(\alpha_i,\alpha_j)} e_{\pm\alpha_j},$$

$$\sum_{r+s=1-a_{i,j}} (-1)^r e_{\pm\alpha_i}^{(r)} e_{\pm\alpha_j} e_{\pm\alpha_i}^{(s)} = 0, \quad i \neq j, \text{ where } e_{\pm\alpha_i}^{(k)} := \frac{e_{\pm\alpha_i}^k}{[k]_{q_i}!},$$

$$\Delta(e_{\alpha_i}) = e_{\alpha_i} \otimes 1 + k_{\alpha_i} \otimes e_{\alpha_i}, \qquad \Delta(e_{-\alpha_i}) = 1 \otimes e_{-\alpha_i} + e_{-\alpha_i} \otimes k_{\alpha_i}^{-1},$$

$$\Delta(k_{\alpha_i}) = k_{\alpha_i} \otimes k_{\alpha_i}, \qquad S(k_{\alpha_i}) = k_{\alpha_i}^{-1},$$

$$S(e_{\alpha_i}) = -k_{\alpha_i}^{-1} e_{\alpha_i}, \qquad S(e_{-\alpha_i}) = -e_{-\alpha_i} k_{\alpha_i}.$$

The adjoint action (3.1) of the Chevalley generators has the form

$$\begin{aligned} \hat{e}_{\alpha_i}(x) &\equiv \mathrm{ad}_{e_{\alpha_i}}(x) = e_{\alpha_i} x - k_{\alpha_i} x k_{\alpha_i}^{-1} e_{\alpha_i}, \\ \hat{e}_{-\alpha_i}(x) &\equiv \mathrm{ad}_{e_{-\alpha_i}}(x) = [e_{-\alpha_i}, x] \cdot k_{\alpha_i}. \end{aligned} \tag{9.1}$$

Let $T_i \equiv T_{s_i} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ be automorphisms of $U_q(\mathfrak{g})$, defined by the relations

$$\begin{aligned} T_i(e_{\alpha_i}) &= -k_{\alpha_i} e_{-\alpha_i}, \qquad T_i(e_{-\alpha_i}) = -e_{\alpha_i} k_{\alpha_i}^{-1}, \qquad T_i(k_{\alpha_j}) = k_{s_{\alpha_i}(\alpha_j)}, \\ T_i(e_{\alpha_j}) &= \sum_{r+s=-a_{i,j}} (-1)^r q_i^r e_{\alpha_i}^{(r)} e_{\alpha_j} e_{\alpha_i}^{(s)}, \quad i \neq j, \\ T_i(e_{-\alpha_j}) &= \sum_{r+s=-a_{i,j}} (-1)^r q_i^{-r} e_{-\alpha_i}^{(s)} e_{-\alpha_j} e_{-\alpha_i}^{(r)}, \quad i \neq j. \end{aligned} \tag{9.2}$$

Here we use the Cartan generators k_γ, $\gamma \in \Delta$. They are defined by the rules $k_{-\alpha} = k_\alpha^{-1}$, and $k_{\alpha+\beta} = k_\alpha k_\beta$.

In Lusztig's notation [L], $T_i \equiv T'_{i,+}$. For any $w \in W$ they define automorphisms $T_w : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$, as in Section 6.1.

Denote by D_q the localization of the commutative algebra $U_q(\mathfrak{h})$ relative to the multiplicative set of denominators, generated by

$$\big\{[h_\alpha + k]_{q_\alpha} \bigm| \alpha \in \Delta,\ k \in \mathbb{Z}\big\}.$$

Denote by $U'_q(\mathfrak{g})$ the extension of $U_q(\mathfrak{g})$ by means of D_q:

$$U'_q(\mathfrak{g}) = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{h})} D_q \approx D_q \otimes_{U_q(\mathfrak{h})} U_q(\mathfrak{g}).$$

As well as in the undeformed case ($q = 1$), there exists an extension $F^q_{\mathfrak{g},\mathfrak{n}}$ of the algebra $U'_q(\mathfrak{g})$ and an element $\mathrm{P} = \mathrm{P}_\mathfrak{n} \in F^q_{\mathfrak{g},\mathfrak{n}}$ (the extremal projector), satisfying the conditions $e_{\alpha_i}\mathrm{P} = \mathrm{P}e_{-\alpha_i} = 0$, $\mathrm{P}^2 = \mathrm{P}$, see [KT].

In particular, for the algebra $U_q(\mathfrak{sl}_2)$, generated by $e_{\pm\alpha}$ and $k_\alpha^{\pm 1}$, we have two projection operators, $\mathrm{P} = \mathrm{P}_\alpha[\rho]$, and $\mathrm{P}_- = \mathrm{P}_{-\alpha}[\rho]$, where

$$\mathrm{P}_{\alpha}[\lambda]=\sum_{n=0}^{\infty}\frac{(-1)^n}{[n]_{q_\alpha}!}q_\alpha^{-\langle h_\alpha,\lambda-\rho\rangle}f_{\alpha,n}[\lambda]e_{-\alpha}^n e_\alpha^n,$$
$$\mathrm{P}_{-\alpha}[\lambda]=\sum_{n=0}^{\infty}\frac{(-1)^n}{[n]_{q_\alpha}!}q_\alpha^{\langle h_\alpha,\lambda-\rho\rangle}g_{\alpha,n}[\lambda]e_{\alpha}^n e_{-\alpha}^n \tag{9.3}$$

and

$$f_{\alpha,n}[\lambda]=\prod_{j=1}^{n}\bigl[h_\alpha+\langle h_\alpha,\lambda\rangle+j\bigr]_{q_\alpha}^{-1},\qquad g_{\alpha,n}[\lambda]=\prod_{j=1}^{n}\bigl[-h_\alpha+\langle h_\alpha,\lambda\rangle+j\bigr]_{q_\alpha}^{-1}.$$

Let $\mathcal{A}$ be an associative algebra, containing $U_q(\mathfrak{g})$. We call $\mathcal{A}$ a $U_q(\mathfrak{g})$-admissible algebra if:

(a) there is a subspace $\mathcal{V}\subset\mathcal{A}$, invariant with respect to the adjoint action of $U_q(\mathfrak{g})$, such that the multiplication m in $\mathcal{A}$ induces isomorphisms of vector spaces

$$\text{(a1)}\quad m:U_q(\mathfrak{g})\otimes\mathcal{V}\to\mathcal{A},\qquad \text{(a2)}\quad m:\mathcal{V}\otimes U_q(\mathfrak{g})\to\mathcal{A};$$

(b) the adjoint action on $\mathcal{V}$ of all real root vectors $e_\gamma\in U_q(\mathfrak{g})$, related to any fixed normal ordering of the root system, is locally nilpotent. The adjoint action of the Cartan subalgebra $U_q(\mathfrak{h})$ on $\mathcal{V}$ is semisimple.

In particular, $\mathcal{A}$ is isomorphic to $U_q(\mathfrak{g})\otimes\mathcal{V}$ and to $\mathcal{V}\otimes U_q(\mathfrak{g})$ as a $U_q(\mathfrak{g})$-module with respect to the adjoint action.

Denote by $\mathfrak{n}$ the linear subspace of $U_q(\mathfrak{g})$ generated by the elements e_{α_i}, $\alpha_i\in\Pi$. Denote by $\mathfrak{n}_-$ the linear subspace of $U_q(\mathfrak{g})$ generated by the elements $e_{-\alpha_i}$, $i\in\Pi$. Let $U_q(\mathfrak{n})$ be the subalgebra of $U_q(\mathfrak{g})$, generated by $\mathfrak{n}$ and $U_q(\mathfrak{n}_-)$ the subalgebra of $U_q(\mathfrak{g})$, generated by $\mathfrak{n}_-$.

For any $U_q(\mathfrak{g})$-admissible algebra $\mathcal{A}$ put $\mathcal{A}'=\mathcal{A}\otimes_{U_q(\mathfrak{h})}D_q$ and define the Mickelsson algebras $S^{\mathfrak{n}}(\mathcal{A})$ and $Z^{\mathfrak{n}}(\mathcal{A})$, as in Section 3.2:

$$S^{\mathfrak{n}}(\mathcal{A})=\mathrm{Nr}(\mathcal{A}\mathfrak{n})/\mathcal{A}\mathfrak{n},\qquad Z^{\mathfrak{n}}(\mathcal{A})=\mathrm{Nr}(\mathcal{A}'\mathfrak{n})/\mathcal{A}'\mathfrak{n},$$

and the double coset algebra ${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}$, see Section 3.3,

$${}_{\mathfrak{n}_-}\mathcal{A}_{\mathfrak{n}}=\mathfrak{n}_-\mathcal{A}'\setminus\mathcal{A}'/\mathcal{A}'\mathfrak{n}\equiv\mathcal{A}'/(\mathfrak{n}_-\mathcal{A}'+\mathcal{A}'\mathfrak{n}),$$

equipped with the multiplication structure (3.7).

With the conditions of Section 3.4, the Mickelsson algebra $Z^{\mathfrak{n}}(\mathcal{A})$ has distinguished generators z_v, z'_v, $v\in\mathcal{V}$, determined by the relations (3.9) and (3.10).

9.2. Basic constructions

Let $\alpha\in\Pi$ be a simple root, $\mathfrak{n}_\alpha=\mathbb{C}e_\alpha$, $\mathfrak{n}_{-\alpha}=\mathbb{C}e_{-\alpha}$. Let $x\in\mathcal{A}$ be an element of $\mathcal{A}$, finite with respect to the adjoint action of e_α. Denote by $\mathrm{q}_\alpha(x)$ the following element of $\mathcal{A}'/\mathcal{A}'\mathfrak{n}_\alpha$:

$$\mathrm{q}_\alpha(x)=\sum_{n\geqslant 0}\frac{(-1)^n}{[n]_{q_\alpha}!}\bigl(\hat{k}_\alpha^{-1}\hat{e}_\alpha\bigr)^n(x)e_{-\alpha}^n g_{n,\alpha}\mod\mathcal{A}'\mathfrak{n}_\alpha, \tag{9.4}$$

where $g_{n,\alpha} = ([h_\alpha]_{q_\alpha}[h_\alpha - 1]_{q_\alpha} \cdots [h_\alpha - n + 1]_{q_\alpha})^{-1}$. The assignment (9.4) has the properties

$$\begin{array}{rl}
\text{(i)} & \mathrm{q}_\alpha(xe_{-\alpha}) = 0, \\
\text{(ii)} & \mathrm{q}_\alpha\big(xk_\alpha^{-1}\big) = \mathrm{q}_\alpha(x)k_\alpha^{-1}q_\alpha^{-2}, \qquad \mathrm{q}_\alpha\big(k_\alpha^{-1}x\big) = q_\alpha^{-2}k_\alpha^{-1}\mathrm{q}_\alpha(x), \\
\text{(iii)} & \mathrm{q}_\alpha(xk_\gamma) = \mathrm{q}_\alpha(x)k_\gamma, \qquad \mathrm{q}_\alpha(k_\gamma x) = k_\gamma \mathrm{q}_\alpha(x), \quad \text{if } \langle h_\alpha, \gamma\rangle = 0, \\
\text{(iv)} & k_\alpha^{-1}e_\alpha \mathrm{q}_\alpha(x) = \mathrm{q}_\alpha\big(k_\alpha^{-1}e_\alpha x\big) = 0.
\end{array}$$

We extend the assignment (9.4) to the map $\mathrm{q}_\alpha : \mathcal{A}' \to \mathcal{A}'/\mathcal{A}'\mathfrak{n}_\alpha$ with a help of the properties (ii) and (iii). It satisfies the properties

$$\mathrm{q}_\alpha(xd) = \mathrm{q}_\alpha(x)\tau_\alpha(d), \qquad \mathrm{q}_\alpha(dx) = \tau_\alpha(d)\mathrm{q}_\alpha(x), \quad d \in D_q, \tag{9.5}$$

where $\tau_\mu : D_q \to D_q$, $\mu \in \mathfrak{h}^*$ is uniquely characterized by the conditions

$$\tau_\alpha(k_\gamma) = q^{(\mu,\gamma)}k_\gamma.$$

Due to the property (iv) the map q_α defines a map $\mathrm{q}_\alpha : {}_{\mathfrak{n}_\alpha}\mathcal{A}_{\mathfrak{n}_{-\alpha}} \to Z^{\mathfrak{n}_\alpha}(\mathcal{A})$. We have an analog of Theorem 5.1:

Proposition 9.1. *The map*

$$\mathrm{q}_\alpha : {}_{\mathfrak{n}_\alpha}\mathcal{A}_{\mathfrak{n}_{-\alpha}} \to Z^{\mathfrak{n}_\alpha}(\mathcal{A})$$

is an isomorphism of algebras.

In the following we assume that the automorphisms (9.2) admit extensions $T_i : \mathcal{A}' \to \mathcal{A}'$, which satisfy the braid group relations (6.1), though, as well as in the case $q = 1$, part of the results below do not depend on such an extension.

Let $w \in W$ be an element of the Weyl group of $\mathfrak{g}$, $\alpha \in \Pi$ a simple root, such that $l(ws_\alpha) = l(w) + 1$. Set $\gamma = w(\alpha)$, $e_{\pm\gamma} = T_w(e_{\pm\alpha})$ and $T_\gamma = T_w T_\alpha T_w^{-1}$. Denote by $\mathfrak{m} = \mathfrak{n}^w$ the linear span of the vectors $T_w(e_{\alpha_i})$, $\alpha_i \in \Pi$ and by $\mathfrak{m}^{s_\gamma}$ the space $T_\gamma(\mathfrak{m})$. Let $\mathrm{q}_{\gamma,\mathfrak{m}}$ be the linear map $\mathrm{q}_{\gamma,\mathfrak{m}} : \mathcal{A}' \to A'/\mathcal{A}'\mathfrak{m}$, defined by the rule:

$$\begin{aligned}
\mathrm{q}_{\gamma,\mathfrak{m}}(x) &= \sum_{n\geqslant 0} \frac{(-1)^n}{[n]_{q_\gamma}!}\big(\hat{k}_\gamma^{-1}\hat{e}_\gamma\big)^n(x)e_{-\gamma}^n g_{n,\gamma} \\
&= \sum_{n\geqslant 0} \frac{(-1)^n}{[n]_{q_\gamma}!} q_\gamma^{-n(\hat{h}_\gamma - n+1)}\hat{e}_\gamma^n(x)e_{-\gamma}^n g_{n,\gamma} \quad \mod \mathcal{A}'\mathfrak{m},
\end{aligned}$$

for all $x \in \mathcal{A}$, which are adjoint finite with respect to e_γ. In general we present any $y \in \mathcal{A}'$ as $y = dx$ with $x \in \mathcal{A}$ being adjoint finite with respect to e_γ and $d \in D_q$ and then use the properties, analogous to (i) and (ii) for the map q_α. Here $g_{n,\gamma} = ([h_\gamma]_{q_\gamma}[h_\gamma - 1]_{q_\gamma} \cdots [h_\gamma - n + 1]_{q_\gamma})^{-1}$.

Proposition 9.2.

(i) *We have* $\mathrm{q}_{\gamma,\mathfrak{m}}(\mathcal{A}'\mathfrak{m}^{s_\gamma})=0$, *so that* $\mathrm{q}_{\gamma,\mathfrak{m}}$ *defines a map* $\mathrm{q}_{\gamma,\mathfrak{m}}:\mathcal{A}'/\mathcal{A}'\mathfrak{m}^{s_\gamma}\to\mathcal{A}'/\mathcal{A}'\mathfrak{m}$;
(ii) *For* $\alpha=w^{-1}(\gamma)$ *we have the equality*

$$\mathrm{q}_{\gamma,\mathfrak{m}}=T_w\mathrm{q}_{\alpha,\mathfrak{n}}T_w^{-1}. \tag{9.6}$$

Remark. Note that the statement (ii) is nontrivial for $q\neq 1$, since

$$T_w\,\mathrm{ad}_x\,T_w^{-1}(y)\neq\mathrm{ad}_{T_w(x)}(y).$$

The proof of (ii) uses coalgebraic properties of Lusztig automorphisms.

Let $\overline{w}=\{w,w=s_{\alpha_{i_1}}s_{\alpha_{i_2}}\cdots s_{\alpha_{i_n}}\}$ be a pair, consisting of an element w of the Weyl group and of a reduced decomposition of w. Let $\gamma_1,\dots,\gamma_n$ be a related sequence of positive roots: $\gamma_1=\alpha_{i_1},\dots,\gamma_k=s_{\alpha_{i_1}}\cdots s_{\alpha_{i_{k-1}}}(\alpha_{i_k}),\dots$. Proposition 9.2(i) implies that there is a well-defined map

$$\mathrm{q}_{\overline{w}}:\mathcal{A}'/\mathcal{A}'\mathfrak{n}^w\to\mathcal{A}'/\mathcal{A}'\mathfrak{n}:\quad \mathrm{q}_{\overline{w}}=\mathrm{q}_{\gamma_1,\mathfrak{n}}\mathrm{q}_{\gamma_2,\mathfrak{n}^{s_{\gamma_1}}}\cdots\mathrm{q}_{\gamma_n,\mathfrak{n}^{s_{\gamma_{n-1}}\cdots s_{\gamma_1}}}.$$

Proposition 9.3. *Let* $\mathfrak{g}$ *be of finite dimension. Then for any reduced decomposition* $\overline{w}_0$ *of the longest element* w_0 *of* W *the map* $\mathrm{q}_{\overline{w}_0}$ *sends a vector* $v\in\mathcal{V}$ *to the generator* z'_v *of the Mickelsson algebra* $Z^{\mathfrak{n}}(\mathcal{A})$.

Proposition 9.3 implies that the maps $q_{\gamma,m}$ satisfy the cocycle conditions, that is, the map $\mathrm{q}_{\overline{w}}$ does not depend on a reduced decomposition $\overline{w}$ of $w\in W$; hence it can be denoted as q_w.

Set $\check{\mathrm{q}}_i=\mathrm{q}_{s_{\alpha_i}}\cdot T_i:\mathcal{A}'/\mathcal{A}'\mathfrak{n}\to\mathcal{A}'/\mathcal{A}'\mathfrak{n}$. Then we have the braid group relations:

$$\underbrace{\check{\mathrm{q}}_i\check{\mathrm{q}}_j\cdots}_{m_{i,j}}=\underbrace{\check{\mathrm{q}}_j\check{\mathrm{q}}_i\cdots}_{m_{i,j}},\quad i\neq j,$$

and, due to Proposition 9.1, the restriction of $\check{\mathrm{q}}_i$ to $Z^{\mathfrak{n}}(\mathcal{A})$ is an automorphism of the Mickelsson algebra, such that

$$\check{\mathrm{q}}_i(dx)=(s_{\alpha_i}\circ d)\cdot\check{\mathrm{q}}_i(x),\qquad \check{\mathrm{q}}_i(xd)=\check{\mathrm{q}}_i(x)\cdot(s_{\alpha_i}\circ d)\quad\text{for } d\in D_q,\ x\in\mathcal{A}'.$$

Here $w\circ d$ is the natural extension of the shifted action of $w\in W$ on $\mathfrak{h}^*$, see (6.8), to the automorphism of D_q, defined by the conditions

$$w\circ k_\gamma=k_{w(\gamma)}\cdot q_\gamma^{\langle\gamma,w(\rho-\rho)\rangle}.$$

9.3. Some calculations

There is a standard construction of the extension of the Hopf algebra $U_q(\mathfrak{g})$ by means of automorphisms T_i. Namely, let $U_q^W(\mathfrak{g})$ be the smash product of $U_q(\mathfrak{g})$ with the algebra, generated by the elements $\mathrm{T}_i^{\pm 1}$, satisfying the braid group relations

$$\underbrace{\mathrm{T}_i\mathrm{T}_j\cdots}_{m_{i,j}}=\underbrace{\mathrm{T}_j\mathrm{T}_i\cdots}_{m_{i,j}},\quad i\neq j.$$

The cross-product relations are

$$\mathrm{T}_i g \mathrm{T}_i^{-1} = T_i(g), \quad g \in U_q(\mathfrak{g}). \tag{9.7}$$

Due to the coalgebraic properties of Lusztig automorphisms [L], the smash product $U_q^W(\mathfrak{g})$ can be equipped with a structure of a Hopf algebra, if we put

$$\Delta(\mathrm{T}_i) = \mathrm{T}_i \otimes \mathrm{T}_i \cdot \tilde{R}_i,$$

where

$$\tilde{R}_i = \exp_{q_i^{-2}}\bigl((q_i - q_i^{-1})e_{-\alpha_i} \otimes e_{\alpha_i}\bigr) = \sum_{n \geqslant 0} \frac{(q_i - q_i^{-1})^n}{(n)_{q_i^{-2}}!} e_{-\alpha_i}^n \otimes e_{\alpha_i}^n.$$

In the same way we extend the algebra $\mathcal{A}'$ to the cross-product $\mathcal{A}^W$, using the relations (9.7). Since U_q^W is a Hopf algebra, the adjoint action $\hat{\mathrm{T}}_i$ of T_i on $\mathcal{A}^W$ is well defined. It preserves the subalgebra $\mathcal{A}' \subset \mathcal{A}^W$: $\hat{\mathrm{T}}_i(\mathcal{A}') \subset \mathcal{A}'$. The following statement is nontrivial for $q \neq 1$ and is important for calculations of the maps $\check{\mathrm{q}}_i$.

Proposition 9.4. *For any $x \in \mathcal{A}'$ we have*

$$\check{\mathrm{q}}_i(x) = \mathrm{q}_i\bigl(\mathrm{T}_i x \mathrm{T}_i^{-1}\bigr) = \mathrm{q}_i\bigl(\hat{\mathrm{T}}_i(x)\bigr).$$

Now we describe the squares of the automorphisms $\check{\mathrm{q}}_i : Z^{\mathfrak{n}}(\mathcal{A}) \to Z^{\mathfrak{n}}(\mathcal{A})$. Assume that the elements $v_{m,j} \in \mathcal{V}$, $m \in \mathbb{Z}_{\geqslant 0}$, $j = 0, 1, \ldots, m$ form a finite-dimensional representation of the algebra $U_q(\mathfrak{sl}_2)$, generated by $e_{\pm\alpha_i}$ and $k_{\alpha_i}^{\pm 1}$ with respect to the adjoint action, such that:

$$\hat{e}_{\alpha_i}^{j+1}(v_{m,j}) = \hat{e}_{-\alpha_i}^{m-j+1}(v_{m,j}) = 0, \qquad \hat{h}_{\alpha_i}(v_{m,j}) = (m-2j)v_{m,j}. \tag{9.8}$$

In particular, $v_m = v_{m,0}$ is the highest weight vector of this representation, and $v_{m,j} = \hat{e}_{-\alpha_i}^{(j)}(v_m)$. Define the operator ε_i evaluating the parity of the dimension of a representation of this $U_q(\mathfrak{sl}_2)$, $\varepsilon_i(v_{m,j}) = (-1)^m v_{m,j}$.

Proposition 9.5. *Assume that $v_{m,j} \in \mathcal{V}$ satisfy* (9.8). *Then*

$$\check{\mathrm{q}}_i^2(v_{m,j}) = q^{-j(m-j+1)-(j+1)(m-j)} \cdot [h_{\alpha_i}+1]_{q_i}^{-1} \cdot \hat{\mathrm{T}}_i^2(v_{m,j}) \cdot [h_{\alpha_i}+1]_{q_i}. \tag{9.9}$$

The property (9.9) simplifies under the natural assumptions on operators $\hat{\mathrm{T}}_i : \mathcal{V} \to \mathcal{V}$. Namely, suppose that the operators $\hat{\mathrm{T}}_i : \mathcal{V} \to \mathcal{V}$ satisfy the properties of Lusztig symmetries $T'_{i,+}$, that is, (see [L, Section 5.2.2]),

$$\hat{\mathrm{T}}_i(v_{m,j}) = (-1)^j q^{j(m+1-j)} v_{m,m-j}. \tag{9.10}$$

Corollary 9.6. *With the conditions* (9.10) *for any $x \in Z^{\mathfrak{n}}(A)$ we have*

$$\check{\mathrm{q}}_i^2(x) = [h_{\alpha_i}+1]_{q_i}^{-1} \cdot \varepsilon_i(x) \cdot [h_{\alpha_i}+1]_{q_i}. \tag{9.11}$$

Keep the notation (3.12) of Section 3.4. For a real root $\alpha \in \Delta^{\mathrm{re}}$, set

$$\overline{f}^{(2)}_{n,\alpha}[\mu] = q_\alpha^{-2n}\big(k_\alpha^{(1)}\big)^{-n}\prod_{k=1}^{n}\big(\big[\hat{h}^{(2)}_\alpha - h^{(1)}_\alpha + \langle h_\alpha, \mu\rangle + k\big]_{q_\alpha}\big)^{-1},$$

$$\overline{g}^{(2)}_{n,\alpha}[\mu] = \big(k_\alpha^{(1)}\big)^{n}\prod_{k=1}^{n}\big(\big[-\hat{h}^{(2)}_\alpha + h^{(1)}_\alpha + \langle h_\alpha, \mu\rangle + k\big]_{q_\alpha}\big)^{-1}.$$

Here $\hat{h}^{(2)}_\alpha = \mathrm{ad}^{(2)}_{h_\alpha}$ is the adjoint action of h_α in $\mathcal{V}$, $h^{(1)}_\alpha$ is the operator of multiplication by h_α in D. For $\mu \in \mathfrak{h}^*$ define operators $\mathrm{C}^{(2)}_\alpha[\mu] : D \otimes \mathcal{V} \to D \otimes \mathcal{V}$ and $\mathrm{B}^{(2)}_{-\alpha}[\mu] : D \otimes \mathcal{V} \to D \otimes \mathcal{V}$ by the relations (3.16):

$$\mathrm{C}^{(2)}_\alpha[\mu] = \sum_{n=0}^{\infty}\frac{(-1)^n}{[n]_{q_\alpha}!}\overline{f}^{(2)}_{n,\alpha}[\mu]\big(\hat{e}^{(2)}_{-\alpha}\big)^n\big(\hat{e}^{(2)}_{\alpha}\big)^n,$$

$$\mathrm{B}^{(2)}_{-\alpha}[\mu] = \sum_{n=0}^{\infty}\frac{(-1)^n}{[n]_{q_\alpha}!}\overline{g}^{(2)}_{n,\alpha}[\mu]\big(\hat{e}^{(2)}_{\alpha}\big)^n\big(\hat{e}^{(2)}_{-\alpha}\big)^n.$$

Proposition 9.7. *For any $v \in \mathcal{V}$ we have*

$$\check{\mathrm{q}}_i\big(z'_v\big) = z'_{\mathrm{B}^{(2)}_{-\alpha_i}[\rho](1\otimes T_i(v))}, \qquad \check{\mathrm{q}}_i(z_v) = z_{\mathrm{C}^{(2)}_{\alpha_i}[-\rho](1\otimes T_i(v))}.$$

9.4. Another adjoint action

In this section we sketch the modifications of the above constructions for the second adjoint action.

Let $U_q^{op}(\mathfrak{g})$ be the Hopf algebra $U_q(\mathfrak{g})$, described in Section 9.1, with the same multiplication and opposite comultiplication,

$$\Delta^{op}(e_{\alpha_i}) = k_{\alpha_i}\otimes e_{\alpha_i} + e_{\alpha_i}\otimes 1, \qquad \Delta^{op}(e_{-\alpha_i}) = e_{-\alpha_i}\otimes k_{\alpha_i}^{-1} + 1\otimes e_{-\alpha_i}.$$

The adjoint action (3.1) for the algebra $U_q^{op}(\mathfrak{g})$ looks slightly different:

$$\begin{aligned}\hat{e}_{\alpha_i}(x) &\equiv \mathrm{ad}_{e_{\alpha_i}}(x) = [e_{\alpha_i}, x]\cdot k_{\alpha_i}^{-1},\\ \hat{e}_{-\alpha_i}(x) &\equiv \mathrm{ad}_{e_{-\alpha_i}}(x) = e_{-\alpha_i}x - k_{\alpha_i}^{-1}xk_{\alpha_i}e_{-\alpha_i}.\end{aligned} \tag{9.12}$$

Let $\mathcal{A}$ be a $U_q^{op}(\mathfrak{g})$-admissible algebra. We define the Zhelobenko operators, starting from the assignment

$$\begin{aligned}\mathrm{q}_\alpha(x) &= \sum_{n\geqslant 0}\frac{(-1)^n}{[n]_{q_\alpha}!}(\hat{e}_\alpha)^n(x)\big(e^n_{-\alpha}k_\alpha\big)g_{n,\alpha}\\ &= \sum_{n\geqslant 0}\frac{(-1)^n}{[n]_{q_\alpha}!}\hat{e}^n_\alpha(x)e^n_{-\alpha}q_\alpha^{n(h_\alpha-n+1)}g_{n,\alpha} \mod \mathcal{A}'\mathfrak{n}_\alpha.\end{aligned}$$

The corresponding maps $\mathrm{q}_{\alpha,\mathfrak{m}}$ satisfy the cocycle conditions, such that the operators

$$\check{\mathrm{q}}_i = \mathrm{q}_{\alpha_i,\mathfrak{n}} \cdot T_i^{-1}$$

are automorphisms of the Mickelsson algebra $Z^{\mathfrak{n}}(\mathcal{A})$, satisfying the braid group relations.

9.5. *Relations to dynamical Weyl group*

Let $\mathcal{V}$ be a $U_q(\mathfrak{g})$-module algebra with a locally nilpotent action of real root vectors of $U_q(\mathfrak{g})$. It means that $\mathcal{V}$ is an associative algebra and a $U_q(\mathfrak{g})$-module, such that for any $g \in U_q(\mathfrak{g})$, $v_1, v_2 \in \mathcal{V}$, we have the equality

$$\hat{g}(v_1 \cdot v_2) = \sum_i \hat{g}_i'(v_1) \cdot \hat{g}_i''(v_2).$$

Here $\Delta(g) = \sum_i g_i' \otimes g_i''$, and $\hat{g}(v)$ is an action of $g \in U_q(\mathfrak{g})$ on $v \in \mathcal{V}$. Suppose that $\mathcal{V}$ is equipped with an action of operators $\hat{T}_i$, which are automorphisms of the algebra $\mathcal{V}$, satisfy the braid group relations, and form an equivariant family with respect to operators (9.2), that is, for any $v \in \mathcal{V}$, $g \in U_q(\mathfrak{g})$,

$$\hat{T}_i\big(\hat{g}(v)\big) = T_i(g)\hat{T}_i(v),$$

where $T_i(g)$ means the application of operators (9.2).

These conditions imply, that $\mathcal{V}$ is a module algebra over $U_q^W(\mathfrak{g})$. So we have a well-defined smash product $U_q^W(\mathfrak{g}) \ltimes \mathcal{V}$, which contains $U_q(\mathfrak{g}) \ltimes \mathcal{V}$ and the elements T_i. The automorphisms $T_i : U_q^W(\mathfrak{g}) \ltimes \mathcal{V} \to U_q^W(\mathfrak{g}) \ltimes \mathcal{V}$, given as

$$T_i(x) = \mathrm{T}_i x \mathrm{T}_i^{-1},$$

preserve the subalgebra $U_q(\mathfrak{g}) \ltimes \mathcal{V}$. Moreover, the restriction of the adjoint action of T_i on $\mathcal{V}$ coincides with $\hat{T}_i$, that is: $\hat{\mathrm{T}}_i|_{\mathcal{V}} = \hat{T}_i$. Thus the smash product $\mathcal{A} = U_q(\mathfrak{g}) \ltimes \mathcal{V}$ is a $U_q(\mathfrak{g})$-admissible algebra.

The elements of the right $Z^{\mathfrak{n}}(\mathcal{A})$-module $\Phi_\lambda(\mathcal{A})$, defined in (8.6), are intertwining operators $\Phi_\lambda^v : M_\lambda \to \mathcal{V} \otimes M_\lambda$. The operators $\check{\mathrm{q}}_i$ give rise to the operators $\check{\mathrm{q}}_{i,\lambda}$ of the dynamical Weyl group

$$\check{\mathrm{q}}_{i,\lambda}\big(\Phi_\lambda^v\big) = \Phi_{s_{\alpha_i}}^{\check{\mathrm{p}}_{-\alpha_i}[\lambda+\rho](\hat{T}_i(v))}$$

and

$$\check{\mathrm{q}}_{w,\lambda}\big(\Phi_\lambda^v\big) = \Phi_{w\circ\lambda}^{\hat{\mathrm{p}}_{-\gamma_1}[\lambda+\rho]\cdots\hat{\mathrm{p}}_{-\gamma_n}[\lambda+\rho](\hat{T}_w(v))},$$

where $\gamma_1, \ldots, \gamma_n$ is the sequence of positive roots, attached to a reduced decomposition $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_1}}$ by the standard rule $\gamma_1 = \alpha_{i_1}$, $\gamma_2 = s_{\alpha_{i_1}}(\alpha_{i_2})$, $\ldots$, and $\check{\mathrm{p}}_{-\gamma_k}[\lambda + \rho]$ is the adjoint action of the operator (9.3), $e_{\pm\gamma_k} = T_{\alpha_{i_1}} \cdots T_{\alpha_{i_{k-1}}}(e_{\pm\alpha_{i_k}})$ are the Cartan–Weyl generators.

10. Concluding remarks

We conclude with remarks on the assumptions on a $\mathfrak{g}$-admissible algebra $\mathcal{A}$, used in the paper. They are listed in Section 3.1.

The assumption (a) requires an existence of an ad-invariant subspace $\mathcal{V} \subset \mathcal{A}$, such that the multiplication m in $\mathcal{A}$ induces isomorphisms of vector spaces

$$\text{(a1)} \quad m : U_q(\mathfrak{g}) \otimes \mathcal{V} \to \mathcal{A}, \qquad \text{(a2)} \quad m : \mathcal{V} \otimes U_q(\mathfrak{g}) \to \mathcal{A}.$$

The assumptions (a1) and (a2) are not of equal importance. For the Mickelsson algebra $Z^{\mathfrak{n}}(\mathcal{A})$ we need the condition (a1) only when we use the generators z_v, that is, in Proposition 3.3, Theorem 3.6, Corollary 4.7, in Section 6.2 and in the corresponding statements of Section 9. On the contrary, the construction of the Zhelobenko operators for the algebra $Z^{\mathfrak{n}}(\mathcal{A})$ requires the condition (a1) from the very beginning. The condition (a2) is necessary for the existence of the generators z'_v.

For the algebra $Z_{\mathfrak{n}_-}(\mathcal{A})$ the situation is opposite. We need the condition (a1) for the construction of the Zhelobenko maps and generators $\tilde{z}'_v$, while the condition (a2) is related only to the generators $\tilde{z}_v$. Both conditions (a1) and (a2) are satisfied for basic examples, listed in Section 3.1.

The condition (b) requires local nilpotency of the adjoint action of real root vectors in $\mathcal{V}$. It is always satisfied if the space $\mathcal{V}$ is a sum of integrable representations or an affinization $V(z)$ of a locally finite representation of an affine algebra $U'_q(\mathfrak{g})$ with the grading element excluded.

In the latter case the generators z_v and z'_v do not formally exist, since neither the highest weight condition (HW) from Section 3.3 nor the lowest weight condition (LW) from Section 7.1 is satisfied. Nevertheless, in this case the generators of the Mickelsson algebra exist as formal series and could be used with a proper attention to convergence.

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