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Deformed Kac–Moody algebras and their representations ☆

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Abstract

A class of Lie algebras $\mathfrak{G}(A)$ associated to generalized Cartan matrices A is studied. The Lie algebras $\mathfrak{G}(A)$ have much simpler structure than Kac–Moody algebras, but have the same root spaces with $\mathfrak{g}(A)$. In particular, $\mathfrak{G}(A)$ has an abelian subalgebra of "half size." We show that, $\mathfrak{G}(A)$ has a non-degenerate invariant symmetric bilinear form if and only if A is symmetrizable; $\mathfrak{G}(X_1) \cong \mathfrak{G}(X_2)$ if and only if the GCMs X_1 and X_2 are the same up to a permutation of rows and columns.

We study the lowest (respectively highest) weight Verma module $\bar{V}(\lambda)$ (respectively $\tilde{V}(\lambda)$) over $\mathfrak{G}(A)$, and obtain the necessary and sufficient conditions for $\bar{V}(\lambda)$ to be irreducible, and also find its maximal proper submodule when $\bar{V}(\lambda)$ is reducible. Then using graded dual module of $\bar{V}(\lambda)$ we deduce the necessary and sufficient conditions for $\tilde{V}(\lambda)$ to be irreducible.

Keywords: Deformed Kac–Moody algebra; Invariant symmetric bilinear form; Lowest (respectively highest) weight Verma module; Irreducible module

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1. Introduction

Nowadays Kac–Moody algebras are widely used in many other mathematics branches and physics. The theory of finite and affine types of Kac–Moody algebras has been well developed. On the other hand, it is well known that the structure of non-symmetrizable Kac–Moody algebras (indefinite type) is very complicated (see [5,10,11]). Even we do not know the multiplicities of imaginary roots of these Kac–Moody algebras (see [1,2]). Our purpose of this paper is to expect to understand the structure of Kac–Moody algebras by studying related Lie algebras with simpler structure.

Recently, a class of interesting Lie algebras corresponding to symmetrizable Kac–Moody algebras was studied by Lu [6–8] and Zhang [12,13] where they studied finite-dimensional non-degenerate solvable Lie algebras. A finite-dimensional solvable Lie algebra $\mathfrak{g}$ over the field $\mathbb{C}$ of complex numbers is called a *solvable Lie algebra with a non-degenerate invariant bilinear form* (or simply, a *non-degenerate solvable Lie algebra*) if there exists a non-degenerate symmetric bilinear form $(\cdot,\cdot):\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ satisfying $([a,b],c)=(a,[b,c])$ for $a,b,c\in\mathfrak{g}$.

Generalizing their constructions, we define the so-called deformed Kac–Moody algebras $\mathfrak{G}(A)$ associated to generalized Cartan matrices A. For the triangular decomposition of the Kac–Moody algebra $\mathfrak{g}(A)=\mathfrak{n}_+\oplus\mathfrak{h}\oplus\mathfrak{n}_-$, we denote $\mathfrak{b}_+=\mathfrak{n}_+\oplus\mathfrak{h}$. Using the graded dual $\mathfrak{b}_+$-module $\mathfrak{b}_-=(\mathfrak{b}_+)^*$ we have the deformed Kac–Moody algebras $\mathfrak{G}(A)=\mathfrak{b}_+\oplus\mathfrak{b}_-$ where $\mathfrak{b}_-$ is abelian and a $\mathfrak{b}_+$-module.

Note that when A is of finite type $(\mathfrak{G}(A),\mathfrak{b}_+,\mathfrak{b}_-)$ gives the Manin triple associated to the canonical Poisson Lie group structure on $\mathfrak{b}_-\simeq(\mathfrak{b}_+)$ (see e.g. Section 11 of [3]). It will be meaningful to study the triple $(\mathfrak{G}(A),\mathfrak{b}_+,\mathfrak{b}_-)$ for non-finite type A.

The Lie algebra $\mathfrak{G}(A)$ has much simpler structure than the corresponding Kac–Moody algebra $\mathfrak{g}(A)$, but has the same root spaces with $\mathfrak{g}(A)$. In particular, it has the abelian subalgebra $\mathfrak{b}_-$ which is of "half size" of the whole algebra. We hope that some new progress will be achieved on the structure of the Kac–Moody algebras of indefinite types by further studying their corresponding simpler algebras $\mathfrak{G}(A)$. The present paper is just a beginning of this project.

This paper is arranged as follows: In Section 2, associated to a generalized Cartan matrix A, we introduce the deformed Kac–Moody algebra $\mathfrak{G}(A)$. When A is a finite type GCM this algebra was introduced in [12] which is a finite-dimensional non-degenerate solvable Lie algebra. In other cases, $\mathfrak{G}(A)$ is not solvable.

In Section 3, we show that $\mathfrak{G}(A)$ has a non-degenerate invariant symmetric bilinear form if and only if A is symmetrizable.

In Section 4, we prove that $\mathfrak{G}(X_1)\cong\mathfrak{G}(X_2)$ if and only if the GCMs X_1 and X_2 are the same up to a permutation of rows and columns.

In Section 5, we study the lowest weight Verma module $\bar{V}(\lambda)$ over $\mathfrak{G}(A)$, obtain the necessary and sufficient conditions for $\bar{V}(\lambda)$ to be irreducible, and determine the maximal proper submodule of $\bar{V}(\lambda)$ when it is reducible.

In Section 6, we study the highest weight Verma module $\tilde{V}(\lambda)$ over $\mathfrak{G}(A)$. Unlike the lowest weight module $\bar{V}(\lambda)$, we cannot directly obtain the necessary and sufficient conditions for $\tilde{V}(\lambda)$ to be irreducible. Fortunately, we can employ the graded dual module of the irreducible lowest weight module $L(\lambda)$ to get the necessary and sufficient conditions for $\tilde{V}(\lambda)$ to be irreducible. But in this case we are not able to explicitly give the maximal proper submodule of $\tilde{V}(\lambda)$ when it is reducible.

In Section 7, we obtain better form of our results established in previous sections in the case of symmetrizable A.

2. Deformed Kac–Moody algebras $\mathfrak{G}(A)$

Definition 2.1. An $n \times n$ integral matrix $A = (a_{ij})_{i,j=1}^n$ is called a *generalized Cartan matrix* (GCM) if

(C1) $a_{ii} = 2$, for all $i = 1, 2, \ldots, n$;
(C2) $a_{ij} \leqslant 0$, for all $i \neq j$;
(C3) $a_{ij} = 0$ implies $a_{ji} = 0$.

In this paper we always assume that A is an $n \times n$ GCM, unless otherwise stated.

Let $\mathfrak{g}(A)$ be the Kac–Moody algebra associated to A, $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}(A)$, $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \mathfrak{h}^*$ the root basis, $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee\} \subseteq \mathfrak{h}$ the coroot basis, and $e_1, e_2, \ldots, e_n$; $f_1, f_2, \ldots, f_n$ the Chevalley generators of $\mathfrak{g}(A)$. Note that $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$. Denote by Δ, Δ_+ and Δ_- the sets of all roots, positive roots and negative roots respectively. Set $Q = \sum_{i=1}^n \mathbb{Z}\alpha_i$, $Q_\pm = \sum_{i=1}^n \mathbb{Z}_\pm \alpha_i$ where $\mathbb{Z}_+$ (respectively $\mathbb{Z}_-$) is the set of all nonnegative (respectively nonpositive) integers. Then $\Delta = \Delta_+ \cup \Delta_-$ (a disjoint union), $\Delta_- = -\Delta_+$ and $\Delta_\pm = \Delta \cap Q_\pm$. The root space decomposition of $\mathfrak{g}(A)$ with respect to $\mathfrak{h}$ is

$$\mathfrak{g}(A) = \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \sum_{\alpha \in \Delta_+} \mathfrak{g}_{-\alpha}. \tag{2.1}$$

Let $\mathfrak{n}_+ = \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$, $\mathfrak{n}_- = \sum_{\alpha \in \Delta_+} \mathfrak{g}_{-\alpha}$ and $\mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$. Then $\mathfrak{n}_+$ (respectively $\mathfrak{n}_-$) is the subalgebra of $\mathfrak{g}(A)$ generated by $e_1, e_2, \ldots, e_n$ (respectively $f_1, f_2, \ldots, f_n$), and $\mathfrak{g}_\alpha$ is the linear span of the elements of the form $[e_{i_1}, [e_{i_2}, [\cdots [e_{i_{s-1}}, e_{i_s}] \cdots]]]$ (respectively $[f_{i_1}, [f_{i_2}, [\cdots [f_{i_{s-1}}, f_{i_s}] \cdots]]]$) with $\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_s} = \alpha$ (respectively $= -\alpha$). In particular, $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$, $\mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i$, for $i = 1, 2, \ldots, n$.

Let $(\mathfrak{g}(A), \mathrm{ad})$ be the adjoint representation of $\mathfrak{g}(A)$. Under this adjoint action, $\mathfrak{g}(A)$ can be regarded as a $\mathfrak{b}_+$-module. As a $\mathfrak{b}_+$-module, $\mathfrak{g}(A)$ has a submodule $\mathfrak{n}_+$. Hence we can obtain a $\mathfrak{b}_+$-quotient module

$$\mathfrak{b}_- = \mathfrak{g}(A)/\mathfrak{n}_+ = \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_-. \tag{2.2}$$

It is clear that the set of weights of $\mathfrak{b}_-$ with respect to $\mathfrak{h}$ is $P(\mathfrak{b}_-) = \{0\} \cup \Delta_-$ and $\mathfrak{b}_+ \cdot \bar{\mathfrak{h}} = \{0\}$. For simplicity, we write the action of $\mathfrak{b}_+$-module $\mathfrak{b}_-$ as $x \cdot v$, $\forall x \in \mathfrak{b}_+, v \in \mathfrak{b}_-$.

Now let us define the *deformed Kac–Moody algebras* $\mathfrak{G}(A)$ associated to A as follows.

Set

$$\mathfrak{G}(A) = \mathfrak{b}_+ \oplus \mathfrak{b}_-. \tag{2.3}$$

Define the following bracket operator $[\cdot,\cdot]$ on $\mathfrak{G}(A)$:

$$\begin{cases} [x, y] = [x, y]_0, & \forall x, y \in \mathfrak{b}_+; \\ [v_1, v_2] = 0, & \forall v_1, v_2 \in \mathfrak{b}_-; \\ [x, v] = -[v, x] = x \cdot v, & \forall x \in \mathfrak{b}_+,\ \forall v \in \mathfrak{b}_-, \end{cases} \tag{2.4}$$

where $[\cdot,\cdot]_0$ is the bracket operator on $\mathfrak{g}(A)$. It is easy to show that $\mathfrak{G}(A)$ becomes a Lie algebra under the above bracket operator $[\cdot,\cdot]$.

It is clear that, as vector spaces, $\mathfrak{h} \oplus \mathfrak{n}_- \cong \mathfrak{b}_-$. Let π be the canonical homomorphism from $\mathfrak{g}(A)$ onto $\mathfrak{b}_-$. Then $\sigma = \pi|_{\mathfrak{h}\oplus\mathfrak{n}_-}$ is an isomorphism between $\mathfrak{h} \oplus \mathfrak{n}_-$ and $\mathfrak{b}_-$, such that $\bar{\mathfrak{h}} = \sigma(\mathfrak{h})$, $\bar{\mathfrak{n}}_- = \sigma(\mathfrak{n}_-)$. We see that

$$\big[x, \sigma(y)\big] = \sigma\big([x, y]_0\big), \quad \text{for } x \in \mathfrak{g}_\alpha,\ y \in \mathfrak{g}_{-\beta};\ \alpha \leqslant \beta,\ \alpha, \beta \in \Delta_+. \tag{2.5}$$

By the construction of $\mathfrak{g}(A)$, $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a realization of A. We supplement $\alpha_{n+1}^\vee, \ldots, \alpha_{2n-l}^\vee$ to $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee\}$ to form a basis of $\mathfrak{h}$, where l is the rank of GCM A. Denote $z_i = \sigma(\alpha_i^\vee)$, $1 \leqslant i \leqslant 2n-l$. Thus $z_1, z_2, \ldots, z_{2n-l}$ form a basis of $\bar{\mathfrak{h}}$. For any $y \in \mathfrak{n}_-$, we still write its image $\sigma(y)$ in $\bar{\mathfrak{n}}_-$ as y, i.e., write elements in $\bar{\mathfrak{n}}_-$ as elements in the Lie algebra $(\mathfrak{n}_-, [\cdot,\cdot]_0)$ by using $f_1, f_2, \ldots, f_n$. Using these notations, we deduce that

$$[e_i, f_j] = \delta_{ij} z_i, \quad \text{for } 1 \leqslant i \leqslant n. \tag{2.6}$$

Let $\mathfrak{H} = \mathfrak{h} \oplus \bar{\mathfrak{h}}$. We can consider $\mathfrak{H}$ as a Cartan subalgebra of $\mathfrak{G}(A)$. For any $\alpha \in \Delta_+$, define $\tilde{\alpha} \in \mathfrak{H}^*$ such that $\tilde{\alpha}|_{\mathfrak{h}} = \alpha$ and $\tilde{\alpha}|_{\bar{\mathfrak{h}}} = 0$. Thus $\tilde{\Delta} = \{\pm\tilde{\alpha} \in \mathfrak{H}^* \mid \alpha \in \Delta_+\}$ is the set of all roots of $\mathfrak{G}(A)$. In addition, for any $\alpha \in \Delta_+$, the root space attached to $\tilde{\alpha}$ (respectively $-\tilde{\alpha}$) is $\mathfrak{G}_{\tilde{\alpha}} = \mathfrak{g}_\alpha$ (respectively $\mathfrak{G}_{-\tilde{\alpha}} = \mathfrak{g}_{-\alpha}$). Hence, $\tilde{\alpha}$, $-\tilde{\alpha}$ and $\tilde{\Delta}$ can be identified with α, $-\alpha$ and Δ respectively. So we get the root space decomposition of $\mathfrak{G}(A)$ with respect to $\mathfrak{H}$:

$$\mathfrak{G}(A) = \sum_{\alpha\in\Delta_+} \mathfrak{G}_\alpha \oplus \mathfrak{H} \oplus \sum_{\alpha\in\Delta_+} \mathfrak{G}_{-\alpha}. \tag{2.7}$$

Denote $\mathfrak{G}_+ = \sum_{\alpha\in\Delta_+} \mathfrak{G}_\alpha$, $\mathfrak{G}_- = \sum_{\alpha\in\Delta_+} \mathfrak{G}_{-\alpha}$. Then we have the triangular decomposition of $\mathfrak{G}(A)$:

$$\mathfrak{G}(A) = \mathfrak{G}_+ \oplus \mathfrak{H} \oplus \mathfrak{G}_-. \tag{2.8}$$

Hence the universal enveloping algebra $U(\mathfrak{G}(A))$ of $\mathfrak{G}(A)$ can be factored as

$$U\big(\mathfrak{G}(A)\big) = U(\mathfrak{G}_+) \otimes U(\mathfrak{H}) \otimes U(\mathfrak{G}_-). \tag{2.9}$$

Now we collect some properties of $\mathfrak{G}(A)$ in the following lemma.

Lemma 2.1.

(1) *The set of all roots of* $\mathfrak{G}(A)$ *with respect to* $\mathfrak{H}$ *is* Δ;
(2) *as vector spaces,* $\mathfrak{G}_-$ *is isomorphic to* $\mathfrak{n}_-$;
(3) $\bar{\mathfrak{h}} \oplus \mathfrak{G}_-$ *is an abelian subalgebra of* $\mathfrak{G}(A)$, $\bar{\mathfrak{h}}$ *is in the center of* $\mathfrak{G}(A)$;
(4) $\bar{\mathfrak{h}} \oplus \mathfrak{G}_-$ *is a* $\mathfrak{h} \oplus \mathfrak{G}_+$*-module, and* $\mathfrak{G}_+ \cdot \bar{\mathfrak{h}} = \{0\}$.

Recall that Δ has a partial order: $\alpha \geqslant \beta$ iff $\alpha - \beta$ is a sum of positive roots (equivalently, of simple roots) or $\alpha = \beta$.

In order to study the representations of $\mathfrak{G}(A)$, we need the following identities which are easy to prove.

Lemma 2.2. *For* $\alpha, \beta \in \Delta_+$, $r \in \mathbb{Z}_+$, $e_\alpha \in \mathfrak{G}_\alpha$ *and* $e_{-\beta} \in \mathfrak{G}_{-\beta}$, *we have the following identities in* $U(\mathfrak{G}(A))$:

(1) $[e_\alpha, e_{-\beta}^r] = 0$, *if* $\alpha > \beta$, *or* $\alpha - \beta \notin \Delta$ *and* $\alpha \neq \beta$;
(2) $[e_\alpha, e_{-\alpha}^{r+1}] = (r+1)e_{-\alpha}^r[e_\alpha, e_{-\alpha}]$;
(3) $[e_\alpha, e_{-\beta}^{r+1}] = (r+1)e_{\alpha-\beta}e_{-\beta}^r$, *for some* $e_{\alpha-\beta} \in \mathfrak{G}_{\alpha-\beta}$, *if* $\alpha - \beta \in \Delta$ *and* $\alpha < \beta$.

Corollary 2.3. *The following identities hold in* $U(\mathfrak{G}(A))$, *for* $\alpha \in \Delta_+$, $r \in \mathbb{Z}_+$, $1 \leqslant i, j \leqslant n$:

(1) $[e_i, f_j^r] = 0$, *when* $i \neq j$;
(2) $[e_\alpha, f_i^{r+1}] = \delta_{\alpha,\alpha_i}(r+1) f_i^r z_i$;
(3) *if* $\alpha \in \Delta_+$ *is not a simple root, and* $0 \neq e_{-\alpha} \in \mathfrak{G}_{-\alpha}$, *then there exist some* e_i *and* $0 \neq e_{\alpha_i-\alpha} \in \mathfrak{G}_{\alpha_i-\alpha}$ *such that*

$$\left[e_i, e_{-\alpha}^{r+1}\right] = (r+1)e_{\alpha_i-\alpha}e_{-\alpha}^r, \quad \text{for } r \in \mathbb{Z}_+. \tag{2.10}$$

Lemma 2.4. *For* $\alpha, \beta \in \Delta_+$, $r \in \mathbb{Z}_+$, $e_\alpha \in \mathfrak{G}_\alpha$ *and* $e_{-\beta} \in \mathfrak{G}_{-\beta}$, *we have the following identities in* $U(\mathfrak{G}(A))$:

(1) $[e_{-\beta}, e_\alpha^r] = 0$, *if* $\alpha > \beta$, *or* $\alpha - \beta \notin \Delta$ *and* $\alpha \neq \beta$;
(2) $[e_{-\alpha}, e_\alpha^{r+1}] = -(r+1)e_\alpha^r[e_\alpha, e_{-\alpha}]$;
(3) $[e_{-\beta}, e_\alpha^{r+1}] = \sum_{i=0}^r e_\alpha^i e_{\alpha-\beta} e_\alpha^{r-i}$ *for some* $e_{\alpha-\beta} \in \mathfrak{G}_{\alpha-\beta}$, *if* $\alpha - \beta \in \Delta$ *and* $\alpha < \beta$.

Corollary 2.5. *For* $\alpha, \beta, \beta - \alpha \in \Delta_+$, $r \in \mathbb{Z}_+$, *there exist some* $e_{i\alpha-\beta} \in \mathfrak{G}_{i\alpha-\beta}$ *such that we have the following identity in* $U(\mathfrak{G}(A))$:

$$e_{-\beta}e_\alpha^r = \sum_{i=0}^r (-1)^i \binom{r}{i} e_\alpha^{r-i} e_{i\alpha-\beta}, \tag{2.11}$$

where $e_{i\alpha-\beta} = 0$ *if* $\beta - i\alpha \notin \Delta \cup \{0\}$.

Proof. For $x \in \mathfrak{G}(A)$ we write R_x, L_x for the right and left multiplication by x in $U(\mathfrak{G}(A))$ respectively. Then $R_x = L_x - \mathrm{ad}(x)$. Denote $e_{i\alpha-\beta} = (\mathrm{ad}\, e_\alpha)^i e_{-\beta}$. Note that if β is an imaginary root, $e_{i\alpha-\beta}$ may not be 0 even if $\beta - i\alpha$ is 0 or a negative root. Applying binomial formula to $R_{e_\alpha}^r e_{-\beta} = (L_{e_\alpha} - \mathrm{ad}(e_\alpha))^r e_{-\beta}$ we easily obtain (2.11) where $e_{i\alpha-\beta} = 0$ if $\beta - i\alpha \neq \Delta \cup \{0\}$. □

3. Invariant bilinear forms on $\mathfrak{G}(A)$

Recall that a $\mathbb{C}$-valued symmetric bilinear form $(\cdot,\cdot)$ on a complex Lie algebra $\mathfrak{g}$ is said to be *invariant* if

$$\big([x, y], z\big) = \big(x, [y, z]\big), \quad \text{for all } x, y, z \in \mathfrak{g}. \tag{3.1}$$

Also recall that a complex $n \times n$ matrix A is called *symmetrizable* if there exist a non-singular $n \times n$ diagonal matrix D and an $n \times n$ symmetric matrix B such that

$$A = DB. \tag{3.2}$$

Lemma 3.1. *(See [11].) Let $A=(a_{ij})_{i,j=1}^n$ be a symmetrizable GCM. Then the diagonal matrix $D=\operatorname{diag}(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n)$ in (3.2) can be chosen so that $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n$ are positive rational numbers. If, further, A is indecomposable, then the matrix D is unique up to a constant factor.*

Theorem 3.2. *The Lie algebra $\mathfrak{G}(A)$ has a non-degenerate symmetric invariant bilinear form $(\cdot,\cdot)$ if and only if A is symmetrizable.*

Proof. "$\Rightarrow$" Suppose $(\cdot,\cdot)$ is a non-degenerate symmetric invariant bilinear form on $\mathfrak{G}(A)$. It is easy to see that $(\mathfrak{G}_\alpha,\mathfrak{G}_\beta)=\{0\}$ if $\alpha+\beta\neq 0$. Recall that we denote by $[\cdot,\cdot]_0$ the bracket operator on $\mathfrak{g}(A)$, $[\cdot,\cdot]$ the bracket operator on $\mathfrak{G}(A)$.

First, in $\mathfrak{g}(A)$, we have the following identities: for $i\neq j$,

$$\big[e_i,[f_i,f_j]_0\big]_0=\big[[e_i,f_i]_0,f_j\big]_0+\big[f_i,[e_i,f_j]_0\big]_0=\big[\alpha_i^\vee,f_j\big]_0=-\langle\alpha_i^\vee,\alpha_j\rangle f_j=-a_{ij}f_j$$

and

$$\big[e_j,[f_i,f_j]_0\big]_0=\big[[e_j,f_i]_0,f_j\big]_0+\big[f_i,[e_j,f_j]_0\big]_0=\big[f_i,\alpha_j^\vee\big]_0=\langle\alpha_j^\vee,\alpha_i\rangle f_i=a_{ji}f_i.$$

Hence, by the invariance of $(\cdot,\cdot)$, we deduce that

$$\begin{aligned}\big([e_i,e_j],[f_i,f_j]_0\big)&=\big(e_i,\big[e_j,[f_i,f_j]_0\big]\big)=\big(e_i,e_j\cdot[f_i,f_j]_0\big)=\big(e_i,\big[e_j,[f_i,f_j]_0\big]_0\big)\\&=a_{ji}(e_i,f_i);\end{aligned}$$

but

$$\begin{aligned}\big([e_i,e_j],[f_i,f_j]_0\big)&=-\big([e_j,e_i],[f_i,f_j]_0\big)=-\big(e_j,\big[e_i,[f_i,f_j]_0\big]\big)=-\big(e_j,e_i\cdot[f_i,f_j]_0\big)\\&=-\big(e_j,\big[e_i,[f_i,f_j]_0\big]_0\big)=a_{ij}(e_j,f_j).\end{aligned}$$

So, for $i,j=1,2,\dots,n$, we have

$$a_{ji}(e_i,f_i)=a_{ij}(e_j,f_j). \tag{3.3}$$

Let $\varepsilon_i=(e_i,f_i)$, $i=1,2,\dots,n$. Then the matrix

$$A\operatorname{diag}(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n) \tag{3.4}$$

is symmetric. We claim that $\varepsilon_i\neq 0$ for $i=1,2,\dots,n$. Instead, if $\varepsilon_i=(e_i,f_i)=0$ for some i ($1\leqslant i\leqslant n$), then e_i,f_i are in the kernel of $(\cdot,\cdot)$, which is impossible. Thus

$$\operatorname{diag}\big(\varepsilon_1^{-1},\varepsilon_2^{-1},\dots,\varepsilon_n^{-1}\big)A \tag{3.5}$$

is symmetric, then A is symmetrizable.

"$\Leftarrow$" Now we assume that A is symmetrizable. It is well known that the Kac–Moody algebra $\mathfrak{g}(A)$ associated to A has a non-degenerate symmetric invariant bilinear form $(\cdot,\cdot)_0$. Let us use $(\cdot,\cdot)_0$ to define a non-degenerate symmetric invariant bilinear form on $\mathfrak{G}(A)$. Recall that σ is the vector space isomorphism from $\mathfrak{h} \oplus \mathfrak{n}_-$ onto $\bar{\mathfrak{h}} \oplus \mathfrak{G}_-$. We define the bilinear form $(\cdot,\cdot)$ on $\mathfrak{G}(A)$ as follows:

$$(\mathfrak{h} + \mathfrak{G}_+, \mathfrak{h} + \mathfrak{G}_+) = \{0\}, \qquad (\bar{\mathfrak{h}} + \mathfrak{G}_-, \bar{\mathfrak{h}} + \mathfrak{G}_-) = \{0\}, \tag{3.6}$$

$$(u, v) = (v, u) = \big(u, \sigma^{-1}(v)\big)_0, \quad \forall u \in \mathfrak{h} + \mathfrak{G}_+,\ v \in \bar{\mathfrak{h}} + \mathfrak{G}_-. \tag{3.7}$$

It is easy to see that $(\cdot,\cdot)$ is a non-degenerate symmetric bilinear form on $\mathfrak{G}(A)$. We need only to show that $(\cdot,\cdot)$ is invariant.

Note that $(\mathfrak{G}_\alpha, \mathfrak{G}_{-\beta}) = \{0\}$ if $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$. Since all other cases are trivial, we need only to verify that for any $x \in \mathfrak{G}_\alpha$, $y \in \mathfrak{G}_\beta$, $z \in \mathfrak{G}_{-\alpha-\beta}$ where $\alpha, \beta, \alpha + \beta \in \Delta_+$, we have $([x, y], z) = (x, [y, z])$. Let $\sigma(z') = z$ for some $z' \in \mathfrak{n}_-$. Using (2.5) we have

$$\big([x, y], z\big) = \big([x, y], z'\big)_0 = \big(x, [y, z']_0\big)_0 = \big(x, \sigma^{-1}\big([y, z]\big)\big)_0 = \big(x, [y, z]\big).$$

The theorem holds. □

Note that there are actually other non-degenerate symmetric invariant bilinear forms on $\mathfrak{G}(A)$, they are not induced from those of $\mathfrak{g}(A)$ if A is degenerate.

4. Isomorphism problem on $\mathfrak{G}(A)$

In this section we solve the isomorphism problem on $\mathfrak{G}(A)$, i.e., determine the conditions on GCMs X_1 and X_2 for $\mathfrak{G}(X_1) \cong \mathfrak{G}(X_2)$. Here Moody's results in [9] play a crucial role.

For $i = 1, 2$, let X_i be an $n_i \times n_i$ GCM of rank l_i. Then the deformed Kac–Moody algebra $\mathfrak{G}(X_i)$ associated to X_i has the decomposition:

$$\mathfrak{G}(X_i) = \mathfrak{G}^i_+ \oplus \mathfrak{h}^i \oplus \bar{\mathfrak{h}}^i \oplus \mathfrak{G}^i_-, \tag{4.1}$$

where $\bar{\mathfrak{h}}^i = \sum_{j=1}^{2n_i - l_i} \mathbb{C} z^i_j$ is in the center of $\mathfrak{G}(X_i)$. Denote $\mathfrak{G}(X_i)^{(0)} = \mathfrak{G}(X_i)$ and $\mathfrak{G}(X_i)^{(j+1)} = [\mathfrak{G}(X_i)^{(j)}, \mathfrak{G}(X_i)^{(j)}]$ ($j \in \mathbb{Z}_+$). From the definition of $\mathfrak{G}(X_i)$, we know that $\mathfrak{G}(X_i)^{(1)} = \mathfrak{G}^i_+ \oplus \sum_{j=1}^{n_i} \mathbb{C} z^i_j \oplus \mathfrak{G}^i_-$.

First, we give a simple fact in Kac–Moody algebra.

Lemma 4.1. *Let A be an indecomposable non-finite type GCM, and $\mathfrak{g}(A) = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ the Kac–Moody algebra associated to A. Then for each $0 \neq e_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Delta_+$, there exists some i ($1 \leqslant i \leqslant n$) such that $[e_i, e_\alpha] \neq 0$.*

Proof. Suppose that there exists some $0 \neq e_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Delta_+$, such that $[e_i, e_\alpha] = 0$ for all $1 \leqslant i \leqslant n$. Let I be the ideal of $\mathfrak{g}(A)$ generated by e_α. By Lemma 1.5 in [5], there exist some $f_{i_1}, f_{i_2}, \ldots, f_{i_s}$ ($1 \leqslant i_1, \ldots, i_s \leqslant n$) such that $0 \neq [f_{i_1}, [f_{i_2}, [\cdots [f_{i_s}, e_\alpha] \cdots]]] \in \mathbb{C}\alpha^\vee_{i_1}$. Hence $\alpha^\vee_{i_1} \in I$. Since A is indecomposable, we can deduce that $e_1, e_2, \ldots, e_n \in I$. It follows that $\sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \subseteq I$. From the fact that for every $1 \leqslant i \leqslant n$, $[e_i, e_\alpha] = 0$, we see that $\mathfrak{g}_\beta \subsetneq I$ for $\beta \in \Delta_+$ with $\operatorname{ht}\beta > \operatorname{ht}\alpha$, which is a contradiction. Hence for each $0 \neq e_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Delta_+$, there exists some i ($1 \leqslant i \leqslant n$) such that $[e_i, e_\alpha] \neq 0$. This completes the proof of this lemma. □

Table 1

Type of $\mathfrak{g}$	$\dim \mathfrak{g}$	Rank of $\mathfrak{g}$
A_n $(n \geqslant 1)$	$(n+1)^2-1$	n
B_n $(n \geqslant 2)$	$2n^2+n$	n
C_n $(n \geqslant 3)$	$2n^2+n$	n
D_n $(n \geqslant 4)$	$2n^2-n$	n
E_6	78	6
E_7	133	7
E_8	248	8
F_4	52	4
G_2	14	2

Lemma 4.2. *(See Chapter 3 in [4].) The type, the dimension, and the rank of a finite-dimensional simple Lie algebra $\mathfrak{g}$ are as in Table* 1.

Lemma 4.3. *Let X_1 and X_2 be Cartan matrices B_n and C_n $(n \geqslant 3)$ types respectively. Then $\mathfrak{G}(X_1)$ and $\mathfrak{G}(X_2)$ cannot be isomorphic.*

Proof. Suppose that ψ is an isomorphism of Lie algebras from $\mathfrak{G}(X_1)$ onto $\mathfrak{G}(X_2)$. Then we see that $\psi(\mathfrak{G}(X_1)^{(j)}) = \mathfrak{G}(X_2)^{(j)}$ for $j \in \mathbb{Z}_+$. Since $\bar{\mathfrak{h}}^i$ is the center of $\mathfrak{G}(X_i)$, we know that $\psi(\mathfrak{G}(X_1)^{(j)}/\bar{\mathfrak{h}}^1) = \mathfrak{G}(X_2)^{(j)}/\bar{\mathfrak{h}}^2$. It follows that $\dim \mathfrak{G}(X_1)^{(j)}/\bar{\mathfrak{h}}^1 = \dim \mathfrak{G}(X_2)^{(j)}/\bar{\mathfrak{h}}^2$.

Let $\Pi_1 = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and $\Pi_2 = \{\beta_1, \beta_2, \dots, \beta_n\}$ be the root bases of B_n and C_n respectively. Then it is well known that all positive roots of B_n and C_n are:

$$\begin{aligned}\Delta(X_1)_+ = {}&\{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j \mid 1 \leqslant i \leqslant j \leqslant n\}\\ &\cup \{\alpha_i + \alpha_{i+1} \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_n \mid 1 \leqslant i < j \leqslant n\}\end{aligned}$$

and

$$\begin{aligned}\Delta(X_2)_+ = {}&\{\beta_i + \beta_{i+1} + \cdots + \beta_j \mid 1 \leqslant i \leqslant j \leqslant n\}\\ &\cup \{\beta_i + \beta_{i+1} + \cdots + \beta_{j-1} + 2\beta_j + \cdots + 2\beta_{n-1} + \beta_n \mid 1 \leqslant i \leqslant j \leqslant n-1\}.\end{aligned}$$

Denote by θ_1 and θ_2 the highest roots in $\Delta(X_1)_+$ and $\Delta(X_2)_+$ respectively. Then

$$\theta_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \cdots + 2\alpha_n \quad \text{and} \quad \theta_2 = 2\beta_1 + 2\beta_2 + \cdots + 2\beta_{n-1} + \beta_n.$$

Now, for $i = 1, 2$, we can check that

$$\mathfrak{G}(X_i)^{(1)}/\bar{\mathfrak{h}}^i = \mathfrak{G}^i_+ \oplus \mathfrak{G}^i_- = \sum_{\gamma \in \Delta(X_i)_+} \mathfrak{G}^i_\gamma \oplus \sum_{\gamma \in \Delta(X_i)_+} \mathfrak{G}^i_{-\gamma}$$

and

$$\mathfrak{G}(X_i)^{(2)}/\bar{\mathfrak{h}}^i = \sum_{\gamma \in \Delta(X_i)_+,\, \mathrm{ht}\,\gamma \neq 1} \mathfrak{G}^i_\gamma \oplus \sum_{\gamma \in \Delta(X_i)_+,\, \gamma \neq \theta_i} \mathfrak{G}^i_{-\gamma}.$$

Hence $\dim \mathfrak{G}(X_1)^{(1)}/\bar{\mathfrak{h}}^1 = \dim \mathfrak{G}(X_2)^{(1)}/\bar{\mathfrak{h}}^2 = 2n^2$ and $\dim \mathfrak{G}(X_1)^{(2)}/\bar{\mathfrak{h}}^1 = \dim \mathfrak{G}(X_2)^{(2)}/\bar{\mathfrak{h}}^2 = 2n^2 - n - 1$.

Next we can get that

$$\mathfrak{G}(X_2)^{(3)}/\bar{\mathfrak{h}}^2 = \sum_{\beta \in \Delta(X_2)_+,\, \mathrm{ht}\,\beta \geqslant 4} \mathfrak{G}^2_\beta \oplus \sum_{\beta \in \Delta(X_2)_+ \setminus \Delta^2} \mathfrak{G}^2_{-\beta},$$

where $\Delta^2 = \{\beta \in \Delta(X_2)_+ \mid \mathrm{ht}\,\beta = \mathrm{ht}\,\theta_2 - 1 \text{ or } \mathrm{ht}\,\theta_2 - 2\}$. But we notice that there are two special elements in $\Delta(X_1)_+$:

$$\alpha = \alpha_{n-2} + \alpha_{n-1} + 2\alpha_n \quad \text{and} \quad \alpha' = \alpha_2 + \alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{n-1} + 2\alpha_n,$$

where α cannot be written as the sum of two positive roots with height 2 and β cannot be gotten from $\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \cdots + 2\alpha_{n-1} + 2\alpha_n$ by decreasing a positive root with height 2. Let $\Delta^1 = \{\gamma \in \Delta(X_1)_+ \mid \mathrm{ht}\,\gamma = \mathrm{ht}\,\theta_1 - 1 \text{ or } \mathrm{ht}\,\theta_1 - 2\} \cup \{\alpha'\}$. Then

$$\mathfrak{G}(X_1)^{(3)}/\bar{\mathfrak{h}}^1 = \begin{cases} \mathfrak{G}_\theta \oplus \mathfrak{G}_{-\alpha_1} \oplus \mathfrak{G}_{-\alpha_3}, & n = 3; \\ \sum_{\substack{\gamma \in \Delta(X_1)_+, \\ \mathrm{ht}\,\gamma \geqslant 4 \text{ but } \gamma \neq \alpha}} \mathfrak{G}^1_\gamma \oplus \sum_{\gamma \in \Delta(X_1)_+ \setminus \Delta^1} \mathfrak{G}^1_{-\gamma}, & n \geqslant 4. \end{cases}$$

It is straightforward to check that

$$\dim \mathfrak{G}(X_1)^{(3)}/\bar{\mathfrak{h}}^1 = \begin{cases} 3, & n = 3; \\ 2n^2 - 3n - 4, & n \geqslant 4, \end{cases}$$

and

$$\dim \mathfrak{G}(X_2)^{(3)}/\bar{\mathfrak{h}}^2 = \begin{cases} 7, & n = 3; \\ 2n^2 - 3n - 2, & n \geqslant 4. \end{cases}$$

Thus $\dim \mathfrak{G}(X_1)^{(3)}/\bar{\mathfrak{h}}^1 \neq \dim \mathfrak{G}(X_2)^{(3)}/\bar{\mathfrak{h}}^2$. It is a contradiction. Hence there cannot exist an isomorphism of Lie algebras from $\mathfrak{G}(X_1)$ onto $\mathfrak{G}(X_2)$. This lemma holds. □

Theorem 4.4. *For $i = 1, 2$, let X_i be an $n_i \times n_i$ indecomposable GCM of rank l_i. Then $\mathfrak{G}(X_1)$ and $\mathfrak{G}(X_2)$ are isomorphic if and only if X_1 and X_2 are the same up to a permutation of rows and columns.*

Proof. "⇐" Suppose that X_1 and X_2 are the same up to a permutation of rows and columns. For $i = 1, 2$, let $\mathfrak{g}(X_i) = \mathfrak{n}^i_- \oplus \mathfrak{h}^i \oplus \mathfrak{n}^i_+$ be the Kac–Moody algebra associated to X_i. Then there exists a Lie algebra isomorphism $\psi : \mathfrak{g}(X_1) \to \mathfrak{g}(X_2)$ such that $\psi(\mathfrak{h}^1) = \mathfrak{h}^2$ and $\psi(\mathfrak{n}^1_\pm) = \mathfrak{n}^2_\pm$. Apply the construction in Section 2 to $\mathfrak{g}(X_1)$ and $\mathfrak{g}(X_2)$ to get the deformed Kac–Moody algebras $\mathfrak{G}(X_1)$ and $\mathfrak{G}(X_2)$. It follows that $\mathfrak{G}(X_1)$ and $\mathfrak{G}(X_2)$ are isomorphic.

"⇒" Assume that $\mathfrak{G}(X_1)$ and $\mathfrak{G}(X_2)$ have the decompositions as in (4.1) and $\varphi : \mathfrak{G}(X_1) \to \mathfrak{G}(X_2)$ is an isomorphism of Lie algebras.

Case 1. X_1 and X_2 are finite type GCMs.

In this case, $\mathfrak{G}(X_i)^{(1)} = \mathfrak{G}^i_+ \oplus \bar{\mathfrak{h}}^i \oplus \mathfrak{G}^i_-$ for $i = 1, 2$, and $\varphi(\mathfrak{G}(X_1)^{(1)}) = \mathfrak{G}(X_2)^{(1)}$. So $\dim \mathfrak{G}(X_1)^{(1)} = \dim \mathfrak{G}(X_2)^{(1)}$. Since the codimension of $\mathfrak{G}(X_i)^{(1)}$ in $\mathfrak{G}(X_i)$ is $\dim \mathfrak{h}^i = n_i$, we deduce that $n_1 = n_2$, i.e., X_1 and X_2 have the same size $n = n_1 = n_2$. By Lemma 4.2, we know that if X_1 and X_2 are different finite type GCMs (except for B_n and C_n), then $\dim \mathfrak{G}(X_1) \neq \dim \mathfrak{G}(X_2)$. In addition, if X_1 and X_2 are B_n and C_n types respectively, then, by Lemma 4.3, $\mathfrak{G}(X_1) \ncong \mathfrak{G}(X_2)$. So X_1 and X_2 must be the same type GCMs. Hence X_1 and X_2 are the same up to a permutation of rows and columns.

Case 2. X_1 and X_2 are non-finite type GCMs.

Note that $\mathfrak{G}(X_i)^{(1)} = \mathfrak{G}^i_+ \oplus \sum_{j=1}^{n_i} \mathbb{C}z^i_j \oplus \mathfrak{G}^i_-$ for $i = 1, 2$. Since the center of $\mathfrak{G}(X_i)^{(1)}$ is $\sum_{j=1}^{n_i} \mathbb{C}z^i_j$ with dimension n_i, we see that $n_1 = n_2$. Now assume that $n = n_1 = n_2$. Denote

$$K_i = \{x \in \mathfrak{G}(X_i)^{(1)} \mid \dim\langle x\rangle < \infty\},$$

where $\langle x\rangle$ is the ideal of $\mathfrak{G}(X_i)^{(1)}$ generated by x. We shall show that $K_i = \mathfrak{G}^i_- \oplus \sum_{j=1}^{n} \mathbb{C}z^i_j$.

In fact, since $\sum_{j=1}^{n} \mathbb{C}z^i_j$ is in the center of $\mathfrak{G}(X_i)$, and by Lemma 2.2, we can deduce that $\mathfrak{G}^i_- \oplus \sum_{j=1}^{n} \mathbb{C}z^i_j \subseteq K_i$. On the other hand, by Lemma 4.1, we get that for any $x \in \mathfrak{G}^i_+$, $x \notin K_i$. Hence $K_i = \mathfrak{G}^i_- \oplus \sum_{j=1}^{n} \mathbb{C}z^i_j$.

Obviously, $\varphi(K_1) = K_2$. Since K_i is an ideal of $\mathfrak{G}(X_i)^{(1)}$, we deduce that $\varphi(\mathfrak{G}(X_1)^{(1)}/K_1) = \mathfrak{G}(X_2)^{(1)}/K_2$. Hence $\mathfrak{G}^1_+$ and $\mathfrak{G}^2_+$ are isomorphic. By Theorem 8.2 in [9], we deduce that X_1 and X_2 are the same up to a permutation of rows and columns. This completes the proof of this theorem. $\square$

From the proof of the previous theorem, we have obtained the following.

Corollary 4.5. *Let X_1 and X_2 be indecomposable GCMs. Then $\mathfrak{G}(X_1)$ and $\mathfrak{G}(X_2)$ are isomorphic if and only if $\mathfrak{G}(X_1)^{(1)}$ and $\mathfrak{G}(X_2)^{(1)}$ are isomorphic.*

Lemma 4.6. *Let $A = (a_{ij})^n_{i,j=1}$ be an indecomposable GCM of rank l, and $\mathfrak{G}(A)$ the deformed Kac–Moody algebra associated to A. Then $\mathfrak{G}(A)^{(1)}$ is an indecomposable ideal, i.e., $\mathfrak{G}(A)^{(1)}$ cannot be written as $\mathfrak{G}(A)^{(1)} = I_1 \oplus I_2$ where I_1 and I_2 are nonzero ideals of $\mathfrak{G}(A)$.*

Proof. Note that $\mathfrak{G}(A) = \mathfrak{G}_+ \oplus \mathfrak{h} \oplus \bar{\mathfrak{h}} \oplus \mathfrak{G}_-$ where $\bar{\mathfrak{h}} = \sum_{i=1}^{2n-l} \mathbb{C}z_i$. Then $\mathfrak{G}(A)^{(1)} = \mathfrak{G}_+ \oplus \sum_{i=1}^{n} \mathbb{C}z_i \oplus \mathfrak{G}_-$. Suppose that $\mathfrak{G}(A)^{(1)}$ can be written as: $\mathfrak{G}(A)^{(1)} = I_1 \oplus I_2$ where I_1 and I_2 are nonzero ideals of $\mathfrak{G}(A)$. We know that I_1 and I_2 are $\mathfrak{h}^*$ graded ideals. Hence there exist non-empty sets $P_1 = \{e_{i_1}, \dots, e_{i_s}\} \subseteq I_1$ and $P_2 = \{e_{j_1}, \dots, e_{j_t}\} \subseteq I_2$, such that $P_1 \cup P_2 = \{e_1, \dots, e_n\}$ (a disjoint union) and $[e_{i_p}, e_{j_q}] = 0$ for $e_{i_p} \in P_1$, $e_{j_q} \in P_2$. It follows that $a_{i_p j_q} = 0$. This contradicts the fact that A is indecomposable. Hence $\mathfrak{G}(A)^{(1)}$ is indecomposable. $\square$

Now we are ready to give the main result in this section.

Theorem 4.7. *Let X_1 and X_2 be GCMs. Then $\mathfrak{G}(X_1)$ and $\mathfrak{G}(X_2)$ are isomorphic if and only if X_1 and X_2 are the same up to a permutation of rows and columns.*

Proof. The sufficiency is clear. We need only to consider the necessity.

Assume that X_1 and X_2 have the following direct sum decompositions of indecomposable matrices:

$$X_1 = M_1 \oplus M_2 \oplus \cdots \oplus M_s \quad \text{and} \quad X_2 = N_1 \oplus N_2 \oplus \cdots \oplus N_t, \quad \text{for } s, t \geqslant 1.$$

Then we have the corresponding decompositions:

$$\mathfrak{G}(X_1) = \mathfrak{G}(M_1) \oplus \mathfrak{G}(M_2) \oplus \cdots \oplus \mathfrak{G}(M_s) \quad \text{and} \quad \mathfrak{G}(X_2) = \mathfrak{G}(N_1) \oplus \mathfrak{G}(N_2) \oplus \cdots \oplus \mathfrak{G}(N_t).$$

Clearly, $\mathfrak{G}(M_i)$ $(i = 1, 2, \ldots, k)$ are ideals of $\mathfrak{G}(X_1)$, and $[\mathfrak{G}(M_i), \mathfrak{G}(M_j)] = \{0\}$ for $i \neq j$. Hence

$$\mathfrak{G}(X_1)^{(1)} = \mathfrak{G}(M_1)^{(1)} \oplus \mathfrak{G}(M_2)^{(1)} \oplus \cdots \oplus \mathfrak{G}(M_s)^{(1)}.$$

By Lemma 4.6, we can see that $\mathfrak{G}(M_i)^{(1)}$ is an indecomposable ideal of $\mathfrak{G}(X_1)^{(1)}$. It is straightforward to check that

$$\big[\mathfrak{G}(M_i)^{(1)}, \mathfrak{G}(X_1)\big] = \mathfrak{G}(M_i)^{(1)}. \tag{4.2}$$

Moreover, we claim that $\mathfrak{G}(M_i)^{(1)}$ is a maximal indecomposable ideal of $\mathfrak{G}(X_1)^{(1)}$ with the property (4.2). Let us take $\mathfrak{G}(M_1)^{(1)}$ as an example. In fact, suppose that I is an indecomposable ideal of $\mathfrak{G}(X_1)^{(1)}$ with $[I, \mathfrak{G}(X_1)] = I$, which contains $\mathfrak{G}(M_1)^{(1)}$ properly. Then

$$\begin{aligned} I &= \big[I, \mathfrak{G}(M_1) \oplus \mathfrak{G}(M_2) \oplus \cdots \oplus \mathfrak{G}(M_s)\big] = \big[I, \mathfrak{G}(M_1)\big] \oplus \big[I, \mathfrak{G}(M_2)\big] \oplus \cdots \oplus \big[I, \mathfrak{G}(M_s)\big] \\ &= \mathfrak{G}(M_1)^{(1)} \oplus \big[I, \mathfrak{G}(M_2)\big] \oplus \cdots \oplus \big[I, \mathfrak{G}(M_s)\big]. \end{aligned}$$

Thus $\mathfrak{G}(M_1)^{(1)} \neq I$ and $[I, \mathfrak{G}(M_2)] \oplus \cdots \oplus [I, \mathfrak{G}(M_s)] \neq I$ are ideals of $\mathfrak{G}(X_1)^{(1)}$. This contradicts the fact that I is indecomposable.

Similarly,

$$\mathfrak{G}(X_2)^{(1)} = \mathfrak{G}(N_1)^{(1)} \oplus \mathfrak{G}(N_2)^{(1)} \oplus \cdots \oplus \mathfrak{G}(N_t)^{(1)},$$

and $\mathfrak{G}(N_i)^{(1)}$ $(1 \leqslant i \leqslant t)$ are maximal indecomposable ideals of $\mathfrak{G}(X_2)^{(1)}$ with

$$\big[\mathfrak{G}(N_i)^{(1)}, \mathfrak{G}(X_2)\big] = \mathfrak{G}(N_i)^{(1)}.$$

Let φ be an isomorphism from $\mathfrak{G}(X_1)$ onto $\mathfrak{G}(X_2)$. Then we know that

$$\varphi\big(\mathfrak{G}(M_1)^{(1)}\big) \oplus \varphi\big(\mathfrak{G}(M_2)^{(1)}\big) \oplus \cdots \oplus \varphi\big(\mathfrak{G}(M_s)^{(1)}\big) = \mathfrak{G}(N_1)^{(1)} \oplus \mathfrak{G}(N_2)^{(1)} \oplus \cdots \oplus \mathfrak{G}(N_t)^{(1)}.$$

Since $\mathfrak{G}(M_1)^{(1)}$ is a maximal indecomposable ideal of $\mathfrak{G}(X_1)^{(1)}$ with $[\mathfrak{G}(M_1)^{(1)}, \mathfrak{G}(X_1)] = \mathfrak{G}(M_1)^{(1)}$, we can deduce that $\varphi(\mathfrak{G}(M_1)^{(1)})$ is a maximal indecomposable ideal of $\mathfrak{G}(X_2)^{(1)}$ with $[\varphi(\mathfrak{G}(M_1)^{(1)}), \mathfrak{G}(X_2)] = \varphi(\mathfrak{G}(M_1)^{(1)})$. Hence

$$\varphi\big(\mathfrak{G}(M_1)^{(1)}\big) = \big[\varphi\big(\mathfrak{G}(M_1)^{(1)}\big), \mathfrak{G}(N_1)\big] \oplus \cdots \oplus \big[\varphi\big(\mathfrak{G}(M_1)^{(1)}\big), \mathfrak{G}(N_t)\big].$$

Since $\varphi(\mathfrak{G}(M_1)^{(1)})$ is indecomposable, there exists a matrix N_j such that $\varphi(\mathfrak{G}(M_1)^{(1)}) = [\varphi(\mathfrak{G}(M_1)^{(1)}), \mathfrak{G}(N_j)]$ and $[\varphi(\mathfrak{G}(M_1)^{(1)}), \mathfrak{G}(N_k)] = \{0\}$ for $k \neq j$. Hence $\varphi(\mathfrak{G}(M_1)^{(1)}) \subseteq \mathfrak{G}(N_k)^{(1)}$. By the maximality of $\varphi(\mathfrak{G}(M_1)^{(1)})$ we can see that $\varphi(\mathfrak{G}(M_1)^{(1)}) = \mathfrak{G}(N_j)^{(1)}$. By Theorem 4.4 and Corollary 4.5, N_j is the same up to a permutation of rows and columns with M_1. In addition, $\varphi(\mathfrak{G}(X_1)/\mathfrak{G}(M_1)) \cong \mathfrak{G}(X_2)/\mathfrak{G}(N_j)$. By induction on s, we can deduce that $s = t$ and M_i $(i = 1, 2, \dots, s)$ is the same up to a permutation of rows and columns with some N_j. It follows that X_1 and X_2 are the same up to a permutation of rows and columns. This completes the proof of this theorem. □

5. The lowest weight modules over $\mathfrak{G}(A)$

In this section we study the lowest weight Verma modules $\bar{V}(\lambda)$ over $\mathfrak{G}(A)$ for any $\lambda \in \mathfrak{H}^*$. More precisely, we obtain the necessary and sufficient conditions for $\bar{V}(\lambda)$ to be an irreducible $\mathfrak{G}(A)$-module, and determine the maximal proper submodule of $\bar{V}(\lambda)$ when it is reducible.

For any linear function $\lambda \in \mathfrak{H}^*$, we define the 1-dimensional $(\mathfrak{H} \oplus \mathfrak{G}_-)$-module $\mathbb{C}\omega_\lambda$ via

$$\mathfrak{G}_- \cdot \omega_\lambda = 0; \quad x \cdot \omega_\lambda = \lambda(x)\omega_\lambda, \quad \text{if } x \in \mathfrak{H}. \tag{5.1}$$

Thus we have the induced $\mathfrak{G}(A)$-module

$$\bar{V}(\lambda) = \operatorname{Ind}_{\mathfrak{H}\oplus\mathfrak{G}_-}^{\mathfrak{G}(A)} \mathbb{C}\omega_\lambda = U\big(\mathfrak{G}(A)\big) \otimes_{U(\mathfrak{H}\oplus\mathfrak{G}_-)} \mathbb{C}\omega_\lambda. \tag{5.2}$$

This $\mathfrak{G}(A)$-module $\bar{V}(\lambda)$ is called a *lowest weight Verma module* with lowest weight λ.

Note that by (2.9), condition (5.2) can be replaced by

$$\bar{V}(\lambda) = U(\mathfrak{G}_+) \otimes \mathbb{C}\omega_\lambda. \tag{5.3}$$

It is clear that $\bar{V}(\lambda) \cong U(\mathfrak{G}_+)$ as vector spaces. Hence we have the decomposition of weight spaces

$$\bar{V}(\lambda) = \bigoplus_{\mu \in \mathfrak{H}^*} \bar{V}(\lambda)_\mu, \tag{5.4}$$

where $\bar{V}(\lambda)_\mu = \{v \in \bar{V}(\lambda) \mid x \cdot v = \mu(x)v,\ \forall x \in \mathfrak{H}\}$. It is clear that $\bar{V}(\lambda)_\mu = \{0\}$, for $\mu < \lambda$; $\dim \bar{V}(\lambda)_\mu < \infty$, for $\mu \geqslant \lambda$, and $\bar{V}(\lambda)_\lambda = \mathbb{C}\omega_\lambda$.

Since the weight space with weight λ is 1-dimensional, the module $\bar{V}(\lambda)$ has a unique maximal proper submodule J. Then we obtain the irreducible module

$$L(\lambda) = \bar{V}(\lambda)/J. \tag{5.5}$$

It is clear that $L(\lambda)$ is uniquely determined by the linear function λ.

Prior to our main result in this section, we first give a fact in linear algebra.

Lemma 5.1. *Let W and V be two vector spaces over $\mathbb{C}$ with $\dim W = \dim V = r$, and f any bilinear function on $W \times V$. Then for any fixed nonzero vector $x_1 \in W$, there exist a basis $x_1, x_2, \dots, x_r$ of W and a basis $y_1, y_2, \dots, y_r$ of V, such that*

$$f(x_i, y_j) = 0, \quad \text{for } i \neq j;\ i, j = 1, 2, \dots, r.$$

Proof. The result is obvious when f is trivial on $W \times V$, so we can assume that $f(W, V) \neq \{0\}$. We shall prove this lemma by induction on r. Clearly, it is true for $r = 1$.

Suppose that the lemma holds for $\dim W = \dim V < r$ ($r \geqslant 2$). Now consider the case that $\dim W = \dim V = r$. If $f(x_1, V) \neq \{0\}$, there exists some $y_1 \in V$ satisfying $f(x_1, y_1) \neq 0$. Then we can find the complementary subspace W_1 (respectively V_1) of $\mathbb{C}x_1$ (respectively $\mathbb{C}y_1$), such that $f(x_1, V_1) = \{0\}$ and $f(W_1, y_1) = \{0\}$. Since $\dim W_1 = \dim V_1 = r - 1$ and the restriction of f to $W_1 \times V_1$ is also a bilinear function, by inductive hypothesis, there exist a basis $x_2, \dots, x_r$ of W_1 and a basis $y_2, \dots, y_r$ of V_1, such that

$$f(x_i, y_j) = 0, \quad \text{for } i \neq j;\ i, j = 2, \dots, r.$$

Hence $x_1, x_2, \dots, x_r$ of W and $y_1, y_2, \dots, y_r$ of V are the bases desired.

If $f(x_1, V) = \{0\}$, then there exists some $y_1 \in V$ such that $f(W, y_1) = \{0\}$. Thus we can take any complementary subspace W_1 (respectively V_1) of $\mathbb{C}x_1$ (respectively $\mathbb{C}y_1$). Applying the induction, we obtain the result in this case. This completes the proof. □

Corollary 5.2. *Suppose $\alpha \in \Delta_+$, $\lambda \in \mathfrak{H}^*$. For any fixed nonzero $e_\alpha^{(1)} \in \mathfrak{G}_\alpha$, there exist a basis $e_\alpha^{(1)}, e_\alpha^{(2)}, \dots, e_\alpha^{(r)}$ of $\mathfrak{G}_\alpha$ and a basis $e_{-\alpha}^{(1)}, e_{-\alpha}^{(2)}, \dots, e_{-\alpha}^{(r)}$ of $\mathfrak{G}_{-\alpha}$, where $r = \dim \mathfrak{G}_\alpha = \dim \mathfrak{G}_{-\alpha}$, such that*

$$\lambda\big(\big[e_\alpha^{(i)}, e_{-\alpha}^{(j)}\big]\big) = 0, \quad \textit{for } i \neq j;\ i, j = 1, 2, \dots, r. \tag{5.6}$$

Proof. It is clear that $\lambda([\cdot,\cdot])$ is a bilinear function on $\mathfrak{G}_\alpha \times \mathfrak{G}_{-\alpha}$. Applying Lemma 5.1 we can easily obtain the corollary. □

Recall that for $\alpha = \sum_{i=1}^n k_i \alpha_i \in \Delta_+$, the positive integer $\operatorname{ht} \alpha := \sum_{i=1}^n k_i$ is called the *height* of α. For $\alpha = \sum_{i=1}^n k_i \alpha_i$, $\beta = \sum_{i=1}^n l_i \alpha_i \in \Delta_+$, we define a new order $\prec$ on Δ_+:

$$\alpha \prec \beta \quad \text{iff} \quad \operatorname{ht} \alpha < \operatorname{ht} \beta, \text{ or } \operatorname{ht} \alpha = \operatorname{ht} \beta \text{ and the first nonzero } l_i - k_i \text{ is positive.} \tag{5.7}$$

This is a total order on Δ_+. Note that $\alpha_n \prec \alpha_{n-1} \prec \cdots \prec \alpha_1$.

Since $\dim \mathfrak{G}_\alpha = \dim \mathfrak{G}_{-\alpha} < \infty$ for any $\alpha \in \Delta_+$, by Corollary 5.2, we can fix a basis $e_\alpha^{(1)}, e_\alpha^{(2)}, \dots, e_\alpha^{(r)}$ of $\mathfrak{G}_\alpha$ and a basis $e_{-\alpha}^{(1)}, e_{-\alpha}^{(2)}, \dots, e_{-\alpha}^{(r)}$ of $\mathfrak{G}_{-\alpha}$ such that $\lambda([e_\alpha^{(i)}, e_{-\alpha}^{(j)}]) = 0$, for $i \neq j$; $i, j = 1, 2, \dots, r$, where $r = \dim \mathfrak{G}_\alpha = \dim \mathfrak{G}_{-\alpha}$. Thus the order $\prec$ on Δ_+ can induce an order on the basis of $\mathfrak{G}_+$, still write as $\prec$,

$$e_\alpha^{(i)} \prec e_\beta^{(j)} \quad \text{iff} \quad \alpha \prec \beta, \text{ or } \alpha = \beta \text{ and } i < j. \tag{5.8}$$

Set

$$\mathfrak{B}_+ = \big\{e_\alpha^{(1)}, e_\alpha^{(2)}, \dots, e_\alpha^{(r)} \bigm| r = \dim \mathfrak{G}_\alpha,\ \alpha \in \Delta_+\big\}. \tag{5.9}$$

Then $\mathfrak{B}_+$ is an ordered basis of $\mathfrak{G}_+$. Hence, by PBW Theorem,

$$\big\{x_1^{j_1} x_2^{j_2} \cdots x_s^{j_s} \cdot \omega_\lambda \bigm| x_1 \prec x_2 \prec \cdots \prec x_s, x_i \in \mathfrak{B}_+,\ j_i \in \mathbb{Z}_+,\ 1 \leqslant i \leqslant s\big\} \tag{5.10}$$

forms a basis of $\bar{V}(\lambda)$.

Moreover, we can define an order on the above basis of $\bar{V}(\lambda)$. If $x_1 \prec x_2 \prec \cdots \prec x_s$, $x_i \in \mathfrak{B}_+$, define

$$x_1^{i_1} x_2^{i_2} \cdots x_s^{i_s} \cdot \omega_\lambda < x_1^{j_1} x_2^{j_2} \cdots x_s^{j_s} \cdot \omega_\lambda$$
$$\text{iff} \quad i_k = j_k, i_{k+1} = j_{k+1} \cdots, i_s = j_s, \text{ but } i_{k-1} < j_{k-1}, \text{ for some } k, 1 \leqslant k \leqslant s. \tag{5.11}$$

It is clear that ω_λ is the least nonzero element under the above order.

Theorem 5.3. *$\bar{V}(\lambda)$ is irreducible if and only if for each $\alpha \in \Delta_+$, the bilinear form $\lambda([\cdot,\cdot])|_{\mathfrak{G}_\alpha \times \mathfrak{G}_{-\alpha}}$ is non-degenerate.*

Proof. "$\Rightarrow$" Suppose that there exist $\alpha \in \Delta_+$ and $0 \neq e_\alpha \in \mathfrak{G}_\alpha$ such that $\lambda([e_\alpha, \mathfrak{G}_{-\alpha}]) = 0$. Let $U(\mathfrak{G}_+)e_\alpha$ be the left ideal of $U(\mathfrak{G}_+)$ generated by e_α. Denote

$$N = U(\mathfrak{G}_+)e_\alpha \cdot \omega_\lambda.$$

We shall prove that N is a nonzero proper submodule of $\bar{V}(\lambda)$.

First, applying Lemma 2.4 and Corollary 2.5, we deduce that

$$e_{-\beta} x_1^{i_1} x_2^{i_2} \cdots x_s^{i_s} \in U(\mathfrak{G}_+)\bar{\mathfrak{h}} + U(\mathfrak{G}_+)\mathfrak{G}_-, \quad \text{for } \beta \in \Delta_+,\ x_j \in \mathfrak{B}_+,\ i_j \in \mathbb{Z}_+.$$

In addition, from the fact that $\lambda([e_\alpha, e_{-\alpha}]) = 0$ for every $e_{-\alpha} \in \mathfrak{G}_{-\alpha}$, then for all $\beta \in \Delta_+$, $e_{-\beta} \in \mathfrak{G}_{-\beta}$,

$$e_{-\beta} e_\alpha \cdot \omega_\lambda = e_\alpha e_{-\beta} \cdot \omega_\lambda + [e_{-\beta}, e_\alpha] \cdot \omega_\lambda = -\delta_{\beta\alpha} \lambda\big([e_\alpha, e_{-\alpha}]\big) \cdot \omega_\lambda = 0.$$

Hence, for any $v = \sum_{i=1}^k a_i x_1^{i_1} x_2^{i_2} \cdots x_s^{i_s} e_\alpha \cdot \omega_\lambda \in N$, $\beta \in \Delta_+$, we have

$$\begin{aligned} e_{-\beta} \cdot v &= \sum_{i=1}^k a_i e_{-\beta} x_1^{i_1} x_2^{i_2} \cdots x_s^{i_s} e_\alpha \cdot \omega_\lambda \in \sum_{j=1}^s U(\mathfrak{G}_+)\bar{\mathfrak{h}} e_\alpha \cdot \omega_\lambda + U(\mathfrak{G}_+)\mathfrak{G}_- e_\alpha \cdot \omega_\lambda \\ &\subseteq U(\mathfrak{G}_+)e_\alpha \cdot \omega_\lambda = N. \end{aligned}$$

It is clear that $xN \subseteq N$ for any $x \in \mathfrak{G}_+ + \mathfrak{H}$. Since $\omega_\lambda \notin N$, hence N is a proper submodule of $\bar{V}(\lambda)$. This direction follows.

"$\Leftarrow$" Suppose that for each $0 \neq e_\alpha \in \mathfrak{G}_\alpha$, $\alpha \in \Delta_+$, there exists some $e_{-\alpha} \in \mathfrak{G}_{-\alpha}$ such that $\lambda([e_\alpha, e_{-\alpha}]) \neq 0$, and that N is a nonzero submodule of $\bar{V}(\lambda)$. Any nonzero element $v \in N$ can be written as $v = \sum_{i=1}^k a_i x_1^{i_1} x_2^{i_2} \cdots x_s^{i_s} \cdot \omega_\lambda$ satisfying $(x_1^{1_1} x_2^{1_2} \cdots x_s^{1_s} \cdot \omega_\lambda) < \cdots < (x_1^{k_1} x_2^{k_2} \cdots x_s^{k_s} \cdot \omega_\lambda)$ under the order defined in (5.11), where $a_i \in \mathbb{C} \setminus \{0\}$ and $x_i \in \mathfrak{B}_+$ with $x_1 \prec x_2 \prec \cdots \prec x_s$. Let us call $x_1^{k_1} x_2^{k_2} \cdots x_s^{k_s} \cdot \omega_\lambda$ the leading term of v.

We may choose $v \in N$ such that its leading term is minimum among the elements in N. If the leading term of v is ω_λ, then $k = 1$ and $v \in \mathbb{C}\omega_\lambda$. Therefore $\omega_\lambda \in N$ and $N = \bar{V}(\lambda)$.

If the leading term of v is greater than ω_λ, there exists some i $(1 \leqslant i \leqslant k)$ with $i_1 \neq 0$. Assume that $x_1 \in \mathfrak{G}_\alpha$. From Lemma 2.4 and Corollary 5.2, there exists some $y \in \mathfrak{G}_{-\alpha}$ such that $\lambda([x_1, y]) \neq 0$, $\lambda([x_i, y]) = 0$ if $x_i \in \mathfrak{G}_\alpha$ with $i > 1$, and $[x_i, y] = 0$ if $x_i \notin \mathfrak{G}_\alpha$. So we deduce that $0 \neq y \cdot v = \sum_{i=1}^k i_1 \lambda([y, x_1]) x_1^{i_1 - 1} x_2^{i_2} \cdots x_s^{i_s} \cdot \bar{\omega}_\lambda \in N$ whose leading term is clearly less than that of v. This contradicts the choice of v. Hence $N = \bar{V}(\lambda)$. This completes the proof of this theorem. $\square$

Now we can further determine the maximal submodule of $\bar{V}(\lambda)$.

For any $\alpha \in \Delta_+$ and $\lambda \in \mathfrak{H}^*$, by Corollary 5.2, we can fix a basis $e_\alpha^{(1)}, e_\alpha^{(2)}, \dots, e_\alpha^{(r)}$ of $\mathfrak{G}_\alpha$ and a basis $e_{-\alpha}^{(1)}, e_{-\alpha}^{(2)}, \dots, e_{-\alpha}^{(r)}$ of $\mathfrak{G}_{-\alpha}$ such that

$$\lambda\big(\big[e_\alpha^{(i)}, e_{-\alpha}^{(j)}\big]\big) = \delta_{ij}, \quad \text{for } i, j = 1, 2, \dots, t \ (0 \leqslant t \leqslant r), \text{ and}$$
$$\lambda\big(\big[e_\alpha^{(k)}, \mathfrak{G}_{-\alpha}\big]\big) = \lambda\big(\big[\mathfrak{G}_\alpha, e_{-\alpha}^{(k)}\big]\big) = 0, \quad t+1 \leqslant k \leqslant r,$$

where $r = \dim \mathfrak{G}_\alpha = \dim \mathfrak{G}_{-\alpha}$. Denote

$$\mathfrak{B}_\alpha^{(1)}(\lambda) = \big\{e_\alpha^{(1)}, \dots, e_\alpha^{(t)}\big\}, \qquad \mathfrak{B}_\alpha^{(2)}(\lambda) = \big\{e_\alpha^{(t+1)}, \dots, e_\alpha^{(r)}\big\} \quad \text{and}$$
$$\mathfrak{B}_\alpha(\lambda) = \mathfrak{B}_\alpha^{(1)}(\lambda) \cup \mathfrak{B}_\alpha^{(2)}(\lambda).$$

Thus $\mathfrak{B}_\alpha(\lambda)$ is a basis of $\mathfrak{G}_\alpha$. Put

$$\mathfrak{B}_+^{(1)}(\lambda) = \bigcup_{\alpha \in \Delta_+} \mathfrak{B}_\alpha^{(1)}(\lambda) \quad \text{and} \quad \mathfrak{B}_+^{(2)}(\lambda) = \bigcup_{\alpha \in \Delta_+} \mathfrak{B}_\alpha^{(2)}(\lambda).$$

Then the basis of $\mathfrak{G}_+$ given in (5.9) can be written as

$$\mathfrak{B}_+(\lambda) = \mathfrak{B}_+^{(1)}(\lambda) \cup \mathfrak{B}_+^{(2)}(\lambda). \tag{5.12}$$

From the proof of the previous theorem we see that

$$J = \sum_{\substack{e_\alpha \in \mathfrak{G}_\alpha,\ \alpha \in \Delta_+ \\ \lambda([e_\alpha,\ \mathfrak{G}_{-\alpha}])=0}} U(\mathfrak{G}_+)e_\alpha \cdot \omega_\lambda \tag{5.13}$$

is a submodule of $\bar{V}(\lambda)$. We also know that if $\lambda([e_\alpha,\ \mathfrak{G}_{-\alpha}]) = 0$ for some $e_\alpha \in \mathfrak{G}_\alpha$, then e_α can be written as a linear combination of elements in $\mathfrak{B}_\alpha^{(2)}(\lambda)$. Hence the submodule in (5.13) can be simplified as

$$J = \sum_{\alpha \in \Delta_+,\ x \in \mathfrak{B}_\alpha^{(2)}(\lambda)} U(\mathfrak{G}_+)x \cdot \omega_\lambda = \sum_{x \in \mathfrak{B}_+^{(2)}(\lambda)} U(\mathfrak{G}_+)x \cdot \omega_\lambda. \tag{5.14}$$

In the following theorem, we shall prove that J is maximal.

Theorem 5.4. *Let $\lambda \in \mathfrak{H}^*$. Then*

$$J = \sum_{x \in \mathfrak{B}_+^{(2)}(\lambda)} U(\mathfrak{G}_+)x \cdot \omega_\lambda \tag{5.15}$$

is the maximal submodule of $\bar{V}(\lambda)$.

Proof. We only need to prove that J is maximal. Assume that N is a submodule of $\bar{V}(\lambda)$ which contains J properly. By PBW Theorem, every element $v \in \bar{V}(\lambda)$ can be written as

$$v = \sum_{i=1}^{k} a_i x_1^{i_1} \cdots x_t^{i_t} x_{t+1}^{i_{t+1}} \cdots x_s^{i_s} \cdot \omega_\lambda,$$

where $x_i \in \mathfrak{B}_+^{(1)}(\lambda)$, $1 \leqslant i \leqslant t$; $x_j \in \mathfrak{B}_+^{(2)}(\lambda)$, $t+1 \leqslant j \leqslant s$; $a_i \in \mathbb{C} \setminus \{0\}$, $1 \leqslant i \leqslant k$. Then we have a nonzero vector $v = \sum_{x_i \in \mathfrak{B}_+^{(1)}(\lambda)} a_i x_1^{i_1} \cdots x_t^{i_t} \cdot \omega_\lambda \in N$ where $a_i \in \mathbb{C} \setminus \{0\}$, $i_j \in \mathbb{Z}_+$. As in the proof of "$\Leftarrow$" of the previous theorem, we can deduce that $N = \bar{V}(\lambda)$. Therefore J is the maximal submodule of $\bar{V}(\lambda)$. This completes the proof. $\square$

Corollary 5.5. *If $\lambda \in \mathfrak{H}^*$ with $\lambda|_{\bar{\mathfrak{h}}} = 0$, then $\bar{V}(\lambda)$ is reducible, and has the maximal proper submodule J such that $\bar{V}(\lambda)/J$ is the 1-dimensional trivial module.*

6. The highest weight modules over $\mathfrak{G}(A)$

In this section we shall study highest weight Verma modules $\tilde{V}(\lambda)$ over $\mathfrak{G}(A)$ for any $\lambda \in \mathfrak{H}^*$, and employ the graded dual module of the irreducible lowest weight module $L(\lambda)$ to get the necessary and sufficient conditions for $\tilde{V}(\lambda)$ to be irreducible.

For any linear function $\lambda \in \mathfrak{H}^*$, we define the 1-dimensional $(\mathfrak{H} \oplus \mathfrak{G}_+)$-module $\mathbb{C}\tilde{\omega}_\lambda$ via

$$\mathfrak{G}_+ \cdot \tilde{\omega}_\lambda = 0; \quad x \cdot \tilde{\omega}_\lambda = \lambda(x)\tilde{\omega}_\lambda, \quad \text{if } x \in \mathfrak{H}. \tag{6.1}$$

Then the *highest weight Verma module with highest weight* λ is defined as

$$\tilde{V}(\lambda) = \operatorname{Ind}_{\mathfrak{H}\oplus\mathfrak{G}_+}^{\mathfrak{G}(A)} \mathbb{C}\tilde{\omega}_\lambda = U\big(\mathfrak{G}(A)\big) \otimes_{U(\mathfrak{H}\oplus\mathfrak{G}_+)} \mathbb{C}\tilde{\omega}_\lambda. \tag{6.2}$$

It is clear that $\tilde{V}(\lambda) \cong U(\mathfrak{G}_-)$ as vector spaces. Hence we also have the decomposition of weight spaces

$$\tilde{V}(\lambda) = \bigoplus_{\mu \in \mathfrak{H}^*} \tilde{V}(\lambda)_\mu, \tag{6.3}$$

where $\tilde{V}(\lambda)_\mu = \{v \in \tilde{V}(\lambda) \mid x \cdot v = \mu(x)v,\ \forall x \in \mathfrak{H}\}$. It is clear that $\tilde{V}(\lambda)_\mu = \{0\}$, for $\mu > \lambda$; $\dim \tilde{V}(\lambda)_\mu < \infty$, for $\mu \leqslant \lambda$, and $\tilde{V}(\lambda)_\lambda = \mathbb{C}\tilde{\omega}_\lambda$.

Since the weight space with weight λ is 1-dimensional, the module $\tilde{V}(\lambda)$ has a unique maximal proper submodule J. Then we obtain the irreducible module

$$V(\lambda) = \tilde{V}(\lambda)/J. \tag{6.4}$$

It is clear that $V(\lambda)$ is uniquely determined by the linear function λ.

For any $\alpha \in \Delta_+$, since $\dim \mathfrak{G}_{-\alpha} < \infty$, we can fix a basis $e_{-\alpha}^{(1)}, e_{-\alpha}^{(2)}, \ldots, e_{-\alpha}^{(r)}$ of $\mathfrak{G}_{-\alpha}$, where $r = \dim \mathfrak{G}_{-\alpha}$. Thus the order $\prec$ on Δ_+ also induces an order on the basis of $\mathfrak{G}_-$, still write as $\prec$,

$$e_{-\alpha}^{(i)} \prec e_{-\beta}^{(j)} \quad \text{iff} \quad \alpha \prec \beta, \text{ or } \alpha = \beta \text{ and } i < j. \tag{6.5}$$

Set

$$\mathfrak{B}_- = \{e_{-\alpha}^{(1)}, e_{-\alpha}^{(2)}, \ldots, e_{-\alpha}^{(r)} \mid r = \dim \mathfrak{G}_{-\alpha},\ \alpha \in \Delta_+\}. \tag{6.6}$$

Then $\mathfrak{B}_-$ is an ordered basis of $\mathfrak{G}_-$. Since $\mathfrak{G}_-$ is an abelian subalgebra of $\mathfrak{G}(A)$, $U(\mathfrak{G}_-)$ is isomorphic to polynomial algebra with infinitely many indeterminates $e_{-\alpha}^{(1)}, e_{-\alpha}^{(2)}, \ldots, e_{-\alpha}^{(r)}$, $r = \dim \mathfrak{G}_{-\alpha}$, $\alpha \in \Delta_+$. Hence

$$\{y_1^{j_1} y_2^{j_2} \cdots y_s^{j_s} \cdot \tilde{\omega}_\lambda \mid y_1 \prec y_2 \prec \cdots \prec y_s,\ y_i \in \mathfrak{B}_-,\ j_i \in \mathbb{Z}_+,\ 1 \leqslant i \leqslant s\} \tag{6.7}$$

forms a basis of $\tilde{V}(\lambda)$.

Unlike Theorem 5.3, we are not able to give a direct proof of a result similar to that. We have to employ the graded dual module of $L(\lambda)$, where $L(\lambda)$ is the irreducible lowest weight module with lowest weight λ defined in (5.5). The dual space $L(\lambda)^*$ of $L(\lambda)$ can be made into a $\mathfrak{G}(A)$-module under the following action

$$(x \cdot f)(v) = -f(x \cdot v), \quad \text{for all } f \in L(\lambda)^*,\ x \in \mathfrak{G}(A) \text{ and } v \in L(\lambda). \tag{6.8}$$

In general, we call $L(\lambda)^*$ the $\mathfrak{G}(A)$-module contragredient to $L(\lambda)$.

Recall that we have the weight space decomposition of $L(\lambda)$:

$$L(\lambda) = \bigoplus_{\mu \geqslant \lambda} L(\lambda)_\mu, \tag{6.9}$$

with $L(\lambda)_\lambda = \mathbb{C}\omega_\lambda$ and $\dim L(\lambda)_\mu < \infty$. Denote by $P(\lambda) =: \{\mu \in \mathfrak{H}^* \mid L(\lambda)_\mu \neq \{0\}\}$ the weight set of $L(\lambda)$. For each $\mu \in P(\lambda)$, we can identify the dual space $L(\lambda)_\mu^*$ of $L(\lambda)_\mu$ with $\{f \in L(\lambda)^* \mid f(L(\lambda)_\nu) = 0,\ \text{for all } \nu \neq \mu\}$. Then the subspace

$$L^*(\lambda) = \bigoplus_{\mu \geqslant \lambda} L(\lambda)_\mu^* \tag{6.10}$$

of $L(\lambda)^*$ is a weight submodule over $\mathfrak{G}(A)$. We call $L^*(\lambda)$ the *graded dual module* of $L(\lambda)$.

It is straightforward to check that $L^*(\lambda)$ is irreducible. In addition, for any $0 \neq \omega_\lambda^* \in L(\lambda)_\lambda^*$, one has

$$\mathfrak{G}_+ \cdot \omega_\lambda^* = 0; \qquad x \cdot \omega_\lambda^* = -\lambda(x)\omega_\lambda^*, \quad \text{for } x \in \mathfrak{H}. \tag{6.11}$$

Hence $L^*(\lambda)$ is an irreducible highest weight module over $\mathfrak{G}(A)$ with highest weight $-\lambda$. By the universal property of Verma module, there exists an epimorphism from $\tilde{V}(-\lambda)$ onto $L^*(\lambda)$. We deduce that $\tilde{V}(-\lambda)$ is irreducible iff $L^*(\lambda) \cong \tilde{V}(-\lambda)$. Comparing the dimension of weight spaces of these two modules, since $\dim \mathfrak{G}_\alpha = \dim \mathfrak{G}_{-\alpha}$, we see that $\dim \tilde{V}(-\lambda)_{-\lambda-\alpha} = \dim \bar{V}(\lambda)_{\lambda+\alpha}$ for any $\alpha \in Q_+$. It follows that $\tilde{V}(-\lambda)$ is irreducible iff $\bar{V}(\lambda)$ is irreducible. Hence we have proved the following theorem:

Theorem 6.1. *$\tilde{V}(\lambda)$ is irreducible if and only if $\bar{V}(-\lambda)$ is irreducible, if and only if for each $\alpha \in \Delta_+$, the bilinear form $\lambda([\cdot,\cdot])|_{\mathfrak{G}_\alpha \times \mathfrak{G}_{-\alpha}}$ is non-degenerate.*

Question. What is the maximal proper submodule of $\tilde{V}(\lambda)$ when it is reducible?

7. The case that A is a symmetrizable GCM

In this section, we assume that A is an $n \times n$ symmetrizable GCM.

We use the construction in the proof of Theorem 3.2. For any fixed non-degenerate symmetric invariant bilinear form $(\cdot,\cdot)_0$ on $\mathfrak{g}(A)$, we have a non-degenerate symmetric invariant bilinear form $(\cdot,\cdot)$ on $\mathfrak{G}(A)$ such that

$$(x, y)_0 = \big(x, \sigma(y)\big), \quad \forall x \in \mathfrak{g}_\alpha,\ y \in \mathfrak{g}_{-\beta},\ \alpha, \beta \in \Delta_+; \tag{7.1}$$

$$(h_1, h_2)_0 = \big(h_1, \sigma(h_2)\big), \quad \forall h_1, h_2 \in \mathfrak{h}. \tag{7.2}$$

Clearly, for any $x \in \mathfrak{H}$, we can uniquely determine an element in $\mathfrak{H}^*$, denote by $\mu(x)$, such that

$$\big\langle y, \mu(x)\big\rangle = (y, x), \quad \forall y \in \mathfrak{H}. \tag{7.3}$$

Thus we have a vector space isomorphism

$$\mu : \mathfrak{H} \to \mathfrak{H}^*, \quad x \mapsto \mu(x). \tag{7.4}$$

Now μ induces in a natural way a non-degenerate bilinear form on $\mathfrak{H}^*$, still denote by $(\cdot,\cdot)$, satisfying

$$(\lambda_1, \lambda_2) = \big(\mu^{-1}(\lambda_1), \mu^{-1}(\lambda_2)\big), \quad \forall \lambda_1, \lambda_2 \in \mathfrak{H}^*. \tag{7.5}$$

It is clear that for any $\alpha, \beta \in \Delta$, we have

$$(\alpha, \beta) = \big(\mu^{-1}(\alpha), \mu^{-1}(\beta)\big) = 0. \tag{7.6}$$

This is very different from the case of Kac–Moody algebras (see Chapter 2 in [5]).

Similar to the Kac–Moody algebra case, we have the following lemma.

Lemma 7.1. *Let A be a symmetrizable GCM, $\mathfrak{G}(A)$ the deformed Kac–Moody algebra associated to A, $(\cdot,\cdot)$ a non-degenerate symmetric invariant bilinear form on $\mathfrak{G}(A)$. Then for any $\alpha \in \Delta$, $x \in \mathfrak{G}_\alpha$, $y \in \mathfrak{G}_{-\alpha}$, we have*

$$[x, y] = (x, y)\mu^{-1}(\alpha). \tag{7.7}$$

Proof. It is clear that two sides of the above identity are in $\mathfrak{H}$. Hence we only need to check that

$$\big(h + z, [x, y]\big) = \big(h + z, (x, y)\mu^{-1}(\alpha)\big), \quad \forall h \in \mathfrak{h}, z \in \bar{\mathfrak{h}}.$$

We have

$$\big(h + z, [x, y]\big) = \big([h + z, x], y\big) = \alpha(h)(x, y)$$

and

$$\big(h + z, (x, y)\mu^{-1}(\alpha)\big) = (x, y)\big(h + z, \mu^{-1}(\alpha)\big) = \alpha(h)(x, y). \qquad \square$$

It is clear that $(\cdot,\cdot)|_{\mathfrak{G}_\alpha\times\mathfrak{G}_{-\alpha}}$ is non-degenerate and $(\cdot,\cdot)|_{\mathfrak{G}_\alpha\times\mathfrak{G}_{-\beta}}$ is trivial if $\alpha,\beta\in\Delta$ with $\alpha+\beta\neq 0$. For any $\alpha=\sum_{i=1}^n k_i\alpha_i\in\Delta_+$, we can get a basis $e_\alpha^{(1)}, e_\alpha^{(2)},\dots,e_\alpha^{(r)}$ of $\mathfrak{G}_\alpha$ and a basis $e_{-\alpha}^{(1)}, e_{-\alpha}^{(2)},\dots,e_{-\alpha}^{(r)}$ of $\mathfrak{G}_{-\alpha}$, where $r=\dim\mathfrak{G}_\alpha=\dim\mathfrak{G}_{-\alpha}$, such that

$$\big(e_\alpha^{(i)}, e_{-\alpha}^{(j)}\big)=\delta_{ij},\quad i,j=1,2,\dots,r. \tag{7.8}$$

Thus, by the previous lemma,

$$\big[e_\alpha^{(i)}, e_{-\alpha}^{(j)}\big]=\delta_{ij}\mu^{-1}(\alpha)\in\bar{\mathfrak{h}},\quad i,j=1,2,\dots,r. \tag{7.9}$$

We see that $\lambda([\mathfrak{G}_\alpha,\mathfrak{G}_{-\alpha}])=\mathbb{C}\lambda(\mu^{-1}(\alpha))=\mathbb{C}(\lambda,\alpha)$.

Hence we can simplify the theorems in the previous two sections.

Theorem 7.2. *Let A be a symmetrizable GCM, $\mathfrak{G}(A)$ the deformed Kac–Moody algebras associated to A, $\lambda\in\mathfrak{H}^*$.*

(a) *The lowest weight Verma module $\bar{V}(\lambda)$ over $\mathfrak{G}(A)$ is irreducible if and only if for every $\alpha\in\Delta_+$, $(\lambda,\alpha)=\lambda(\mu^{-1}(\alpha))\neq 0$. If $\bar{V}(\lambda)$ is reducible, then the maximal proper submodule of $\bar{V}(\lambda)$ is*

$$J=\sum_{(\lambda,\alpha)=0,\ \alpha\in\Delta_+} U(\mathfrak{G}_+)\mathfrak{G}_\alpha\omega_\lambda.$$

(b) *The highest weight Verma module $\tilde{V}(\lambda)$ over $\mathfrak{G}(A)$ is irreducible if and only if for every $\alpha\in\Delta_+$, $(\lambda,\alpha)=\lambda(\mu^{-1}(\alpha))\neq 0$.*

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